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Research article

An optimal investment strategy for DC pension plans with costs and the return of premium clauses under the CEV model

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Abstract: This paper presents a novel optimization model that explores the optimal investment strategies for DC pension plans with return of premium clauses. We have assumed that the financial market consists of a risk-free asset and a risky asset, where the price of the risky asset follows the CEV model. Under the expected utility criterion, the optimal investment strategies were derived by employing stochastic optimal control theory and the Legendre transformation method. Explicit expressions of the optimal investment strategy were provided when the utility function was specified as exponential, power, or logarithmic. Finally, numerical analysis was conducted to examine the impact of factors such as interest rate, return rate, and volatility of the risky asset on the optimal strategies.

Keywords: constant elasticity of variance model; defined contribution pension plan; return of premium clauses; expected utility; tax; trading fee
Mathematics Subject Classification: 91B16, 91B70

1. Introduction

Pension funds are one of the most important institutions in the financial markets because they can help to secure one's life after retirement. Therefore, studying the optimal management strategy for a pension fund is a research topic of theoretical and practical importance. Typically, there are mainly two kinds of pension plans, which are defined benefit (DB) and defined contribution (DC) pension plans. In recent years, DC pension plans have become more popular in the market because DC pension plans can meet the demographic development and the financial market. Obviously, DC pension plans are worth studying.

Over the last two decades, DC pension plans have been extensively studied in the literature. From the perspective of financial markets, for the case where the price of the stock is modeled by the geometric Brownian motion (GBM), see Boulier et al. [1], Deelstra et al. [2], Han and Hung [3], Guan and Liang [4], Guan and Liang [5], Chen et al. [6], Fombellida and Casado [7], Dong and Zheng [8], and the references therein. For the case where the stock price is assumed to be the constant elasticity variance (CEV) model, see Xiao et al. [9], Gao [10], Gao [11], Zhang et al. [12], Li et al. [13], Li et al. [14], and Chen et al. [15]. Since the GBM can not suitably capture the volatility smile of the stock, as the CEV model can do well, thus the CEV model is more suitable to model the prices of stock. Meanwhile, the GMB can be considered as a special case of the CEV model.

Recently, from a practical point of view, many researchers studied the optimal investment strategies for DC pension plans with the return of premium clauses. In a DC pension plan without return of premium clauses, if a pension fund member dies during the accumulation phase, then the pension plan is automatically terminated. Meanwhile, the premiums paid by the pension member will not be returned to the member. In contrast to DC pension plans without return of premium clauses, in a DC pension plan with return of premium clauses, the pension members withdraw their premiums when they die during the accumulation phase and the difference between the premium and the accumulation (negative or positive) is distributed among the surviving members. Obviously, DC pension plans with return of premium clauses can protect the rights and interests of pensioners well. Therefore, it is of great importance to study the optimal management strategies for DC pension plans with the return of premium clauses. In this direction, He and Liang [16] first introduced the return of premium clauses into a DC pension plan model. Under the mean-variance criterion, they studied the optimal investment strategies where the stock price was modeled by a GBM. Under the mean-variance criterion, Sheng and Rong [17] studied the optimal investment strategies for a DC pension plan with an annuity contract, where the price of the stock was modeled by Heston's stochastic volatility model. Under the mean-variance criterion, Sun and Li [18] investigated the optimal investment strategy for a DC pension plan, where the price of the stock was modeled by a jump-diffusion process. Under the multi-period mean-variance criterion, Bian et al. [19] studied the investment strategy for a DC pension plan in a discrete time setting. By maximising the mean-variance of terminal wealth, Li and Rong [14] studied the optimal investment strategy for DC pension plans, where the price of the stock was modeled by the CEV model and the invested bond was defaultable.

All of the above references to optimal investment strategies for DC pension plans with return of premium clauses used mean-variance criteria. However, it is well-known that besides the popular mean-variance criterion, another popular optimality criterion is maximizing the expected utility of the terminal wealth. Therefore, this paper will focus on the expected utility criterion. On the other hand, in reality, financial markets usually involve costs such as trading fees and taxes, while these economic factors have not been discussed in the literature for a DC pension plan with return of premium clauses. Consequently, this paper is mainly motivated by the following two concerns:

(1) How can we best incorporate the trading fees and taxes into the model of DC pension plans with the return of premiums clauses?

(2) Furthermore, under the above first concern, what is the optimal investment strategy by that maximizes the expected utility of the terminal wealth?

To the best of our knowledge, the above two concerns have not been discussed in the literature. In the present paper, we aim to address the above two concerns. We assume that the financial market consists of a risk-free asset and a risky asset (called a stock), where the price of the stock is modeled by the CEV model. We also assume that there are trading fees and taxes in the financial market, which may affect the asset process. Instead of mean-variance criterion, by maximizing the expected utility of the terminal wealth, we obtain the optimal investment strategy for a DC pension plan with the return of premium clauses via stochastic optimal control theory and the Legendre transformation method. When the utility function is specified in terms of exponential, power, and logarithmic utility functions, the explicit expressions of the optimal investment strategies are given. Finally, the effect of the involved parameters on the optimal investment strategies is analyzed, and these parameters are the rate of interest and the return rate and volatility of the stock.

The remainder of this paper is organized as follows. In Section 2, we introduce the financial market and the optimization model for a DC pension plan with return of premium clauses. In Section 3, we provide a general solution to the optimization model by solving the Hamilton-Jacobi-Bellman (HJB) equation. In Section 4, we give the explicit expressions of the optimal investment strategies when the utility function is specified in terms of the exponential, power, and logarithmic utility functions. In Section 5, we provide a numerical analysis of the effect of parameters on the optimal investment strategies. These parameters include the interest rate and the return and volatility of the stock. Finally, Section 6 summarizes the conclusions.

2. Model formulation

In this section, we introduce the financial market and the optimization program.

2.1. The financial market

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions and $\mathscr{F}_t$ represents the information up to time t in the market. The accumulation phase of the DC pension plan occurs within $[0, T]$. The DC pension fund starts at time 0 and $T > 0$ is the retirement time. We assume that all of the processes introduced below are well-defined and also adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$.

Suppose that the market structure consists of two financial assets, a risk-free asset and a risky asset. We denote the price of the risk-free asset (i.e., cash) at time t by $B(t)$, and the price process by

$$\frac{\mathrm{d}B(t)}{B(t)} = r\mathrm{d}t, \tag{2.1}$$

where r is a constant rate of interest.

The price process of the risky asset (i.e., stock) at time t by $S(t)$, is described by the CEV model

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\mathrm{d}t + kS^{\beta}(t)\mathrm{d}W(t), \tag{2.2}$$

where μ is an instantaneous expected return on the stock and satisfies the general condition $\mu > r$, $kS^{\beta}(t)$ is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta \geq 0$. $\{W_t : t \geq 0\}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

Remark 2.1. *Note that if the elasticity parameter $\beta = 0$ in(2), then it reduces to a GBM. If $\beta < 0$, the instantaneous volatility kS_t^{β} increases as the stock price decreases, and can generate a distribution with a fatter left tail. If $\beta > 0$, the situation is reversed.*

2.2. The optimization problem

For a DC pension, the holder of the pension fund in the accumulation stage needs to pay a certain amount of premium in advance. In the model, the premium paid per unit of time is P (P is constant) and the starting point of the accumulation phase is ω_0. The final moment is $\omega_0 + T$. According to the rules of endowment insurance, if a member dies before retirement, the balance of the individual contribution in the personal account and the interest earned are returned to the heir in a lump sum. In order to protect the rights and interests of pensioners who die during the accumulation phase, this paper introduces the premium return clause into the pension plan. Under actuarial terms, a deceased annuitant can withdraw premiums. ${}_{\Delta t}\mathrm{q}_{\omega_0+t}$ is an actuarial notation that denotes the probability of dying at age $\omega_0 + t$ and in time $[\omega_0 + t, \omega_0 + t + \Delta t]$. tP denotes the accumulation of premiums at time t, so that the return of the death annuity at age $\omega_0 + t$ during the period $[0, \Delta t]$ is $tP\,{}_{\Delta t}\mathrm{q}_{\omega_0+t}$ and the difference between the income after the premiums are returned and the the difference between the accumulation phase will be distributed equally to the survivors.

To increase the wealth value of the pension fund, the manager of the pension fund invests the insurance premium in a risky asset (stock) and a risk-free asset (cash), where the proportion allocated of π and $(1 - \pi)$ respectively. ρ is the trading fees of the stock trading, and θ is the tax when the holder of the pension fund withdraws.

Consider the discrete form of wealth in $[t, t + \frac{1}{n}]$:

$$\begin{aligned} X^{\pi}\left(t + \frac{1}{n}\right) = \Bigg[X^{\pi}(t)\left(\pi\frac{S(t + \frac{1}{n})}{S(t)} + (1 - \pi)\frac{B(t + \frac{1}{n})}{B(t)}\right) - X^{\pi}(t)(\pi\rho + \theta)\frac{1}{n} \\ +P\frac{1}{n} - aPt\,{}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t}\Bigg] \times \left(\frac{1}{1 - {}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t}}\right). \end{aligned} \tag{2.3}$$

To compare the optimal strategies of pension funds with and without the return of premium clauses, we introduce parameters, the parameters $a = 1$, which means there is a return of premium clause, and $a = 0$, which means there is no return of premium clause. The last item of the above formula means that the difference between the income and the accumulate phase amount will be distributed equally to the survivors.

Denote

$$\Delta\delta_t^{\frac{1}{n}} := \pi\frac{S(t + \frac{1}{n}) - S(t)}{S(t)} + (1 - \pi)\frac{B(t + \frac{1}{n}) - B(t)}{B(t)}, \tag{2.4}$$

then

$$\begin{aligned} X^{\pi}\left(t + \frac{1}{n}\right) = \Bigg[X(t)^{\pi}(1 + \Delta\delta_t^{\frac{1}{n}}) - X^{\pi}(t)(\pi\rho + \theta)\frac{1}{n} + P\frac{1}{n} \\ -aPt\,{}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t}\Bigg] \times \left(1 + \frac{{}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t}}{1 - {}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t}}\right). \end{aligned} \tag{2.5}$$

In (2.5) $u(t)$ is the death force function, and ${}_t\mathrm{q}_x$ is an actuarial symbol standing for the probability that the person who is alive at the age of x will be dead in the following t time period, and satisfies ${}_t\mathrm{q}_x = 1 - {}_t\mathrm{p}_x = 1 - e^{-\int_0^t u(x+s)\mathrm{d}s}$. So

$${}_{\frac{1}{n}}\mathrm{q}_{\omega_0+t} = 1 - e^{-\int_0^{\frac{1}{n}} u(\omega_0+t+s)\mathrm{d}s} \approx u(\omega_0 + t)\frac{1}{n} = O(\frac{1}{n}). \tag{2.6}$$

Let $t \to 0$, then $u(\omega_0 + t)$ is small in the accumulation stage and

$$\frac{{}_{\frac{1}{n}}q_{\omega_0+t}}{1 - {}_{\frac{1}{n}}q_{\omega_0+t}} = \frac{1 - e^{-\int_0^{\frac{1}{n}} u(\omega_0+t+s)\mathrm{d}s}}{e^{-\int_0 u(\omega_0+t+s)\mathrm{d}s}} = e^{\int_0^{\frac{1}{n}} u(\omega_0+t+s)\mathrm{d}s} - 1 \approx u(\omega_0 + t)\frac{1}{n} = O(\frac{1}{n}). \tag{2.7}$$

Hence, it is easy to get

$$\Delta\delta_t \cdot O(\frac{1}{n}) = o(\frac{1}{n}),\ {}_{\frac{1}{n}}q_{\omega_0+t} \cdot O(\frac{1}{n}) = o(\frac{1}{n}).$$

Then (2.5) can be written as follows

$$X^\pi\left(t + \frac{1}{n}\right) = X^\pi(t)(1 + \Delta\delta_t^{\frac{1}{n}}) + X^\pi(t)u(\omega_0 + t)\frac{1}{n} - X^\pi(t)(\pi\rho + \theta)\frac{1}{n} + P\frac{1}{n} - aPtu(\omega_0 + t)\frac{1}{n} + o(\frac{1}{n}). \tag{2.8}$$

When $t \to 0$, and we have

$$\begin{aligned} \mathrm{d}X^\pi(t) =& X^\pi(t)\left[\pi\frac{\mathrm{d}S(t)}{S(t)} + (1 - \pi)\frac{\mathrm{d}B(t)}{B(t)}\right] + X(t)u(\omega_0 + t)\mathrm{d}t \\ &- X^\pi(t)(\pi\rho + \theta)\mathrm{d}t + P\mathrm{d}t - aPtu(\omega_0 + t)\mathrm{d}t. \end{aligned} \tag{2.9}$$

Substitute (2.1) and (2.2) into (2.9), we have

$$\mathrm{d}X^\pi(t) = X^\pi(t)[\pi(\mu - \rho - r) + (r - \theta) + u(\omega_0 + t)]\mathrm{d}t + P\mathrm{d}t - aPtu(\omega_0 + t)\mathrm{d}t + X^\pi(t)\pi k S^\beta(t)\mathrm{d}W(t). \tag{2.10}$$

To simplify the model, we have chosen the Abraham De Moivre model (refer to Kohler and Kohler [20]) to characterize the force of mortality function, $u(t)$, in the following form:

$$u(t) = \frac{1}{\omega - t}, 0 \le t < \omega, \tag{2.11}$$

where ω is the limit age of the person, (2.10) of the wealth process of the pension fund can be written as follows

$$\begin{cases} \mathrm{d}X^\pi(t) = X^\pi(t)[\pi(\mu - \rho - r) + (r - \theta) + \dfrac{1}{\omega - \omega_0 - t}]\mathrm{d}t \\ \qquad\qquad + P\dfrac{\omega - \omega_0 - (1 + a)t}{\omega - \omega_0 - t}\mathrm{d}t + X^\pi(t)\pi k S^\beta(t)\mathrm{d}W(t), \\ X(0) = x_0. \end{cases} \tag{2.12}$$

In this paper, a dynamic asset allocation strategy $\pi(t)$ is an asset allocation proportion invested in the stock at time t. A strategy $\Pi = \{\pi(t) \in [0, \infty], (0 \le t \le T)\}$ is called admissible. We denote the set of all admissible strategies as the following.

Definition 2.1. *For any fixed $t \in [0, T]$, a strategy π is said to be admissible if*
(1) π is $\mathcal{F}$-predictable;
(2) For any $v \in [t, T]$, $E[\int_t^T (x(v)\pi(v)ks^{2\beta})^2] < +\infty$;
(3) For any $v \in R \times R$, Eq (2.12) has a unique solution $X^\pi(v)_{v\in[t,T]}$ with $X^\pi(t) = x, S(t) = s$.

Let $U(x) : \mathbf{R}_+ \rightarrow \mathbf{R}$ be a utility function. Assume that $U(x)$ has the finite first- and second-order derivatives, and assume that $U'(x) > 0$ and $U''(x) < 0$ for any x. Denote by RAC(x) the risk aversion coefficient of the utility function $U(x)$, that is RAC(x):= $-\frac{U''(x)}{U'(x)}$. The RAC(x) reflects the risk preference of an investor. The larger the RAC(x) is, the more risk-averse the investor . In other words, the smaller the RAC(x) , the more risk-appetite the investor is.

To study a DC pension plan is to choose an optimal investment strategy $\pi(t)$ that maximizes the expected utility of the terminal wealth $E[U(X(T))]$. In other words, the optimization problem studied in this paper is described as following:

$$\max_{\pi_t \in \Pi} E[U(X(T))]. \tag{2.13}$$

3. Solution of the model

This section begins by deriving the general framework of the optimization problem (2.13) using the maximum principle. It then proceeds to find the explicit solution of the optimal investment strategy that maximizes the terminal expected utility function using the Legendre transformation.

3.1. General framework

By using the classical tools of stochastic optimal control, we define the value function

$$H(t, s, x) = \sup_{\pi_t \in \Pi} E\{U(X(T))|S(t) = s, X(t) = x\}. \tag{3.1}$$

Assume that $V(t, s, x) \in \mathbb{C}^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R})$, then we can define the following variational operator on any function $V(t, s, x)$:

$$\mathscr{A}^{\pi(t)}V(t, s, x) = H_t + \left\{x\left[\pi(\mu - \rho - r) + (r - \theta) + \frac{1}{\omega - \omega_0 - t}\right] + P\frac{\omega - \omega_0 - (1 + a)t}{\omega - \omega_0 - t}\right\} H_x$$
$$+\mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}x^2\pi^2 k^2 s^{2\beta} H_{xx} + x\pi k^2 s^{2\beta+1} H_{xs},$$

where V_t, V_x, V_{xx}, V_s, V_{ss}, and V_{sx} represent first-order and second-order partial derivatives with respect to the variables t, s, and x respectively.

According to the principle of stochastic dynamic programming, we can easily derive the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} &H_t + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \left\{x\left[(r - \theta) + \frac{1}{\omega - \omega_0 - t}\right] + P\frac{\omega - \omega_0 - (1 + a)t}{\omega - \omega_0 - t}\right\} H_x \\ &\sup_{\pi_t \in \Pi}\left\{x\pi(\mu - \rho - r)H_x + \frac{1}{2}x^2\pi^2 k^2 s^{2\beta} H_{xx} + x\pi k^2 s^{2\beta+1} H_{xs}\right\} = 0. \end{aligned} \tag{3.2}$$

The optimal investment strategy π^* satisfies

$$\pi^* = -\frac{(\mu - \rho - r)H_x + k^2 s^{2\beta+1} H_{xs}}{x k^2 s^{2\beta} H_{xx}}. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain another partial differential equation for the value function H

$$\begin{aligned} &H_t + \mu s H_s + \left\{ x \left[(r-\theta) + \frac{1}{\omega - \omega_0 - t} \right] + P\frac{\omega - \omega_0 - (1+a)t}{\omega - \omega_0 - t} \right\} H_x \\ &\qquad + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} - \frac{1}{2}\frac{[(\mu - \rho - r)H_x + k^2 s^{2\beta+1} H_{xs}]^2}{k^2 s^{2\beta} H_{xx}} = 0. \end{aligned} \tag{3.4}$$

(3.4) represents a non-linear second-order partial differential equation, which can be challenging to solve. However, the use of the Legendre transformation can transform it into a linear partial differential equation, making it easier to solve.

The following theorem verifies that the investment strategy given by (3.3) is indeed optimal for optimization problem (2.13) and its proof can be found in the Appendix.

Theorem 3.1. *If $V(t,s,x) \in \mathbb{C}^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R})$ is a solution of the HJB equation (3.2), i.e., $V(t,s,x)$ satisfies the following HJB equation*

$$\sup_{\pi_t \in \Pi} \{\mathcal{A}^{\pi(t)} V(t,s,x)\} = 0, V(T,s,x) = U(x(T)),$$

then for all admissible strategies $\pi(t) \in \Pi$, one has $H(t,s,x) \leq V(t,s,x)$; if $\widetilde{\pi}(t)$ satisfies

$$\widetilde{\pi}(t) \in \arg\sup_{\pi_t \in \Pi} E\{U(X(T))|S(t) = s, X(t) = x\},$$

then one has $H(t,s,x) = V(t,s,x)$, and it implies that $\widetilde{\pi}(t)$ is the optimal solution of optimization problem (2.13). Consequently, the investment strategy given by (3.3) is the optimal solution to the optimization problem (2.13).

3.2. Legendre transformation

Definition 3.1. *Let $f : R_n \longrightarrow R$ be a convex function. For $z > 0$, define the Legendre transform*

$$L(z) = \max_x f(x) - zx,$$

and the function $L(z)$ is called the Legendre dual of the function $f(x)$.

If $f(x)$ is strictly convex, the maximum in the above equation will be attained at just one point, which we denote by x_0. It is attained at the unique solution to the first-order condition

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} - z = 0.$$

So we can write

$$L(z) = f(x_0) - zx_0.$$

We also use the assumed convexity of the value function to define the Legendre transform

$$\hat{H}(t,s,z) = \sup_{x>0} \{H(t,s,x) - zx\}, \tag{3.5}$$

where $z > 0$ denotes the variable dual to x. The value of x where this optimum is denoted by $g(t,s,z)$, so that

$$g(t,s,z) = \inf_{x>0}\{x|H(t,s,x) \geq zx + \hat{H}(t,s,z)\}. \tag{3.6}$$

The function $\hat{H}$ is related to g by

$$g(t,s,z) = -\hat{H}_z(t,s,z). \tag{3.7}$$

Hence we can take either one of the two functions g and $\hat{H}$ as the dual of H.

As the Legendre transform describes, we have that

$$H_x(t,s,g(t,s,z)) = z, \tag{3.8}$$

and this leads to

$$\hat{H}(t,s,z) = H(t,s,g) - zg. \tag{3.9}$$

By differentiating (3.5) and (3.6) with respect to t, s, and z, the transformation rules for the derivatives of the value function H and the dual function $\hat{H}$ can be given as (e.g., Jonsson and Sircar [21])

$$H_x = z, \quad H_t = \hat{H}_t, \quad H_{ss} = \hat{H}_{ss} - \frac{\hat{H}_{sz}^2}{\hat{H}_{zz}},$$
$$H_s = \hat{H}_s, H_{xs} = -\frac{\hat{H}_{sz}}{\hat{H}_{zz}}, H_{xx} = -\frac{1}{\hat{H}_{zz}}. \tag{3.10}$$

At the terminal time, we define

$$\hat{U}(z) = \sup_{x>0}\{U(x) - zx\}, G(z) = \inf_{x>0}\{X|U(x) \geq zx + \hat{U}(z)\}. \tag{3.11}$$

In this way, the problem can be turned into a dual problem.

Substituting expression (3.10), we rewrite (3.4) as

$$\hat{H}_t + \mu s\hat{H}_s + x\left(r - \theta + \frac{1}{\omega - \omega_0 - t}\right)z + P\left(\frac{\omega - \omega_0 - (1+a)t}{\omega - \omega_0 - t}\right)z$$
$$+ \frac{1}{2}k^2 s^{2\beta+2}(\hat{H}_{ss} - \frac{\hat{H}_{sz}^2}{\hat{H}_{zz}}) - \frac{1}{2}\frac{[(\mu - \rho - r)z + k^2 s^{2\beta+1}(-\frac{\hat{H}_{sz}}{\hat{H}_{zz}})]^2}{k^2 s^{2\beta}(-\frac{1}{\hat{H}_{zz}})} = 0, \tag{3.12}$$

where $\varphi_1 := r - \theta + \frac{1}{\omega-\omega_0-t}$, $\varphi_2 := P\left(\frac{\omega-\omega_0-(1+a)t}{\omega-\omega_0-t}\right)$, $\varphi_3 := \mu - \rho - r$.

Now we have the following partial differential equation

$$\hat{H}_t + \mu s\hat{H}_s + x\varphi_1 z + \varphi_2 z + \frac{1}{2}k^2 s^{2\beta+2}\hat{H}_{ss} + \frac{1}{2}\frac{\varphi_3^2 z^2}{k^2 s^{2\beta}}\hat{H}_{zz} - \varphi_3 sz\hat{H}_{sz} = 0. \tag{3.13}$$

The derivative with respect to z is

$$\hat{H}_{tz} + (\mu - \varphi_3)s\hat{H}_{sz} + x\varphi_1 + \varphi_1 z x_z + \varphi_2 + \frac{1}{2}k^2 s^{2\beta+2}\hat{H}_{ssz} + \frac{1}{2}\frac{\varphi_3^2 z^2}{k^2 s^{2\beta}}\hat{H}_{zzz} + \frac{\varphi_3^2 z}{k^2 s^{2\beta}}\hat{H}_{zz} - \varphi_3 sz\hat{H}_{szz} = 0. \tag{3.14}$$

For any t, s, let $x = g(t, x, z)$ in (3.14), and by using (3.7), we have

$$g_t + (\mu - \varphi_3)sg_s + (\frac{\varphi_3^2}{k^2 s^{2\beta}} - \varphi_1)zg_z - g\varphi_1 - \varphi_2 + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} + \frac{1}{2}\frac{\varphi_3^2 z^2}{k^2 s^{2\beta}} g_{zz} - \varphi_3 szg_{sz} = 0. \quad (3.15)$$

The focus is typically on the optimal investment strategy rather than the value function. Using Eq (3.4), we can calculate the optimal stock holding through a feedback formula based on the derivatives of the value function. The optimal stock holding can be expressed in terms of the dual function g.

$$\pi^* = -\frac{\varphi_3 H_x + k^2 s^{2\beta+1} H_{xs}}{xk^2 s^{2\beta} H_{xx}} = \frac{-\varphi_3 zg_z + k^2 s^{2\beta+1} g_s}{gk^2 s^{2\beta}}. \quad (3.16)$$

Thus, for a given optimal problem, we solve the linear PDE for g and obtain the investment amount π^*.

4. Explicit solutions

This section presents the explicit expressions for the solutions to the optimization problem (2.13) when the utility function $U(x)$ is specified as the exponential (constant absolute risk aversion (CARA)), power (constant relative risk aversion (CRRA)), and logarithmic utility functions.

4.1. Explicit expression for the optimal solution for the CARA utility

The exponential utility function is given by

$$U(x) = -\frac{1}{q}e^{-qx},$$

where $q > 0$ is the risk aversion coefficient.

The Eq (3.15) is then solved.

$$g(T, s, z) = (U')^{-1}(z) = -\frac{1}{q}\ln z.$$

The solution to Eq (3.15) is sought as follows.

$$g(t, s, z) = -\frac{1}{q}[b(t)(\ln z + m(t, s))] + \alpha(t) \quad (4.1)$$

with the boundary conditions given by $m(T, s) = 0, \alpha(T) = 0$, and $b(T) = 1$.

Then

$$\begin{gathered} g_t = -\frac{1}{q}\{b'(t)[\ln z + m(t, s)] + b(t)m_t\} + \alpha'(t), \quad g_s = -\frac{1}{q}b(t)m_s, \quad g_z = -\frac{b(t)}{qz}, \\ g_{ss} = -\frac{1}{q}b(t)m_{ss}, \quad g_{sz} = 0, \quad g_{zz} = \frac{b(t)}{qz^2}. \end{gathered} \quad (4.2)$$

Plugging (4.1) in (3.15), we have

$$
\begin{aligned}
&-\frac{1}{q}\ln z\{b'(t)-b(t)\varphi_1\}-\frac{1}{q}b(t)\{m\frac{b'(t)}{b(t)}+m_t+(u-\varphi_3)sm_s-m\varphi_1-\varphi_1\\
&\quad+\frac{1}{2}k^2s^{2\beta+2}m_{ss}+\frac{1}{2}\frac{\varphi_2^3}{k^2s^{2\beta}}\}+\alpha'(t)-\alpha(t)\varphi_1-\varphi_2=0.
\end{aligned}
\tag{4.3}
$$

The equation can be split into three separate equations.

$$\alpha'(t)-\alpha(t)\varphi_1-\varphi_2=0, \tag{4.4}$$

$$b'(t)-b(t)\varphi_1=0, \tag{4.5}$$

$$m_t+(\rho+r)sm_s+\frac{1}{2}k^2s^{2\beta+2}m_{ss}+\frac{1}{2}\frac{\varphi_3^2}{k^2s^{2\beta}}-\varphi_1=0. \tag{4.6}$$

Taking into account the boundary condition $\alpha(T)=0$, we have

$$\alpha(t)=\frac{P(1+a)[1-e^{(r-\theta)(t-T)}]}{(\omega-\omega_0-t)(r-\theta)^2}-\frac{P(\omega-\omega_0-(1+a)t)}{(\omega-\omega_0-t)(r-\theta)}+\frac{P(\omega-\omega_0-(1+a)T)}{(\omega-\omega_0-t)(r-\theta)}e^{(r-\theta)(t-T)}. \tag{4.7}$$

Taking into account the boundary condition $b(T)=1$, we have

$$b(t)=\frac{\omega-\omega_0-t}{\omega-\omega_0-T}e^{(r-\theta)(t-T)}. \tag{4.8}$$

Note that since (4.5) includes coefficient variables $s, s^{2\beta+2}$, and $s^{2\beta}$, it is difficult to solve this equation. Therefore, we can use a power transformation and a variable change technique to transform the non-linear equation into a linear one.

$$m(t,s)=h(t,y), y=s^{-2\beta},$$

so that

$$m_t=h_t,\quad m_s=-2\beta s^{-2\beta-1}h_y,\quad m_{ss}=4\beta^2s^{-4\beta-2}h_{yy}+2\beta(2\beta+1)s^{-2\beta-2}h_y. \tag{4.9}$$

Substituting these derivatives into (4.5), we obtain the following linear PDE

$$h_t-2\beta(\rho+r)yh_y+2\beta^2k^2yh_{yy}+k^2\beta(2\beta+1)h_y+\frac{1}{2}\frac{\varphi_3^2y}{k^2}-\varphi_1=0. \tag{4.10}$$

We try to find a solution of (4.9) with the following structure

$$h(t,y)=\widetilde{A}(t)+\widetilde{B}(t)y, \tag{4.11}$$

with $\widetilde{A}(T)=0, \widetilde{B}(T)=0$.

Then

$$h_t=\widetilde{A}_t+\widetilde{B}_ty,\quad h_y=\widetilde{B},\quad h_{yy}=0, \tag{4.12}$$

and introducing this in (4.9), we derive

$$\widetilde{A}_t+k^2\beta(2\beta+1)\widetilde{B}-\varphi_1+y\{\widetilde{B}_t-2\beta(\rho+r)\widetilde{B}+\frac{\varphi_3^2}{2k^2}\}=0. \tag{4.13}$$

We can decompose (4.12) into two conditions in order to eliminate the dependence in y and t:

$$\widetilde{A}_t + k^2\beta(2\beta+1)\widetilde{B} - \varphi_1 = 0, \tag{4.14}$$

$$\widetilde{B}_t - 2\beta(\rho+r)\widetilde{B} + \frac{\varphi_3^2}{2k^2} = 0. \tag{4.15}$$

Taking into account the boundary condition $\widetilde{B}(T) = 0$, the solution to (4.15) is

$$\widetilde{B}(t) = \frac{\varphi_3^2}{4k^2\beta(\rho+r)}(1 - e^{2\beta(\rho+r)(t-T)}). \tag{4.16}$$

Taking into account the boundary condition $\widetilde{A}(T) = 0$ and $\widetilde{B}(t)$, the solution to (4.13) is

$$\widetilde{A}(t) = \left(-\frac{(2\beta+1)\varphi_3^2}{4(\rho+r)} + r - \theta\right)(t-T) + \frac{(2\beta+1)\varphi_3^2}{8\beta(\rho+r)}e^{2\beta(\rho+r)(t-T)} + \ln\frac{\omega-\omega_0-T}{\omega-\omega_0-t}. \tag{4.17}$$

Finally, we can obtain the optimal solution of problem (2.13) under the CARA utility. The optimal strategy invested in the stock is given by

$$\pi^*(t) = \frac{\varphi_3 b(t) + 2\beta k^2 b(t)\widetilde{B}(t)}{qxk^2 s^{2\beta}}, \tag{4.18}$$

which is summarized in the following theorem,

Theorem 4.1. *For the optimization problem (2.13) with wealth process (2.12) and the exponential utility function, the optimal investment strategy is given by*

$$\pi^*(t) = \frac{\varphi_3 b(t) + 2\beta k^2 b(t)\widetilde{B}(t)}{qxk^2 s^{2\beta}},$$

where

$$\varphi_3 = \mu - \rho - r$$

$$b(t) = \frac{\omega-\omega_0-t}{\omega-\omega_0-T}e^{(r-\theta)(t-T)},$$

$$\widetilde{B}(t) = \frac{\varphi_3^2}{4k^2\beta(\rho+r)}(1 - e^{2\beta(\rho+r)(t-T)}).$$

Remark 4.1. *For the CARA utility function, the return of premium clauses have no effect on the optimal investment strategy. If elasticity parameter $\beta = 0$, the CEV model reduces to the GBM model, and the optimal investment strategy is given by $\pi^*(t) = \frac{\varphi_3 b(t)}{qxk^2}$, which also implies that the optimal investment strategy has nothing to do with the price of the stock.*

4.2. *Explicit expression for the optimal solution for the CRRA utility*

4.2.1. Explicit solutions for the power utility

The power utility function is given by

$$U(x) = \frac{1}{\gamma}x^{\gamma},$$

where $\gamma < 1, \gamma \neq 0$.

Next, we solve Eq (3.15).

$$g(T, s, z) = (U')^{-1}(z) = z^{\frac{1}{\gamma-1}}.$$

We try to find a solution to (3.15) in the following way

$$g(t, s, z) = z^{\frac{1}{\gamma-1}}h(t, s) + \alpha(t), \tag{4.19}$$

with the boundary conditions given by $h(T, s) = 1, \alpha(T) = 0$.

Then

$$\begin{aligned} g_t &= h_t z^{\frac{1}{\gamma-1}} + \alpha'(t), \qquad g_s = h_s z^{\frac{1}{\gamma-1}}, \qquad g_z = \frac{h}{\gamma-1}z^{\frac{1}{\gamma-1}-1}, \\ g_{ss} &= h_{ss} z^{\frac{1}{\gamma-1}}, \qquad g_{sz} = \frac{h_s}{\gamma-1}z^{\frac{1}{\gamma-1}-1}, \qquad g_{zz} = \frac{(2-\gamma)h}{(\gamma-1)^2}z^{\frac{1}{\gamma-1}-2}. \end{aligned} \tag{4.20}$$

Plugging (4.19) in (3.15), we have

$$\begin{aligned} z^{\frac{1}{\gamma-1}} \Bigg[& h_t + (u - \frac{\gamma}{\gamma-1}\varphi_3)sh_s - \frac{\gamma}{\gamma-1}h\varphi_1 + \frac{1}{2}k^2 s^{2\beta+2}h_{ss} \\ & + \frac{1}{2}\frac{\varphi_3^2}{k^2 s^{2\beta}}\frac{(2-\gamma)h}{(\gamma-1)^2} + \frac{\varphi_3^2}{k^2 s^{2\beta}}\frac{h}{\gamma-1} \Bigg] + \alpha'(t) - \alpha(t)\varphi_1 - \varphi_2 = 0. \end{aligned} \tag{4.21}$$

We can split (4.20) into equations in order to eliminate the dependence on $z^{\frac{1}{\gamma-1}}$:

$$\alpha'(t) - \alpha(t)\varphi_1 - \varphi_2 = 0, \tag{4.22}$$

$$h_t + (u - \frac{\gamma}{\gamma-1}\varphi_3)sh_s - h\frac{\gamma}{\gamma-1}\varphi_1 + \frac{1}{2}k^2 s^{2\beta+2}h_{ss} + \frac{\varphi_3^2}{k^2 s^{2\beta}}\frac{\gamma h}{2(\gamma-1)^2}. \tag{4.23}$$

Taking into account the boundary condition $\alpha(T) = 0$, we have

$$\alpha(t) = \frac{P(1+a)[1 - e^{(r-\theta)(t-T)}]}{(\omega - \omega_0 - t)(r-\theta)^2} - \frac{P(\omega - \omega_0 - (1+a)t)}{(\omega - \omega_0 - t)(r-\theta)} + \frac{P(\omega - \omega_0 - (1+a)T)}{(\omega - \omega_0 - t)(r-\theta)}e^{(r-\theta)(t-T)}. \tag{4.24}$$

To solve (4.23), similar to the exponential case, we define let $h(t, s) = f(t, y)$ and $y = s^{-2\beta}$ so that

$$h_t = f_t, \quad h_s = -2\beta s^{-2\beta-1}f_y, \quad h_{ss} = 4\beta^2 s^{-4\beta-2}f_{yy} + 2\beta(2\beta+1)s^{-2\beta-2}f_y. \tag{4.25}$$

Substituting (4.25) into (4.23), we obtain the following:

$$f_t - \frac{\gamma f}{\gamma-1}\left(\varphi_1 - \frac{\varphi_3^2}{2k^2(\gamma-1)}\right) + 2k^2\beta^2 y f_{yy} + \beta f_y\left\{-2uy + k^2(2\beta+1) + \frac{\gamma}{\gamma-1}2\varphi_3 y\right\} = 0. \tag{4.26}$$

We try to find a solution of (4.26) with the following:

$$f(t,y) = A(t)e^{B(t)y}, \tag{4.27}$$

with $A(T) = 1, B(T) = 0$.

Then

$$f_t = A'(t)e^{B(t)y} + A(t)B'(t)ye^{B(t)y}, \quad f_y = A(t)B(t)e^{B(t)y}, \quad f_{yy} = A(t)B^2(t)e^{B(t)y}. \tag{4.28}$$

Introducing this in (4.26), we derive

$$\begin{aligned}\frac{A_t}{A} + B_t y - \frac{\gamma}{\gamma-1}\left(\varphi_1 - \frac{\varphi_3^2}{2k^2(\gamma-1)}\right) + 2k^2\beta^2 B^2 y \\ + \beta B\{-2\mu y + k^2(2\beta+1) + \frac{\gamma}{\gamma-1}2\varphi_3 y\} = 0.\end{aligned} \tag{4.29}$$

We can decompose (4.29) into two conditions in order to eliminate the dependence in y and t:

$$\frac{A_t}{A} - \frac{\gamma}{\gamma-1}\varphi_1 + Bk^2\beta(2\beta+1) = 0, A(T) = 1, \tag{4.30}$$

$$B_t + \frac{\varphi_3^2\gamma}{2k^2(\gamma-1)^2} + 2k^2\beta^2 B^2 + 2\beta B\frac{\mu - \gamma(\rho+r)}{\gamma-1} = 0, B(T) = 0. \tag{4.31}$$

The solution to (4.30) and (4.31) is

$$A(t) = (\frac{\omega - \omega_0 - T}{\omega - \omega_0 - t})^{\frac{\gamma}{\gamma-1}}(\lambda_2 - \lambda_1)^{\frac{\beta(2\beta+1)}{2\beta^2}}\left[\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}\right]^{\frac{-\beta(2\beta+1)}{2\beta^2}} e^{[\frac{\gamma}{\gamma-1}(r-\theta)-\beta(2\beta+1)\lambda_1](t-T)}, \tag{4.32}$$

$$B(t) = k^{-2}I(t), \tag{4.33}$$

where $I(t) = \frac{\lambda_1 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}{1-\frac{\lambda_1}{\lambda_2}e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}$, $\lambda_{1,2} = \frac{\mu-\gamma(\rho+r)\pm\sqrt{(1-\gamma)(\mu^2-(\rho+r)^2\gamma)}}{2\beta(1-\gamma)}$.

Finally, we can obtain the optimal solution of problem (2.13) under the CRRA utility. The optimal strategy invested in the stock is given by

$$\pi^*(t) = -\frac{\varphi_3(x-\alpha(t))}{(\gamma-1)xk^2 s^{2\beta}} - \frac{2\beta(x-\alpha(t))I(t)}{xk^2 s^{2\beta}}, \tag{4.34}$$

which is summarized in the following theorem.

Theorem 4.2. *For the optimization problem (2.13) with wealth process (2.12) and the power utility function, the optimal investment strategy is given by*

$$\pi^*(t) = -\frac{\varphi_3(x-\alpha(t))}{(\gamma-1)xk^2 s^{2\beta}} - \frac{2\beta(x-\alpha(t))I(t)}{xk^2 s^{2\beta}},$$

where

$$\varphi_3 = \mu - \rho - r$$

$$\alpha(t) = \frac{P(1+a)[1-e^{(r-\theta)(t-T)}]}{(\omega-\omega_0-t)(r-\theta)^2} - \frac{P(\omega-\omega_0-(1+a)t)}{(\omega-\omega_0-t)(r-\theta)} + \frac{P(\omega-\omega_0-(1+a)T)}{(\omega-\omega_0-t)(r-\theta)}e^{(r-\theta)(t-T)},$$

$$I(t) = \frac{\lambda_1 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}{1-\frac{\lambda_1}{\lambda_2}e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}, \lambda_{1,2} = \frac{u-\gamma(\rho+r)\pm\sqrt{(1-\gamma)(u^2-(\rho+r)^2\gamma)}}{2\beta(1-\gamma)}.$$

Remark 4.2. *If elasticity parameter $\beta = 0$, the CEV model reduces to the GBM model, and the optimal investment strategy is given by $\pi^*(t) = \frac{\varphi_3(x-\alpha(t))}{(1-\gamma)xk^2}$, which also implies that the optimal investment strategy has nothing to do with the price of the stock.*

4.2.2. Explicit solutions for the logarithm utility

The logarithm utility function is given by

$$U(x) = \ln x.$$

Next, we solve the Eq (3.15).

$$g(T, s, z) = (U')^{-1}(z) = \frac{1}{z}.$$

We try to find a solution of (3.15) in the following way:

$$g(t, s, z) = \frac{1}{z} f(s) + \alpha(t), \tag{4.35}$$

where we denote the s value of the terminal time T as s_T, with $f(s_T) = 1, \alpha(T) = 0$, and we can obtain

$$\begin{aligned} g_t &= \alpha'(t), \quad g_s = \frac{1}{z} f'(s), \quad g_z = -\frac{1}{z^2} f(s); \\ g_{ss} &= \frac{1}{z} f''(s), \quad g_{zz} = \frac{2}{z^3} f(s), \quad g_{sz} = -\frac{1}{z^2} f'(s). \end{aligned} \tag{4.36}$$

Plugging (4.35) in (3.15), we have

$$\alpha'(t) + \mu s \frac{1}{z} f'(s) - \varphi_1 \alpha(t) - \varphi_2 + \frac{1}{2} k^2 s^{2\beta+2} \frac{1}{z} f''(s) = 0. \tag{4.37}$$

We can decompose this equation into two conditions in order to eliminate the dependence on s:

$$\begin{cases} \alpha'(t) - \varphi_1 \alpha(t) - \varphi_2 = 0, \\ \mu s \frac{1}{z} f'(s) + \frac{1}{2} k^2 s^{2\beta+2} \frac{1}{z} f''(s) = 0. \end{cases} \tag{4.38}$$

The solution to (4.37), which takes into account the initial conditions

$$\varphi(T) = 0, \quad f(s_T) = 1,$$

is then

$$\alpha(t) = \frac{P(1+a)[1 - e^{(r-\theta)(t-T)}]}{(\omega - \omega_0 - t)(r-\theta)^2} - \frac{P(\omega - \omega_0 - (1+a)t)}{(\omega - \omega_0 - t)(r-\theta)} + \frac{P(\omega - \omega_0 - (1+a)T)}{(\omega - \omega_0 - t)(r-\theta)} e^{(r-\theta)(t-T)}, \tag{4.39}$$

$$f(s) = 1. \tag{4.40}$$

Finally, this leads to

$$\begin{aligned} g = \frac{1}{z} + &\frac{P(1+a)[1 - e^{(r-\theta)(t-T)}]}{(\omega - \omega_0 - t)(r-\theta)^2} - \frac{P(\omega - \omega_0 - (1+a)t)}{(\omega - \omega_0 - t)(r-\theta)} \\ &+ \frac{P(\omega - \omega_0 - (1+a)T)}{(\omega - \omega_0 - t)(r-\theta)} e^{(r-\theta)(t-T)}. \end{aligned} \tag{4.41}$$

Introducing (4.41) in (3.16) and using (4.36), we obtain the optimal solution in the case of the logarithm utility

$$\pi^*(t) = \frac{\varphi_3 [x - \alpha(t)]}{x k^2 s^{2\beta}}, \tag{4.42}$$

which is summarized in the following theorem.

Corollary 1. *For the optimization problem (2.13) with wealth process (2.12) and the logarithm utility function, the optimal investment strategy is given by*

$$\pi^*(t) = \frac{\varphi_3(x - \alpha(t))}{xk^2 s^{2\beta}},$$

where

$$\varphi_3 = \mu - \rho - r,$$

$$\alpha(t) = \frac{P(1+a)[1 - e^{(r-\theta)(t-T)}]}{(\omega - \omega_0 - t)(r-\theta)^2} - \frac{P(\omega - \omega_0 - (1+a)t)}{(\omega - \omega_0 - t)(r-\theta)} + \frac{P(\omega - \omega_0 - (1+a)T)}{(\omega - \omega_0 - t)(r-\theta)} e^{(r-\theta)(t-T)}.$$

Remark 4.3. *If elasticity parameter $\beta = 0$, the CEV model reduces to the GBM model, and the optimal investment strategy is given by $\pi^*(t) = \frac{\varphi_3[x-\alpha(t)]}{xk^2}$. Therefore, the optimal investment strategy has nothing to do with the price of the stock.*

5. Numerical studies

In this section, we will numerically analyze the effect of parameters on the optimal investment strategies. Throughout this section, unless otherwise stated, the following parameters are given as $P = 1,\ s = 5,\ x = 1,\ \omega = 100,\ \omega_0 = 20,\ T = 40,\ r = 0.02,\ u = 0.1,\ k = 0.5,\ \beta = 1,\ \theta = 0.005,\ \rho = 0.01,\ q = 5$, and $\gamma = 0.5$.

5.1. The impact of return of premium clauses on optimal investment strategies

Figures 1 and 2 show that, compared with the case without return of premium clauses, the proportion of money invested into the stock becomes less under the case with return of premium clauses. This phenomenon can be economically interpreted as follows. The possible return of premium during the accumulation phase could reduce the fund size level. Hence, under the expected utility investment criterion and to hedge the loss of the premium income, the fund manager would prefer to invest less money in the stock. In other words, under the expected utility investment criterion, the risk preference of the fund manager is risk-averse.

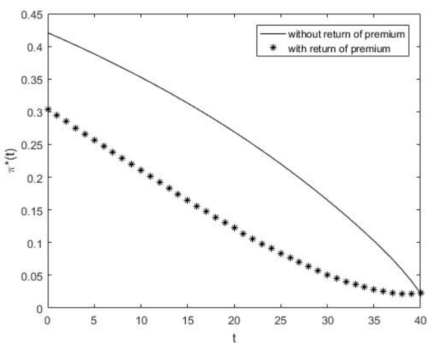


Figure 1. Relationship between $\pi^*(t)$ and t for the power utility function.

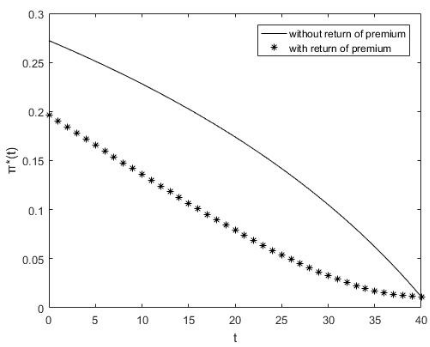


Figure 2. Relationship between $\pi^*(t)$ and t for the logarithm utility function.

Figure 3 shows that the proportion of money invested into the stock is the same for both of the cases of with and without return of premiums. This phenomenon can be economically interpreted as follows.

In contrast to the power and logarithmic utility functions, the risk aversion coefficient RAC(x) of the exponential utility function is a constant, which implies that the risk preference of an investor does not depend on its wealth. Therefore, in the case of the exponential utility function, the proportions of the money invested into the stock stay the same regardless of being with or without return of premiums.

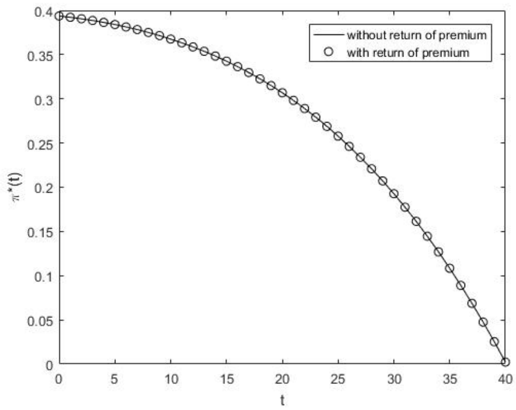


Figure 3. Relationship between $\pi^*(t)$ and t for the CARA utility.

5.2. The impact of utility functions on the optimal investment strategy

Figure 4 shows that the proportion of money invested into the stock under the logarithmic utility function is less than that under the power utility function. This phenomenon coincides with the following intuition. Since the RAC(x) is $\frac{1}{x}$ for the logarithmic utility function and $\frac{1-\gamma}{x}$ for the power utility function, hence the intuition is that the fund manager is more risk-averse under the logarithmic utility function than that under the power utility function. Therefore, the fund manager would prefer to invest less money in the stock in the case of the logarithmic utility function. Note also that the risk aversion coefficients for the logarithmic and power utility functions become almost the same when the wealth is large enough. Meanwhile, the fund size should become larger and larger when the time moves into the future. Consequently, when the time moves into the future, the proportions of money invested into the stock under both the logarithmic and power utility functions become the same.

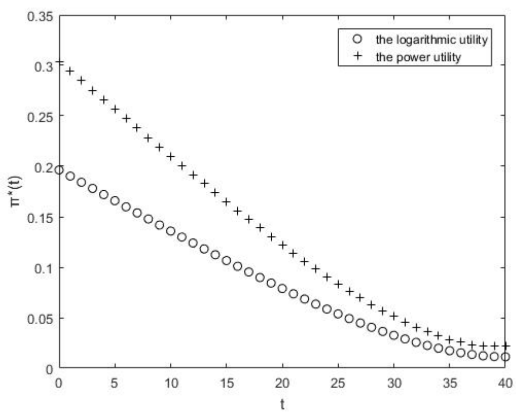


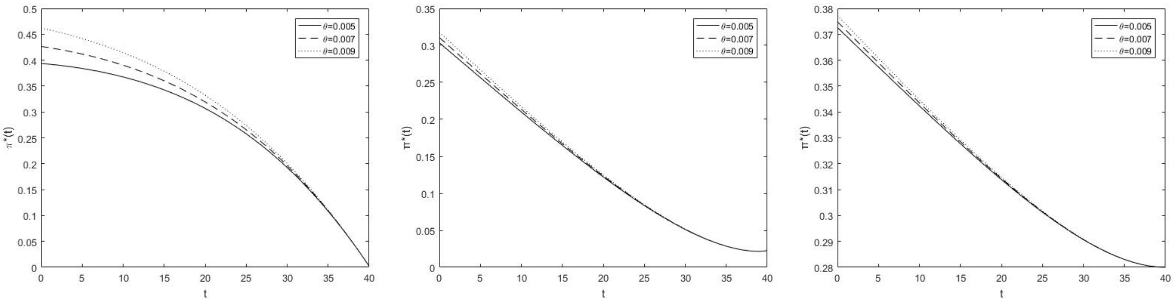
Figure 4. Comparison of $\pi^*(t)$ for the power and logarithmic utility functions.

5.3. The impact of parameters on the optimal investment strategy

In this subsection, we examine the sensitivity of the parameters involved in the model. These parameters include taxes, trading fees, the rate of interest, and the return rate and volatility of the stock.

5.3.1. The impact of taxes on the optimal investment strategy

Figure 5 shows that under any one of the exponential, power, and logarithmic utility functions, the larger the tax, the more money invested into the stock. This phenomenon can be economically interpreted as follows. Tax is a kind of income tax. Therefore, in order to hedge this loss of income tax, the fund manager would like to invest more money into the stock, because the stock is of higher return although it is of higher risk.

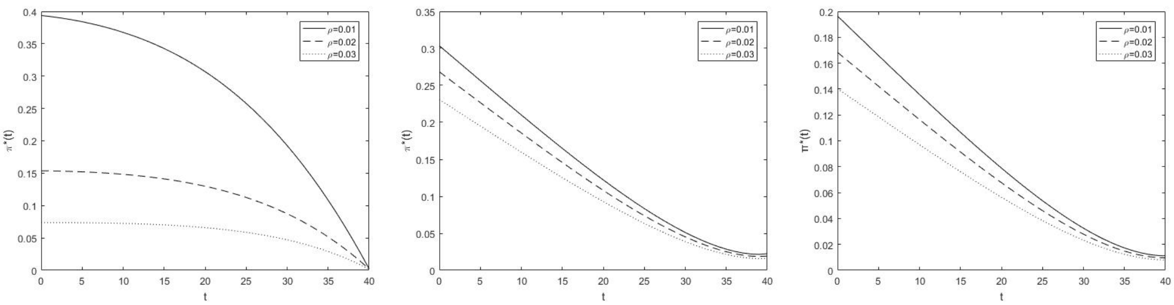


(a) The exponential utility (b) The power utility (c) The logarithmic utility

Figure 5. Impact of tax on the optimal investment strategy.

5.3.2. The impact of trading fees on the optimal investment strategy

Figure 6 shows that under any one of the exponential, power, and logarithmic utility functions, the larger the trading fee, the less the money invested into the stock. This phenomenon can be interpreted from the behavioral finance point of view as follows. The trading fee is a kind of market management cost. Therefore, in order to reduce this loss of market management cost, the fund manager would like to invest less money into the stock.



(a) The exponential utility (b) The power utility (c) The logarithmic utility

Figure 6. Impact of trading fees on the optimal investment strategy.

5.3.3. The impact of the parameters of the utility functions on the optimal investment strategy

Figure 7 shows that in the case of the exponential utility function, the larger the risk aversion coefficient q , the less proportion of money invested into the stock. This phenomenon also coincides with the fact that the RAC(x) is now just the risk aversion coefficient q. On the other hand, since the RAC(x) does not depend on the wealth, the impacts of the different values of q on the proportion of money invested into the stock appear more sensitive when the wealth is small, but less sensitive when the wealth is large enough.

Figure 8 shows that under the power utility function, the proportion of money invested into the stock increases when γ increases. This phenomenon can be interpreted as follows. Since the RAC(x) is $\frac{1-\gamma}{x}$, the fund manager is more risk-appetite when γ becomes large. Hence the fund manager would prefer to invest more money into the stock when γ increases. Note also that the RAC(x) becomes almost the same when the wealth x is larger enough. Meanwhile, the fund size should become larger when the time moves into the future. Consequently, when the time moves into the future, the proportion of money invested into the stock under the power utility function becomes the same.

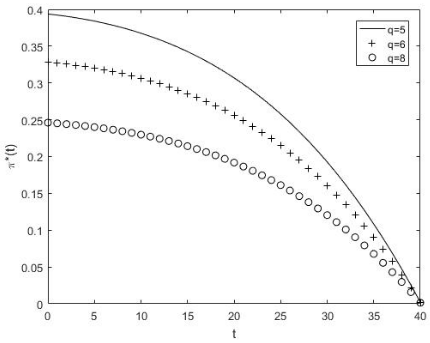


Figure 7. The parameters of the exponential utility.

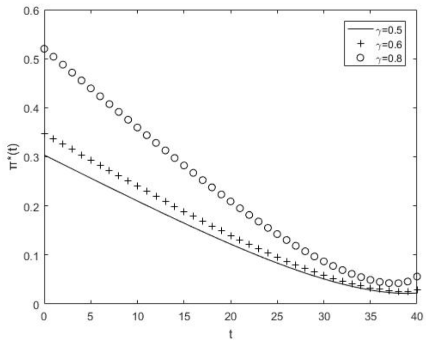
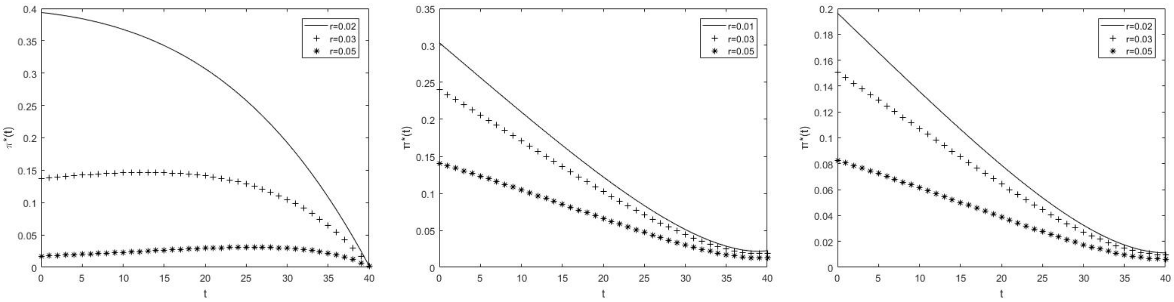


Figure 8. The parameters of the power utility.

5.3.4. The impact of interest rate on the optimal investment strategy

Figure 9 shows that under any one of the exponential, power, and logarithmic utility functions, the larger the interest rate, the smaller the proportion of money invested into the stock. This phenomenon coincides with people's common behavior. Indeed, when the interest rate increases, the fund manager would have no incentive to invest more money into the stock in order to maintain the profit target. In other words, the fund manager can simply invest more money into the risk-free asset to reach his/her profit target.

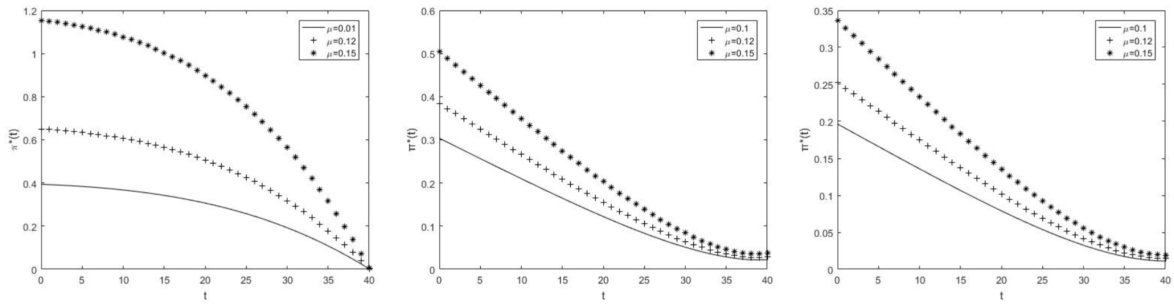


(a) The exponential utility (b) The power utility (c) The logarithmic utility

Figure 9. The impact of interest rate on the optimal investment strategy.

5.3.5. The impact the return rate and volatility of the stock on the optimal investment strategy

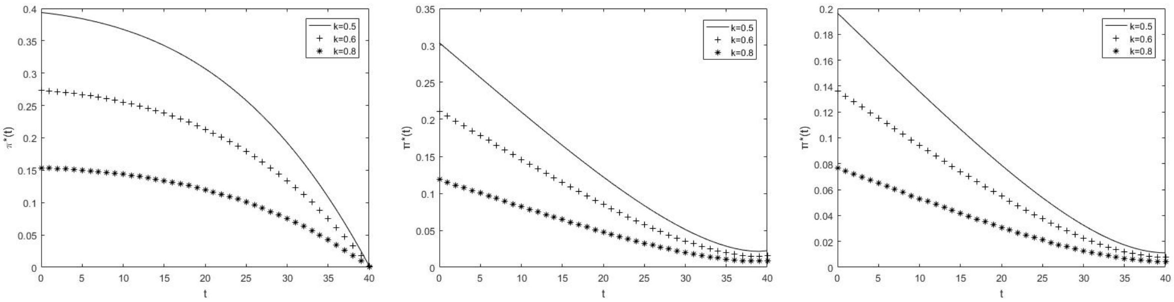
Figure 10 shows that under any one of the exponential, power, and logarithmic utility functions, the larger the return rate of the stock, the greater the proportion of money invested into the stock. This phenomenon coincides with people's common behavior. Indeed, when the return rate of the stock increases, the fund manager would have the incentive to invest more money into the stock in order to achieve higher profit.



(a) The exponential utility (b) The power utility (c) The logarithmic utility

Figure 10. The impact of the return rate of the stock on the optimal investment strategy.

Figure 11 shows that under any one of the exponential, power, and logarithmic utility functions, the larger the volatility of the stock, the smaller the proportion of money invested into the stock. This phenomenon can be interpreted as follows. Larger volatility of the stock represents higher risk. Hence, fund managers who are not risk-appetite would like to invest less money into the stock, and at the same time invest more money into the risk-free asset.



(a) The exponential utility (b) The power utility (c) The logarithmic utility

Figure 11. The impact of volatility of the stock on the optimal investment strategy.

6. Conclusions

In this paper, we have proposed a new optimal DC pension model with the return of premium by incorporating two practical factors, tax and trading fee, into the model. In the model, the pension is allowed to invest in a risk-free asset and a risky asset, where the price of a risky asset is assumed to follow the CEV model. The optimal investment strategy is obtained by maximizing the expected utility of the terminal wealth. Explicit expressions of the optimal investment strategies are also given when the utility function is specified to be the exponential, the power, and the logarithmic utility functions, respectively. Finally, numerical studies are provided to show the impacts of the parameters involved on the optimal investment strategies with the interpretations in the context of economics. In the future, the return of premium clauses can be considered in more complex financial models to obtain the optimal investment strategy for DC pensions.

Appendix

Let $Q = R \times R+$, and we choose a sequence of bounded open sets Q_i satisfying $Q_i \subset Q_{i+1} \subset Q$, $i = 1, 2, \cdots$, and $Q = \bigcup_{i=1}^{\infty} Q_i$. For $(s, x) \in Q_i$, assume that the exit time of $(S(t), X(t))$ from Q_i is denoted by τ_i. When $i \to \infty$, we get $\tau_i \wedge T \to T$.

(i) Consider an arbitrary admissible strategy $\pi(t)$. By applying $It\hat{o}$'s formula for $V(t, s, x)$ on $[t, T]$, we obtain

$$\begin{aligned} V(T, S(T), X(T)) = V(t, s, x) &+ \int_t^T \mathcal{A}^{\pi(t)} V(v, s(v), x(v)) \mathrm{d}v \\ &+ \int_t^T x\pi k s^{\beta} V_x(v, s(v), x(v)) \mathrm{d}W(v) \\ &+ \int_t^T k s^{\beta} V_s(v, s(v), x(v)) \mathrm{d}W(v). \end{aligned}$$

Taking $\sup_{\pi(t)\in\Pi}\{\mathcal{A}^{\pi(t)} V(t, s, x)\} = 0$ into consideration, which displays that the variational inequality

$\mathcal{A}^{\pi(t)}V(t,s,x) \le 0$, we have

$$V(T,S(T),X(T)) \le V(t,s,x) + \int_t^T x\pi k s^{\beta} V_x(v,s(v),x(v)))\mathrm{d}W(v) + \int_t^T k s^{\beta} V_s(v,s(v),x(v)))\mathrm{d}W(v).$$

The last three terms on the right-hand side of the above inequality are square-integrable martingales and their expectations are equal to zero. Hence, we get

$$E(V(T,S(T),X(T)) \mid S(t)=s, X(t)=x) \le V(t,s,x).$$

Further, taking the supremum, we obtain

$$\sup_{\pi_t\in\Pi} E(V(T,S(T),X(T)) \mid S(t)=s, X(t)=x) \le V(t,s,x),$$

and it implies that

$$H(t,r,x) \le V(t,r,x).$$

(ii) $E(V(\tau_i\wedge T, S(\tau_i\wedge T), X(\tau_i\wedge T))) < \infty$ for a specific strategy $\pi^*(t)$. Applying *Itô*'s formula to $V(t,s,x)$ on $[0,\tau_i\wedge T]$ once again, we have

$$\begin{aligned} V(\tau_i\wedge T, S(\tau_i\wedge T), X(\tau_i\wedge T)) = V(0,s_0,x_0) &+ \int_0^{\tau_i\wedge T} \mathcal{A}^{\pi^*(t)}V(v,s(v),x(v))\mathrm{d}v \\ &+ \int_0^{\tau_i\wedge T} x\pi k s^{\beta} V_x(v,s(v),x(v))\mathrm{d}W(v) \\ &+ \int_0^{\tau_i\wedge T} k s^{\beta} V_s(v,s(v),x(v))\mathrm{d}W(v). \end{aligned}$$

For a specific strategy $\pi^*(t)$ satisfies (15), i.e., $\mathcal{A}^{\pi^*(t)}V(t,S(v),X(v)) = 0$, and the last three terms are also square-integrable martingales. Hence, taking the expectation on both sides on the above equation, we obtain

$$E(V(\tau_i\wedge T, S(\tau_i\wedge T), X(\tau_i\wedge T))) = V(0,s_0,x_0) < \infty.$$

(iii) $H(t,s,x) = V(t,s,x)$ for a specific strategy $\pi^*(t)$. Using *Itô*'s formula for $V(t,s,x)$ on $[t,\tau_i\wedge T]$ once more, similarly, we derive

$$\begin{aligned} V(\tau_i\wedge T, S(\tau_i\wedge T), X(\tau_i\wedge T)) = V(t,s,x) &+ \int_0^{\tau_i\wedge T} \mathcal{A}^{\pi^*(t)}V(v,s(v),x(v))\mathrm{d}v \\ &+ \int_0^{\tau_i\wedge T} x\pi k s^{\beta} V_x(v,s(v),x(v))\mathrm{d}W(v) \\ &+ \int_0^{\tau_i\wedge T} k s^{\beta} V_s(v,s(v),x(v))\mathrm{d}W(v). \end{aligned}$$

Taking the conditional expectation, we yield

$$V(t,s,x) = E(V(\tau_i\wedge T, S(\tau_i\wedge T), X(\tau_i\wedge T)) \mid S(t)=s, X(t)=x).$$

Taking the limitation once more, we get

$$V(t, s, x) = \lim_{i\to\infty} E(V(\tau_i \wedge T, S(\tau_i \wedge T), X(\tau_i \wedge T)) \mid S(t) = s, X(t) = x).$$

In addition, we derive

$$\begin{aligned} H(t, s, x) &= \sup_{\pi_t\in\Pi} E(V(\tau_i \wedge T, S(\tau_i \wedge T), X(\tau_i \wedge T)) \mid S(t) = s, X(t) = x) \\ &= \lim_{i\to\infty} E(V(\tau_i \wedge T, S(\tau_i \wedge T), X(\tau_i \wedge T)) \mid S(t) = s, X(t) = x) \\ &= V(t, s, x). \end{aligned}$$

Therefore, it implies that $\pi^*(t)$ is indeed the optimal investment strategy for the problem (2.13).

Author contributions

Xiaoyi Tang: Conceptualization; Wei Liu: Supervision; Wanyin Wu: Validation; Yijun Hu: Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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