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Research article

Yabu's formulae for hypergeometric ${}_3F_2$-series through Whipple's quadratic transformations

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Abstract: By means of Whipple's quadratic transformations, two classes of hypergeometric ${}_3F_2$-series are expressed in terms of the Lerch transcendent function. Several difficult series with a free variable are explicitly evaluated in closed form, including Yabu's three remarkable identities.

Keywords: hypergeometric series; Whipple's quadratic transformations; the Lerch transcendent function
Mathematics Subject Classification: Primary 33C20, Secondary 33C90

1. Introduction and outline

Denote by $\mathbb{Z}$ and $\mathbb{N}$, respectively, the sets of integers and natural numbers. For an indeterminate x, define the rising factorials by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

According to Bailey [3, §2.1], the classical hypergeometric series, for $m \in \mathbb{N}$ and an indeterminate z, reads as

$${}_{1+m}F_m\left[\begin{matrix} a_0, & a_1, & \cdots, & a_m \\ & b_1, & \cdots, & b_m \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_m)_k}{k!\,(b_1)_k\cdots(b_m)_k} z^k.$$

When $|z| < 1$ and none of the numerator and denominator parameters results in a non-positive integer, the corresponding series is not only convergent, but also well-defined and nonterminating.

There exist numerous hypergeometric series identities in the literature (see [4, Chapter 8] and [6,7]). Recently, algebraic expressions for certain classes of ${}_3F_2$-series arose much attention (see [1,2,5]). In particular, Yabu [9] succeeded in evaluating explicitly the following series with a free variable x in terms of the logarithmic function:

$$ {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix}\middle|x\right],\ {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{7}{4}\end{matrix}\middle|x\right],\ {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{matrix}\middle|x\right]. $$

The formulae for these series are remarkable, since it is rare that a hypergeometric series of higher order beyond Gauss' classical ${}_2F_1$-series with a free variable turns into a closed algebraic expression. Motivated by Yabu's formulae, we shall investigate two general classes of the ${}_3F_2$-series as below:

$$ \mathcal{F}(m,\lambda,y) := {}_3F_2\left[\begin{matrix}1,\ \frac{m}{2},\ \frac{1+m}{2}\\ 1+\lambda, 1+m-\lambda\end{matrix}\middle|y\right], $$

$$ \mathcal{G}(m,\lambda,y) := {}_3F_2\left[\begin{matrix}1,\ \frac{1+m}{2},\ 1+\frac{m}{2}\\ 1+\lambda, 1+m-\lambda\end{matrix}\middle|y\right], $$

where $\lambda \in (0,1)$ and $m \in \mathbb{N}$, with y being a free variable subject to $|y| < 1$ such that the series are convergent. Instead of algebraic-geometric approach employed in [1, 2, 9], we find that the quadratic transformations due to Whipple [8] (cf. Bailey [3, page 97]) are more efficient. To facilitate their subsequent use, they are reproduced as follows:

$$ {}_3F_2\left[\begin{matrix}\frac{a}{2},\ \frac{1+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\middle|y\right] = (1-x)^a {}_3F_2\left[\begin{matrix}a,\ b,\ c\\ 1+a-b, 1+a-c\end{matrix}\middle|x\right], \tag{1} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1+a}{2}, 1+\frac{a}{2}, 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\middle|y\right] = \frac{(1-x)^{1+a}}{1+x}{}_4F_3\left[\begin{matrix}a,\ 1+\frac{a}{2},\ b,\ c\\ \frac{a}{2}, 1+a-b, 1+a-c\end{matrix}\middle|x\right], \tag{2} $$

where the two variables are related by equations

$$ y = \frac{-4x}{(1-x)^2} \qquad \rightleftharpoons \qquad x = \frac{(1-\sqrt{1-y})^2}{-y}, \tag{3} $$

with the domain $y \in (-1,1)$ and the codomain $x \in (-1, 3-2\sqrt{2})$, respectively.

In the next section, we shall first reformulate $\mathcal{F}(m,\lambda,y)$ by means of (1) and then evaluate the resulting series by the Lerch transcendent function. Then, in Section 3, the series $\mathcal{G}(m,\lambda,y)$ will be treated analogously via the second quadratic transformation (2). The two main theorems (Theorems 1 and 2) state that the series $\mathcal{F}(m,\lambda,y)$ (also $\mathcal{G}(m,\lambda,y)$) results in a two-term linear combination of the Lerch transcendent function plus a remainder polynomial. Finally, the paper will end in Section 4, where several difficult series are explicitly evaluated in closed form as applications. Compared with the algebraic method adopted by Yabu [9], the authors believe that the approach presented in this paper is simpler and more accessible.

2. Evaluation of series $\mathcal{F}(m,\lambda,y)$

In Whipple's first transformation (1), by specifying the parameters

$$ a = m,\ c = \lambda,\ b = a-c, $$

we can reformulate the series $\mathcal{F}(m,\lambda,y)$ as

$$ {}_3F_2\left[\begin{matrix} 1,\ \frac{m}{2},\ \frac{1+m}{2} \\ 1+\lambda, 1+m-\lambda \end{matrix}\middle| y\right] = (1-x)^m {}_3F_2\left[\begin{matrix} m,\ \lambda,\ m-\lambda \\ 1+\lambda, 1+m-\lambda \end{matrix}\middle| x\right]. \tag{4} $$

The rightmost ${}_3F_2(x)$-series can be explicitly expressed as

$$ {}_3F_2\left[\begin{matrix} m,\ \lambda,\ m-\lambda \\ 1+\lambda, 1+m-\lambda \end{matrix}\middle| x\right] = \frac{\lambda(m-\lambda)}{(m-1)!}\sum_{n=0}^{\infty}\frac{(n+1)_{m-1}x^n}{(n+\lambda)(n+m-\lambda)}. \tag{5} $$

Keeping in mind that $\lambda \in (0,1)$ and $m \in \mathbb{N}$, it suffices to examine the case "$m \neq 2\lambda$". Otherwise, the only case exists for "$m = 2\lambda = 1$", in which we have a simpler series

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} 1,\ 1,\ \frac{1}{2} \\ \frac{3}{2},\ \frac{3}{2} \end{matrix}\middle| y\right] =&(1-x)\times {}_3F_2\left[\begin{matrix} 1,\ \frac{1}{2},\ \frac{1}{2} \\ \frac{3}{2},\ \frac{3}{2} \end{matrix}\middle| x\right] \\ =&\sum_{n=0}^{\infty}\frac{(1-x)x^n}{(2n+1)^2} = \frac{1-x}{4}\Phi\Big(x,2,\frac{1}{2}\Big), \end{aligned} $$

where $\Phi(\cdots)$ stands for the Lerch transcendent function:

$$ \Phi(z,\sigma,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^{\sigma}}, \text{ for } |z|<1,\ \Re(\sigma)>0 \text{ and } \alpha \notin \mathbb{Z}\backslash\mathbb{N}. $$

Now rewrite the rational function by

$$ R(n) = \frac{(n+1)_{m-1}}{(n+\lambda)(n+m-\lambda)} = \frac{(n+1)_{m-1}}{m-2\lambda}\left\{\frac{1}{n+\lambda} - \frac{1}{n+m-\lambda}\right\}. \tag{6} $$

According to the Chu-Vandermonde convolution formula

$$ \begin{aligned} (n+1)_{m-1} &= \sum_{i=1}^{m}\binom{m-1}{i-1}(n+\lambda)_{i-1}(1-\lambda)_{m-i} \\ &= \sum_{i=1}^{m}\binom{m-1}{i-1}(n+m-\lambda)_{i-1}(1-m+\lambda)_{m-i}, \end{aligned} $$

we can express

$$ \begin{aligned} R(n) = \frac{1}{m-2\lambda}\bigg\{&\frac{(1-\lambda)_{m-1}}{n+\lambda} + \sum_{i=2}^{m}\binom{m-1}{i-1}(1+n+\lambda)_{i-2}(1-\lambda)_{m-i} \\ &- \frac{(1-m+\lambda)_{m-1}}{n+m-\lambda} - \sum_{i=2}^{m}\binom{m-1}{i-1}(1+n+m-\lambda)_{i-2}(1-m+\lambda)_{m-i}\bigg\}. \end{aligned} $$

Rewrite further the shifted factorials

$$(1+n+\lambda)_{i-2}=\sum_{k=2}^{i}\binom{i-2}{k-2}(n+1)_{k-2}(\lambda)_{i-k},$$
$$(1+n+m-\lambda)_{i-2}=\sum_{k=2}^{i}\binom{i-2}{k-2}(n+1)_{k-2}(m-\lambda)_{i-k},$$

and then making substitutions, we can manipulate the double series

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} m,\ \lambda,\ m-\lambda \\ 1+\lambda, 1+m-\lambda \end{matrix}\middle| x\right]=\frac{\lambda(m-\lambda)}{(m-1)!}\sum_{n=0}^{\infty}R(n)x^n\\
=&\frac{\lambda(m-\lambda)}{(m-1)!(m-2\lambda)}\sum_{n=0}^{\infty}\left\{\frac{(1-\lambda)_{m-1}}{n+\lambda}x^n-\frac{(1-m+\lambda)_{m-1}}{n+m-\lambda}x^n\right\}\\
+&\frac{\lambda(m-\lambda)}{(m-1)!(m-2\lambda)}\sum_{n=0}^{\infty}x^n\sum_{i=2}^{m}\binom{m-1}{i-1}(1-\lambda)_{m-i}\sum_{k=2}^{i}\binom{i-2}{k-2}(n+1)_{k-2}(\lambda)_{i-k}\\
-&\frac{\lambda(m-\lambda)}{(m-1)!(m-2\lambda)}\sum_{n=0}^{\infty}x^n\sum_{i=2}^{m}\binom{m-1}{i-1}(1-m+\lambda)_{m-i}\sum_{k=2}^{i}\binom{i-2}{k-2}(n+1)_{k-2}(m-\lambda)_{i-k}.
\end{aligned}
$$

- The sum in the first line results in

$$\frac{\lambda(1-\lambda)_m}{(m-1)!(m-2\lambda)}\Phi(x,1,\lambda)-\frac{(m-\lambda)(1-m+\lambda)_m}{(m-1)!(m-2\lambda)}\Phi(x,1,m-\lambda).$$

- The double sum in the middle line can be simplified into a finite sum

$$
\begin{aligned}
&\frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\binom{m-i-\lambda}{m-i}\sum_{k=2}^{i}(-1)^{i-k}\binom{-\lambda}{i-k}\sum_{n=0}^{\infty}\frac{x^n}{i-1}\binom{n+k-2}{k-2}\\
=&\frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{m-i-\lambda}{m-i}\binom{-\lambda}{i-k}\frac{(1-x)^{1-k}}{i-1}.
\end{aligned}
$$

- The double sum in the ultimate line can be reduced analogously to a finite sum

$$
\begin{aligned}
&\frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\binom{\lambda-i}{m-i}\sum_{k=2}^{i}(-1)^{i-k}\binom{\lambda-m}{i-k}\sum_{n=0}^{\infty}\frac{x^n}{i-1}\binom{n+k-2}{k-2}\\
=&\frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{\lambda-i}{m-i}\binom{\lambda-m}{i-k}\frac{(1-x)^{1-k}}{i-1}.
\end{aligned}
$$

Summing up, we have established the following theorem:

Theorem 1 ($m \neq 2\lambda$)**.** *For two variables x and y related by* (3)*, we have*

$$
{}_3F_2\left[\begin{matrix}1,\ \frac{m}{2},\ \frac{1+m}{2}\\ 1+\lambda, 1+m-\lambda\end{matrix}\middle| y\right] = (1-x)^m {}_3F_2\left[\begin{matrix}m,\ \lambda,\ m-\lambda\\ 1+\lambda, 1+m-\lambda\end{matrix}\middle| x\right]
$$
$$
= (1-x)^m\Delta(m,\lambda;x) + \frac{\lambda(1-\lambda)_m}{(m-1)!(m-2\lambda)}(1-x)^m\Phi(x,1,\lambda)
$$
$$
- \frac{(m-\lambda)(1-m+\lambda)_m}{(m-1)!(m-2\lambda)}(1-x)^m\Phi(x,1,m-\lambda),
$$

where the remainder term is given by two finite sums

$$
\Delta(m,\lambda;x) = \frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{m-i-\lambda}{m-i}\binom{-\lambda}{i-k}\frac{(1-x)^{1-k}}{i-1}
$$
$$
-\frac{\lambda(m-\lambda)}{m-2\lambda}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{\lambda-i}{m-i}\binom{\lambda-m}{i-k}\frac{(1-x)^{1-k}}{i-1}.
$$

3. Evaluation of series $\mathcal{G}(m,\lambda,y)$

Alternatively, by specifying the parameters in Whipple's second transformation (2)

$$
a = m,\ c = \lambda,\ b = a - c,
$$

we can transform the series $\mathcal{G}(m,\lambda,y)$ as

$$
{}_3F_2\left[\begin{matrix}1,\ \frac{1+m}{2},\ 1+\frac{m}{2}\\ 1+\lambda, 1+m-\lambda\end{matrix}\middle| y\right] = \frac{(1-x)^{m+1}}{1+x}{}_4F_3\left[\begin{matrix}m,\ 1+\frac{m}{2},\ \lambda,\ m-\lambda\\ \frac{m}{2}, 1+\lambda, 1+m-\lambda\end{matrix}\middle| x\right]. \tag{7}
$$

The ${}_4F_3(x)$-series on the right can be explicitly restated as

$$
{}_4F_3\left[\begin{matrix}m,\ 1+\frac{m}{2},\ \lambda,\ m-\lambda\\ \frac{m}{2}, 1+\lambda, 1+m-\lambda\end{matrix}\middle| x\right] = \frac{\lambda(m-\lambda)}{m!}\sum_{n=0}^{\infty}\frac{(n+1)_{m-1}(m+2n)}{(n+\lambda)(n+m-\lambda)}x^n. \tag{8}
$$

Analogously, the only series with "$m = 2\lambda = 1$" is the following reduced one:

$$
{}_2F_1\left[\begin{matrix}1,\ 1\\ \frac{3}{2}\end{matrix}\middle| y\right] = \frac{(1-x)^2}{1+x}\times {}_2F_1\left[\begin{matrix}1,\ \frac{1}{2}\\ \frac{3}{2}\end{matrix}\middle| x\right] = \frac{(1-x)^2\text{arctanh}\sqrt{x}}{(1+x)\sqrt{x}}.
$$

Let $\mathcal{R}(n)$ be a rational function subject to with "$m \neq 2\lambda$"

$$
\mathcal{R}(n) := \frac{(n+1)_{m-1}(n+\frac{m}{2})}{(n+\lambda)(n+m-\lambda)} = \frac{(n+1)_{m-1}}{2}\left\{\frac{1}{n+\lambda} + \frac{1}{n+m-\lambda}\right\}.
$$

Observe that the above $\mathcal{R}(n)$ resembles almost identically that $R(n)$ in (6) under replacements "$m-2\lambda \to 2$" for denominator factors and "$- \to +$" inside braces "$\{\cdots\}$". By applying the same procedure used to prove Theorem 1, we derive the formula presented in the following theorem.

Theorem 2 ($m \neq 2\lambda$)**.** *For two variables x and y related by* (3)*, we have*

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1,\ \frac{1+m}{2},\ 1+\frac{m}{2} \\ 1+\lambda, 1+m-\lambda \end{matrix}\middle| y\right] &= \frac{(1-x)^{m+1}}{1+x}\,{}_4F_3\left[\begin{matrix} m,\ 1+\frac{m}{2},\ \lambda,\ m-\lambda \\ \frac{m}{2}, 1+\lambda, 1+m-\lambda \end{matrix}\middle| x\right] \\ &= \frac{(1-x)^{m+1}}{1+x}\nabla(m,\lambda;x) + \frac{\lambda(1-\lambda)_m}{m!}\frac{(1-x)^{m+1}}{1+x}\Phi(x,1,\lambda) \\ &+ \frac{(m-\lambda)(1-m+\lambda)_m}{m!}\frac{(1-x)^{m+1}}{1+x}\Phi(x,1,m-\lambda), \end{aligned}$$

where the remainder term is given by two finite sums

$$\begin{aligned} \nabla(m,\lambda;x) = &\frac{\lambda(m-\lambda)}{m}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{m-i-\lambda}{m-i}\binom{-\lambda}{i-k}\frac{(1-x)^{1-k}}{i-1} \\ +&\frac{\lambda(m-\lambda)}{m}\sum_{i=2}^{m}\sum_{k=2}^{i}(-1)^{i-k}\binom{\lambda-i}{m-i}\binom{\lambda-m}{i-k}\frac{(1-x)^{1-k}}{i-1}. \end{aligned}$$

4. Closed formulae $\mathcal{F}(m,\lambda,y)$ and $\mathcal{G}(m,\lambda,y)$

According to Theorems 1 and 2, both series $\mathcal{F}(m,\lambda,y)$ and $\mathcal{G}(m,\lambda,y)$ can be expressed in terms of the Lerch transcendent function $\Phi(x,m,\lambda)$ plus a remainder polynomial. When the involved $\Phi(x,m,\lambda)$ admit explicit expressions in terms of logarithmic and arctan functions, we then find closed formulae for the corresponding series $\mathcal{F}(m,\lambda,y)$ and $\mathcal{G}(m,\lambda,y)$.

Throughout this section, x and y are two variables related by (3). For $m = 1$ and $\lambda \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$, we are first going to review three formulae due to Yabu [9]. Then, for $m = 1$ and irreducible rational numbers $\lambda = p/q \in \mathbb{Q}$ with $q \in \{5, 8, 10, 12\}$, several closed formulae will be shown in pairs for series $\mathcal{F}(1,p/q,y)$ and $\mathcal{G}(1,p/q,y)$. Finally, when $m \neq 1$, we shall record a few expressions, as examples, for $\mathcal{F}(m,p/q,y)$ and $\mathcal{G}(m,p/q,y)$ in terms of the Lerch transcendent function.

4.1. Review of Yabu's three formulae

We first review the explicit formulae for three particular ${}_3F_2$-series in terms of the logarithmic function, obtained by Yabu in his thesis [9].

- Yabu's first formula (cf. [9, Theorem 1.4]) can be reproduced as below:

$${}_3F_2\left[\begin{matrix} 1,\ 1,\ \frac{1}{2} \\ \frac{4}{3},\ \frac{5}{3} \end{matrix}\middle| t^6\right] = \frac{-4\mathbf{i}}{3\sqrt{3}t^3}\left\{\mathrm{A}(e^{\frac{\pi \mathbf{i}}{3}}t)\ln\left(1+\frac{3t^3}{2}\mathrm{B}(t)\right) + \mathrm{A}(t)\ln\left(1-\frac{3t^3}{2}\mathrm{B}(e^{\frac{\pi \mathbf{i}}{3}}t)\right)\right\},$$

where

$$\mathrm{A}(t) = \frac{t}{(1+\sqrt{1-t^6})^{\frac{1}{3}}} - \frac{(1+\sqrt{1-t^6})^{\frac{1}{3}}}{t},$$

$$\mathrm{B}(t) = \frac{t}{(1+\sqrt{1-t^6})^{\frac{1}{3}}} + \frac{(1+\sqrt{1-t^6})^{\frac{1}{3}}}{t}.$$

By making use of the trisection series (or Mathematica command "FunctionExpand"), we have the explicit expressions

$$\Phi(x,1,\tfrac{1}{3})=\frac{1}{2\sqrt[3]{x}}\left\{3\ln\left(\frac{\sqrt[3]{1-x}}{1-\sqrt[3]{x}}\right)+2\sqrt{3}\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{2+\sqrt[3]{x}}\right)\right\},$$

$$\Phi(x,1,\tfrac{2}{3})=\frac{1}{2\sqrt[3]{x^2}}\left\{3\ln\left(\frac{\sqrt[3]{1-x}}{1-\sqrt[3]{x}}\right)-2\sqrt{3}\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{2+\sqrt[3]{x}}\right)\right\}.$$

According to Theorems 1 and 2, we obtain the following two closed formulae:

$$\begin{aligned}\mathcal{F}(1,\tfrac{1}{3},y)={}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\middle|y\right]&=\frac{2}{3}(1-x)\left\{\Phi(x,1,\tfrac{1}{3})-\Phi(x,1,\tfrac{2}{3})\right\}\\ &=\frac{1-x}{3\sqrt[3]{x^2}}\left\{3(1-\sqrt[3]{x})\ln\left(\frac{1-\sqrt[3]{x}}{\sqrt[3]{1-x}}\right)+2\sqrt{3}(1+\sqrt[3]{x})\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{2+\sqrt[3]{x}}\right)\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{G}(1,\tfrac{1}{3},y)={}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\middle|y\right]&=\frac{2(1-x)^2}{9(1+x)}\left\{\Phi(x,1,\tfrac{1}{3})+\Phi(x,1,\tfrac{2}{3})\right\}\\ &=\frac{(1-x)^2}{9\sqrt[3]{x^2}}\left\{3\frac{1+\sqrt[3]{x}}{1+x}\ln\left(\frac{\sqrt[3]{1-x}}{1-\sqrt[3]{x}}\right)-2\sqrt{3}\frac{1-\sqrt[3]{x}}{1+x}\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{2+\sqrt[3]{x}}\right)\right\}.\end{aligned}$$

Without involving the imaginary root of unity, these expressions have advantages over Yabu's.

- Yabu's second formula reads as (see [9, Theorem 1.5])

$${}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix}\middle|t^4\right]=-\frac{3\mathbf{i}\sqrt{1-t^2}}{2t^3}\ln(\sqrt{1-t^2}-\mathbf{i}t)-\frac{3\sqrt{1+t^2}}{2t^3}\ln(\sqrt{1+t^2}-t).$$

Recalling Theorems 1 and 2, and then applying two equalities:

$$\Phi(x,1,\tfrac{1}{4})=\frac{1}{\sqrt[4]{x}}\left\{\ln\left(\frac{1+x^{\frac{1}{4}}}{1-x^{\frac{1}{4}}}\right)+2\arctan\left(x^{\frac{1}{4}}\right)\right\},$$

$$\Phi(x,1,\tfrac{3}{4})=\frac{1}{\sqrt[4]{x^3}}\left\{\ln\left(\frac{1+x^{\frac{1}{4}}}{1-x^{\frac{1}{4}}}\right)-2\arctan\left(x^{\frac{1}{4}}\right)\right\},$$

we can directly write down two elegant closed formulae:

$$\begin{aligned}\mathcal{F}(1,\tfrac{1}{4},y)={}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix}\middle|y\right]&=\frac{3}{8}(1-x)\left\{\Phi(x,1,\tfrac{1}{4})-\Phi(x,1,\tfrac{3}{4})\right\}\\ &=\frac{3(1-x)}{8\sqrt[4]{x^3}}\left\{(1-\sqrt{x})\ln\left(\frac{1-x^{\frac{1}{4}}}{1+x^{\frac{1}{4}}}\right)+2(1+\sqrt{x})\arctan\left(x^{\frac{1}{4}}\right)\right\},\end{aligned}$$

$$\mathcal{G}(1,\tfrac{1}{4},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{5}{4},\ \frac{7}{4}\end{matrix}\middle|\,y\right] = \frac{3(1-x)^2}{16(1+x)}\Big\{\Phi(x,1,\tfrac{1}{4})+\Phi(x,1,\tfrac{3}{4})\Big\}$$
$$= \frac{3(1-x)^2}{16\sqrt[4]{x^3}}\left\{\frac{1+\sqrt{x}}{1+x}\ln\left(\frac{1+x^{\frac{1}{4}}}{1-x^{\frac{1}{4}}}\right)-2\frac{1-\sqrt{x}}{1+x}\arctan\left(x^{\frac{1}{4}}\right)\right\}.$$

These formulae look more transparent than Yabu's formula.

- The third formula due to Yabu [9, Theorem 1.6] is given by

$${}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{matrix}\middle|\,t^6\right] = \frac{5\mathbf{i}}{12t^3}\left\{\mathcal{A}(e^{\frac{\pi\mathbf{i}}{3}}t)\ln\left(\frac{2-\sqrt{3}t^3(\mathcal{B}(t)-1)}{2+\sqrt{3}t^3(\mathcal{B}(t)-1)}\right)+\mathcal{A}(t)\ln\left(\frac{2+\sqrt{3}t^3(\mathcal{B}(e^{\frac{\pi\mathbf{i}}{3}}t)-1)}{2-\sqrt{3}t^3(\mathcal{B}(e^{\frac{\pi\mathbf{i}}{3}}t)-1)}\right)\right\},$$

where

$$\mathcal{A}(t) = \frac{t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}}-\frac{(1+\sqrt{1-t^6})^{\frac{2}{3}}}{t^2},$$

$$\mathcal{B}(t) = \frac{t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}}+\frac{(1+\sqrt{1-t^6})^{\frac{2}{3}}}{t^2}.$$

By employing the two explicit expressions:

$$\Phi(x,1,\tfrac{1}{6}) = \frac{1}{2\sqrt[6]{x}}\left\{\ln\left(\frac{(1-\sqrt{x})(1+x^{\frac{1}{6}})^3}{(1+\sqrt{x})(1-x^{\frac{1}{6}})^3}\right)+2\sqrt{3}\arctan\left(\frac{\sqrt{3}x^{\frac{1}{6}}}{1-x^{\frac{1}{3}}}\right)\right\},$$

$$\Phi(x,1,\tfrac{5}{6}) = \frac{1}{2\sqrt[6]{x^5}}\left\{\ln\left(\frac{(1-\sqrt{x})(1+x^{\frac{1}{6}})^3}{(1+\sqrt{x})(1-x^{\frac{1}{6}})^3}\right)-2\sqrt{3}\arctan\left(\frac{\sqrt{3}x^{\frac{1}{6}}}{1-x^{\frac{1}{3}}}\right)\right\},$$

and then, from Theorems 1 and 2, we derive the following closed formulae:

$$\mathcal{F}(1,\tfrac{1}{6},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{matrix}\middle|\,y\right] = \frac{5}{24}(1-x)\Big\{\Phi(x,1,\tfrac{1}{6})-\Phi(x,1,\tfrac{5}{6})\Big\}$$
$$= \frac{5(1-x)}{48\sqrt[6]{x^5}}\left\{(1-x^{\frac{2}{3}})\ln\left(\frac{(1+\sqrt{x})(1-x^{\frac{1}{6}})^3}{(1-\sqrt{x})(1+x^{\frac{1}{6}})^3}\right)+2\sqrt{3}(1+x^{\frac{2}{3}})\arctan\left(\frac{\sqrt{3}x^{\frac{1}{6}}}{1-x^{\frac{1}{3}}}\right)\right\},$$

$$\mathcal{G}(1,\tfrac{1}{6},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{matrix}\middle|\,y\right] = \frac{5(1-x)^2}{36(1+x)}\Big\{\Phi(x,1,\tfrac{1}{6})+\Phi(x,1,\tfrac{5}{6})\Big\}$$
$$= \frac{5(1-x)^2}{72\sqrt[6]{x^5}}\left\{\frac{1+x^{\frac{2}{3}}}{1+x}\ln\left(\frac{(1-\sqrt{x})(1+x^{\frac{1}{6}})^3}{(1+\sqrt{x})(1-x^{\frac{1}{6}})^3}\right)-2\sqrt{3}\frac{1-x^{\frac{2}{3}}}{1+x}\arctan\left(\frac{\sqrt{3}x^{\frac{1}{6}}}{1-x^{\frac{1}{3}}}\right)\right\}.$$

They look simpler than Yabu's original formula.

4.2. Further closed formulae

By carrying out the same procedure as in §4.1, we can establish further closed formulae for series $\mathcal{F}(1,\lambda,y)$ and $\mathcal{G}(1,\lambda,y)$.

- $\boxed{\mathcal{F}(1,\frac{1}{5},y) \text{ and } \mathcal{G}(1,\frac{1}{5},y).}$ Applying the explicit expressions:

$$\Phi(x,1,\tfrac{1}{5}) = \frac{1}{4\sqrt[5]{x}}\Big\{5\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big) + 2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big) + 2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{4}{5}) = \frac{1}{4\sqrt[5]{x^4}}\Big\{5\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big) - 2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big) - 2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

we derive the following two closed formulae:

$$\mathcal{F}(1,\tfrac{1}{5},y) = {}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{6}{5},\frac{9}{5}\end{matrix}\Big| y\right] = \frac{4}{15}(1-x)\Big\{\Phi(x,1,\tfrac{1}{5}) - \Phi(x,1,\tfrac{4}{5})\Big\}$$
$$= \frac{1-x}{15\sqrt[5]{x^4}}\Big\{5(1-x^{\frac{3}{5}})\ln\Big(\frac{1-x^{\frac{1}{5}}}{(1-x)^{\frac{1}{5}}}\Big) + 2\sqrt{10+2\sqrt{5}}(1+x^{\frac{3}{5}})\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}(1-x^{\frac{3}{5}})\ln\Big(\frac{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big) + 2\sqrt{10-2\sqrt{5}}(1+x^{\frac{3}{5}})\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{1}{5},y) = {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{6}{5},\frac{9}{5}\end{matrix}\Big| y\right] = \frac{4(1-x)^2}{25(1+x)}\Big\{\Phi(x,1,\tfrac{1}{5}) + \Phi(x,1,\tfrac{4}{5})\Big\}$$
$$= \frac{(1-x)^2}{25\sqrt[5]{x^4}}\Big\{5\frac{1+x^{\frac{3}{5}}}{1+x}\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big) - 2\sqrt{10+2\sqrt{5}}\frac{1-x^{\frac{3}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}\frac{1+x^{\frac{3}{5}}}{1+x}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big) - 2\sqrt{10-2\sqrt{5}}\frac{1-x^{\frac{3}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\frac{2}{5},y) \text{ and } \mathcal{G}(1,\frac{2}{5},y).}$ By employing the two equalities:

$$\Phi(x,1,\tfrac{2}{5}) = \frac{1}{4\sqrt[5]{x^2}}\Big\{5\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big) + 2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$

$$+\sqrt{5}\ln\Big(\frac{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big)-2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{3}{5})=\frac{1}{4\sqrt[5]{x^3}}\Big\{5\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big)-2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}\ln\Big(\frac{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big)+2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

we can establish the following two closed formulae:

$$\mathcal{F}(1,\tfrac{2}{5},y)={}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{7}{5},\frac{8}{5}\end{matrix}\Big|y\right]=\frac{6}{5}(1-x)\Big\{\Phi(x,1,\tfrac{2}{5})-\Phi(x,1,\tfrac{3}{5})\Big\}$$
$$=\frac{3(1-x)}{10\sqrt[5]{x^3}}\Big\{5(1-x^{\frac{1}{5}})\ln\Big(\frac{1-x^{\frac{1}{5}}}{(1-x)^{\frac{1}{5}}}\Big)+2\sqrt{10-2\sqrt{5}}(1+x^{\frac{1}{5}})\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}(1-x^{\frac{1}{5}})\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big)-2\sqrt{10+2\sqrt{5}}(1+x^{\frac{1}{5}})\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{2}{5},y)={}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{7}{5},\frac{8}{5}\end{matrix}\Big|y\right]=\frac{6(1-x)^2}{25(1+x)}\Big\{\Phi(x,1,\tfrac{2}{5})+\Phi(x,1,\tfrac{3}{5})\Big\}$$
$$=\frac{3(1-x)^2}{50\sqrt[5]{x^3}}\Big\{5\frac{1+x^{\frac{1}{5}}}{1+x}\ln\Big(\frac{(1-x)^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}\Big)-2\sqrt{10-2\sqrt{5}}\frac{1-x^{\frac{1}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{5}}}{4+(1-\sqrt{5})x^{\frac{1}{5}}}\Big)$$
$$+\sqrt{5}\frac{1+x^{\frac{1}{5}}}{1+x}\ln\Big(\frac{2+(1-\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{5}}+2x^{\frac{2}{5}}}\Big)+2\sqrt{10+2\sqrt{5}}\frac{1-x^{\frac{1}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{5}}}{4+(1+\sqrt{5})x^{\frac{1}{5}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\frac{1}{8},y)\text{ and }\mathcal{G}(1,\frac{1}{8},y).}$ By utilizing the two explicit expressions:

$$\Phi(x,1,\tfrac{1}{8})=\frac{1}{2\sqrt[8]{x}}\Big\{2\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)+\sqrt{2}\ln\Big(\frac{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)+4\arctan\Big(x^{\frac{1}{8}}\Big)+2\sqrt{2}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{7}{8})=\frac{1}{2\sqrt[8]{x^7}}\Big\{2\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)+\sqrt{2}\ln\Big(\frac{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)-4\arctan\Big(x^{\frac{1}{8}}\Big)-2\sqrt{2}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

we find the following two closed formulae:

$$\mathcal{F}(1,\tfrac{1}{8},y)={}_3F_2\left[\begin{matrix}1,1,\frac{1}{2}\\ \frac{9}{8},\frac{15}{8}\end{matrix}\Big|y\right]=\frac{7}{48}(1-x)\Big\{\Phi(x,1,\tfrac{1}{8})-\Phi(x,1,\tfrac{7}{8})\Big\}$$

$$= \frac{7(1-x)}{96\sqrt[8]{x^7}}\Big\{2(1-x^{\frac{3}{4}})\ln\Big(\frac{1-x^{\frac{1}{8}}}{1+x^{\frac{1}{8}}}\Big)+4(1+x^{\frac{3}{4}})\arctan\Big(x^{\frac{1}{8}}\Big)$$

$$+\ \sqrt{2}(1-x^{\frac{3}{4}})\ln\Big(\frac{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)+2\sqrt{2}(1+x^{\frac{3}{4}})\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{1}{8},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{9}{8},\ \frac{15}{8}\end{matrix}\Big|y\right] = \frac{7(1-x)^2}{64(1+x)}\Big\{\Phi(x,1,\tfrac{1}{8})+\Phi(x,1,\tfrac{7}{8})\Big\}$$

$$= \frac{7(1-x)^2}{128\sqrt[8]{x^7}}\Big\{2\frac{1+x^{\frac{3}{4}}}{1+x}\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)-4\frac{1-x^{\frac{3}{4}}}{1+x}\arctan\Big(x^{\frac{1}{8}}\Big)$$

$$+\ \sqrt{2}\frac{1+x^{\frac{3}{4}}}{1+x}\ln\Big(\frac{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)-2\sqrt{2}\frac{1-x^{\frac{3}{4}}}{1+x}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\tfrac{3}{8},y) \text{ and } \mathcal{G}(1,\tfrac{3}{8},y).}$ By employing the two equalities:

$$\Phi(x,1,\tfrac{3}{8}) = \frac{1}{2\sqrt[8]{x^3}}\Big\{2\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)+\sqrt{2}\ln\Big(\frac{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)-4\arctan\Big(x^{\frac{1}{8}}\Big)+2\sqrt{2}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{5}{8}) = \frac{1}{2\sqrt[8]{x^5}}\Big\{2\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)+\sqrt{2}\ln\Big(\frac{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)+4\arctan\Big(x^{\frac{1}{8}}\Big)-2\sqrt{2}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

we deduce the following two closed formulae:

$$\mathcal{F}(1,\tfrac{3}{8},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{11}{8},\ \frac{13}{8}\end{matrix}\Big|y\right] = \frac{15}{16}(1-x)\Big\{\Phi(x,1,\tfrac{3}{8})-\Phi(x,1,\tfrac{5}{8})\Big\}$$

$$= \frac{15(1-x)}{32\sqrt[8]{x^5}}\Big\{2(1-x^{\frac{1}{4}})\ln\Big(\frac{1-x^{\frac{1}{8}}}{1+x^{\frac{1}{8}}}\Big)-4(1+x^{\frac{1}{4}})\arctan\Big(x^{\frac{1}{8}}\Big)$$

$$+\ \sqrt{2}(1-x^{\frac{1}{4}})\ln\Big(\frac{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)+2\sqrt{2}(1+x^{\frac{1}{4}})\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{3}{8},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{11}{8},\ \frac{13}{8}\end{matrix}\Big|y\right] = \frac{15(1-x)^2}{64(1+x)}\Big\{\Phi(x,1,\tfrac{3}{8})+\Phi(x,1,\tfrac{5}{8})\Big\}$$

$$= \frac{15(1-x)^2}{128\sqrt[8]{x^5}}\Big\{2\frac{1+x^{\frac{1}{4}}}{1+x}\ln\Big(\frac{1+x^{\frac{1}{8}}}{1-x^{\frac{1}{8}}}\Big)+4\frac{1-x^{\frac{1}{4}}}{1+x}\arctan\Big(x^{\frac{1}{8}}\Big)$$

$$+\ \sqrt{2}\frac{1+x^{\frac{1}{4}}}{1+x}\ln\Big(\frac{1-\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}{1+\sqrt{2}x^{\frac{1}{8}}+x^{\frac{1}{4}}}\Big)-2\sqrt{2}\frac{1-x^{\frac{1}{4}}}{1+x}\arctan\Big(\frac{\sqrt{2}x^{\frac{1}{8}}}{1-x^{\frac{1}{4}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\frac{1}{10},y) \text{ and } \mathcal{G}(1,\frac{1}{10},y).}$ By utilizing the two explicit expressions:

$$\Phi(x,1,\tfrac{1}{10}) = \frac{1}{4\sqrt[10]{x}}\Big\{(5-\sqrt{5})\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) + (1-\sqrt{5})\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) + 2\sqrt{5}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$
$$+2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big) + 2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{9}{10}) = \frac{1}{4\sqrt[10]{x^9}}\Big\{(5-\sqrt{5})\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) + (1-\sqrt{5})\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) + 2\sqrt{5}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$
$$-2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big) - 2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\};$$

we establish the following closed formulae:

$$\mathcal{F}(1,\tfrac{1}{10},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{11}{10},\ \frac{19}{10}\end{matrix}\Big|y\right] = \frac{9}{80}(1-x)\Big\{\Phi(x,1,\tfrac{1}{10}) - \Phi(x,1,\tfrac{9}{10})\Big\}$$
$$= \frac{9(1-x)}{320\sqrt[10]{x^9}}\Big\{2\sqrt{5}(1-x^{\frac{4}{5}})\ln\Big(\frac{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$
$$+(5-\sqrt{5})(1-x^{\frac{4}{5}})\ln\Big(\frac{1-x^{\frac{1}{10}}}{1+x^{\frac{1}{10}}}\Big) + 2\sqrt{10+2\sqrt{5}}(1+x^{\frac{4}{5}})\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)$$
$$+(1-\sqrt{5})(1-x^{\frac{4}{5}})\ln\Big(\frac{1+\sqrt{x}}{1-\sqrt{x}}\Big) + 2\sqrt{10-2\sqrt{5}}(1+x^{\frac{4}{5}})\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{1}{10},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{11}{10},\ \frac{19}{10}\end{matrix}\Big|y\right] = \frac{9(1-x)^2}{100(1+x)}\Big\{\Phi(x,1,\tfrac{1}{10}) + \Phi(x,1,\tfrac{9}{10})\Big\}$$
$$= \frac{9(1-x)^2}{400\sqrt[10]{x^9}}\Big\{2\sqrt{5}\frac{1+x^{\frac{4}{5}}}{1+x}\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$
$$+(5-\sqrt{5})\frac{1+x^{\frac{4}{5}}}{1+x}\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) - 2\sqrt{10+2\sqrt{5}}\frac{1-x^{\frac{4}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)$$
$$+(1-\sqrt{5})\frac{1+x^{\frac{4}{5}}}{1+x}\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) - 2\sqrt{10-2\sqrt{5}}\frac{1-x^{\frac{4}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\frac{3}{10},y) \text{ and } \mathcal{G}(1,\frac{3}{10},y).}$ By employing the two equalities:

$$\Phi(x,1,\tfrac{3}{10}) = \frac{1}{4\sqrt[10]{x^3}}\Big\{(5+\sqrt{5})\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) + (1+\sqrt{5})\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) + 2\sqrt{5}\ln\Big(\frac{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$

$$-2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big) + 2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{7}{10}) = \frac{1}{4\sqrt[10]{x^7}}\Big\{(5+\sqrt{5})\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) + (1+\sqrt{5})\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) + 2\sqrt{5}\ln\Big(\frac{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$

$$+2\sqrt{10-2\sqrt{5}}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big) - 2\sqrt{10+2\sqrt{5}}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\},$$

we find the following two closed formulae:

$$\mathcal{F}(1,\tfrac{3}{10},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{13}{10},\ \frac{17}{10}\end{matrix}\Big| y\right] = \frac{21}{40}(1-x)\Big\{\Phi(x,1,\tfrac{3}{10}) - \Phi(x,1,\tfrac{7}{10})\Big\}$$

$$= \frac{21(1-x)}{160\sqrt[10]{x^7}}\Big\{2\sqrt{5}(1-x^{\frac{2}{5}})\ln\Big(\frac{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$

$$+(5+\sqrt{5})(1-x^{\frac{2}{5}})\ln\Big(\frac{1-x^{\frac{1}{10}}}{1+x^{\frac{1}{10}}}\Big) - 2\sqrt{10-2\sqrt{5}}(1+x^{\frac{2}{5}})\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)$$

$$+(1+\sqrt{5})(1-x^{\frac{2}{5}})\ln\Big(\frac{1+\sqrt{x}}{1-\sqrt{x}}\Big) + 2\sqrt{10+2\sqrt{5}}(1+x^{\frac{2}{5}})\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{3}{10},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{13}{10},\ \frac{17}{10}\end{matrix}\Big| y\right] = \frac{21(1-x)^2}{100(1+x)}\Big\{\Phi(x,1,\tfrac{3}{10}) + \Phi(x,1,\tfrac{7}{10})\Big\}$$

$$= \frac{21(1-x)^2}{400\sqrt[10]{x^7}}\Big\{2\sqrt{5}\frac{1+x^{\frac{2}{5}}}{1+x}\ln\Big(\frac{2-(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}{2+(1+\sqrt{5})x^{\frac{1}{10}}+2x^{\frac{1}{5}}}\Big)$$

$$+(5+\sqrt{5})\frac{1+x^{\frac{2}{5}}}{1+x}\ln\Big(\frac{1+x^{\frac{1}{10}}}{1-x^{\frac{1}{10}}}\Big) + 2\sqrt{10-2\sqrt{5}}\frac{1-x^{\frac{2}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10+2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)$$

$$+(1+\sqrt{5})\frac{1+x^{\frac{2}{5}}}{1+x}\ln\Big(\frac{1-\sqrt{x}}{1+\sqrt{x}}\Big) - 2\sqrt{10+2\sqrt{5}}\frac{1-x^{\frac{2}{5}}}{1+x}\arctan\Big(\frac{\sqrt{10-2\sqrt{5}}x^{\frac{1}{10}}}{2-2x^{\frac{1}{5}}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\tfrac{1}{12},y) \text{ and } \mathcal{G}(1,\tfrac{1}{12},y).}$ By utilizing the two explicit expressions:

$$\Phi(x,1,\tfrac{1}{12}) = \frac{1}{2\sqrt[12]{x}}\Big\{\ln\Big(\frac{(1-x^{\frac14})(1+x^{\frac1{12}})^3}{(1+x^{\frac14})(1-x^{\frac1{12}})^3}\Big) + \sqrt{3}\ln\Big(\frac{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$
$$+2\sqrt{3}\arctan\Big(\frac{\sqrt{3}x^{\frac1{12}}}{1-x^{\frac16}}\Big) + 2\pi + 2\arctan\Big(\frac{3x^{\frac1{12}}-3x^{\frac14}}{1-4x^{\frac16}+x^{\frac13}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{11}{12}) = \frac{1}{2\sqrt[12]{x^{11}}}\Big\{\ln\Big(\frac{(1-x^{\frac14})(1+x^{\frac1{12}})^3}{(1+x^{\frac14})(1-x^{\frac1{12}})^3}\Big) + \sqrt{3}\ln\Big(\frac{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$
$$-2\sqrt{3}\arctan\Big(\frac{\sqrt{3}x^{\frac1{12}}}{1-x^{\frac16}}\Big) - 2\pi - 2\arctan\Big(\frac{3x^{\frac1{12}}-3x^{\frac14}}{1-4x^{\frac16}+x^{\frac13}}\Big)\Big\},$$

we derive the following two closed formulae:

$$\mathcal{F}(1,\tfrac{1}{12},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac12\\ \frac{13}{12},\ \frac{23}{12}\end{matrix}\Big| y\right] = \frac{11}{120}(1-x)\Big\{\Phi(x,1,\tfrac{1}{12}) - \Phi(x,1,\tfrac{11}{12})\Big\}$$
$$= \frac{11(1-x)}{240\sqrt[12]{x^{11}}}\Big\{(1-x^{\frac56})\ln\Big(\frac{(1+x^{\frac14})(1-x^{\frac1{12}})^3}{(1-x^{\frac14})(1+x^{\frac1{12}})^3}\Big) + \sqrt{3}(1-x^{\frac56})\ln\Big(\frac{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$
$$+2\pi(1+x^{\frac56}) + 2\sqrt{3}(1+x^{\frac56})\arctan\Big(\frac{\sqrt{3}x^{\frac1{12}}}{1-x^{\frac16}}\Big) + 2(1+x^{\frac56})\arctan\Big(\frac{3x^{\frac1{12}}-3x^{\frac14}}{1-4x^{\frac16}+x^{\frac13}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{1}{12},y) = {}_3F_2\left[\begin{matrix}1,\ 1,\ \frac32\\ \frac{13}{12},\ \frac{23}{12}\end{matrix}\Big| y\right] = \frac{11(1-x)^2}{144(1+x)}\Big\{\Phi(x,1,\tfrac{1}{12}) + \Phi(x,1,\tfrac{11}{12})\Big\}$$
$$= \frac{11(1-x)^2}{288\sqrt[12]{x^{11}}}\Big\{\frac{1+x^{\frac56}}{1+x}\ln\Big(\frac{(1-x^{\frac14})(1+x^{\frac1{12}})^3}{(1+x^{\frac14})(1-x^{\frac1{12}})^3}\Big) + \sqrt{3}\frac{1+x^{\frac56}}{1+x}\ln\Big(\frac{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$
$$-2\pi\frac{1-x^{\frac56}}{1+x} - 2\sqrt{3}\frac{1-x^{\frac56}}{1+x}\arctan\Big(\frac{\sqrt{3}x^{\frac1{12}}}{1-x^{\frac16}}\Big) - 2\frac{1-x^{\frac56}}{1+x}\arctan\Big(\frac{3x^{\frac1{12}}-3x^{\frac14}}{1-4x^{\frac16}+x^{\frac13}}\Big)\Big\}.$$

- $\boxed{\mathcal{F}(1,\tfrac{5}{12},y) \text{ and } \mathcal{G}(1,\tfrac{5}{12},y).}$ By employing the two equalities:

$$\Phi(x,1,\tfrac{5}{12}) = \frac{1}{2\sqrt[12]{x^{5}}}\Big\{\ln\Big(\frac{(1-x^{\frac14})(1+x^{\frac1{12}})^3}{(1+x^{\frac14})(1-x^{\frac1{12}})^3}\Big) + \sqrt{3}\ln\Big(\frac{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$
$$-2\sqrt{3}\arctan\Big(\frac{\sqrt{3}x^{\frac1{12}}}{1-x^{\frac16}}\Big) + 2\pi + 2\arctan\Big(\frac{3x^{\frac1{12}}-3x^{\frac14}}{1-4x^{\frac16}+x^{\frac13}}\Big)\Big\},$$

$$\Phi(x,1,\tfrac{7}{12}) = \frac{1}{2\sqrt[12]{x^{7}}}\Big\{\ln\Big(\frac{(1-x^{\frac14})(1+x^{\frac1{12}})^3}{(1+x^{\frac14})(1-x^{\frac1{12}})^3}\Big) + \sqrt{3}\ln\Big(\frac{1-\sqrt{3}x^{\frac1{12}}+x^{\frac16}}{1+\sqrt{3}x^{\frac1{12}}+x^{\frac16}}\Big)$$

$$+2\sqrt{3}\arctan\Big(\frac{\sqrt{3}x^{\frac{1}{12}}}{1-x^{\frac{1}{6}}}\Big)-2\pi-2\arctan\Big(\frac{3x^{\frac{1}{12}}-3x^{\frac{1}{4}}}{1-4x^{\frac{1}{6}}+x^{\frac{1}{3}}}\Big)\Big\},$$

we find the following two closed formulae:

$$\mathcal{F}(1,\tfrac{5}{12},y)={}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{17}{12},\ \frac{19}{12}\end{matrix}\Big|y\right]=\frac{35}{24}(1-x)\Big\{\Phi(x,1,\tfrac{5}{12})-\Phi(x,1,\tfrac{7}{12})\Big\}$$

$$=\frac{35(1-x)}{48\sqrt[12]{x^7}}\Big\{(1-x^{\frac{1}{6}})\ln\Big(\frac{(1+x^{\frac{1}{4}})(1-x^{\frac{1}{12}})^3}{(1-x^{\frac{1}{4}})(1+x^{\frac{1}{12}})^3}\Big)+\sqrt{3}(1-x^{\frac{1}{6}})\ln\Big(\frac{1+\sqrt{3}x^{\frac{1}{12}}+x^{\frac{1}{6}}}{1-\sqrt{3}x^{\frac{1}{12}}+x^{\frac{1}{6}}}\Big)$$

$$+2\pi(1+x^{\frac{1}{6}})-2\sqrt{3}(1+x^{\frac{1}{6}})\arctan\Big(\frac{\sqrt{3}x^{\frac{1}{12}}}{1-x^{\frac{1}{6}}}\Big)+2(1+x^{\frac{1}{6}})\arctan\Big(\frac{3x^{\frac{1}{12}}-3x^{\frac{1}{4}}}{1-4x^{\frac{1}{6}}+x^{\frac{1}{3}}}\Big)\Big\},$$

$$\mathcal{G}(1,\tfrac{5}{12},y)={}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2}\\ \frac{17}{12},\ \frac{19}{12}\end{matrix}\Big|y\right]=\frac{35(1-x)^2}{144(1+x)}\Big\{\Phi(x,1,\tfrac{5}{12})+\Phi(x,1,\tfrac{7}{12})\Big\}$$

$$=\frac{35(1-x)^2}{288\sqrt[12]{x^7}}\Big\{\frac{1+x^{\frac{1}{6}}}{1+x}\ln\Big(\frac{(1-x^{\frac{1}{4}})(1+x^{\frac{1}{12}})^3}{(1+x^{\frac{1}{4}})(1-x^{\frac{1}{12}})^3}\Big)+\sqrt{3}\frac{1+x^{\frac{1}{6}}}{1+x}\ln\Big(\frac{1-\sqrt{3}x^{\frac{1}{12}}+x^{\frac{1}{6}}}{1+\sqrt{3}x^{\frac{1}{12}}+x^{\frac{1}{6}}}\Big)$$

$$-2\pi\frac{1-x^{\frac{1}{6}}}{1+x}+2\sqrt{3}\frac{1-x^{\frac{1}{6}}}{1+x}\arctan\Big(\frac{\sqrt{3}x^{\frac{1}{12}}}{1-x^{\frac{1}{6}}}\Big)-2\frac{1-x^{\frac{1}{6}}}{1+x}\arctan\Big(\frac{3x^{\frac{1}{12}}-3x^{\frac{1}{4}}}{1-4x^{\frac{1}{6}}+x^{\frac{1}{3}}}\Big)\Big\}.$$

4.3. Examples of $\mathcal{F}(m,\lambda,y)$ and $\mathcal{G}(m,\lambda,y)$ with $m\neq 1$

A few explicit expressions for $\mathcal{F}(m,\lambda,y)$ and $\mathcal{G}(m,\lambda,y)$ are recorded as examples, in particular, those for $\lambda=1/2$.

$$\mathcal{F}(2,\tfrac{1}{2},y)={}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{3}{2},\frac{5}{2}\end{matrix}\Big|y\right]=\frac{3}{8}(1-x)^2\Big\{\Phi(x,1,\tfrac{1}{2})+\Phi(x,1,\tfrac{3}{2})\Big\},$$

$$\mathcal{G}(2,\tfrac{1}{2},y)={}_3F_2\left[\begin{matrix}1,2,\frac{3}{2}\\ \frac{3}{2},\frac{5}{2}\end{matrix}\Big|y\right]=\frac{3(1-x)^2}{4(1+x)}+\frac{3(1-x)^3}{16(1+x)}\Big\{\Phi(x,1,\tfrac{1}{2})-\Phi(x,1,\tfrac{3}{2})\Big\};$$

$$\mathcal{F}(3,\tfrac{1}{2},y)={}_3F_2\left[\begin{matrix}1,2,\frac{3}{2}\\ \frac{3}{2},\frac{7}{2}\end{matrix}\Big|y\right]=\frac{5}{8}(1-x)^2+\frac{15}{64}(1-x)^3\Big\{\Phi(x,1,\tfrac{1}{2})-\Phi(x,1,\tfrac{5}{2})\Big\},$$

$$\mathcal{G}(3,\tfrac{1}{2},y)={}_3F_2\left[\begin{matrix}1,2,\frac{5}{2}\\ \frac{3}{2},\frac{7}{2}\end{matrix}\Big|y\right]=\frac{5(3-x)(1-x)^2}{24(1+x)}+\frac{5(1-x)^4}{32(1+x)}\Big\{\Phi(x,1,\tfrac{1}{2})+\Phi(x,1,\tfrac{5}{2})\Big\};$$

$$\mathcal{F}(2,\tfrac{1}{3},y)={}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{4}{3},\frac{8}{3}\end{matrix}\Big|y\right]=\frac{5}{18}(1-x)^2\Big\{\Phi(x,1,\tfrac{1}{3})+\Phi(x,1,\tfrac{5}{3})\Big\},$$

$$\mathcal{G}(2,\tfrac{1}{3},y) = {}_3F_2\left[\begin{matrix}1,2,\tfrac{3}{2}\\ \tfrac{4}{3},\tfrac{8}{3}\end{matrix}\Big| y\right] = \frac{5(1-x)^2}{9(1+x)} + \frac{5(1-x)^3}{27(1+x)}\Big\{\Phi(x,1,\tfrac{1}{3}) - \Phi(x,1,\tfrac{5}{3})\Big\};$$

$$\mathcal{F}(2,\tfrac{1}{4},y) = {}_3F_2\left[\begin{matrix}1,1,\tfrac{3}{2}\\ \tfrac{5}{4},\tfrac{11}{4}\end{matrix}\Big| y\right] = \frac{7}{32}(1-x)^2\Big\{\Phi(x,1,\tfrac{1}{4}) + \Phi(x,1,\tfrac{7}{4})\Big\},$$

$$\mathcal{G}(2,\tfrac{1}{4},y) = {}_3F_2\left[\begin{matrix}1,2,\tfrac{3}{2}\\ \tfrac{5}{4},\tfrac{11}{4}\end{matrix}\Big| y\right] = \frac{7(1-x)^2}{16(1+x)} + \frac{21(1-x)^3}{128(1+x)}\Big\{\Phi(x,1,\tfrac{1}{4}) - \Phi(x,1,\tfrac{7}{4})\Big\};$$

$$\mathcal{F}(2,\tfrac{2}{5},y) = {}_3F_2\left[\begin{matrix}1,1,\tfrac{3}{2}\\ \tfrac{7}{5},\tfrac{13}{5}\end{matrix}\Big| y\right] = \frac{8}{25}(1-x)^2\Big\{\Phi(x,1,\tfrac{2}{5}) + \Phi(x,1,\tfrac{8}{5})\Big\},$$

$$\mathcal{G}(2,\tfrac{1}{5},y) = {}_3F_2\left[\begin{matrix}1,2,\tfrac{3}{2}\\ \tfrac{6}{5},\tfrac{14}{5}\end{matrix}\Big| y\right] = \frac{9(1-x)^2}{25(1+x)} + \frac{18(1-x)^3}{125(1+x)}\Big\{\Phi(x,1,\tfrac{1}{5}) - \Phi(x,1,\tfrac{9}{5})\Big\}.$$

5. Conclusions

By combining the bisection series approach with Whipple's quadtratic transformation formulae, we succeeded in evaluating several remarkable ${}_3F_2(y)$-series in terms of Lerch's transcendental function, including Yabu's results as very initial examples. However, the remaining problem is how to extend these methods to the generalized hypergeometric series of higher order. The interested reader is encouraged to make further attempts to evaluate the related series explicitly.

Author contributions

Marta Na Chen: Computation, Writing, and Editing; Wenchang Chu: Original draft, Review, and Supervision. Both authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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