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Research article

Decision-making in diagnosing heart failure problems using basic rough sets

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Abstract: This manuscript introduces novel rough approximation operators inspired by topological structures, which offer a more flexible approach than existing methods by extending the scope of applications through a reliance on a general binary relation without constraints. Initially, four distinct types of neighborhoods, termed basic-minimal neighborhoods, are generated from any binary relation. The relationships between these neighborhoods and their properties are elucidated. Subsequently, new rough set models are constructed from these neighborhoods, outlining the main characteristics of their lower and upper approximations. These approximations are applied to classify the subset regions and to compute the accuracy measures. The primary advantages of this approach include its ability to achieve the highest accuracy values compared to all approaches in the published literature and to maintain the monotonicity property of the accuracy and roughness measures. Furthermore, the efficacy of the proposed technique is demonstrated through the analysis of heart failure diagnosis data, showcasing a 100% accuracy rate compared to previous methods, thus highlighting its clinical significance. Additionally, the topological properties of the proposed approaches and the topologies generated from the suggested neighborhoods are discussed, positioning these methods as a bridge to more topological applications in the rough set theory. Finally, an algorithm and flowchart are developed to illustrate the determination and utilization of basic-minimal exact sets in decision-making problems.

Keywords: basic minimal-neighborhoods; rough sets; topology; heart failure problems
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1. Introduction

The rough set (RS) theory and its extensions have garnered increasing attention, particularly in computer science, artificial intelligence, and medical applications. Though other uncertainty theories, such as fuzzy or grey, and hybrid theories such as fuzzy-rough, have their merits, RS offers distinct advantages. Unlike the fuzzy theory, which relies on membership degrees, RS handles incomplete knowledge by classifying objects based on equivalence relations, thus allowing for the determination of information completeness within a set. Similarly, the RS provides a structured approach to handle uncertainty, which distinguishes it from the grey theory. Moreover, the RS theory's flexibility in dealing with imprecise and uncertain data sets outshines hybrid theories such as fuzzy-rough, which may face challenges in balancing complexity and interpretability. By focusing on these theories, we aim to underscore the novelty and significance of our research. We provide a comparative analysis in Table 1, outlining the advantages and limitations of the RS theory against other uncertainty theories, thereby emphasizing the unique contributions and potential applications of our proposed approach.

Table 1. Comparison of advantages and limitations of rough set theory with other uncertainty theories and hybrid theories like fuzzy-rough.

Theory	Advantages	Limitations
Rough set (RS)	Handles incomplete knowledge effectively	Strict requirement of equivalence relations
Fuzzy theory	Deals with membership degrees	May struggle with handling uncertainty
Grey theory	Addresses uncertainty effectively	Limited flexibility in data representation
Fuzzy-rough	Integrates fuzzy and rough set concepts	Complexity may hinder interpretability

Pawlak's pioneering work in 1982 [1,2] established RS theory as an effective tool to handle incomplete knowledge by classifying objects based on equivalence relations, thus allowing for the determination of information completeness within a set. The fundamental principles of this theory include approximation operators and accuracy measures, which provide crucial insights to decision-makers regarding the structures and sizes of the boundary regions. However, the strict requirement of an equivalence relation limits the applicability of the traditional RS theory. These limitations have driven researchers to propose various generalizations that use either arbitrary or specific relations to broaden the scope of the theory. Yao [3] initiated this line of research in 1996, which led to the emergence of numerous proposals that present generalized rough sets, such as tolerance [4], similarity [5,6], quasi-order [7], and general relations [8–10]. These advancements have significantly expanded the applicability of the RS theory.

In 2014, Abd El-Monsef et al. introduced the concept of the k-neighborhood space (k-$\boldsymbol{NS}$), which provided a generalized framework derived from binary relations [11]. This expansion extended Pawlak's model by incorporating various induced topologies. Subsequently, many researchers used these models to increase the applications of topology in RS's, resulting in various generalizations of this theory from a topological perspective. For instance, in 2020, Nawar et al. [12] applied the k-$\boldsymbol{NS}$ concept to develop and establish the concept of adhesion neighborhoods within the context of generalized covering approximation spaces [13], thereby building upon the concept of adhesion sets [14].

Additionally, in the same year, Atef et al. [15] developed the k-$\boldsymbol{NS}$ concept to introduce eight generalized types of neighborhoods (k-adhesion neighborhoods), thereby proposing different models to generalize Pawlak's theory. However, El-Bably et al. identified numerous errors in this research and provided corrections and significant results in [16]. In another trajectory of k-neighborhood exploration, El-Bably and Al-shami introduced core minimal-neighborhoods [17], extending Pawlak RSs into various generalized forms. They applied these forms in a significant medical context related to lung cancer diseases. The topologies derived from Abd El-Monsef et al.'s method have enabled diverse topological applications in RS's approaches, particularly in medicine [18–20] and economics [21,22]. In 2022, El-Bably et al. [23] explored new generalized closure spaces via binary relations using the k-$\boldsymbol{NS}$ concept, leading to new RS formulations and an enhanced granularity, significantly impacting real-life applications using soft RS's methodologies [24–28]. Several subsequent works, explored fuzzy RS's approaches [29,30], ideal structures [31], and corrections to prior methodologies [32], have expanded on these foundations, employing topological properties to define RS's methodologies and their medical applications [33–37]. These explorations underscore the potential of k-$\boldsymbol{NS}$ in advancing neighborhood-based concepts.

Despite these advancements, gaps remain in fully leveraging topological concepts within the RS theory. While powerful, traditional RS methods face limitations due to the stringent requirements of equivalence relations. Researchers have sought to address these limitations through generalizations and by introducing new structures. For instance, the pursuit of more general results and valid solutions has led to the development of structures such as Aczel-Alsina power Bonferroni aggregation operators for picture fuzzy information and decision analyses [38], hesitant fuzzy linguistic multigranulation decision-theoretic RS's [39], generalized Z-fuzzy soft β-covering [40], rough neutrosophic matrices [41], and rough set-based bipolar approaches [42].

Building on this trajectory and the models of Abd El-Monsef et al. (k-$\boldsymbol{NS}$), this study introduces new rough approximation operators inspired by topological structures to enhance the applicability of (RS) models inspired by general binary relations, free from constraints but inherently topological in nature. These models aim to bridge rough set theory and topology, addressing critical challenges and broadening the applicability of RS theory. The proposed methods are based on the concept of "basic-minimal neighborhoods", which extends the idea of "basic neighborhoods". The notion of "basic-neighborhoods" was initially introduced by Abu-Gdairi et al. [9] as a counterpart to "initial neighborhoods" [19] that proposed in 2021. Additionally, in the same year, a related idea referred to as "containment neighborhoods" was described in [43], where rough sets were applied within the framework of the k-NS model developed by Abd El-Monsef et al. [11]. However, the study in [43] did not thoroughly examine the associated topological properties or provide a detailed construction of corresponding topologies.

Subsequently, El-Gayar et al. [21] expanded the concept of "basic-neighborhoods" by defining four distinct types of neighborhoods, analyzing their topological properties, and proposing methods for constructing associated topologies. These advancements were also applied in economic decision-making contexts. Moreover, the term “initial-neighborhoods” was redefined as "subset neighborhoods" in 2022, as noted in [44].

The key motivations for exploring RS models from a topological perspective are as follows:

1) To alleviate some conditions imposed on the topological RS models, thereby expanding their applications;
2) To preserve most of Pawlak's properties of approximation operators, which were often lost in

previous topologically derived methods [3,6,8,9,11];

3) To ensure that the values of accuracy and roughness satisfy the monotonic property, making the approach more suitable to analyze large samples; and
4) To demonstrate that the approximation operators obtained are superior to those defined by either the topological structures or binary relations, as well as the models presented in the literature in aiding decision-making for medical diagnoses and other applications.

Initially, we introduce four new types of neighborhoods, which are extensions of other neighborhoods, such as those in the method by Abd El-Monsef et al. in [11]. These neighborhoods are called "basic-minimal neighborhoods" and are fundamentally based on the concept of "basic-neighborhoods" introduced by Abu-Gdairi et al. in [9]. Then, we examine the properties of these neighborhoods and their relationships with other types. Based on these neighborhoods, we propose four different approximation models, study their fundamental properties and mutual relationships, and identify the best and strongest among them in terms of the highest accuracy factor. These approximations are compared with the previous methods mentioned in the references through counterexamples and theoretical proofs, demonstrating their accuracy and robustness. It is worth noting that the proposed models can be compared with recent advancements, including ternary models [39,45], thus highlighting improvements in the accuracy and generalization.

One of the principal contributions of this paper is the introduction of topological structures for these approximations, thus linking the approximation theory to topology and its applications. This connection facilitates further applications of topological concepts within the RS theory. We discuss and study methods to generate different topologies from the neighborhoods (basic-minimal neighborhoods) and prove that the basic-minimal lower and basic-minimal upper approximations represent the interior and closure operators of these topologies. Therefore, we initiate a new bridge to apply more topological concepts using the suggested approaches in the RS theory.

In the realm of medical diagnostics, particularly in the context of heart failure diagnoses, our primary focus revolves around the development of an accurate diagnostic methodology. This endeavor finds its application in the medical field, thereby leveraging data gathered from a study that involved 12 patients conducted at Al-Azhar University's Cardiology Department, within the premises of Sayed Galal University Hospital in Egypt [46]. Through this application, we demonstrate the effectiveness of our proposed methodology, showcasing a remarkable 100% accuracy coefficient, seamlessly aligning with the diagnoses made by physicians as documented in the dataset. In stark contrast, previous methods have faltered in delivering precise diagnoses for this condition. Our work signifies a notable breakthrough in mathematical modeling. Not only does it bolster the accuracy of decision-making processes, but it also furnishes a comprehensive framework for deciphering medical data pertinent to heart failure diagnoses. By implementing our methodology on the provided dataset, we achieve results that mirror the diagnoses rendered by medical professionals, thus accurately discerning patients with heart failure from their healthy counterparts. While other methods failed to accurately identify the infected patients from healthy ones, this reflects the superiority of our methods in medical diagnoses. Hence, it is evident that the methodologies elucidated in this paper hold promise for revolutionizing medical diagnostics, potentially streamlining processes, and conserving invaluable time and resources for patients and healthcare providers alike.

The rest of this paper is organized as follows.

Section 2 discusses the fundamental principles and results of the RS's and their generalizations.

Section 3 introduces the new concept of "basic-minimal neighborhoods", detailing their properties and interrelationships.

Section 4 is the principal section of this paper, presenting the primary contributions by introducing four distinct RS approximation approaches, referred to as basic-minimal approximations, utilizing these neighborhoods. This section is structured into three subsections:

Subsection 4.1 proposes four different RS approximations. Their fundamental properties are analyzed, demonstrating that they satisfy Pawlak's core axioms without any constraints or conditions. Additionally, the relationships among these approximations are examined, and the best method is identified based on the higher accuracy factor.

Subsection 4.2 explores the generation of different topologies from basic-minimal neighborhoods and proves that the approximations introduced in 4.1 represent the closure and interior operators for these topologies. This establishes connections between the RS theory and topology, thereby facilitating further topological applications. Moreover, these topological structures and their properties are studied.

Subsection 4.3 compares the proposed approximations with previous ones, such as those by Yao [3], Dai et al. [6], Allam et al. [8], and Abd El-Monsef et al. [11]. This comparison demonstrates the superiority of the proposed methods over the previous approaches through counterexamples and proven theorems in various specific and general cases.

Section 5 investigates the effectiveness of our approach to analyze heart failure data, showcasing its clinical significance. Additionally, an algorithm and flowchart are developed to illustrate how basic-minimal exact sets are determined and used in decision-making problems.

Section 6 presents the conclusions, strengths, and advantages of the proposed approaches and discusses potential future research directions.

2. Basic concepts

In this section, we review the principles and results related to RS's concepts and k-$\boldsymbol{NS}$ that are essential to understand the context of this manuscript. Additionally, we discuss the historical development of several previous approaches and the motivations behind their study. Additionally, we provide proofs for some key results and properties of these approaches.

2.1. Abd El-Monsef et al. approaches

Definition 2.1. [3,8,11] The k-neighborhood of $\wp \in \mho$, indicated by $\mathbb{N}_k(\wp)$ for all $k \in \mathcal{K}$, induced by a binary relation $\mathcal{R}$ on a non-empty finite set $\mho$, where $\mathcal{K} = \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, is given by the following:

a. r-neighborhood [3]: $\mathbb{N}_r(\wp) = \{q \in \mho : \wp \mathcal{R} q\}$;

b. l-neighborhood [3]: $\mathbb{N}_l(\wp) = \{q \in \mho : q \mathcal{R} \wp\}$;

c. $\langle r\rangle$-neighborhood [8]: $\mathbb{N}_{\langle r\rangle}(\wp) = \begin{cases} \bigcap_{\wp \in \mathbb{N}_r(q)} \mathbb{N}_r(q), & \text{if } \wp \in \mathbb{N}_r(q) \\ \Phi, & \text{Otherwise} \end{cases}$;

d. $\langle l\rangle$-neighborhood[8]: $\mathbb{N}_{\langle l\rangle}(\wp) = \begin{cases} \bigcap_{\wp \in \mathbb{N}_l(q)} \mathbb{N}_l(q), & \text{if } \wp \in \mathbb{N}_l(q) \\ \Phi, & \text{Otherwise} \end{cases}$;

e. i-neighborhood [11]: $\mathbb{N}_i(\wp) = \mathbb{N}_r(\wp) \cap \mathbb{N}_l(\wp)$;

f. u-neighborhood [11]: $\mathbb{N}_u(\wp) = \mathbb{N}_r(\wp) \cup \mathbb{N}_l(\wp)$;

g. $\langle i\rangle$-neighborhood [11]: $\mathbb{N}_{\langle i\rangle}(\wp) = \mathbb{N}_{\langle r\rangle}(\wp) \cap \mathbb{N}_{\langle l\rangle}(\wp)$; and

h. $\langle u\rangle$-neighborhood [11]: $\mathbb{N}_{\langle u\rangle}(\wp) = \mathbb{N}_{\langle r\rangle}(\wp) \cup \mathbb{N}_{\langle l\rangle}(\wp)$.

Note that: The neighborhood $\mathbb{N}_r(\wp)$ (resp. $\mathbb{N}_l(\wp)$) is called the 'right' (resp. 'left') neighborhood of an element $\wp \in \mho$, which was first provided by Yao [3]. Moreover, the neighborhood $\mathbb{N}_{\langle r\rangle}(\wp)$ (resp. $\mathbb{N}_{\langle l\rangle}(\wp)$) is called the 'minimal-right' (resp. 'minimal-left') neighborhood of an element $\wp \in \mho$, which was first provided by Allam et al. [8].

Definition 2.2. [11] Let $\mathcal{R}$ be a binary relation defined on $\mho$ and $\mathcal{F}_k: \mho \longrightarrow P(\mho)$ be a mapping that assigns its k-neighborhood in $P(\mho)$ to each $\wp \in \mho$. Then, the triple $(\mho, \mathcal{R}, \mathcal{F}_k)$ is termed a k-neighborhood space, and is abbreviated as k-$\boldsymbol{NS}$.

Theorem 2.1. [11] Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$; then, for each $k \in \mathcal{K}$, the collection $\mathfrak{T}_k = \{O \subseteq \mho : \forall\, \wp \in O, \mathbb{N}_k(\wp) \subseteq O\}$ represents a topology on $\mho$.

Definition 2.3. [11] If $(\mho, \mathcal{R}, \mathcal{F}_k)$ is a k-$\boldsymbol{NS}$. A subset $O \subseteq \mho$ is considered an k-open set if $O \in \mathfrak{T}_k$, and its complement is termed a k-closed set. The family $\mathcal{C}_k$ of all k-closed sets of a k-$\boldsymbol{NS}$ is defined as $\mathcal{C}_k = \{O \subseteq \mho : O^c \in \mathfrak{T}_k\}$.

Definition 2.4. [11] Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$. Hence, the k-lower and k-upper approximations of $O \subseteq \mho$ are assumed, respectively, as:

$$\underline{\mathcal{R}}_k(O) = \cup\{\mathcal{G} \in \mathfrak{T}_k : \mathcal{G} \subseteq O\} = \boldsymbol{int}_k(O) \text{ and } \overline{\mathcal{R}}_k(O) = \cap\{\mathcal{H} \in \mathcal{C}_k : O \subseteq \mathcal{H}\} = \boldsymbol{cl}_k(O).$$

Here, $\boldsymbol{int}_k(O)$ (resp. $\boldsymbol{cl}_k(O)$ **)** represents the k-interior (resp. k-closure) of O.

The k-boundary, k-positive, and k-negative regions of O are provided, respectively, as follows:

$$\mathfrak{B}_j(O) = \overline{\mathcal{R}}_k(O) - \underline{\mathcal{R}}_k(O),\ pos_k(O) = \underline{\mathcal{R}}_k(O), \text{ and } neg_k(O) = \mho - \overline{\mathcal{R}}_k(O).$$

The k-accuracy of the approximations is given by the following:

$$\gamma_k(O) = \frac{|\underline{\mathcal{R}}_k(O)|}{|\overline{\mathcal{R}}_k(O)|}, where\ |\overline{\mathcal{R}}_k(O)| \neq 0.$$

Definition 2.5. [11] Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ and $O \subseteq \mho$. Then, O is called a k-exact set if $\underline{\mathcal{R}}_k(O) = \overline{\mathcal{R}}_k(O) = O$. If not, it is k-rough. It is clear that $0 \leq \gamma_k(O) \leq 1$ and $\gamma_j(O) = 1$ if O is a k-exact set. Otherwise, it is k-rough.

2.2. Yao approach

Definition 2.6. [3] Consider a k-$\boldsymbol{NS}$ $(\mho, \mathcal{R}, \mathcal{F}_k)$. We describe the Yao-lower, denoted as $Y_*(O)$, and the Yao-upper, $Y^*(O)$, approximations of a subset $O \subseteq \mho$ as follows:

$$Y_*(O) = \{\wp \in \mho |\ \mathbb{N}_r(\wp) \subseteq O\},$$

$$Y^*(O)\{\wp \in \mho |\ \mathbb{N}_r(\wp) \cap O \neq \Phi\}.$$

The Yao-boundary, Yao-positive, and Yao-negative regions of O are defined as follows:

- The Yao-boundary $\mathfrak{B}(O)$ comprises points in O whose neighborhoods partially intersect with O and partially lie outside of it.
- The Yao-positive region $pos(O)$ includes points in O for which the r-neighborhood is entirely contained within O.

- The Yao-negative region $neg(\mathcal{O})$ encompasses points in $\mho$ outside of $\mathcal{O}$ whose neighborhoods do not intersect with $\mathcal{O}$.

The k-accuracy of the approximations is given by the following:

$$\gamma(\mathcal{O}) = \frac{|Y_*(\mathcal{O})|}{|Y^*(\mathcal{O})|}, where\ |Y^*(\mathcal{O})| \neq 0.$$

2.3. Allam et al. approach

Definition 2.7. [8] Suppose we have a k-$\boldsymbol{NS}$ $(\mho, \mathcal{R}, \mathcal{F}_k)$, where each $k = \langle r \rangle$. In this context, we describe the Minimal-lower and Minimal-upper approximations of a subset $\mathcal{O} \subseteq \mho$ by the following:

- The Minimal-lower approximation, denoted as $\underline{\mathcal{A}}_k(\mathcal{O})$, comprises points p in $\mho$ for which the k-neighborhood of p is entirely contained within $\mathcal{O}$.
- Similarly, the Minimal-upper approximation, denoted as $\overline{\mathcal{A}}_k(\mathcal{O})$, consists of points p in $\mho$ for which the k-neighborhood of p intersects with $\mathcal{O}$.

Now, let's define the regions associated with $\mathcal{O}$:

- The Minimal-boundary region, denoted as $\mathcal{B}_k(\mathcal{O})$, encompasses points in $\mathcal{O}$ whose k-neighborhoods partially intersect with $\mathcal{O}$ and partially lie outside of it;
- The Positive region, denoted as $POS_k(\mathcal{O})$, includes points in $\mathcal{O}$ for which the k-neighborhood is entirely contained within $\mathcal{O}$;
- The Negative region, denoted as $NEG_k(\mathcal{O})$, covers points in in $\mho$ outside of $\mathcal{O}$ whose k-neighborhoods do not intersect with $\mathcal{O}$; and
- The Maximal-accuracy of the approximations is given by the following:

$$\mu_k(\mathcal{O}) = \frac{|\underline{\mathcal{A}}_k(\mathcal{O})|}{|\overline{\mathcal{A}}_k(\mathcal{O})|}, where\ |\overline{\mathcal{A}}_k(\mathcal{O})| \neq 0.$$

2.4. Dai et al. approach

Definition 2.8. [6] Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$. Then, the maximal-neighborhood of $p \in \mho$ is well-defined as follows:

$$\mathbb{N}_m(p) = \begin{cases} \bigcup_{p \in \mathbb{N}_r(q)} \mathbb{N}_r(q), & \text{if } p \in \mathbb{N}_r(q) \\ \Phi, & \text{Otherwise} \end{cases}.$$

Definition 2.9. [6] Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$. Then, the Maximal-lower and Maximal-upper approximations of $\mathcal{O} \subseteq \mho$ are well-defined, respectively, as follows:

$$\underline{\mathcal{R}}_m(\mathcal{O}) = \{p \in \mho : \mathbb{N}_m(p) \subseteq \mathcal{O}\} \text{ and } \overline{\mathcal{R}}_m(\mathcal{O}) = \{p \in \mho : \mathbb{N}_m(p) \cap \mathcal{O} \neq \Phi\}.$$

The Maximal-boundary (resp. positive and negative) regions of $\mathcal{O}$ are given, respectively, as follows:

$$\mathfrak{B}_m(\mathcal{O}) = \overline{\mathcal{R}}_m(\mathcal{O}) - \underline{\mathcal{R}}_m(\mathcal{O}),\ pos_m(\mathcal{O}) = \underline{\mathcal{R}}_m(\mathcal{O}), \text{ and } neg_m(\mathcal{O}) = \mho - \overline{\mathcal{R}}_m(\mathcal{O}).$$

The Maximal-accuracy of the approximations is given by the following:

$$\gamma_m(\mathcal{O}) = \frac{|\underline{\mathcal{R}}_m(\mathcal{O})|}{|\overline{\mathcal{R}}_m(\mathcal{O})|}, where\ |\overline{\mathcal{R}}_m(\mathcal{O})| \neq 0.$$

Lemma 2.1. [6,8] Let $\mathcal{R}$ be a binary relation on $\mho$:
(i) If $q \in \mathbb{N}_k(p)$, then $\mathbb{N}_k(q) \subseteq \mathbb{N}_k(p)$, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle\}$; and
(ii) If $q \in \mathbb{N}_m(p)$, then $\mathbb{N}_m(q) \subseteq \mathbb{N}_m(p)$.
Proof.
(i) In the paper [8], the property was proven for each $k = \langle r\rangle$. Therefore, we can similarly prove the other cases; and
(ii) The property was proven in [6].
Lemma 2.2. If $\mathcal{R}$ constitutes a reflexive relation on $\mho$, then for every $p \in \mho$, the following holds:
(i) $p \in \mathbb{N}_k(p)$ for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$; and
(ii) $p \in \mathbb{N}_m(p)$.
Proof. We will explain the first statement in a case of $k = \langle r\rangle$ and the others similarly.

If $\mathcal{R}$ is reflexive, then $p \in \mathbb{N}_r(p)$, $\forall p \in \mho$. Thus $\bigcap_{p\in\mathbb{N}_r(q)} \mathbb{N}_r(q) \neq \Phi$ and we get:

$$p \in \bigcap_{p\in\mathbb{N}_r(q)} \mathbb{N}_r(q) = \mathbb{N}_{\langle r\rangle}(p).$$

Lemma 2.3. If $\mathcal{R}$ is a reflexive relation on $\mho$, then for every $p \in \mho$, the following holds:
(i) $\mathbb{N}_{\langle r\rangle}(p) \subseteq \mathbb{N}_r(p) \subseteq \mathbb{N}_m(p)$; and
(ii) $\mathbb{N}_{\langle k\rangle}(p) \subseteq \mathbb{N}_k(p)$, for each $k \in \{r, l, i, u\}$.
Proof. By using Definitions 2.1 and 2.8, the proof is clear.

According to Theorem 2.1, we can generate a general topology by using the maximal-neighborhoods as the following result illustrates.
Theorem 2.2. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$; then, the class $\mathfrak{T}_m = \{O \subseteq \mho: \forall\, p \in O, \mathbb{N}_m(p) \subseteq O\}$ represents a topology on $\mho$.

3. Basic minimal-neighborhoods and their properties

This section is dedicated to the generalization of the idea of the 'basic neighborhood' [9] into new types, thereby yielding four distinct topologies derived from these neighborhoods. It is worth noting that this definition was also referred to as a "containment neighborhood" in [43].
Definition 3.1. [43] Assume that $\mathcal{R}$ is a binary relation on $\mho$. Then, we define the following neighborhoods of $p \in \mho$:

(i) Basic $\langle r\rangle$-neighborhood: $\mathbb{N}^{b}_{\langle r\rangle}(p) = \{q \in \mho : \mathbb{N}_{\langle r\rangle}(q) \subseteq \mathbb{N}_{\langle r\rangle}(p)\}$;

(ii) Basic $\langle l\rangle$-neighborhood: $\mathbb{N}^{b}_{\langle l\rangle}(p) = \{q \in \mho : \mathbb{N}_{\langle l\rangle}(q) \subseteq \mathbb{N}_{\langle l\rangle}(p)\}$;

(iii) Basic $\langle i\rangle$-neighborhood: $\mathbb{N}^{b}_{\langle i\rangle}(p) = \mathbb{N}^{b}_{\langle r\rangle}(p) \cap \mathbb{N}^{b}_{\langle l\rangle}(p)$; and

(iv)Basic $\langle u\rangle$-neighborhood: $\mathbb{N}^{b}_{\langle u\rangle}(p) = \mathbb{N}^{b}_{\langle r\rangle}(p) \cup \mathbb{N}^{b}_{\langle l\rangle}(p)$.

In the following, we illustrate some characteristics of the aforementioned neighborhoods.

Lemma 3.1. Suppose there exists a binary relation $\mathcal{R}$ defined on $\mho$. Thus, the following holds:

(i) $p \in \mathbb{N}_k^{b}(p)$, $\forall k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$;

(ii) $\mathbb{N}_k^{b}(p) \neq \Phi$, $\forall k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$; and

(iii) If $q \in \mathbb{N}_k^{b}(p)$, then $\mathbb{N}_k^{b}(q) \subseteq \mathbb{N}_k^{b}(p)$, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

Proof. First, (i) and (ii) are obvious. Now, we prove (iii) in a case $k = \langle r\rangle$, and the others similarly.

If $q \in \mathbb{N}_{\langle r\rangle}^{b}(p)$, then

$$\mathbb{N}_{\langle r\rangle}(q) \subseteq \mathbb{N}_{\langle r\rangle}(p). \tag{3.1}$$

Let $w \in \mathbb{N}_{\langle r\rangle}^{b}(q)$; then, $\mathbb{N}_{\langle r\rangle}(w) \subseteq \mathbb{N}_{\langle r\rangle}(q)$. Therefore, by Eq (3.1), $\mathbb{N}_{\langle r\rangle}(w) \subseteq \mathbb{N}_{\langle r\rangle}(p)$ implies $w \in \mathbb{N}_{\langle r\rangle}^{b}(p)$. Hence, $\mathbb{N}_{\langle r\rangle}^{b}(q) \subseteq \mathbb{N}_{\langle r\rangle}^{b}(p)$.

Remark 3.1. Example 3.1 highlights the following observations:

(i) The statement (iii) of Lemma 3.1 does not hold true in the case of $k = \langle u\rangle$.

(ii) In the general case, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the basic k-neighborhoods and the k-neighborhoods are independent (non-comparable) when $\mathcal{R}$ is a binary relation on $\mho$.

Example 3.1. Let $\mho = \{z_1, z_2, z_3, z_4\}$ and $\mathcal{R} = \{(z_1, z_1), (z_1, z_4), (z_2, z_1), (z_2, z_3), (z_3, z_4), (z_3, z_1)\}$ be a binary relation on $\mho$. Consequently, we obtain the following tables (Tables 2–4) which contain all neighborhoods generated by $\mathcal{R}$.

Table 2. k-neighborhoods of $p \in \mho$.

x	$\mathbb{N}_r(x)$	$\mathbb{N}_l(x)$	$\mathbb{N}_i(x)$	$\mathbb{N}_u(x)$
z_1	$\{z_1, z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1\}$	$\mho$
z_2	$\{z_1, z_3\}$	Φ	Φ	$\{z_1, z_3\}$
z_3	$\{z_1, z_4\}$	$\{z_2\}$	Φ	$\{z_1, z_2, z_4\}$
z_4	Φ	$\{z_1, z_3\}$	Φ	$\{z_1, z_3\}$

Table 3. $\langle k\rangle$-neighborhoods of $p \in \mho$.

x	$\mathbb{N}_{\langle r\rangle}(x)$	$\mathbb{N}_{\langle l\rangle}(x)$	$\mathbb{N}_{\langle i\rangle}(x)$	$\mathbb{N}_{\langle u\rangle}(x)$
z_1	$\{z_1\}$	$\{z_1, z_3\}$	$\{z_1\}$	$\{z_1, z_3\}$
z_2	Φ	$\{z_2\}$	Φ	$\{z_2\}$
z_3	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\{z_1, z_3\}$
z_4	$\{z_1, z_4\}$	Φ	Φ	$\{z_1, z_4\}$

Table 4. Basic $\langle k\rangle$-neighborhoods of $p \in \mho$.

x	$\mathbb{N}_{\langle r\rangle}^{b}(x)$	$\mathbb{N}_{\langle l\rangle}^{b}(x)$	$\mathbb{N}_{\langle i\rangle}^{b}(x)$	$\mathbb{N}_{\langle u\rangle}^{b}(x)$
z_1	$\{z_1, z_2\}$	$\{z_1, z_3, z_4\}$	$\{z_1\}$	$\mho$
z_2	$\{z_2\}$	$\{z_2, z_4\}$	$\{z_2\}$	$\{z_2, z_4\}$
z_3	$\{z_1, z_2, z_3\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_3\}$	$\mho$
z_4	$\{z_1, z_2, z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_1, z_2, z_4\}$

The proof of the following lemma is easy; therefore, we omit it.

Lemma 3.2. Let $\mathcal{R}$ be a binary relation on $\mho$. Then, for every $p \in \mho$, the following holds:

(i) $\mathbb{N}^{b}_{\langle i\rangle}(p) \subseteq \mathbb{N}^{b}_{\langle r\rangle}(p) \subseteq \mathbb{N}^{b}_{\langle u\rangle}(p)$; and

(ii) $\mathbb{N}^{b}_{\langle i\rangle}(p) \subseteq \mathbb{N}^{b}_{\langle l\rangle}(p) \subseteq \mathbb{N}^{b}_{\langle u\rangle}(p)$.

The following lemma examines the connection between basic the k-neighborhoods and the k-neighborhoods, where k ranges over $\{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

Lemma 3.3. In a k-$\boldsymbol{NS}$ $(\mho, \mathcal{R}, \mathcal{F}_k)$, where $\mathcal{R}$ is a reflexive relation, $\forall k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the basic k-neighborhoods $\mathbb{N}^{b}_{k}(p)$ are equivalent to the k-neighborhoods $\mathbb{N}_{k}(p)$ for all $p \in \mho$.

Proof. We demonstrate the lemma for $k = \langle r\rangle$, with similar reasoning applicable to other cases.

First, according to Definition 3.1, if $q \in \mathbb{N}^{b}_{\langle r\rangle}(p)$, then

$$\mathbb{N}_{\langle r\rangle}(q) \subseteq \mathbb{N}_{\langle r\rangle}(p). \tag{3.2}$$

Given that $\mathcal{R}$ is reflexive, $q \in \mathbb{N}_{\langle r\rangle}(q)$. Hence, Eq (3.2), $q \in \mathbb{N}_{\langle r\rangle}(p)$, which implies that $\mathbb{N}^{b}_{\langle r\rangle}(p) \subseteq \mathbb{N}_{\langle r\rangle}(p)$, for all p in $\mho$.

Conversely, utilizing Lemma 2.1, if $q \in \mathbb{N}_{\langle r\rangle}(p)$, then $\mathbb{N}_{\langle r\rangle}(q) \subseteq \mathbb{N}_{\langle r\rangle}(p)$, implying $q \in \mathbb{N}^{b}_{\langle r\rangle}(p)$. Therefore, $\mathbb{N}_{\langle r\rangle}(p) \subseteq \mathbb{N}^{b}_{\langle r\rangle}(p)$, for all p in $\mho$.

Corollary 3.1. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ where $\mathcal{R}$ is an equivalence relation; then, $\mathbb{N}^{b}_{k}(p) = \mathbb{N}_{k}(p) = [p]_{\mathcal{R}}$, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, and $[p]_{\mathcal{R}}$ signifies the equivalence class of $p \in \mho$.

Lemma 3.4. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ where $\mathcal{R}$ is a reflexive relation; then, $\forall k \in \{r, l, i, u\}$ and the following holds:

(i) $\mathbb{N}^{b}_{\langle k\rangle}(p) \subseteq \mathbb{N}_{k}(p)$, $\forall p \in \mho$; and

(ii) $\mathbb{N}^{b}_{\langle k\rangle}(p) \subseteq \mathbb{N}_{m}(p)$, $\forall p \in \mho$.

Proof. Utilizing Lemmas 2.3 and 3.3, the proof becomes self-evident.

Remark 3.2. Example 3.2 illustrates that the following:

(i) Illustrating Lemma 3.3.

(ii) The converse of Lemma 3.4 is not generally true.

Example 3.2. Let $\mho = \{z_1, z_2, z_3, z_4\}$ be a set, and let $\mathcal{R}$ be a reflexive relation on $\mho$ defined as follows: $\mathcal{R} = \{(z_1, z_1), (z_2, z_2), (z_3, z_3), (z_4, z_4), (z_1, z_2), (z_2, z_3)\}$. The undermentioned Tables 5–7 illustrate the neighborhoods generated by $\mathcal{R}$.

Table 5. k-neighborhoods of $p \in \mho$.

$\boldsymbol{x}$	$\mathbb{N}_{\boldsymbol{r}}(\boldsymbol{x})$	$\mathbb{N}_{\boldsymbol{l}}(\boldsymbol{x})$	$\mathbb{N}_{\boldsymbol{i}}(\boldsymbol{x})$	$\mathbb{N}_{\boldsymbol{u}}(\boldsymbol{x})$
$\boldsymbol{z_1}$	$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1\}$	$\{z_1, z_2\}$
$\boldsymbol{z_2}$	$\{z_2, z_3\}$	$\{z_1, z_2\}$	$\{z_2\}$	$\{z_1, z_2, z_3\}$
$\boldsymbol{z_3}$	$\{z_3\}$	$\{z_2, z_3\}$	$\{z_3\}$	$\{z_2, z_3\}$
$\boldsymbol{z_4}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$

Table 6. $\langle k \rangle$-neighborhoods of $\wp \in \mho$.

x	$\mathbb{N}_{\langle r \rangle}(x)$	$\mathbb{N}_{\langle l \rangle}(x)$	$\mathbb{N}_{\langle i \rangle}(x)$	$\mathbb{N}_{\langle u \rangle}(x)$
z_1	$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1\}$	$\{z_1, z_2\}$
z_2	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$
z_3	$\{z_3\}$	$\{z_2, z_3\}$	$\{z_3\}$	$\{z_2, z_3\}$
z_4	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$

Table 7. Basic $\langle k \rangle$-neighborhoods of $\wp \in \mho$.

x	$\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(x)$	$\mathbb{N}^{\mathfrak{b}}_{\langle l \rangle}(x)$	$\mathbb{N}^{\mathfrak{b}}_{\langle i \rangle}(x)$	$\mathbb{N}^{\mathfrak{b}}_{\langle u \rangle}(x)$
z_1	$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1\}$	$\{z_1, z_2\}$
z_2	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$
z_3	$\{z_3\}$	$\{z_2, z_3\}$	$\{z_3\}$	$\{z_2, z_3\}$
z_4	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$

4. Basic-minimal rough approximations and topological applications

In this section, which is divided into three subsections, we delve into a comprehensive analysis of various RS approximations (basic-minimal approximations) and their implications. Through a systematic exploration, we aim to elucidate the relationships between the RS theory and topology, paving the way for an enhanced understanding and practical applications in both domains.

4.1. Generalized rough sets based on basic minimal-neighborhoods

In this subsection, we introduce four distinct RS approximations (called basic-minimal approximations), dissecting their core properties and establishing their adherence to Pawlak's fundamental axioms. Furthermore, we conduct a comparative analysis to identify the most effective approximation method based on its accuracy factor.

Definition 4.1. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ where $k \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$. The basic-minimal lower and upper approximations of a subset $\mathcal{O} \subseteq \mho$ are formally defined as follows:

$$\underline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}) = \{\wp \in \mho : \mathbb{N}^{\mathfrak{b}}_k(\wp) \subseteq \mathcal{O}\} \text{ and } \overline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}) = \{\wp \in \mho : \mathbb{N}^{\mathfrak{b}}_k(\wp) \cap \mathcal{O} \neq \Phi\}.$$

Furthermore, the basic-minimal boundary, basic-minimal positive, and basic-minimal negative regions of $\mathcal{O}$, respectively, are defined as follows:

$$\mathfrak{B}^{\mathfrak{b}}_k(\mathcal{O}) = \overline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}) - \underline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}),\ pos^{\mathfrak{b}}_k(\mathcal{O}) = \underline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}), \text{ and } neg^{\mathfrak{b}}_k(\mathcal{O}) = \mho - \overline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O}).$$

The basic-minimal accuracy of the approximations is given by the following:

$$\gamma^{\mathfrak{b}}_k(\mathcal{O}) = \frac{\left|\underline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O})\right|}{\left|\overline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O})\right|}, where \ \left|\overline{\mathcal{R}}^{\mathfrak{b}}_k(\mathcal{O})\right| \neq 0.$$

It is evident that $0 \leq \gamma^{\mathfrak{b}}_k(\mathcal{O}) \leq 1$, and if $\gamma^{\mathfrak{b}}_k(\mathcal{O}) = 1$, then $\mathcal{O}$ is termed a basic k-definable (basic k-

exact) set; otherwise, it is considered basic k-rough.

The proposition asserts certain properties of the j-basic approximations.

Proposition 4.1. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ and $\mathcal{Q}, \mathcal{S} \subseteq \mho$. Thus, the following holds:

(i) $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \subseteq \mathcal{Q} \subseteq \overline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$;

(ii) $\underline{\mathcal{R}}_k^{\flat}(\mho) = \overline{\mathcal{R}}_k^{\flat}(\mho) = \mho$, $\underline{\mathcal{R}}_k^{\flat}(\Phi) = \overline{\mathcal{R}}_k^{\flat}(\Phi) = \Phi$;

(iii) If $\mathcal{Q} \subseteq \mathcal{S}$, then $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \subseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(iv) If $\mathcal{Q} \subseteq \mathcal{S}$, then $\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \subseteq \overline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(v) $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S}) = \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cap \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(vi) $\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cup \mathcal{S}) = \overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cup \overline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(vii) $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cup \mathcal{S}) \supseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cup \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(viii) $\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S}) \subseteq \overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cap \overline{\mathcal{R}}_k^{\flat}(\mathcal{S})$;

(ix) $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) = \left[\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}^c)\right]^c$, where $\mathcal{Q}^c$ represents a complement of $\mathcal{Q}$;

(x) $\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) = \left[\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}^c)\right]^c$;

(xi) $\underline{\mathcal{R}}_k^{\flat}\left(\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})\right) = \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$; and

(xii) $\overline{\mathcal{R}}_k^{\flat}\left(\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q})\right) = \overline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$.

Proof. The validity of (i), (ii), (iii), and (iv) is readily apparent by using Definition 4.1. Therefore, we will prove the remaining items (v)-(xii) as follows.

(v) Since $(\mathcal{Q} \cap \mathcal{S}) \subseteq \mathcal{Q}$ and $(\mathcal{Q} \cap \mathcal{S}) \subseteq \mathcal{S}$, then $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S}) \subseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$ and $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S}) \subseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})$.

Now, let $\varpi \in [\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cap \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})]$. Then, $\varpi \in \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$ and $\varpi \in \underline{\mathcal{R}}_k^{\flat}(\mathcal{S})$, which implies $\mathbb{N}_k^{\flat}(\varpi) \subseteq \mathcal{Q}$ and $\mathbb{N}_k^{\flat}(\varpi) \subseteq \mathcal{S}$. Thus, $\mathbb{N}_k^{\flat}(\varpi) \subseteq \mathcal{Q} \cap \mathcal{S}$ which that $\varpi \in \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S})$.

Therefore, $\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q}) \cap \underline{\mathcal{R}}_k^{\flat}(\mathcal{S}) \subseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q} \cap \mathcal{S})$;

(vi) Similar to (v), using a comparable approach;
(vii)Similar to (v), using a comparable approach;
(viii) Similar to (v), using a comparable approach;

(ix) $\left[\overline{\mathcal{R}}_k^{\flat}(\mathcal{Q}^c)\right]^c = \left[\{\wp \in \mho : \mathbb{N}_k^{\flat}(\wp) \cap \mathcal{Q}^c \neq \Phi\}\right]^c = \{\wp \in \mho : \mathbb{N}_k^{\flat}(\wp) \cap \mathcal{Q}^c = \Phi\}$
$= \{\wp \in \mho : \mathbb{N}_k^{\flat}(\wp) \subseteq \mathcal{Q}\} = \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$;

(x) By a similar way such as (ix);

(xi) First, by (i), $\underline{\mathcal{R}}_k^{\flat}\left(\underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})\right) \subseteq \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$.

Now, let $\varpi \in \underline{\mathcal{R}}_k^{\flat}(\mathcal{Q})$. Then,

$$\mathbb{N}_k^{\mathfrak{b}}(\varpi) \subseteq \mathcal{Q}. \tag{4.1}$$

We need to prove that $\mathbb{N}_k^{\mathfrak{b}}(\varpi) \subseteq \underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q})$ as follows:

If $z \in \mathbb{N}_k^{\mathfrak{b}}(\varpi)$, then $\mathbb{N}_k^{\mathfrak{b}}(z) \subseteq \mathbb{N}_k^{\mathfrak{b}}(\varpi)$, which implies that $\mathbb{N}_k^{\mathfrak{b}}(z) \subseteq \mathcal{Q}$ from Eq (4.1). Therefore, $z \in \underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q})$, which means that $\mathbb{N}_k^{\mathfrak{b}}(\varpi) \subseteq \underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q})$, which implies $\varpi \in \underline{\mathcal{R}}_k^{\mathfrak{b}}\left(\underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q})\right)$. Hence, $\underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q}) \subseteq \underline{\mathcal{R}}_k^{\mathfrak{b}}\left(\underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{Q})\right)$; and

(xii)By a similar way such as (xi).

The subsequent findings, which elucidate the connections between the proposed approximations (basic-minimal approximations), are straightforward to demonstrate with Lemma 3.2, hence the proof is omitted.

Proposition 4.2. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$ and $\mathcal{O} \subseteq \mho$. Then, the following holds:

(i) $\underline{\mathcal{R}}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \underline{\mathcal{R}}_{\langle r\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \underline{\mathcal{R}}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(ii) $\underline{\mathcal{R}}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \underline{\mathcal{R}}_{\langle l\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \underline{\mathcal{R}}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(iii) $\overline{\mathcal{R}}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \overline{\mathcal{R}}_{\langle r\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \overline{\mathcal{R}}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O})$; and

(iv)$\overline{\mathcal{R}}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \overline{\mathcal{R}}_{\langle l\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \overline{\mathcal{R}}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O})$.

Corollary 4.1. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ is a k-$\boldsymbol{NS}$ and $\mathcal{O} \subseteq \mho$. Then, the following holds:

(i) $\mathfrak{B}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathfrak{B}_{\langle r\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathfrak{B}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(ii) $\mathfrak{B}_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathfrak{B}_{\langle l\rangle}^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathfrak{B}_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(iii) $\gamma_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O}) \le \gamma_{\langle r\rangle}^{\mathfrak{b}}(\mathcal{O}) \le \gamma_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(iv)$\gamma_{\langle u\rangle}^{\mathfrak{b}}(\mathcal{O}) \le \gamma_{\langle l\rangle}^{\mathfrak{b}}(\mathcal{O}) \le \gamma_{\langle i\rangle}^{\mathfrak{b}}(\mathcal{O})$;

(v) If $\mathcal{O}$ is a basic $\langle u\rangle$-exact set, then it follows that $\mathcal{O}$ is also a basic $\langle r\rangle$-exact set, which in turn implies that $\mathcal{O}$ is a basic $\langle i\rangle$-exact set; and

(vi)If $\mathcal{O}$ is a basic $\langle u\rangle$-exact set, then it follows that $\mathcal{O}$ is also a basic $\langle l\rangle$-exact set, which in turn implies that $\mathcal{O}$ is a basic $\langle i\rangle$-exact set.

Remark 4.1. Example 4.1 serves to illustrate that the converse of the aforementioned results is not universally valid.

Example 4.1. Let $\mho = \{z_1, z_2, z_3\}$ and consider the binary relation $\mathcal{R}$ on $\mho$ defined as follows: $\mathcal{R} = \{(z_1, z_1), (z_2, z_2), (z_3, z_3), (z_2, z_3), (z_3, z_1)\}$.

Consequently, we construct Tables 8 and 9 to represent the basic k-lower and basic k-upper approximations, along with the basic k-accuracies of the approximations for all subsets $\mho$.

Table 8. Comparison of various types of basic k-approximations.

$\mathcal{O} \subseteq \mho$	basic $\langle r\rangle$-approximations		basic $\langle l\rangle$-approximations		basic $\langle i\rangle$-approximations		basic $\langle u\rangle$-approximations	
	$\underline{\mathcal{R}}^{\mathrm{b}}_{\langle r\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathrm{b}}_{\langle r\rangle}(\mathcal{O})$	$\underline{\mathcal{R}}^{\mathrm{b}}_{\langle l\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathrm{b}}_{\langle l\rangle}(\mathcal{O})$	$\underline{\mathcal{R}}^{\mathrm{b}}_{\langle i\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathrm{b}}_{\langle i\rangle}(\mathcal{O})$	$\underline{\mathcal{R}}^{\mathrm{b}}_{\langle u\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathrm{b}}_{\langle u\rangle}(\mathcal{O})$
$\{z_1\}$	$\{z_1\}$	$\{z_1\}$	Φ	$\{z_1\}$	$\{z_1\}$	$\{z_1\}$	Φ	$\{z_1\}$
$\{z_2\}$	Φ	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$	$\{z_2\}$	Φ	$\{z_2\}$
$\{z_3\}$	$\{z_3\}$	$\{z_2, z_3\}$	$\{z_3\}$	$\{z_1, z_3\}$	$\{z_3\}$	$\{z_3\}$	$\{z_3\}$	$\mho$
$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1, z_2\}$	$\{z_2\}$	$\{z_1, z_2\}$	$\{z_1, z_2\}$	$\{z_1, z_2\}$	Φ	$\{z_1, z_2\}$
$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\mho$	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\{z_1, z_3\}$	$\mho$
$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\mho$	$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\mho$
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$

Table 9. Comparison of various types of basic k-accuracies.

$\mathcal{O} \subseteq \mho$	$\gamma^{\mathrm{b}}_{\langle r\rangle}(\mathcal{O})$	$\gamma^{\mathrm{b}}_{\langle l\rangle}(\mathcal{O})$	$\gamma^{\mathrm{b}}_{\langle i\rangle}(\mathcal{O})$	$\gamma^{\mathrm{b}}_{\langle u\rangle}(\mathcal{O})$
$\{z_1\}$	1	0	1	0
$\{z_2\}$	0	1	1	0
$\{z_3\}$	$1/2$	$1/2$	1	$1/3$
$\{z_1, z_2\}$	$1/2$	$1/2$	1	0
$\{z_1, z_3\}$	$2/3$	1	1	$2/3$
$\{z_2, z_3\}$	1	$2/3$	1	$2/3$
$\mho$	1	1	1	1

Remark 4.2. Based on Proposition 4.2, Corollary 4.1 and Example 4.1, the optimal method for approximating rough sets is the use of basic $\langle i\rangle$-approximations, which provide the highest accuracy measures.

4.2. Different topological structures via basic-minimal neighborhoods

Moving beyond mere approximations, this subsection investigates the generation of diverse topologies derived from basic-minimal neighborhoods. We rigorously demonstrate that the approximations proposed in Subsection 4.1 serve as closure and interior operators for these newly formed topologies, forging vital connections between the RS theory and topology for future explorations and applications.

Theorem 4.1. Given a k-$\boldsymbol{NS}$ $(\mho, \mathcal{R}, \mathcal{F}_k)$, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the set $\mathfrak{T}^{\mathrm{b}}_k$ where: $\mathfrak{T}^{\mathrm{b}}_k = \{\mathcal{O} \subseteq \mho : \forall\, \wp \in \mathcal{O}, \mathbb{N}^{\mathrm{b}}_k(\wp) \subseteq \mathcal{O}\}$ forms a topology on $\mho$.

Proof.

(T1) It is evident that $\mho$ and Φ belong to $\mathfrak{T}_k^{\mathrm{b}}$.

(T2) Suppose $\{Q_\beta : \beta \in \mathcal{J}\}$ is a class of members in $\mathfrak{T}_k^{\mathrm{b}}$, and let $q \in \bigcup_\beta Q_\beta$. Then, there exists $\beta_o \in \mathcal{J}$ such that $q \in Q_{\beta_o}$. Therefore, $\mathbb{N}_k^{\mathrm{b}}(q) \subseteq Q_{\beta_o}$, which implies that $\mathbb{N}_k^{\mathrm{b}}(q) \subseteq \bigcup_\beta Q_\beta$. Hence, $\bigcup_\beta Q_\beta \in \mathfrak{T}_k^{\mathrm{b}}$.

(T3) Let $Q_1, Q_2 \in \mathfrak{T}_k^{\mathrm{b}}$ and $q \in Q_1 \cap Q_2$. Then, $q \in Q_1$ and $q \in Q_2$, which implies $\mathbb{N}_k^{\mathrm{b}}(q) \subseteq Q_1$ and $\mathbb{N}_k^{\mathrm{b}}(q) \subseteq Q_2$. Thus, $\mathbb{N}_k^{\mathrm{b}}(q) \subseteq (Q_1 \cap Q_2)$, and hence $(Q_1 \cap Q_2) \in \mathfrak{T}_k^{\mathrm{b}}$.

By (T1), (T2), and (T3), we conclude that $\mathfrak{T}_k^{\mathrm{b}}$ forms a topology on $\mho$.

By employing Lemma 3.2, we can readily establish the subsequent result, elucidating the relationships among various topologies $\mathfrak{T}_k^{\mathrm{b}}$.

Proposition 4.3. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$. Then, the following holds:

(i) $\mathfrak{T}_{\langle u\rangle}^{\mathrm{b}} \subseteq \mathfrak{T}_{\langle r\rangle}^{\mathrm{b}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathrm{b}}$; and

(ii) (ii)$\mathfrak{T}_{\langle u\rangle}^{\mathrm{b}} \subseteq \mathfrak{T}_{\langle l\rangle}^{\mathrm{b}} \subseteq \mathfrak{T}_{\langle i\rangle}^{\mathrm{b}}$.

The negation of Proposition 4.3 is shown to be incorrect in Example 4.2.

Example 4.2. Considering Example 3.1, we generate the following topologies:
$\mathfrak{T}_{\langle r\rangle}^{\mathrm{b}} = \{\mho, \Phi, \{z_2\}, \{z_1, z_2\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}\}$,
$\mathfrak{T}_{\langle l\rangle}^{\mathrm{b}} = \{\mho, \Phi, \{z_4\}, \{z_2, z_4\}, \{z_1, z_3, z_4\}\}$,
$\mathfrak{T}_{\langle i\rangle}^{\mathrm{b}} = \{\mho, \Phi, \{z_1\}, \{z_2\}, \{z_4\}, \{z_1, z_2\}, \{z_1, z_3\}, \{z_1, z_4\}, \{z_2, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_1, z_3, z_4\}\}$,
and $\mathfrak{T}_{\langle u\rangle}^{\mathrm{b}} = \{\mho, \Phi\}$.
However, $\mathfrak{T}_{\langle r\rangle} = \{\mho, \Phi, \{z_1\}, \{z_2\}, \{z_1, z_2\}, \{z_1, z_3\}, \{z_1, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_1, z_3, z_4\}\}$.

Remark 4.3. Based on Example 4.2, the following observations can be made:

(i) The topologies $\mathfrak{T}_k$ and $\mathfrak{T}_k^{\mathrm{b}}$ are generally independent, for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

(ii) The topologies $\mathfrak{T}_{\langle r\rangle}^{\mathrm{b}}$ and $\mathfrak{T}_{\langle l\rangle}^{\mathrm{b}}$ are generally non-comparable.

The next proposition illustrates the relationships between the topologies generated by the basic-minimal neighborhoods and those induced by the preceding neighborhoods.

Proposition 4.4. Let $(\mho, \mathcal{R}, \mathcal{F}_k)$ be a k-$\boldsymbol{NS}$, where $\mathcal{R}$ is a reflexive relation. Then, $\forall k \in \{r, l, i, u\}$, and the following holds:

(i) $\mathfrak{T}_{\langle k\rangle} = \mathfrak{T}_{\langle k\rangle}^{\mathrm{b}}$;

(ii) $\mathfrak{T}_k \subseteq \mathfrak{T}_{\langle k\rangle}^{\mathrm{b}}$; and

(iii) $\mathfrak{T}_m \subseteq \mathfrak{T}_{\langle k\rangle}^{\mathrm{b}}$.

Proof. Utilizing Lemmas 3.3 and 3.4, the proof becomes evident.

The next example proves that the opposite of Proposition 4.4 does not hold in general.

Example 4.3. By using Example 3.2, we compute the topologies $\mathfrak{T}_k$, $\mathfrak{T}_m$, $\mathfrak{T}_{\langle k\rangle}$, and $\mathfrak{T}_{\langle k\rangle}^{\mathrm{b}}$ in the case where $k = r$, and similarly for the other cases.
$\mathfrak{T}_m = \{\mho, \Phi, \{z_4\}, \{z_1, z_2, z_3\}\}$, $\mathfrak{T}_r = \{\mho, \Phi, \{z_3\}, \{z_4\}, \{z_2, z_3\}, \{z_3, z_4\}, \{z_1, z_2, z_3\}, \{z_2, z_3, z_4\}\}$,
and $\mathfrak{T}_{\langle r\rangle} = \mathfrak{T}_{\langle r\rangle}^{\mathrm{b}} = \{\mho, \Phi, \{z_2\}, \{z_3\}, \{z_4\}, \{z_1, z_2\}, \{z_2, z_3\}, \{z_2, z_4\}, \{z_3, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_2, z_3, z_4\}\}$.

The following theory is highly significant, as it serves as the link between the set-theoretic theory of RS's approach on one hand, and topological science on the other. Consequently, it stands as the cornerstone to apply all the concepts of topology and its applications within the theory of RS's. This would greatly benefit those interested in topology applications, yet not fundamentally specialized in topology itself. On the other hand, the following theorem introduces another method to compute the basic-minimal approximations in view of topology.

Theorem 4.2. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ is a k-***NS***, then for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the basic-minimal lower and the basic-minimal upper approximations of $\mathcal{O} \subseteq \mho$ are well-defined, respectively, as follows:

$$\underline{\mathcal{R}}_k^{\flat}(\mathcal{O}) = \cup \left\{\mathcal{G} \in \mathfrak{T}_{\langle k\rangle}^{\flat} : \mathcal{G} \subseteq \mathcal{O}\right\} \text{ and } \overline{\mathcal{R}}_k^{\flat}(\mathcal{O}) = \cap \left\{\mathcal{H} \in \mathcal{F}_{\langle k\rangle}^{\flat} : \mathcal{O} \subseteq \mathcal{H}\right\}, \text{ where } \mathcal{F}_{\langle k\rangle}^{\flat} = \left(\mathfrak{T}_{\langle k\rangle}^{\flat}\right)^c.$$

Proof. We will prove the first statement and the other by the duality property.

Necessity condition:

Let $x \in \cup \left\{\mathcal{G} \in \mathfrak{T}_{\langle k\rangle}^{\flat} : \mathcal{G} \subseteq \mathcal{O}\right\}$; then, $\exists \mathfrak{D} \in \mathfrak{T}_{\langle k\rangle}^{\flat}$ such that $x \in \mathfrak{D} \subseteq \mathcal{O}$. Hence, $\mathbb{N}_k^{\flat}(x) \subseteq \mathfrak{D}$, which implies that $x \in \left\{p \in \mho : \mathbb{N}_k^{\flat}(p) \subseteq \mathcal{O}\right\}$.

Sufficiency condition:

Let $x \in \underline{\mathcal{R}}_k^{\flat}(\mathcal{O})$; then, $\mathbb{N}_k^{\flat}(x) \subseteq \mathcal{O}$. However, from Lemma 3.1, $\forall y \in \mathbb{N}_k^{\flat}(x)$, $\mathbb{N}_k^{\flat}(y) \subseteq \mathbb{N}_k^{\flat}(x)$, which implies that $\mathbb{N}_k^{\flat}(x) = \mathcal{G} \in \mathfrak{T}_{\langle k\rangle}^{\flat}$ such that $x \in \mathcal{G} \subseteq \mathcal{O}$. Thus, $x \in \cup \left\{\mathcal{G} \in \mathfrak{T}_{\langle k\rangle}^{\flat} : \mathcal{G} \subseteq \mathcal{O}\right\}$.

Remark 4.4. From Theorem 4.2, we observe that the basic-minimal lower and basic-minimal upper approximations correspond to the interior and closure operators of $\mathcal{O} \subseteq \mho$, respectively. This connection underscores the significance of the proposed approaches, highlighting their role as a crucial bridge to subsequent topological applications in the RS theory.

4.3. Comparisons between the suggested methods (basic-minimal approximations) and some of the others studies

As a dedicated comparison, Subsection 4.3 scrutinizes the proposed approximations against prior methodologies, including those by Yao [3], Dai et al. [6], Allam et al. [8], and Abd El-Monsef et al. [11]. Through a meticulous analysis, bolstered by counterexamples and established theorems, we showcase the superiority of our proposed methods across various specific and general scenarios.

First, we present comparative analyses between the proposed approaches in the current paper and some other methods in the case of a general binary relation.

Example 4.4. Referring to Example 3.1, we proceed to calculate the approximations for all subsets of $\mho$ using both the current technique and the preceding methods (Yao, Allam, Abd El-Monsef et al., and Dai et al. approaches), as presented in Tables 10 and 11.

Remark 4.5 Upon examination of Tables 10 and 11, the following observations can be made:

(i) The Yao, Allam, and Dai methods are generally to approximate RS's due to their inability to be generally applied across relations, lacking key properties necessary for approximations. Consequently, these limitations confine the scope of the RS theory applications, exemplified by the highlighted cells in Tables 10 & 11. Consequently, these methods introduce inconsistencies within the RS theory. Furthermore, it is evident that the proposed method demonstrates a superior accuracy compared to the approaches by Abd El-Monsef et al. Additionally, according to the preceding methods, all subsets are categorized as rough, thus indicating an inherent vagueness in the data (see the highlighted cells in Tables 10 and 11).

(ii) Conversely, the methods outlined in our current paper stand out as the optimal approaches to approximate sets across general cases. This is because the basic-approximations fulfill all of Pawlak's RS properties unconditionally, devoid of any limitations or prerequisites. Additionally, our approaches encompass exact subsets, signifying the potential of our suggested method in unveiling the inherent vagueness within the data.

The next results elucidate the relationships among the current approaches and the methodologies proposed by Yao [3], Abd El-Monsef et al. [11], Allam et al. [8], and Dai et al. [6].

Theorem 4.3. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ is a k-$\boldsymbol{NS}$, where $\mathcal{R}$ is a reflexive relation, then for all $\mathcal{O} \subseteq \mho$ and $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the following holds:

(i) $\underline{\mathcal{R}}_k(\mathcal{O}) \subseteq \underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathcal{O} \subseteq \overline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{O}) \subseteq \overline{\mathcal{R}}_k(\mathcal{O})$;

(ii) $\mathfrak{B}_k^{\mathfrak{b}}(\mathcal{O}) \subseteq \mathfrak{B}_k(\mathcal{O})$ and $\gamma_k(\mathcal{O}) \leq \gamma_k^{\mathfrak{b}}(\mathcal{O})$; and

(iii) If $\mathcal{O}$ is k-exact, then it is basic k-exact.

Proof. By employing Proposition 4.4, the proof becomes evident.

Table 10. Comparison between Yao, Allam, and the current approach in general case.

$\mathcal{O} \subseteq \mho$	Yao's method [3]		Allam et al.'s method [8]		Current method	
	$Y_*(\mathcal{O})$	$Y^*(\mathcal{O})$	$\underline{\mathcal{A}}_{\langle r\rangle}(\mathcal{O})$	$\overline{\mathcal{A}}_{\langle r\rangle}(\mathcal{O})$	$\underline{\mathcal{R}}^{\mathfrak{b}}_{\langle r\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathfrak{b}}_{\langle r\rangle}(\mathcal{O})$
$\{z_1\}$	$\{z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2\}$	$\{z_4\}$	Φ	$\{z_2\}$	Φ	$\{z_2\}$	$\mho$
$\{z_3\}$	$\{z_4\}$	$\{z_2\}$	$\{z_2\}$	$\{z_3\}$	Φ	$\{z_3\}$
$\{z_4\}$	$\{z_4\}$	$\{z_1, z_3\}$	$\{z_2\}$	$\{z_4\}$	Φ	$\{z_4\}$
$\{z_1, z_2\}$	$\{z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_2\}$	$\mho$
$\{z_1, z_3\}$	$\{z_2, z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_1, z_4\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_4\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2, z_3\}$	$\{z_4\}$	$\{z_2\}$	$\{z_2\}$	$\{z_3\}$	$\{z_2\}$	$\mho$
$\{z_2, z_4\}$	$\{z_4\}$	$\{z_1, z_3\}$	$\{z_2\}$	$\{z_4\}$	$\{z_2\}$	$\mho$
$\{z_3, z_4\}$	$\{z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_2\}$	$\{z_3, z_4\}$	Φ	$\{z_3, z_4\}$
$\{z_1, z_2, z_3\}$	$\{z_2, z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_2, z_3\}$	$\mho$
$\{z_1, z_2, z_4\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_4\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_2, z_4\}$	$\mho$
$\{z_1, z_3, z_4\}$	$\mho$	$\{z_1, z_2, z_3\}$	$\mho$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2, z_3, z_4\}$	$\{z_4\}$	$\{z_1, z_2, z_3\}$	$\{z_2\}$	$\{z_3, z_4\}$	$\{z_2\}$	$\mho$
$\mho$	$\mho$	$\{z_1, z_2, z_3\}$	$\mho$	$\{z_1, z_3, z_4\}$	$\mho$	$\mho$
Φ	$\{z_4\}$	Φ	$\{z_2\}$	Φ	Φ	Φ

Table 11. Comparison between the methods of Abd El-Monsef et al., Dai et al. and the current method in general case.

$\mathcal{O} \subseteq \mho$	Dai et al.'s method [6]		Abd El-Monsef et al. method [11]		Current method	
	$\underline{\mathcal{R}}_m(\mathcal{O})$	$\overline{\mathcal{R}}_m(\mathcal{O})$	$\underline{\mathcal{R}}_r(\mathcal{O})$	$\overline{\mathcal{R}}_r(\mathcal{O})$	$\underline{\mathcal{R}}^{\flat}_{\langle r\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\flat}_{\langle r\rangle}(\mathcal{O})$
$\{z_1\}$	$\{z_2\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_2, z_3\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2\}$	$\{z_2\}$	Φ	Φ	$\{z_2\}$	$\{z_2\}$	$\mho$
$\{z_3\}$	$\{z_2\}$	$\{z_1, z_3\}$	Φ	$\{z_2, z_3\}$	Φ	$\{z_3\}$
$\{z_4\}$	$\{z_2\}$	$\{z_1, z_4\}$	$\{z_4\}$	$\mho$	Φ	$\{z_4\}$
$\{z_1, z_2\}$	$\{z_2\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_2, z_3\}$	$\{z_1, z_2\}$	$\mho$
$\{z_1, z_3\}$	$\{z_2, z_3\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_2, z_3\}$	Φ	$\{z_1, z_3, z_4\}$
$\{z_1, z_4\}$	$\{z_2, z_4\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_4\}$	$\mho$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2, z_3\}$	$\{z_2\}$	$\{z_1, z_3\}$	Φ	$\{z_2, z_3\}$	$\{z_2\}$	$\mho$
$\{z_2, z_4\}$	$\{z_2\}$	$\{z_1, z_4\}$	$\{z_4\}$	$\mho$	$\{z_2\}$	$\mho$
$\{z_3, z_4\}$	$\{z_2\}$	$\{z_1, z_3, z_4\}$	$\{z_4\}$	$\mho$	Φ	$\{z_3, z_4\}$
$\{z_1, z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_1, z_3, z_4\}$	Φ	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\mho$
$\{z_1, z_2, z_4\}$	$\{z_2, z_4\}$	$\{z_1, z_3, z_4\}$	$\{z_1, z_4\}$	$\mho$	$\{z_1, z_2, z_4\}$	$\mho$
$\{z_1, z_3, z_4\}$	$\mho$	$\{z_1, z_3, z_4\}$	$\{z_1, z_3, z_4\}$	$\mho$	Φ	$\{z_1, z_3, z_4\}$
$\{z_2, z_3, z_4\}$	$\{z_2\}$	$\{z_1, z_3, z_4\}$	$\{z_4\}$	$\mho$	$\{z_2\}$	$\mho$
$\mho$	$\mho$	$\{z_1, z_3, z_4\}$	$\mho$	$\mho$	$\mho$	$\mho$
Φ	$\{z_2\}$	Φ	Φ	Φ	Φ	Φ

Remark 4.6. The following are observed from Example 4.4:

(i) The converse of Theorem 4.3 does not hold generally; and

(ii) The basic-minimal approaches demonstrate greater accuracy compared to the methods proposed by Abd El-Monsef et al. [11].

The subsequent results delineate the connections between the proposed basic-minimal approximations and previous methodologies, encompassing Yao [3], Allam et al. [8], and Dai et al. [6], especially concerning a reflexive relation.

Utilizing Lemmas 3.3 and 3.4, the subsequent theorem can be established. Hence, the proof is deleted.

Theorem 4.4. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ constitutes a k-***NS*** with $\mathcal{R}$ being a reflexive relation, then for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the following holds:

(i) $\underline{\mathcal{A}}_k(\mathcal{O}) = \underline{\mathcal{R}}^{\flat}_k(\mathcal{O})$ and $\overline{\mathcal{A}}_k(\mathcal{O}) = \overline{\mathcal{R}}^{\flat}_k(\mathcal{O})$;

(ii) $Y_*(\mathcal{O}) \subseteq \underline{\mathcal{R}}^{\flat}_k(\mathcal{O}) \subseteq \mathcal{O} \subseteq \overline{\mathcal{R}}^{\flat}_k(\mathcal{O}) \subseteq Y^*(\mathcal{O})$; and

(iii) $\underline{\mathcal{R}}_m(\mathcal{O}) \subseteq \underline{\mathcal{R}}^{\flat}_k(\mathcal{O}) \subseteq \mathcal{O} \subseteq \overline{\mathcal{R}}^{\flat}_k(\mathcal{O}) \subseteq \overline{\mathcal{R}}_m(\mathcal{O})$.

Corollary 4.2. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ constitutes a k-***NS*** with $\mathcal{R}$ being a reflexive relation, then for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the following holds:

(i) $\mathfrak{B}^{\flat}_k(\mathcal{O}) = \mathcal{B}_k(\mathcal{O})$;

(ii) $\mathfrak{B}^{\flat}_k(\mathcal{O}) \subseteq \mathfrak{B}(\mathcal{O}) \subseteq \mathfrak{B}_m(\mathcal{O})$;

(iii) $\gamma^{\flat}_k(\mathcal{O}) = \mu_k(\mathcal{O})$; and

(iv) $\mu_m(\mathcal{O}) \leq \gamma(\mathcal{O}) \leq \gamma^{\flat}_k(\mathcal{O})$.

Corollary 4.3. If $(\mho, \mathcal{R}, \mathcal{F}_k)$ constitutes a k-***NS*** with $\mathcal{R}$ being a reflexive relation, then for each $k \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, the following holds:

(i) If $\mathcal{O}$ is a maximal-exact set, then it implies that $\mathcal{O}$ is Yao-exact, consequently making it a basic-exact set; and
(ii) If $\mathcal{O}$ is a maximal-exact set, then it implies that $\mathcal{O}$ is Yao-exact, consequently rendering it a minimal-exact set.

Note: It should be noted that the converse of the preceding results is not generally true, as demonstrated by Example 4.5.

Example 4.5. According to Example 3.2, we calculate the approximations for all subsets of $\mho$ using the current technique and the preceding methods (Yao [3] technique and Dai et al. [6] approach), as shown in Table 12.

Table 12. Comparison between the Yao technique, Dai approach and the current method in the case of a reflexive relation.

$\mathcal{O} \subseteq \mho$	Yao's method [3]		Dai et al.'s method		Current method	
	$Y_*(\mathcal{O})$	$Y^*(\mathcal{O})$	$\underline{\mathcal{R}}_m(\mathcal{O})$	$\overline{\mathcal{R}}_m(\mathcal{O})$	$\underline{\mathcal{R}}^{\mathfrak{b}}_{\langle r\rangle}(\mathcal{O})$	$\overline{\mathcal{R}}^{\mathfrak{b}}_{\langle r\rangle}(\mathcal{O})$
$\{z_1\}$	Φ	$\{z_1\}$	Φ	$\{z_1, z_2\}$	Φ	$\{z_1\}$
$\{z_2\}$	Φ	$\{z_1, z_2\}$	Φ	$\{z_1, z_2, z_3\}$	$\{z_2\}$	$\{z_1, z_2\}$
$\{z_3\}$	$\{z_3\}$	$\{z_2, z_3\}$	Φ	$\{z_2, z_3\}$	$\{z_3\}$	$\{z_3\}$
$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$	$\{z_4\}$
$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1, z_2\}$	$\{z_1\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2\}$	$\{z_1, z_2\}$
$\{z_1, z_3\}$	$\{z_3\}$	$\{z_1, z_2, z_3\}$	Φ	$\{z_1, z_2, z_3\}$	$\{z_3\}$	$\{z_1, z_3\}$
$\{z_1, z_4\}$	$\{z_4\}$	$\{z_1, z_4\}$	$\{z_4\}$	$\{z_1, z_2, z_4\}$	$\{z_4\}$	$\{z_1, z_4\}$
$\{z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_1, z_2, z_3\}$
$\{z_2, z_4\}$	$\{z_4\}$	$\{z_1, z_2, z_4\}$	$\{z_4\}$	$\mho$	$\{z_2, z_4\}$	$\{z_1, z_2, z_4\}$
$\{z_3, z_4\}$	$\{z_3, z_4\}$	$\{z_2, z_3, z_4\}$	$\{z_4\}$	$\{z_2, z_3, z_4\}$	$\{z_3, z_4\}$	$\{z_3, z_4\}$
$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$	$\{z_1, z_2, z_3\}$
$\{z_1, z_2, z_4\}$	$\{z_1, z_4\}$	$\{z_1, z_2, z_4\}$	$\{z_1, z_4\}$	$\mho$	$\{z_1, z_2, z_4\}$	$\{z_1, z_2, z_4\}$
$\{z_1, z_3, z_4\}$	$\{z_3, z_4\}$	$\mho$	$\{z_4\}$	$\mho$	$\{z_3, z_4\}$	$\{z_1, z_3, z_4\}$
$\{z_2, z_3, z_4\}$	$\{z_2, z_3, z_4\}$	$\mho$	$\{z_3, z_4\}$	$\mho$	$\{z_2, z_3, z_4\}$	$\mho$
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$
Φ	Φ	Φ	Φ	Φ	Φ	Φ

Remark 4.7. As observed from Theorem 4.4, Corollaries 4.2 and 4.3, and Example 4.5, the basic-minimal approaches demonstrate a greater accuracy compared to the methods proposed by Yao and Dai et al.

5. Decision-making in diagnosing heart failure using basic-minimal approaches

In this section, we emphasize the crucial role of a minimally structured framework in medical science, particularly in addressing decision-making complexities. Our focus is on applying this framework within the context of heart failure. The dataset includes the outcomes of five symptoms observed in twelve patients. This study was conducted at the Cardiology Department of Al-Azhar University, located at Sayed Galal University Hospital in Egypt [46]. The research involved twelve patients with diverse symptoms, all of whom underwent thorough medical assessments, including comprehensive medical histories, physical examinations, extensive laboratory analyses, resting electrocardiograms (ECGs), and traditional echocardiographic evaluations. Based on these assessments, the diagnosis of heart failure was either confirmed or excluded. This study analyzed the experimental results of an initial investigation that examined five symptoms correlated with heart disease, as delineated by Dickstein et al. [47].

Table 13 provides an overview of the heart failure issue, where the columns signify symptoms ('Yes' meaning symptom presence and 'No' indicating absence) associated with heart failure diagnoses (considered as condition attributes '$\mathcal{C}$'). Specifically, $\mathcal{H}_1$ stands for breathlessness, $\mathcal{H}_2$ for orthopnea, $\mathcal{H}_3$ for paroxysmal nocturnal dyspnea, $\mathcal{H}_4$ for a reduced exercise tolerance, and $\mathcal{H}_5$ for ankle swelling. The '$\mathcal{D}$' attribute represents the decision regarding heart failure. Within Table 13, the rows designated as $\mathcal{P} = \{\wp_1, \wp_2, \wp_3, \ldots, \wp_{12}\}$ correspond to the individual patients

Table 13. Original medical information system [46].

Person ($\mathcal{P}$)	symptoms ($\mathcal{C}$)					Decision ($\mathcal{D}$)
	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	
$\wp_1$	Yes	Yes	Yes	Yes	No	Yes
$\wp_2$	No	No	No	Yes	Yes	No
$\wp_3$	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_4$	No	No	No	Yes	No	No
$\wp_5$	Yes	No	No	Yes	Yes	No
$\wp_6$	No	No	No	Yes	No	No
$\wp_7$	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_8$	Yes	Yes	No	Yes	Yes	Yes
$\wp_9$	Yes	No	Yes	Yes	No	Yes
$\wp_{10}$	No	No	No	Yes	Yes	No
$\wp_{11}$	Yes	No	Yes	Yes	No	Yes
$\wp_{12}$	Yes	No	No	Yes	Yes	No

We initiate the application process by transforming the descriptive attributes (condition attributes) $\mathcal{C} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5\}$ into qualitative terms, as presented in Table 13. This table encapsulates the resemblances among the patient symptoms, where the degree of similarity $\psi\,(\wp_i, \wp_j)$ is defined by:

$$\psi\,(\wp_i, \wp_j) = \frac{\sum_{g=1}^{n}[a_g(\wp_i) = a_g(\wp_j)]}{n},$$

where:

- i, $j \in \{1,2,3,\ldots,12\}$;
- a_g represents an attribute, i.e., $a_g \in \mathcal{C}$;
- n represents the number of condition attributes.

Therefore, we compute the similarities between the symptoms of the 12 patients as follows:

For $\wp_1$: It is evident that $\wp_1$ and $\wp_2$ share the same value for symptom $\mathcal{H}_4$, thus the similarity between $\wp_1$ and $\wp_2$ is $\frac{1}{5}$.

Similarly, p_1 and p_3 share the same values for symptoms $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, thus the similarity between p_1 and p_3 is $\frac{4}{5}$.

Using the same method, we evaluate the similarities between all the patients, as illustrated in Table 14.

Table 14. Similarities between symptoms of twelve of patients.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}
p_1	1	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$
p_2	$\frac{1}{5}$	1	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{4}{5}$
p_3	$\frac{4}{5}$	$\frac{2}{5}$	1	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	1	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
p_4	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	1	$\frac{3}{5}$	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
p_5	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	1	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	1
p_6	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	1	$\frac{3}{5}$	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
p_7	$\frac{4}{5}$	$\frac{2}{5}$	1	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	1	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
p_8	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	1	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
p_9	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	1	$\frac{2}{5}$	1	$\frac{3}{5}$
p_{10}	$\frac{1}{5}$	1	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{4}{5}$
p_{11}	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	1	$\frac{2}{5}$	1	$\frac{3}{5}$
p_{12}	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	1	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	1

Our next step involves constructing a minimal structured space based on the relationship that aligns with the inherent nature of the problem under study. It's important to highlight that we describe the connection within each issue constructed by the criteria specified by the experts. In this context, we denote $p_i \mathcal{R} p_j \Leftrightarrow \psi\,(p_i, p_j) \geq \frac{4}{5}$, $\forall i$, $j = \{1,2,3,\ldots,12\}$, where $\psi\,(p_i, p_j)$ represents the sum of similar symptoms between 'p_i' and 'p_j' divided by the total number of symptoms.

Note: The above process suggests a relation based on the requirements of system experts' perspective. It is assumed that this relation, along with the number 4/5, represents a similar degree, with a higher number indicating an increased similarity, thus providing more accurate results. Furthermore, both this relation and the number 4/5 can be adjusted according to the concepts of system experts. It is evident that the suggested relation is reflexive and symmetric, but not transitive, which renders the Pawlak approximations space inadequate to describe system.

Therefore, to compute all the r-neighborhoods, for each patient, we proceed as follows:

For $\boldsymbol{\wp_1}$**:** $\mathbb{N}_r(\wp_1) = \{\wp_i \in \mathcal{P} |\ \psi\ (\wp_1, \wp_i) \geq \frac{4}{5}\} = \{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$. In a similar way, the r-neighborhoods for the other patients are determined and presented in Table 15.

Next, we construct all the m-neighborhoods for each patient as follows:

For $\boldsymbol{\wp_1}$**:** $\mathbb{N}_m(\wp_1) = \cup_{\wp_1 \in \mathbb{N}_r(\wp_i)} \mathbb{N}_r(\wp_i) = \{\wp_1, \wp_3, \wp_7, \wp_8, \wp_9, \wp_{11}\}$.

Similarly, the m-neighborhoods for the other patients are derived and shown in Table 15.

Now, to compute the basic-minimal neighborhoods $\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(\boldsymbol{x})$, we first determine the minimal-neighborhoods $\mathbb{N}_{\langle r \rangle}(\boldsymbol{x})$ for each patient as follows:

For $\boldsymbol{\wp_1}$**:** $\mathbb{N}_{\langle r \rangle}(\wp_1) = \cap_{\wp_1 \in \mathbb{N}_r(\wp_i)} \mathbb{N}_r(\wp_i) = \{\wp_1\}$. Following the same procedure, the $\mathbb{N}_{\langle r \rangle}(\boldsymbol{x})$ for the other patients is calculated.

Consequently, the basic-minimal neighborhoods $\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(\boldsymbol{x})$ for each patient as follows:

For $\boldsymbol{\wp_1}$**:** $\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(\wp_1) = \{\wp_i \in \mathcal{P}\text{: } \mathbb{N}_{\langle r \rangle}(\wp_i) \subseteq \mathbb{N}_{\langle r \rangle}(\wp_1)\} = \{\wp_1\}$.

Similarly, the $\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(\boldsymbol{x})$ for the other patients are calculated and listed in Table 15.

Therefore, we proceed to construct the right neighborhoods, the maximal neighborhoods, and the basic-minimal right neighborhoods for each patient within the universe, as displayed in Table 15. These constructions utilize the relationship that corresponds to the specific nature of the problem under study.

Table 15. r-neighborhoods, m-neighborhoods, and basic $\langle r \rangle$-neighborhoods of each patient.

	$\mathbb{N}_r(\boldsymbol{x})$	$\mathbb{N}_m(\boldsymbol{x})$	$\mathbb{N}^{\mathfrak{b}}_{\langle r \rangle}(\boldsymbol{x})$
$\boldsymbol{\wp_1}$	$\{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$	$\{\wp_1, \wp_3, \wp_7, \wp_8, \wp_9, \wp_{11}\}$	$\{\wp_1\}$
$\boldsymbol{\wp_2}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_8, \wp_{10}, \wp_{12}\}$	$\{\wp_2\}$
$\boldsymbol{\wp_3}$	$\{\wp_1, \wp_3, \wp_7, \wp_8\}$	$\{\wp_1, \wp_3, \wp_5, \wp_7, \wp_8, \wp_9, \wp_{11}, \wp_{12}\}$	$\{\wp_3, \wp_7\}$
$\boldsymbol{\wp_4}$	$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$
$\boldsymbol{\wp_5}$	$\{\wp_2, \wp_5, \wp_8, \wp_{10}, \wp_{12}\}$	$\mathcal{P}$-$\{\wp_1, \wp_9, \wp_{11}\}$	$\{\wp_5, \wp_{12}\}$
$\boldsymbol{\wp_6}$	$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$
$\boldsymbol{\wp_7}$	$\{\wp_1, \wp_3, \wp_7, \wp_8\}$	$\{\wp_1, \wp_2, \wp_3, \wp_5, \wp_7, \wp_8, \wp_{12}\}$	$\{\wp_3, \wp_7\}$
$\boldsymbol{\wp_8}$	$\{\wp_3, \wp_5, \wp_7, \wp_{8,}, \wp_{12}\}$	$\mathcal{P}$-$\{\wp_4, \wp_6, \wp_9, \wp_{11}\}$	$\{\wp_8\}$
$\boldsymbol{\wp_9}$	$\{\wp_1, \wp_9, \wp_{11}\}$	$\{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$	$\{\wp_1, \wp_9, \wp_{11}\}$
$\boldsymbol{\wp_{10}}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_8, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_{10}\}$
$\boldsymbol{\wp_{11}}$	$\{\wp_1, \wp_9, \wp_{11}\}$	$\{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$	$\{\wp_1, \wp_9, \wp_{11}\}$
$\boldsymbol{\wp_{12}}$	$\{\wp_2, \wp_5, \wp_{8,}, \wp_{12}\}$	$\mathcal{P}$-$\{\wp_1, \wp_{11}\}$	$\{\wp_5, \wp_{12}\}$

From Table 13, the universe is divided into the following two independent sets are:

- The group of patients diagnosed with the disease: $\mathbb{S} = \{\boldsymbol{\wp_1}, \boldsymbol{\wp_3}, \boldsymbol{\wp_7}, \boldsymbol{\wp_8}, \boldsymbol{\wp_9}, \boldsymbol{\wp_{11}}\}$; and
- The group of patients without a diagnosis of heart failure: $\mathbb{T} = \{\boldsymbol{\wp_2}, \boldsymbol{\wp_4}, \boldsymbol{\wp_5}, \boldsymbol{\wp_6}, \boldsymbol{\wp_{10}}, \boldsymbol{\wp_{12}}\}$.

Therefore, by employing the suggested approximations (basic-minimal approximations) alongside previous approaches (Yao [3] and Dai et al. [6]), we can assess the accuracy of the decision-making for the two patient groups, as illustrated in Table 16. Following this, we present the Discussions section, which summarizes the concluding remarks and provides an analysis of this application. The discussion on the results has been expanded to offer a more in-depth analysis. Additionally, the validation section includes comparisons with the existing applications in the field and discusses the advantages of our approach.

Table 16. Comparison among the present technique and the alternative methods.

Set		$\mathbb{S}$	$\mathbb{T}$
		$\{\wp_1, \wp_3, \wp_7, \wp_8, \wp_9, \wp_{11}\}$	$\{\wp_2, \wp_4, \wp_5, \wp_6, \wp_{10}, \wp_{12}\}$
Yao method	$Y_*(\mathcal{O})$	$\{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$	$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$
	$Y^*(\mathcal{O})$	$\mathcal{P}$-$\{\wp_2, \wp_4, \wp_6, \wp_{10}\}$	$\mathcal{P}$-$\{\wp_1, \wp_3, \wp_7, \wp_9, \wp_{11}\}$
	$\mathfrak{B}(\mathcal{O})$	$\{\wp_5, \wp_8, \wp_{12}\}$	$\{\wp_5, \wp_8, \wp_{12}\}$
	$\gamma(\mathcal{O})$	$\frac{5}{8}$	$\frac{4}{7}$
Dai et al. method	$\underline{\mathcal{R}}_m(\mathcal{O})$	$\{\wp_1, \wp_9, \wp_{11}\}$	$\{\wp_4, \wp_6\}$
	$\overline{\mathcal{R}}_m(\mathcal{O})$	$\mathcal{P}$- $\{\wp_4, \wp_6\}$	$\mathcal{P}$-$\{\wp_1, \wp_9, \wp_{11}\}$
	$\mathfrak{B}_m(\mathcal{O})$	$\{\wp_2, \wp_3, \wp_5, \wp_7, \wp_8, \wp_{10}, \wp_{12}\}$	$\{\wp_2, \wp_3, \wp_5, \wp_7, \wp_8, \wp_{10}, \wp_{12}\}$
	$\gamma_m(\mathcal{O})$	$\frac{3}{10}$	$\frac{2}{9}$
Current method	$\underline{\mathcal{R}}_k^{\flat}(\mathcal{O})$	$\mathbb{S}$	$\mathbb{T}$
	$\overline{\mathcal{R}}_k^{\flat}(\mathcal{O})$	$\mathbb{S}$	$\mathbb{T}$
	$\mathfrak{B}_k^{\flat}(\mathcal{O})$	Φ	Φ
	$\gamma_k^{\flat}(\mathcal{O})$	1	1

6. Discussions

In the realm of medical science, effective decision-making frameworks play a pivotal role in navigating the complexities inherent in diagnoses, particularly in conditions such as heart failure. Our focus in this study was to highlight the application of a minimally structured framework within the context of diagnosing heart failure, leveraging data obtained from the Cardiology Department of Al-Azhar University, situated at the Sayed Galal University Hospital in Egypt.

The dataset encompassed observations from twelve patients exhibited a spectrum of symptoms associated with heart failure. Thorough medical assessments, including detailed medical histories, physical examinations, laboratory analyses, electrocardiograms, and echocardiographic evaluations, were conducted to ascertain the diagnosis. Through a structured inquiry, we sought to elucidate the efficacy of our proposed methodologies to enhance diagnostic accuracy within this medical domain.

The initial investigation focused on analyzing five key symptoms correlated with heart disease, as identified by Dickstein et al. [47]. The subsequent transformation of the descriptive attributes into qualitative terms facilitated the computation of similarities among patient symptoms, which was a critical step in our diagnostic approach. By constructing the minimal structured spaces based on these relationships, we aimed to delineate distinct the patient groups based on their symptom profiles.

From the constructed structured spaces, it became apparent that the universe could be divided into two independent sets: patients diagnosed with heart failure and those without. This segmentation provided a foundational basis for further analyses, enabling the evaluation of the diagnostic accuracy across patient groups.

The application of our proposed methodologies, particularly the basic-minimal approximations, alongside traditional approaches such as those by Yao [3] and Dai et al. [6], yielded insightful comparisons. Notably, our methodologies exhibited high accuracy coefficients, which closely aligned with the medical diagnoses derived from empirical data. In contrast, the previous methods demonstrated limitations in accurately identifying patients with heart failure, underscoring the need for more refined diagnostic frameworks (see the highlighted cells in Table 16).

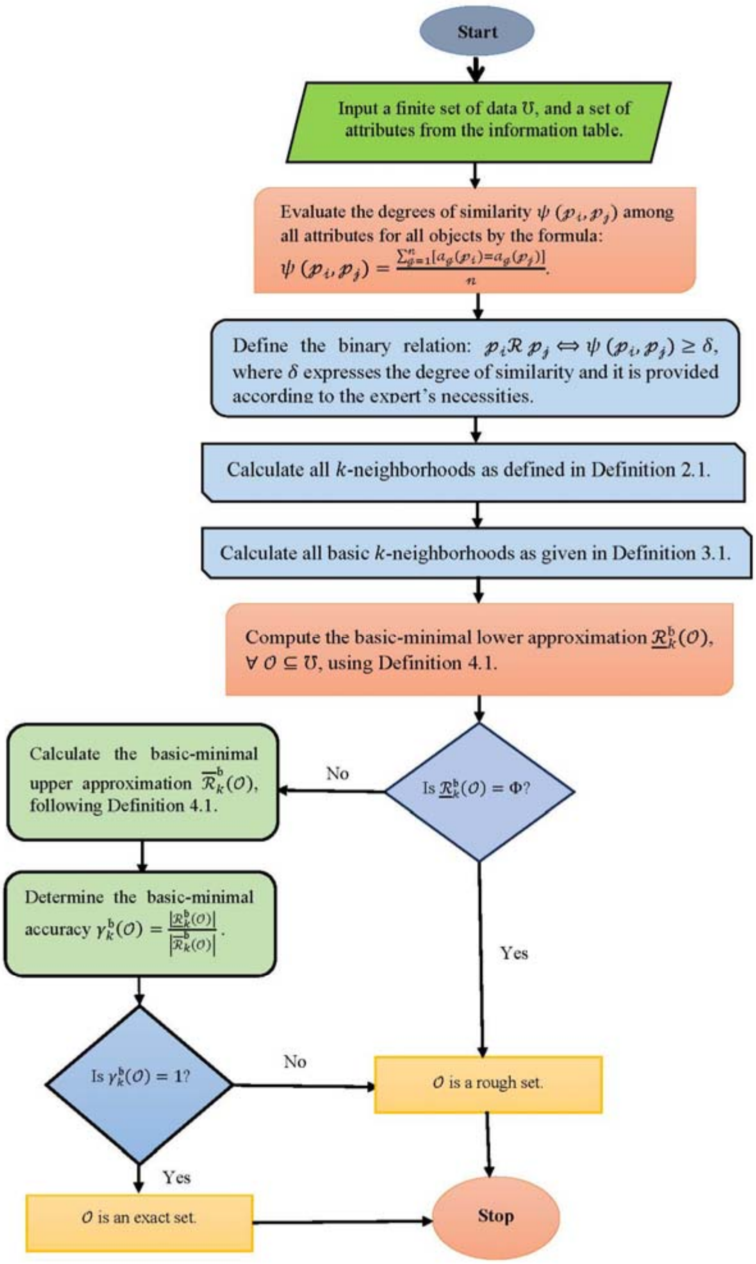

Figure 1. Flowchart for using basic-minimal approximations in decision-making problems.

For instance, the set of patients diagnosed with the disease according to the doctor's decision was $\mathbb{S} = \{\wp_1, \wp_3, \wp_7, \wp_8, \wp_9, \wp_{11}\}$. Using the method by Dai et al., the lower approximation was $\{\wp_1, \wp_9, \wp_{11}\}$, which indicated that only patients $\wp_1$, $\wp_9$, and $\wp_{11}$ were identified as having heart failure, which contradicts the decision table and the doctor's decision. Conversely, our methods yielded an accuracy measure of 100%, meaning that the set of patients with heart failure was equivalent to the set $\mathbb{S}$ determined by the doctor's decision.

Moreover, the boundary region, which represents the doubtful or uncertain region of two sets (patients with heart failure and healthy individuals), was the same according to Dai et al.'s technique, namely $\{\wp_2, \wp_3, \wp_5, \wp_7, \wp_8, \wp_{10}, \wp_{12}\}$. This means that these individuals could not be definitively identified as patients with or without heart failure. On the other hand, our approach resulted in an empty boundary region, which provided an accurate measure for diagnosis.

Our findings underscored the superiority of the proposed approaches in enhancing the approximation operators and the accuracy measures under diverse binary relations. Importantly, these methodologies upheld the core principles of Pawlak's framework without imposing restrictive conditions, thereby expanding the scope of the practical problems amenable to effective solutions.

In conclusion, our study not only sheds light on the efficacy of minimally structured frameworks in medical decision-making, but also underscores the transformative potential of advanced methodologies to enhance the diagnostic accuracy and clinical outcomes. Moving forward, further research in this direction holds promise to advance the frontiers of medical diagnostics and improve patient care outcomes.

We present an algorithm and a corresponding flowchart of the proposed techniques (basic-minimal approximations) to aid in decision-making problems. This algorithm (Algorithm 1), illustrated in (Figure 1) serves as a straightforward tool that can be utilized in MATLAB.

Algorithm 1. A Framework for using basic-minimal approximations in decision-making problems.

Input: Table of information data consists of a set of objects $\mho$ in the first column and a set of attributes $\mathcal{C}$ in the first row.

Output: Provide accurate determinations for exactness and roughness.

Step 1: Evaluate the degrees of similarity $\psi(\wp_i, \wp_j)$ among all attributes for each object using the following formula: $\psi(\wp_i, \wp_j) = \frac{\sum_{g=1}^{n}[a_g(\wp_i) = a_g(\wp_j)]}{n}$, where n represents the number of condition attributes. Then, generate the table illustrating similarities among the attributes for all objects.

Step 2: Establish the binary relation $\wp_i \mathcal{R} \wp_j \Leftrightarrow \psi(\wp_i, \wp_j) \geq \delta$, where δ denotes the degree of similarity, which is tailored to the expert requirements.

Step 3: For each $\wp \in \mho$, compute the following:

(i) All k-neighborhoods, where $k \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, as defined in Definition 2.1.

(ii) All basic k-neighborhoods, where $k \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, as defined in Definition 3.1.

Step 4: For every $\mathcal{O} \subseteq \mho$, perform the following:

(i) By using Definition 4.1, calculate the basic-minimal lower approximation $\underline{\mathcal{R}}_k^{\mathfrak{b}}(\mathcal{O})$;

(ii) If $\underline{\mathcal{R}}_k^{\mathrm{b}}(\mathcal{O}) = \Phi$, conclude that $\mathcal{O}$ is a rough set; and

(iii) Otherwise, perform the following steps:

Step 5: Calculate the basic-minimal upper approximation $\overline{\mathcal{R}}_k^{\mathrm{b}}(\mathcal{O})$, following Definition 4.1;

Step 6: Determine the basic-minimal accuracy $\gamma_k^{\mathrm{b}}(\mathcal{O}) = \frac{|\underline{\mathcal{R}}_k^{\mathrm{b}}(\mathcal{O})|}{|\overline{\mathcal{R}}_k^{\mathrm{b}}(\mathcal{O})|}$;

(i) If $\gamma_k^{\mathrm{b}}(\mathcal{O}) = 1$, designate $\mathcal{O}$ as an exact set.

(ii) Otherwise, conclude that $\mathcal{O}$ is a rough set.

Step 7: End.

7. Conclusions and future work

The RS theory significantly hinges on the RS operators and precision values, which are crucial elements that underpin its practical applications. These components not only offer insights into the data within subsets, but also gauge the representation of these subsets within the broader dataset. Enhanced operators and precision values can invariably lead to more accurate predictions, thus driving research efforts towards refining these aspects.

In this pursuit, there has been a focused exploration into the "basic-minimal approximations" derived from general binary relations, leveraging novel neighborhood constructions, termed basic-minimal neighborhoods. This endeavor expands the horizons of Pawlak's approximation theory, aiming to more effectively capture nuances in data representation. These advanced approximations, characterized by their minimal basics, have demonstrated a marked superiority over the preceding methodologies, notably surpassing the effectiveness of approaches proposed by the methods of Yao [3], Dai et al. [6], Allam et al. [8], Abu-Gdairi [9], and Abd El-Monsef et al. [11]. Particularly in navigating the diagnostic complexities that arise from the symptom similarity, they have substantially enhanced the diagnostic accuracy. The robustness of these methods is substantiated by rigorous mathematical proofs, as encapsulated in Theorem 4.3, Theorem 4.4, and their accompanying corollaries. These analyses underscore the heightened accuracy achieved by the proposed basic-minimal approximations compared to the alternative methodologies. Furthermore, concrete illustrations provided through Examples 4.4 and 4.5, which were complemented by tabulated data (Table 10–12), served to elucidate and reinforce these findings.

One of the noteworthy contributions of this study lies in its endeavor to bridge the RS theory with topology, thereby unveiling topological structures inherent in these approximations. This interdisciplinary approach opens avenues for deeper explorations into topology within the realm of the RS theory, accentuating the pivotal role played by these approximations in delineating topologies.

In essence, this research not only delves into fortifying the theoretical underpinnings of RS theory, but also accentuates its practical ramifications, especially in scenarios characterized by overlapping symptoms. The exceptional accuracy achieved in diagnosing heart failure, as evidenced by a perfect 100% accuracy rate in a dataset sourced from Al-Azhar University's Cardiology Department, underscores the efficacy and real-world applicability of these methodologies. In contrast, conventional methods, such as Dai et al.'s approach, have exhibited limitations in discerning between patients with heart failure and healthy individuals, thereby engendering uncertainty in decision-making processes. Moving forward, the proposed methodologies hold promise for broader applications beyond medical diagnostics. The formulation of a streamlined algorithm, complemented by a structured workflow to be implemented in programming languages such as MATLAB, paves the way for seamless integration into decision-making frameworks across diverse domains.

• Strengths and advantages of the approaches:

Based on the above, challenges, benefits, and strengths of the presented methods can be summarized as follows:

1) The methodology articulated in this paper hinges on generalized neighborhood systems derived from binary relations, which are devoid of restrictive conditions. This inclusivity not only amplifies its applicability, but also underscores its robustness across varied domains.
2) The versatility of this approach shines through its ability to tackle practical challenges under arbitrary relations, circumventing the constraints associated with equivalence relations, as prevalent in conventional methodologies.
3) The delineation of four distinct approaches, with the model predicated on the basic $\langle i \rangle$-neighborhood emerging as the most accurate, facilitates nuanced comparisons and insights into different approximation techniques and precision values, as demonstrated in the results obtained see (see Proposition 4.2 and Corollary 4.1).
4) Proposition 4.1 serves as a testament to the fidelity of the methodologies proposed herein, preserving the foundational tenets of Pawlak's framework without imposing arbitrary restrictions.
5) The scalability of this approach renders it well-suited to handle large datasets, owing to its reliance on neighborhood constructs that are readily discernible through data classification.
6) Notably, the methodologies outlined in this paper offer a heightened accuracy in decision-making processes, particularly in scenarios where accuracy is paramount, such as in infectious disease management cases such as COVID-19, where precision directly correlates with the sample size.
7) Focusing on diagnosing heart failure, the application described in this paper achieves an exceptional accuracy, reaching 100% in a dataset from Al-Azhar University's Cardiology Department. The mathematical results closely align with the decisions made by physicians, which accurately identified patients with heart failure. In contrast, previous methods (such as Yao, and Dai et al.'s methods) have faltered in distinguishing between patients with heart failure and healthy individuals, thereby introducing uncertainty into decision-making processes.

In conclusion, the methodologies delineated in this study not only advance the theoretical frontiers of the RS theory, but also hold profound implications for practical decision-making across diverse domains, promising an enhanced accuracy and reliability in complex decision-making scenarios.

Future works:

In our forthcoming investigations, we will delve into the following areas:

1) **Conducting comparative studies:** We plan to conduct comprehensive comparative studies of the suggested approaches (basic-minimal approximations) against other methodologies. Notably, we aim to compare our proposed models with recent advancements, including ternary models [39,45], to highlight improvements in the accuracy and generalization.
2) **Exploring expanded domains:** Our research will extend to explore the application of the proposed methods in expanded domains, particularly within medical contexts [48–50] and economic applications [21,22]. By venturing into these diverse domains, we aim to assess the versatility and efficacy of our methodologies across varied application scenarios.

3) **Implementation across various frameworks:** We intend to shed light on the implementation of basic-minimal approximations across various frameworks, including fuzzy sets [49], soft sets [50], soft RS's [51], and their utilization in Multicriteria decision-making applications. Additionally, we aim to explore their application in decision-theoretic RS's [24–27], rough fuzzy sets [29,30], fuzzy topological spaces [52], Fuzzy soft topological structures [53,54], Ideals and girll applications [55–57], as well as Rough lattice, Graph medical applications, and Generalized picture fuzzy soft sets [58–62].In essence, our future endeavors aim to further validate and extend the applicability of our proposed methodologies across diverse domains, paving the way for advancements in both theoretical frameworks and practical applications.

Author contributions

Conceptualization and methodologies were developed by M. K. El-Bably and D. I. Taher. M. A. El-Gayar and R. Abu-Gairi contributed to validation and data curation. All authors collaborated on writing, reviewing, and editing the manuscript. Project administration was conducted by M. K. El-Bably. M. K. El-Bably prepared Figures 1 and Algorithm 1. M. A. El-Gayar and D. I. Taher reviewed the validation of Algorithm 1. R. Abu-Gairi funded this research at Zarqa University, Jordan. All authors have thoroughly reviewed and approved the final version of the manuscript for publication.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare that there is no conflict of interest regarding the publication of this manuscript.

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