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Research article

The geometrical properties of the Smarandache curves on 3-dimension pseudo-spheres generated by null curves

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Abstract: In this paper, we presented the geometrical properties of Smarandache curves on 3-pseudo-spheres. These curves were examined in the context of the lightcone, de Sitter space, and anti-de Sitter space. By leveraging the curvature relationships between the null curve and its corresponding Smarandache curves, we established necessary and sufficient conditions. Additionally, we illustrated our main results through two examples.

Keywords: null curve; Smarandache curve; slant helix; pseudo-spheres
Mathematics Subject Classification: 35A53, 58C20

1. Introduction

With the discovery of Einstein's theory of relativity in 1905, researchers extended the topics of differential geometry on the manifolds in Minkowski space [1–8]. Minkowski space provides the mathematical foundation for formulating physical laws and understanding the behavior of particles and fields [9,10]. A. Ferrandez et al. stated that there existed a geometric particle model associated with the geometry of null curves in Minkowski space-time [11]. Meanwhile, the 3-pseudo-spheres (lightcone, de Sitter space and hyperbolic space) are special surfaces describing the temporal evolution of light in Minkowski space-time. Tunahan, T., Ayyildiz, N. obtained the differential geometry properties of the spacelike curves on three-dimensional de-Sitter space [12]. The authors gave the singularities and locations of pedal points when the pseudo-sphere dual curve germs are nonsingular [13]. N. Abazari, et al. showed that any curves in the 3-light cone were space-like or light-like [14]. J. Sun and D. Pei considered the singularities of null surfaces of the null curves on the 3-null cone [15].

The Smarandache curve, named after the Romanian mathematician Florentin Smarandache, has been the subject of mathematical research and exploration since its introduction. The research history of the Smarandache curve includes investigations into its construction, geometric properties and applications in various fields. Mathematicians have studied its self-similarity and the relationship between its construction algorithm and its resulting geometric properties [5,14–21]. Additionally, the differential geometry properties of the Smarandache curve have been a topic of interest, with researchers exploring its curvature, torsion, and other differential geometric properties [6,16,18,22,23]. Furthermore, the Smarandache curve in Minkowski space may have implications for relativistic physics and geometric interpretations of space-time [7, 8]. A. T. Ali obtained the curvature relationships between the regular curve and its Smarandache $\boldsymbol{TN}$ curve [16]. The centers of the osculating spheres and the curvature spheres of the Smarandache curves were found with the Bishop frame [23]. O. Bektaş and S. Yüce discussed the Smarandache $\boldsymbol{Tg}$, $\boldsymbol{Tn}$, $\boldsymbol{gn}$, and $\boldsymbol{Tgn}$ curves with respect to the Darboux frame. Furthermore, they gave the curvatures and torsion of the special curves [18]. H. S. Abdel-Aziza and M. K. Saad illustrated the curvature conditions when the Smarandache curves of time-like curves were geodesic, asymptotic lines, and principal lines, respectively [7]. The Frenet invariants of the spacial Smarandache curves were considered in [8]. S. Ouarab received the necessary and sufficient conditions for the $\boldsymbol{TN}$-Smarandache ruled surface to be a minimal surface by combining the Smarandache curves [21].

To date, there has been limited literature on Smarandache curves on 3-pseudo-spheres. This paper aims to derive the geometrical invariants and establish necessary and sufficient conditions for these curves. By exploring the application of Smarandache curves in geometry, we anticipate offering insights that could benefit researchers in theoretical physics, particularly in the realm of relativity theory.

This paper is organized in the following way. In Section 2, we consider some basic notions about $\mathbb{R}^4_1$, and obtain the geometrical invariants of the null curve $\boldsymbol{\gamma}(s)$ in view of the frame $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$. In the view of the Frenet frame $\{\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{T}}(s), \overline{\boldsymbol{B}}_1(s), \overline{\boldsymbol{B}}_2(s)\}$, we obtain more geometric properties of the Smarandache $\boldsymbol{TNB}_1$ curves on the three-lightcone. Section 4 shows the some results about Smarandache $\boldsymbol{NB}_2$ curves on de Sitter 3-space. The geometrical characteristics of the Smarandache $\boldsymbol{TB}_1$ curves on the hyperbolic three-space are obtained in Section 5. Furthermore, in the last section, we give some examples and its graphics to illustrate our results.

2. Preliminaries

Minkowski space is a fundamental concept in the field of special relativity, developed by the mathematician Hermann Minkowski. Minkowski space serves as the mathematical framework for Einstein's theory of the relativity. Research in this area involves applications of Minkowski space to understand the geometry of space-time, relativistic kinematics, and the concept of causality [3]. It is a generalization of the standard Euclidean space, where the metric or distance function is defined differently to account for the effects of relativity [4,5]. The vector space $\mathbb{R}^4$ is endowed with the metric induced by the pseudo-scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

which is defined as Minkowski spaces $\mathbb{R}^4_1$.

Minkowski spaces have several important properties and applications in physics and mathematics. They are the foundation for Einstein's theory of relativity, as they provide a geometric interpretation of the concepts of space, time, and the invariance of the speed of light. In the following, we will give some basic definitions and computation of Minkowski spaces.

The pseudo vector product of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}_1^4$ is defined as

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the canonical basis of $\mathbb{R}_1^4$. The norm of a vector $\boldsymbol{x}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

In order to better study the contact relationship of curves, it is necessary to use sub-manifolds in Minkowski space. There are many special surface manifolds in Minkowski 4-space $\mathbb{R}_1^4$, but the most common manifolds (3-pseudo-spheres) are as follows, [2]:

• de Sitter 3-space by

$$\mathbb{S}_1^3 = \{\boldsymbol{x} \in \mathbb{R}_1^4 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

• 3-hyperbolic space by

$$\mathbb{H}_0^3 = \{\boldsymbol{x} \in \mathbb{R}_1^4 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

• 3-light cone by

$$\mathbb{LC} = \{\boldsymbol{x} \in \mathbb{R}_1^4 \setminus \{\mathbf{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

Curve is a fundamental area of mathematics with a wide range of applications in various scientific and engineering fields. Curve is closely related to the field of differential geometry, which studies the properties of curves and surfaces using the tools of calculus. Insights from curve research contribute to our understanding of the intrinsic geometric properties of objects and spaces. Meanwhile, curves are used to model and analyze various physical phenomena, such as the motion of fluids, the deformation of materials, and the propagation of waves. Advancements in curve theory can lead to improved simulations and predictions in fields like fluid dynamics, solid mechanics, and wave optics. Overall, the importance of curve theory research lies in its fundamental role in mathematics and its widespread applications in science, engineering, and technology. Continued research in this field can drive innovation and progress in a variety of domains. In this section, we mainly focus on the study of the most special curve (null curve) in Minkowski space. To begin, we provide the definition of the null curve as follows.

Definition 2.1. *For a curve $\boldsymbol{\gamma}(s) \in \mathbb{R}_1^4$, we call $\boldsymbol{\gamma}(s)$ a null curve, if $\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s) \rangle = 0$, where $\boldsymbol{\gamma}'(s)$ is the nonzero tangent vector.*

The Frenet equation is a fundamental tool in curve theory and differential geometry, and research in this area has significant implications for both theoretical and applied domains, ranging from computer graphics and engineering to physics and mathematics. For a null curve $\boldsymbol{\gamma}(s)$, we denote the pseudo arc-length parameter such that $\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle = 1$. One can define the moving Frenet frame $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$ of $\boldsymbol{\gamma}(s)$, and

$$\boldsymbol{\gamma}'(s) = \boldsymbol{T}(s),$$

$$\langle \boldsymbol{T}(s), \boldsymbol{T}(s) \rangle = \langle \boldsymbol{B}_1(s), \boldsymbol{B}_1(s) \rangle = 0,$$

$$\langle \boldsymbol{N}(s), \boldsymbol{N}(s)\rangle = \langle \boldsymbol{B}_2(s), \boldsymbol{B}_2(s)\rangle = \langle \boldsymbol{T}(s), \boldsymbol{B}_1(s)\rangle = 1,$$

$$\langle \boldsymbol{T}(s), \boldsymbol{N}(s)\rangle = \langle \boldsymbol{T}(s), \boldsymbol{B}_2(s)\rangle = \langle \boldsymbol{N}(s), \boldsymbol{B}_1(s)\rangle = 0, \langle \boldsymbol{N}(s), \boldsymbol{B}_2(s)\rangle = \langle \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\rangle = 0.$$

The Frenet formulas are given by [24]

$$\begin{cases} \boldsymbol{T}'(s) = \boldsymbol{N}(s) \\ \boldsymbol{N}'(s) = \kappa(s)\boldsymbol{T}(s) - \boldsymbol{B}_1(s) \\ \boldsymbol{B}_1'(s) = -\kappa(s)\boldsymbol{N}(s) + \tau(s)\boldsymbol{B}_2(s) \\ \boldsymbol{B}_2'(s) = -\tau(s)\boldsymbol{T}(s) \end{cases}, \tag{2.1}$$

where $\kappa(s) = \langle \boldsymbol{N}'(s), \boldsymbol{B}_1(s)\rangle$, and $\tau(s) = \langle \boldsymbol{B}_1'(s), \boldsymbol{B}_2(s)\rangle$ are the second and third curvature of the null curve $\boldsymbol{\gamma}(s)$, respectively.

As a relatively new and actively researched area of curve theory, the study of Smarandache curves contributes to the overall advancement of mathematical knowledge. Uncovering the fundamental properties, classifications, and relationships of Smarandache curves can lead to new mathematical insights and the expansion of our understanding of curves and their underlying structures.

Definition 2.2. *[6] A regular curve in Minkowski 4-space, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Meanwhile, the null slant helix is an active area of research in curve theory and differential geometry. Investigating the properties, classifications, and generalizations of null slant helices can contribute to the overall advancement of our understanding of curves and their underlying mathematical structures. The definition is as follows.

Definition 2.3. *[25] Let $\boldsymbol{\gamma}(s)$ be a null curve parameterized by pseudo arc-length with the Frenet frame $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$, $\boldsymbol{\gamma}(s)$ is called a (k-type (k=1, 2, 3, respectively)) null (slant) helix if and only if there exists a nonzero constant vector field $\boldsymbol{V}$ in $\mathbb{R}^4_1$ such that $\langle \boldsymbol{T}(s), \boldsymbol{V}\rangle \neq 0$ (respectively, $\langle \boldsymbol{N}(s), \boldsymbol{V}\rangle \neq 0$, $\langle \boldsymbol{B}_1(s), \boldsymbol{V}\rangle \neq 0$, $\langle \boldsymbol{B}_2(s), \boldsymbol{V}\rangle \neq 0$) is a constant for all $s \in I$.*

Let $\boldsymbol{V}$ be a nonzero constant vector field of a k-type null slant helix $\boldsymbol{\gamma}(s)$. Then $\boldsymbol{V}$ can be decomposed as

$$\boldsymbol{V} = v_1\boldsymbol{T}(s) + v_2\boldsymbol{N}(s) + v_3\boldsymbol{B}_1(s) + v_4\boldsymbol{B}_2(s),$$

where $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$ is the Frenet frame of $\boldsymbol{\gamma}(s)$, and $v_i = v_i(s) (i = 1, 2, 3, 4)$ are differentiable functions for s. Thus

$$v_1 = \langle \boldsymbol{B}_1(s), \boldsymbol{V}\rangle, \quad v_2 = \langle \boldsymbol{N}(s), \boldsymbol{V}\rangle, \quad v_3 = \langle \boldsymbol{T}(s), \boldsymbol{V}\rangle, \quad v_4 = \langle \boldsymbol{B}_2(s), \boldsymbol{V}\rangle.$$

Lemma 2.4. *[25] Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}^4_1$. Then*

(i) $\boldsymbol{\gamma}(s)$ is a null helix if and only if

$$\kappa' = -\tau \int \tau ds. \tag{2.2}$$

(ii) $\boldsymbol{\gamma}(s)$ is a 1-type null slant helix if and only if

$$2\kappa + \kappa' s = -\tau \int s\tau ds. \tag{2.3}$$

(iii) $\boldsymbol{\gamma}(s)$ is a 2-type null slant helix if and only if

$$\frac{\tau}{\kappa} = \pm\frac{v_3'}{\sqrt{-2c_0v_3 - v_3'^2 + a}}, \tag{2.4}$$

where $c_0 = \langle \boldsymbol{B}_1(s), \boldsymbol{V}\rangle \neq 0, a \in R$.

(iv) $\boldsymbol{\gamma}(s)$ is a 3-type null slant helix if and only if the null third curvature vanishes and

$$v_3''' = 2v_3'\kappa + v_3\kappa'. \tag{2.5}$$

3. The Smarandache curves on the 3-light cone

In this section, we define the Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\boldsymbol{\gamma}}$ of the null curve $\boldsymbol{\gamma}(s)$, and $\hat{\boldsymbol{\gamma}}$ lies on the 3-light cone. We denote $\hat{s}$ as the arc-length parameter of $\hat{\boldsymbol{\gamma}}$, where

$$\hat{s}(s) = \int_{s_0}^{s} |\hat{\boldsymbol{\gamma}}'(s)|ds,\ s_0 \in I(open\ interval).$$

We construct the Frenet frame that is employed to discuss the geometric relationships between $\boldsymbol{\gamma}(s)$ and its Smarandache $\boldsymbol{TNB}_1$ curve.

Definition 3.1. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}_1^4$ with $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$ as the moving frame. The curve*

$$\hat{\boldsymbol{\gamma}}(s) = \boldsymbol{T}(s) + \sqrt{2}\boldsymbol{N}(s) - \boldsymbol{B}_1(s), \tag{3.1}$$

is called the Smarandache $\boldsymbol{TNB}_1$ curve of $\boldsymbol{\gamma}(s)$.

Obviously, $\hat{\boldsymbol{\gamma}}(s)$ fully lies on the 3-light cone. By taking the derivative of Eq (3.1), we have

$$\hat{\boldsymbol{T}}(s) = \sqrt{2}\kappa\boldsymbol{T}(s) + (1+\kappa)\boldsymbol{N}(s) - \sqrt{2}\boldsymbol{B}_1(s) - \tau\boldsymbol{B}_2(s), \tag{3.2}$$

and

$$\langle \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{T}}(s)\rangle = (1-\kappa)^2 + \tau^2 = 1.$$

Proposition 3.2. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}_1^4$, and the Smarandache $\boldsymbol{TNB}_1$ curve located on the 3-light cone can only be space-like.*

Hence, we can denote $\{\hat{\boldsymbol{\gamma}}(s), \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{N}}(s), \hat{\boldsymbol{B}}(s)\}$ as the moving Frenet frame along the curve $\hat{\boldsymbol{\gamma}}(s)$, where

$$\langle \hat{\boldsymbol{N}}(s), \hat{\boldsymbol{\gamma}}(s)\rangle = 1,$$

$$\langle \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{T}}(s)\rangle = 1,\ \langle \hat{\boldsymbol{B}}(s), \hat{\boldsymbol{B}}(s)\rangle = 1,$$

$$\langle \hat{\boldsymbol{\gamma}}(s), \hat{\boldsymbol{\gamma}}(s)\rangle = \langle \hat{\boldsymbol{N}}(s), \hat{\boldsymbol{N}}(s)\rangle = \langle \hat{\boldsymbol{\gamma}}(s), \hat{\boldsymbol{T}}(s)\rangle = 0,$$

$$\langle \hat{\boldsymbol{\gamma}}(s), \hat{\boldsymbol{B}}(s)\rangle = \langle \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{N}}(s)\rangle = \langle \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{B}}(s)\rangle = \langle \hat{\boldsymbol{N}}(s), \hat{\boldsymbol{B}}(s)\rangle = 0.$$

Suppose that $\hat{\boldsymbol{N}}(s)$ satisfies

$$\hat{\boldsymbol{N}}(s) = a\boldsymbol{T}(s) + b\boldsymbol{N}(s) + c\boldsymbol{B}_1(s) + d\boldsymbol{B}_2(s),$$

and the conditions $\langle \hat{\boldsymbol{N}}(s), \hat{\boldsymbol{N}}(s)\rangle = \langle \hat{\boldsymbol{T}}(s), \hat{\boldsymbol{N}}(s)\rangle = 0$, $\langle \hat{\boldsymbol{\gamma}}(s), \hat{\boldsymbol{N}}(s)\rangle = 1$. We can obtain

$$\begin{cases} 2ac + b^2 + d^2 = 0 \\ \sqrt{2}c\kappa + b(1+\kappa) - \sqrt{2}a - d\tau = 0 \\ c + \sqrt{2}b - a = 1 \end{cases}.$$

We take one set of solutions

$$\begin{cases} a = -\frac{\kappa^2}{(1-\kappa)^2} \\ b = -\frac{\sqrt{2}\kappa}{(1-\kappa)^2} \\ c = \frac{1}{(1-\kappa)^2} \\ d = 0 \end{cases}.$$

Thus, the expression of $\hat{\boldsymbol{N}}(s)$ is

$$\hat{\boldsymbol{N}}(s) = -\frac{\kappa^2}{(1-\kappa)^2}\boldsymbol{T}(s) - \frac{\sqrt{2}\kappa}{(1-\kappa)^2}\boldsymbol{N}(s) + \frac{1}{(1-\kappa)^2}\boldsymbol{B}_1(s).$$

Based on the above conditions, we can obtain

$$\begin{aligned} \hat{\boldsymbol{B}}(s) &= \hat{\boldsymbol{\gamma}}(s) \wedge \hat{\boldsymbol{T}}(s) \wedge \hat{\boldsymbol{N}}(s) \\ &= \begin{vmatrix} \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}_1(s) & \mathbf{B}_2(s) \\ 1 & \sqrt{2} & -1 & 0 \\ \sqrt{2}\kappa & 1+\kappa & -\sqrt{2} & -\tau \\ -\frac{\kappa^2}{(1-\kappa)^2} & -\frac{\sqrt{2}\kappa}{(1-\kappa)^2} & \frac{1}{(1-\kappa)^2} & 0 \end{vmatrix} \\ &= \frac{\sqrt{2}\tau}{1-\kappa}\boldsymbol{T}(s) - \frac{(1+\kappa)\tau}{1-\kappa}\boldsymbol{N}(s) - \frac{\sqrt{2}\kappa\tau}{1-\kappa}\boldsymbol{B}_1(s) - (1-\kappa)\boldsymbol{B}_2(s). \end{aligned} \tag{3.3}$$

By taking the derivative of Eqs (3.2) and (3.3),

$$\begin{aligned} \hat{\boldsymbol{T}}'(s) =& (\sqrt{2}\kappa' + \kappa + \kappa^2 + \tau^2)\boldsymbol{T}(s) + (2\sqrt{2}\kappa + \kappa')\boldsymbol{N}(s) \\ & - (1+\kappa)\boldsymbol{B}_1(s) - (\sqrt{2}\tau + \tau')\boldsymbol{B}_2(s) \end{aligned},$$

$$\begin{aligned} \hat{\boldsymbol{N}}'(s) =& -\frac{2\kappa\kappa' + \sqrt{2}\kappa^2(1-\kappa)}{(1-\kappa)^3}\boldsymbol{T}(s) - \frac{\sqrt{2}\kappa'(1+\kappa) + \kappa(1-\kappa^2)}{(1-\kappa)^3}\boldsymbol{N}(s) \\ & + \frac{2\kappa' + \sqrt{2}\kappa(1-\kappa)}{(1-\kappa)^3}\boldsymbol{B}_1(s) + \frac{\tau}{(1-\kappa)^2}\boldsymbol{B}_2(s). \end{aligned}$$

Hence, we can receive the following Frenet formulas of $\hat{\boldsymbol{\gamma}}(s)$

$$\begin{cases} \hat{\boldsymbol{\gamma}}'(s) = \hat{\boldsymbol{T}}(s) \\ \hat{\boldsymbol{T}}'(s) = \hat{\kappa}(s)\hat{\boldsymbol{\gamma}} - \hat{\boldsymbol{N}}(s) \\ \hat{\boldsymbol{N}}'(s) = -\hat{\kappa}(s)\hat{\boldsymbol{T}}(s) + \hat{\tau}(s)\hat{\boldsymbol{B}}(s) \\ \hat{\boldsymbol{B}}'(s) = -\hat{\tau}(s)\hat{\boldsymbol{\gamma}}(s) \end{cases},$$

where

$$\begin{aligned} \hat{\kappa}(s) &= \langle \hat{\boldsymbol{T}}'(s), \hat{\boldsymbol{N}}(s)\rangle \\ &= \frac{\sqrt{2}\kappa'(1-\kappa) + \kappa(1-\kappa)^2 + \tau^2}{(1-\kappa)^2}, \end{aligned} \tag{3.4}$$

and

$$\hat{\tau}(s) = \langle \hat{\boldsymbol{N}}'(s), \hat{\boldsymbol{B}}(s) \rangle = \frac{[(1-\kappa)^3(3\kappa-1) - \sqrt{2}\kappa'(1-4\kappa-\kappa^2)]\tau}{(1-\kappa)^4}. \tag{3.5}$$

Definition 3.3. *[26] Let $\hat{\boldsymbol{\gamma}}(\hat{s})$ be a space-like curve parameterized by arc-length with the Frenet frame $\{\hat{\boldsymbol{\gamma}}(\hat{s}), \hat{\boldsymbol{T}}(\hat{s}), \hat{\boldsymbol{N}}(\hat{s}), \hat{\boldsymbol{B}}(\hat{s})\}$ is called a k-type (k=0,1, 2, respectively) slant helix if and only if there exists a nonzero constant vector field $\boldsymbol{U}$ in $\mathbb{R}^4_1$ such that $\langle \hat{\boldsymbol{T}}(\hat{s}), \boldsymbol{U} \rangle \neq 0$ (respectively, $\langle \hat{\boldsymbol{N}}(\hat{s}), \boldsymbol{U} \rangle \neq 0$, $\langle \hat{\boldsymbol{B}}(\hat{s}), \boldsymbol{U} \rangle \neq 0$) is a constant for all $\hat{s} \in I$.*

Let $\boldsymbol{U}$ be a nonzero constant vector field of a k-type slant helix $\hat{\boldsymbol{\gamma}}(\hat{s})$. Then $\boldsymbol{U}$ can be decomposed as

$$\boldsymbol{U} = u_1\hat{\boldsymbol{\gamma}}(\hat{s}) + u_2\hat{\boldsymbol{T}}(\hat{s}) + u_3\hat{\boldsymbol{N}}(\hat{s}) + u_4\hat{\boldsymbol{B}}(\hat{s}), \tag{3.6}$$

where $\{\hat{\boldsymbol{\gamma}}(\hat{s}), \hat{\boldsymbol{T}}(\hat{s}), \hat{\boldsymbol{N}}(\hat{s}), \hat{\boldsymbol{B}}(\hat{s})\}$ is the Frenet frame of $\hat{\boldsymbol{\gamma}}(\hat{s})$, and $u_i = u_i(\hat{s})(i = 1, 2, 3, 4)$ are differentiable functions for arc-length $\hat{s}$. Thus

$$u_1 = \langle \hat{\boldsymbol{N}}(\hat{s}), \boldsymbol{U} \rangle, \quad u_2 = \langle \hat{\boldsymbol{T}}(\hat{s}), \boldsymbol{U} \rangle, \quad u_3 = \langle \hat{\boldsymbol{\gamma}}(\hat{s}), \boldsymbol{U} \rangle, \quad u_4 = \langle \hat{\boldsymbol{B}}(\hat{s}), \boldsymbol{U} \rangle.$$

By taking derivative on both sides of Eq (3.6), we get

$$(u_1' + u_2\hat{\kappa} - u_4\hat{\tau})\hat{\boldsymbol{\gamma}}(\hat{s}) + (u_1 + u_2' - u_3\hat{\kappa})\hat{\boldsymbol{T}}(\hat{s}) + (u_3' - u_2)\hat{\boldsymbol{N}}(\hat{s}) + (u_3\hat{\tau} + u_4')\hat{\boldsymbol{B}}(\hat{s}) = 0. \tag{3.7}$$

Theorem 3.4. *Let $\boldsymbol{\gamma}(s) : I \longrightarrow \mathbb{R}^4_1$ be a null curve. The Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\boldsymbol{\gamma}}(\hat{s}(s))$ of $\boldsymbol{\gamma}(s)$ is a 0-type slant helix if and only if*

$$2\hat{\kappa} + s\hat{\kappa}' = -\hat{\tau}\int s\hat{\tau}ds + c_0\hat{\tau}. \tag{3.8}$$

Proof. Based on the definition of 0-type slant helix, we have

$$u_2 = \langle \boldsymbol{U}, \hat{\boldsymbol{T}}(s) \rangle = C_0,$$

where C_0 is a nonzero constant. Substituting it into Eq (3.7), we get

$$\begin{cases} u_1' + C_0\hat{\kappa} - u_4\hat{\tau} = 0 \\ u_1 - u_3\hat{\kappa} = 0 \\ u_3' = C_0 \\ u_3\hat{\tau} + u_4' = 0 \end{cases}. \tag{3.9}$$

By the third equation in (3.9), we have $u_3 = C_0s$. Then, the coefficients of $\boldsymbol{U}$ are expressed by

$$\begin{cases} u_1 = C_0s\hat{\kappa} \\ u_2 = C_0 \\ u_3 = C_0s \\ u_4 = -C_0\int s\hat{\tau}ds = \dfrac{u_1' + C_0\hat{\kappa}}{\hat{\tau}} \end{cases}. \tag{3.10}$$

Because $C_0 \neq 0$, we get

$$2\hat{\kappa} + s\hat{\kappa}' = -\hat{\tau}\int s\hat{\tau}ds + c_0\hat{\tau}, \tag{3.11}$$

where $c_0 \in \mathbb{R}$.

Conversely, assume Eq (3.8) holds for Smarandache $\boldsymbol{TNB}_1$ curve. We can define a vector field $\boldsymbol{U}$ as follows:

$$U = C_0 s\hat{\kappa}\hat{\gamma}(s) + C_0\hat{T}(s) + C_0 s\hat{N}(s) - C_0\int s\hat{\tau}ds\hat{B}(s).$$

Then, we have $U' = 0$ and $\langle U, \hat{T}(s)\rangle = C_0 \setminus \{0\}$. □

Corollary 3.5. *Let $\gamma(s) : I \longrightarrow \mathbb{R}^4_1$ be a null curve. The Smarandache TNB_1 curve $\hat{\gamma}(s)$ is a 1-type slant helix if and only if*

$$\hat{\kappa}(\hat{\kappa}^2 + \hat{\tau}^2)u_4' + \hat{\tau}(\hat{\kappa}\hat{\tau}' - \hat{\kappa}'\hat{\tau})u_4 + \hat{\kappa}^2\hat{\tau}C_1 = 0. \tag{3.12}$$

Proof. From the definition of 1-type slant helix, we have

$$u_1 = \langle U, \hat{N}(s)\rangle = C_1,$$

where C_1 is a nonzero constant. Substituting it into Eq (3.7), we get

$$\begin{cases} u_2\hat{\kappa} - u_4\hat{\tau} = 0 \\ C_1 + u_2' - u_3\hat{\kappa} = 0 \\ u_3' - u_2 = 0 \\ u_3\hat{\tau} + u_4' = 0 \end{cases}, \tag{3.13}$$

from which we have

$$\hat{\kappa}(\hat{\kappa}^2 + \hat{\tau}^2)u_4' + \hat{\tau}(\hat{\kappa}\hat{\tau}' - \hat{\kappa}'\hat{\tau})u_4 + \hat{\kappa}^2\hat{\tau}C_1 = 0. \tag{3.14}$$

Solving the above ordinary differential equation, we can obtain

$$u_4 = e^{\int \frac{\hat{\tau}(\hat{\kappa}'\hat{\tau} - \hat{\kappa}\hat{\tau}')}{\hat{\kappa}(\hat{\kappa}^2+\hat{\tau}^2)}ds} ds \int -\frac{C_1\hat{\kappa}\hat{\tau}}{\hat{\kappa}^2+\hat{\tau}^2} e^{-\int \frac{\hat{\tau}(\hat{\kappa}'\hat{\tau} - \hat{\kappa}\hat{\tau}')}{\hat{\kappa}(\hat{\kappa}^2+\hat{\tau}^2)}ds} ds.$$

Conversely, assume Eq (3.12) holds for Smarandache TNB_1 curve. We can define a vector field U as follows:

$$U = C_1\hat{\gamma}(s) + \frac{\hat{\tau}}{\hat{\kappa}}u_4\hat{T}(s) - \frac{1}{\hat{\tau}}u_4'\hat{N}(s) + u_4\hat{B}(s), \text{ when } \hat{\tau} \neq 0,$$

or

$$U = C_1\hat{\gamma}(s) + \frac{C_1}{\hat{\kappa}}\hat{N}(s) + C_3\hat{B}(s), \text{ where } C_3 \in \mathbb{R}, \text{ when } \hat{\tau} = 0.$$

Then, we have $U' = 0$ and $\langle U, \hat{N}(s)\rangle = C_1 \setminus \{0\}$. □

Corollary 3.6. *The Smarandache TNB_1 curve is a 2-type slant helix if and only if the torsion vanishes and*

$$u_3''' = \hat{\kappa}'u_3 + 2\hat{\kappa}u_3'. \tag{3.15}$$

Proof. From the definition of the 2-type slant helix, we have

$$u_4 = \langle U, \hat{B}(s)\rangle = C_2,$$

where C_2 is a nonzero constant. From Eq (3.7), we can obtain

$$u_1' + u_2\hat{\kappa} - u_4\hat{\tau} = 0;$$

$$u_1 + u_2' - u_3\hat{\kappa} = 0;$$

$$u_3' - u_2 = 0;$$

$$u_3\hat{\tau} + u_4' = 0.$$

For $u_4' = 0$, we get

$$\begin{cases} u_1' + u_2\hat{\kappa} - C_2\hat{\tau} = 0 \\ u_1 + u_2' - u_3\hat{\kappa} = 0 \\ u_3' - u_2 = 0 \\ u_3\hat{\tau} = 0 \end{cases}. \tag{3.16}$$

By the third equation in Eq (3.16), we can obtain $u_3 = 0$ or $\hat{\tau} = 0$.

(i) $u_3 = 0$ implies $u_i = 0 (i = 1, 2, 3, 4)$. This is a contradiction.

(ii) When $\hat{\tau} = 0$, we hold

$$\begin{cases} u_1' + u_2\hat{\kappa} = 0 \\ u_1 + u_2' - u_3\hat{\kappa} = 0 \\ u_3' - u_2 = 0 \end{cases}, \tag{3.17}$$

from which we have

$$u_3''' = \hat{\kappa}' u_3 + 2\hat{\kappa} u_3'. \tag{3.18}$$

According to [27], this ordinary differential equation has a solution.

Conversely, assume Eq (3.15) holds for the Smarandache $\boldsymbol{TNB}_1$ curve. We can define a vector field $\boldsymbol{U}$ as follows:

$$\boldsymbol{U} = (\hat{\kappa}u_3 - u_3'')\hat{\boldsymbol{\gamma}}(s) + u_3'\hat{\boldsymbol{T}}(s) + u_3\hat{\boldsymbol{N}}(s) + C_2\boldsymbol{B}(s).$$

Then, we have $\boldsymbol{U}' = 0$ and $\langle \boldsymbol{U}, \hat{\boldsymbol{B}}(s)\rangle = C_2 \setminus \{0\}$. □

Corollary 3.7. *Let $\boldsymbol{\gamma}(s)$ be a 3-type null slant helix in $\mathbb{R}^4_1$. The Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\boldsymbol{\gamma}}(\hat{s}) = \hat{\boldsymbol{\gamma}}(\varphi(s))$ of $\boldsymbol{\gamma}(s)$ is*

(i) 0-type slant helix if and only if it is a straight line;

(ii) a 1-type slant helix;

(iii) a 2-type slant helix.

Proof. Based on the the Lemma 2.4 of 3-type null slant helix, we have that the null third curvature vanishes and

$$v_3''' = 2v_3'\kappa + v_3\kappa'. \tag{3.19}$$

From the Eqs (3.4) and (3.5), we have

$$\hat{\kappa} = \kappa, \hat{\tau} = 0. \tag{3.20}$$

(i) If $\hat{\boldsymbol{\gamma}}(s)$ is a 0-type slant helix, bring the formula (3.20) into Eq (3.8), then we have $\hat{\kappa} = 0$, that is, it is a straight line.

Conversely, if $\hat{\boldsymbol{\gamma}}(s)$ is a straight line, there exists a nonzero constant vector field $\boldsymbol{U} = C_0\hat{\boldsymbol{T}}(s) + C_0 s\hat{\boldsymbol{N}}(s) - C_0C_4\hat{\boldsymbol{B}}(s)$ such that $\langle \boldsymbol{U}, \hat{\boldsymbol{T}}(s)\rangle = C_0$, where $C_4 \in \mathbb{R} \setminus \{0\}$.

(ii) If $\hat{\boldsymbol{\gamma}}(s)$ is a 1-type slant helix, bring the formula (3.20) into Eq (3.12), then we receive $\kappa^3 u_4' = 0$. We can define a vector field $\boldsymbol{U}$ as follows:

$$\boldsymbol{U} = C_1\hat{\boldsymbol{\gamma}}(s) + \frac{C_1}{\hat{\kappa}}\hat{\boldsymbol{N}}(s) + C_3\hat{\boldsymbol{B}}(s), \text{ where } C_3 \in \mathbb{R}, \text{ when } \hat{\tau} = 0.$$

(iii) If $\hat{\boldsymbol{\gamma}}(s)$ is a 2-type slant helix, bring the formula (3.20) into Eq (3.15), then we gain $u_3''' = 2\hat{\kappa}u_3'$. Since $\hat{\kappa} \geq 0$, we get

$$u_3' = ce^{\int \sqrt{2\hat{\kappa}}\frac{\cosh\theta}{\sinh\theta}ds}, \tag{3.21}$$

where c is a constant and the newly defined function $\theta(s)$ satisfies

$$\theta' = \sqrt{2\hat{\kappa}}.$$

We can define the vector field $\boldsymbol{U}$ as

$$\boldsymbol{U} = (\hat{\kappa}u_3 - u_3'')\hat{\boldsymbol{\gamma}}(s) + u_3'\hat{\boldsymbol{T}}(s) + u_3\hat{\boldsymbol{N}}(s) + C_2\boldsymbol{B}(s),$$

where $\langle \boldsymbol{U}, \hat{\boldsymbol{B}}(s)\rangle = C_2 \in \mathbb{R} \setminus \{0\}$.

□

4. The Smarandache curves on the de Sitter 3-Space

In this section, we define the Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}$ of the null curve $\boldsymbol{\gamma}(s)$, and $\tilde{\boldsymbol{\gamma}}$ lies on the de Sitter 3-space. We denote $\tilde{s}$ as the (pseudo) arc-length parameter of $\tilde{\boldsymbol{\gamma}}$, where

$$\tilde{s}(s) = \int_{s_0}^{s} |\tilde{\boldsymbol{\gamma}}'(s)| ds,\ s_0 \in I(open\ interval).$$

We construct the Frenet frame that is employed to discuss the geometric relationships between $\boldsymbol{\gamma}(s)$ and its Smarandache $\boldsymbol{NB}_2$ curve.

Definition 4.1. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}^4_1$ with $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$ as the moving frame. The curve*

$$\tilde{\boldsymbol{\gamma}}(s) = \frac{1}{\sqrt{2}}(\boldsymbol{N}(s) - \boldsymbol{B}_2(s)), \tag{4.1}$$

is called the Smarandache $\boldsymbol{NB}_2$ curve of $\boldsymbol{\gamma}(s)$.

Obviously, $\hat{\boldsymbol{\gamma}}(s)$ fully lies on the de Sitter 3-space. By taking the derivative of Eq (4.1), we have

$$\tilde{\boldsymbol{T}}(s) = \frac{1}{\sqrt{2}}[(\kappa + \tau)\boldsymbol{T}(s) - \boldsymbol{B}_1(s)], \tag{4.2}$$

and

$$\langle \tilde{\boldsymbol{T}}(s), \tilde{\boldsymbol{T}}(s)\rangle = -\sqrt{2}(\kappa + \tau).$$

Proposition 4.2. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}^4_1$, and the Smarandache $\boldsymbol{NB}_2$ curve located on the de Sitter 3-space can only be light-like, space-like or time-like.*

Case 1. The Smarandache $\boldsymbol{NB}_2$ curve is a light-like curve.

Hence, we can denote $\{\tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_1(s), \tilde{\boldsymbol{N}}_1(s), \tilde{\boldsymbol{B}}_1(s)\}$ as the moving Frenet frame along the curve $\tilde{\boldsymbol{\gamma}}(s)$, where

$$\langle \tilde{\boldsymbol{T}}_1(s), \tilde{\boldsymbol{B}}_1(s)\rangle = 1,$$

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{\gamma}}(s)\rangle = 1,\ \langle \tilde{\boldsymbol{B}}_1(s), \tilde{\boldsymbol{B}}_1(s)\rangle = 1,$$

$$\langle \tilde{\boldsymbol{T}}_1(s), \tilde{\boldsymbol{T}}_1(s)\rangle = \langle \tilde{\boldsymbol{B}}_1(s), \tilde{\boldsymbol{B}}_1(s)\rangle = 0,$$

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_1(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_1(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{B}}_1(s)\rangle = \langle \tilde{\boldsymbol{T}}_1(s), \tilde{\boldsymbol{N}}_1(s)\rangle = \langle \tilde{\boldsymbol{N}}_1(s), \tilde{\boldsymbol{B}}_1(s)\rangle = 0.$$

Suppose that $\tilde{\boldsymbol{N}}_1(s)$ satisfies

$$\tilde{\boldsymbol{N}}_1(s) = \tilde{a}_1\boldsymbol{T}(s) + \tilde{b}_1\boldsymbol{N}(s) + \tilde{c}_1\boldsymbol{B}_1(s) + \tilde{d}_1\boldsymbol{B}_2(s),$$

and the conditions $\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_1(s)\rangle = \langle \tilde{\boldsymbol{T}}_1(s), \tilde{\boldsymbol{N}}_1(s)\rangle = 0$, $\langle \tilde{\boldsymbol{N}}_1(s), \tilde{\boldsymbol{N}}_1(s)\rangle = 1$. We can obtain

$$\begin{cases} 2\tilde{a}_1\tilde{c}_1 + \tilde{b}_1^2 + \tilde{d}_1^2 = 1 \\ \tilde{b}_1 - \tilde{d}_1 = 0 \\ \tilde{c}_1(\kappa + \tau) - \tilde{a}_1 = 0 \end{cases}.$$

We take the solution

$$\begin{cases} \tilde{a}_1 = 0 \\ \tilde{b}_1 = \dfrac{1}{\sqrt{2}} \\ \tilde{c}_1 \; isarbitrary \\ \tilde{d}_1 = \dfrac{1}{\sqrt{2}} \end{cases}.$$

Thus, we can choose the expression of $\tilde{\boldsymbol{N}}_1(s)$ as

$$\tilde{\boldsymbol{N}}_1(s) = \frac{1}{\sqrt{2}}(\boldsymbol{N}(s) + \boldsymbol{B}_2(s)).$$

Based on the above conditions, we can obtain

$$\begin{aligned} \tilde{\boldsymbol{B}}_1(s) &= \tilde{\boldsymbol{\gamma}}(s) \wedge \tilde{\boldsymbol{T}}_1(s) \wedge \tilde{\boldsymbol{N}}_1(s) \\ &= \begin{vmatrix} \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}_1(s) & \mathbf{B}_2(s) \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} \\ &= -\frac{1}{\sqrt{2}}\boldsymbol{T}(s). \end{aligned} \tag{4.3}$$

By taking the derivative of Eqs (4.2) and (4.3),

$$\tilde{\boldsymbol{T}}_1'(s) = -\frac{1}{\sqrt{2}}(-\kappa\boldsymbol{N}(s) + \tau\boldsymbol{B}_2(s)),$$

$$\tilde{\boldsymbol{N}}_1'(s) = \frac{1}{\sqrt{2}}[(\kappa - \tau)\boldsymbol{T}(s) - \boldsymbol{B}_1(s)].$$

Hence, we can receive the following Frenet formulas of $\tilde{\boldsymbol{\gamma}}(s)$

$$\begin{cases} \tilde{\boldsymbol{\gamma}}'(s) = \tilde{\boldsymbol{T}}_1(s) \\ \tilde{\boldsymbol{T}}_1'(s) = \tilde{\kappa}_1(s)\tilde{\boldsymbol{N}}_1(s) \\ \tilde{\boldsymbol{N}}_1'(s) = -\tilde{\kappa}_1(s)\tilde{\boldsymbol{B}}_1(s) + \tilde{\tau}_1(s)\tilde{\boldsymbol{T}}_1(s) \\ \tilde{\boldsymbol{B}}_2'(s) = -\tilde{\tau}_1(s)\tilde{\boldsymbol{N}}_1(s) - \tilde{\boldsymbol{\gamma}} \end{cases},$$

where

$$\tilde{\kappa}_1(s) = \langle \tilde{\boldsymbol{T}}_1'(s), \tilde{\boldsymbol{N}}_1(s)\rangle = \kappa, \tag{4.4}$$

and

$$\tilde{\tau}_1(s) = \langle \tilde{\boldsymbol{N}}_1'(s), \tilde{\boldsymbol{B}}_1(s)\rangle = \frac{1}{2}. \tag{4.5}$$

If $\tilde{\kappa}_1(s) = 0$, then $\tilde{\boldsymbol{\gamma}}(s)$ is a straight line, and the form is too simple. We will not consider this situation here, but only consider the case of $\tilde{\kappa}_1(s) \neq 0$.

Let $\tilde{\boldsymbol{U}}$ be an axis (or a slope axis) of a null helix (or a k-type null slant helix) $\tilde{\boldsymbol{\gamma}}(\tilde{s})$. Then $\tilde{\boldsymbol{U}}$ can be decomposed as

$$\tilde{\boldsymbol{U}} = \tilde{u}_1\tilde{\boldsymbol{\gamma}}(\tilde{s}) + \tilde{u}_2\tilde{\boldsymbol{T}}_1(\tilde{s}) + \tilde{u}_3\tilde{\boldsymbol{N}}_1(\tilde{s}) + \tilde{u}_4\tilde{\boldsymbol{B}}_1(\tilde{s}), \tag{4.6}$$

where $\{\tilde{\boldsymbol{\gamma}}(\tilde{s}), \tilde{\boldsymbol{T}}_1(\tilde{s}), \tilde{\boldsymbol{N}}_1(\tilde{s}), \tilde{\boldsymbol{B}}_1(\tilde{s})\}$ is the Frenet frame of $\tilde{\boldsymbol{\gamma}}(\tilde{s})$, and $\tilde{u}_i = \tilde{u}_i(\tilde{s})(i = 1, 2, 3, 4)$ are differentiable functions for arc-length $\tilde{s}$. Thus

$$\tilde{u}_1 = \langle \tilde{\boldsymbol{\gamma}}(\tilde{s}), \tilde{\boldsymbol{U}}\rangle, \quad \tilde{u}_2 = \langle \tilde{\boldsymbol{B}}_1(\tilde{s}), \tilde{\boldsymbol{U}}\rangle, \quad \tilde{u}_3 = \langle \tilde{\boldsymbol{N}}_1(\tilde{s}), \tilde{\boldsymbol{U}}\rangle, \quad \tilde{u}_4 = \langle \tilde{\boldsymbol{T}}_1(\tilde{s}), \tilde{\boldsymbol{U}}\rangle.$$

By taking derivative on both sides of Eq (4.6), we get

$$(\tilde{u}_1' - \tilde{u}_4)\tilde{\boldsymbol{\gamma}}(\tilde{s}) + (\tilde{u}_2' + \tilde{u}_1 + \tilde{u}_3\tilde{\tau}_1)\tilde{\boldsymbol{T}}_1(\tilde{s}) + (\tilde{u}_3' + \tilde{u}_2\tilde{\kappa}_1 - \tilde{u}_4\tilde{\tau}_1)\tilde{\boldsymbol{N}}_1(\tilde{s}) + (\tilde{u}_4' - \tilde{u}_3\tilde{\kappa}_1)\tilde{\boldsymbol{B}}_1(\tilde{s}) = 0. \tag{4.7}$$

Theorem 4.3. *Let $\boldsymbol{\gamma}(s) : I \longrightarrow \mathbb{R}_1^4$ be a null curve. The Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}(s))$ of $\boldsymbol{\gamma}(s)$ is a null helix if and only if*

$$\tilde{\tau}_1(s) = \tilde{C}(s)\tilde{\kappa}_1(s). \tag{4.8}$$

Proof. Based on the definition of null helix, we have

$$\tilde{u}_4 = \langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{T}}_1(s)\rangle = \tilde{C}_0,$$

where $\tilde{C}_0$ is a nonzero constant. Substituting it into Eq (4.7), we get

$$\begin{cases} \tilde{u}_1' - \tilde{C}_0 = 0 \\ \tilde{u}_2' + \tilde{u}_1 + \tilde{u}_3\tilde{\tau}_1 = 0 \\ \tilde{u}_3' + \tilde{u}_2\tilde{\kappa}_1 - \tilde{C}_0\tilde{\tau}_1 = 0 \\ \tilde{u}_3\tilde{\kappa}_1 = 0 \end{cases}. \tag{4.9}$$

By the fourth equation in (4.9), we have $\tilde{u}_3 = 0$. Thus, we can get

$$\tilde{\kappa}_1\tilde{\tau}_1' - \tilde{\kappa}_1'\tilde{\tau}_1 + s\tilde{\kappa}_1^2 = 0,$$

that is,

$$\tilde{\tau}_1(s) = \tilde{C}(s)\tilde{\kappa}_1(s).$$

where $\tilde{C}(s) = -\frac{1}{2}s^2 + \tilde{c}, \tilde{c} \in \mathbb{R}$.

Conversely, assume Eq (4.8) holds for Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{U}}$ as follows:

$$\tilde{\boldsymbol{U}} = \tilde{C}_0 s\tilde{\boldsymbol{\gamma}}(s) + \tilde{C}_0\frac{\tilde{\tau}_1}{\tilde{\kappa}_1}\tilde{\boldsymbol{T}}_1(s) + \tilde{C}_0\tilde{\boldsymbol{B}}_1(s).$$

Then, we have $\tilde{\boldsymbol{U}}' = 0$ and $\langle \boldsymbol{U}, \tilde{\boldsymbol{T}}_1(s)\rangle = \tilde{C}_0 \in \mathbb{R} \setminus \{0\}$. □

Theorem 4.4. *Let $\boldsymbol{\gamma}(s) : I \longrightarrow \mathbb{R}^4_1$ be a null curve. The Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(s)$ is a 1-type null slant helix if and only if*

$$2\tilde{\tau}_1 + (\frac{\tilde{\tau}_1}{\tilde{\kappa}_1})' \int \tilde{\kappa}_1 ds + \int (\int \tilde{\kappa}_1 ds) ds = 0. \tag{4.10}$$

Proof. From the definition of 1-type null slant helix, we have

$$\tilde{u}_3 = \langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{N}}_1(s) \rangle = \tilde{C}_1,$$

where $\tilde{C}_1$ is a nonzero constant. Substituting it into Eq (4.7), we get

$$\begin{cases} \tilde{u}_1' - \tilde{u}_4 = 0 \\ \tilde{u}_2' + \tilde{u}_1 + \tilde{C}_1 \tilde{\tau}_1 = 0 \\ \tilde{u}_2 \tilde{\kappa}_1 - \tilde{u}_4 \tilde{\tau}_1 = 0 \\ \tilde{u}_4' - \tilde{C}_1 \tilde{\kappa}_1 = 0 \end{cases}.$$

From which we have

$$\begin{cases} \tilde{u}_1' = \tilde{u}_4 = \tilde{C}_1 \int \tilde{\kappa}_1 ds \\ \tilde{u}_2' + \tilde{u}_1 + \tilde{C}_1 \tilde{\tau}_1 = 0 \\ \tilde{u}_2 = \frac{\tilde{\tau}_1}{\tilde{\kappa}_1} \tilde{u}_4 \end{cases}. \tag{4.11}$$

By taking the derivative of the third formula of (4.11) and substituting it into the second formula of (4.11), we can receive

$$2\tilde{\tau}_1 + (\frac{\tilde{\tau}_1}{\tilde{\kappa}_1})' \int \tilde{\kappa}_1 ds + \int (\int \tilde{\kappa}_1 ds) ds = 0.$$

Conversely, assume Eq (4.10) holds for the Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{U}}$ as follows:

$$\tilde{\boldsymbol{U}} = \tilde{C}_1 \int (\int \tilde{\kappa}_1 ds) ds \tilde{\boldsymbol{\gamma}}(s) + \tilde{C}_1 \frac{\tilde{\tau}_1}{\tilde{\kappa}_1} \int \tilde{\kappa}_1 ds \tilde{\boldsymbol{T}}_1(s) + \tilde{C}_0 \tilde{\boldsymbol{N}}_1(s) + \tilde{C}_1 \int \tilde{\kappa}_1 ds \tilde{\boldsymbol{B}}_1(s).$$

Then, we have $\tilde{\boldsymbol{U}}' = 0$ and $\langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{N}}_1(s) \rangle = \tilde{C}_1 \setminus \{0\}$. □

Corollary 4.5. *The Smarandache $\boldsymbol{NB}_2$ curve is a 2-type null slant helix if and only if*

$$\tilde{u}_3'' + \tilde{\kappa}_1 \tilde{\tau}_1' \tilde{u}_3' + (\tilde{\tau}_1'^2 - \tilde{\kappa}_1 \tilde{\tau}_1) \tilde{u}_3 + \tilde{C}_2 \tilde{\kappa}_1' = 0. \tag{4.12}$$

Proof. From the definition of the 2-type null slant helix, we have

$$\tilde{u}_2 = \langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{B}}_1(s) \rangle = \tilde{C}_2,$$

where $\tilde{C}_2$ is a nonzero constant. Substituting it into Eq (4.7), we get

$$\begin{cases} \tilde{u}_1' - \tilde{u}_4 = 0 \\ \tilde{u}_1 + \tilde{u}_3 \tilde{\tau}_1 = 0 \\ \tilde{u}_3' + \tilde{C}_2 \tilde{\kappa}_1 - \tilde{u}_4 \tilde{\tau}_1 = 0 \\ \tilde{u}_4' - \tilde{u}_3 \tilde{\kappa}_1 = 0 \end{cases}, \tag{4.13}$$

from which we have

$$\tilde{u}_3'' + \tilde{\kappa}_1 \tilde{\tau}_1' \tilde{u}_3' + (\tilde{\tau}_1'^2 - \tilde{\kappa}_1 \tilde{\tau}_1) \tilde{u}_3 + \tilde{C}_2 \tilde{\kappa}_1' = 0.$$

According to [27], this ordinary differential equation has solutions.

Conversely, assume Eq (4.12) holds for Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{U}}$ as follows:

$$\tilde{\boldsymbol{U}} = -\tilde{\tau}_1\tilde{u}_2\tilde{\boldsymbol{\gamma}}(s) + \tilde{C}_2\tilde{\boldsymbol{T}}_1(s) + \tilde{u}_3\tilde{\boldsymbol{N}}_1(s) + \frac{\tilde{u}_3' + \tilde{C}_2\tilde{\kappa}_1}{\tilde{\tau}_1}\tilde{\boldsymbol{B}}_1(s).$$

Then, we have $\tilde{\boldsymbol{U}}' = 0$ and $\langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{B}}_1(s)\rangle = \tilde{C}_2 \in \mathbb{R} \setminus \{0\}$. □

Corollary 4.6. *Let $\boldsymbol{\gamma}(s)$ be a 3-type null slant helix in $\mathbb{R}^4_1$. The Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \tilde{\boldsymbol{\gamma}}(\varphi(s))$ of $\boldsymbol{\gamma}(s)$ is curve with $\tilde{\kappa}_1 = 0,\ \tilde{\tau}_1 = \frac{1}{2}$.*

Case 2. The Smarandache $\boldsymbol{NB}_2$ curve is a space-like curve.

Hence, we can denote $\{\tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_2(s), \tilde{\boldsymbol{N}}_2(s), \tilde{\boldsymbol{B}}_2(s)\}$ as the moving Frenet frame along the curve $\tilde{\boldsymbol{\gamma}}(s)$, where

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{\gamma}}(s)\rangle = \langle \tilde{\boldsymbol{T}}_2(s), \tilde{\boldsymbol{T}}_2(s)\rangle = \langle \tilde{\boldsymbol{N}}_2(s), \tilde{\boldsymbol{N}}_2(s)\rangle = -\langle \tilde{\boldsymbol{B}}_2(s), \tilde{\boldsymbol{B}}_2(s)\rangle = 1,$$

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_2(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_2(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{B}}_2(s)\rangle = \langle \tilde{\boldsymbol{T}}_2(s), \tilde{\boldsymbol{N}}_2(s)\rangle = \langle \tilde{\boldsymbol{T}}_2(s), \tilde{\boldsymbol{B}}_2(s)\rangle = \langle \tilde{\boldsymbol{N}}_2(s), \tilde{\boldsymbol{B}}_2(s)\rangle = 0.$$

Suppose that $\tilde{\boldsymbol{N}}_2(s)$ satisfies

$$\tilde{\boldsymbol{N}}_2(s) = \tilde{a}_2\boldsymbol{T}(s) + \tilde{b}_2\boldsymbol{N}(s) + \tilde{c}_2\boldsymbol{B}_1(s) + \tilde{d}_2\boldsymbol{B}_2(s),$$

and the conditions $\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_2(s)\rangle = \langle \tilde{\boldsymbol{T}}_2(s), \tilde{\boldsymbol{N}}_2(s)\rangle = 0$, $\langle \tilde{\boldsymbol{N}}_2(s), \tilde{\boldsymbol{N}}_2(s)\rangle = 1$. We can obtain

$$\begin{cases} 2\tilde{a}_2\tilde{c}_2 + \tilde{b}_2^2 + \tilde{d}_2^2 = 1 \\ \tilde{b}_2 - \tilde{d}_2 = 0 \\ \tilde{c}_2(\kappa + \tau) - \tilde{a}_2 = 0 \end{cases}.$$

We take one set of solutions

$$\begin{cases} \tilde{a}_2 = 0 \\ \tilde{b}_2 = \dfrac{1}{\sqrt{2}} \\ \tilde{c}_2 = 0 \\ \tilde{d}_2 = \dfrac{1}{\sqrt{2}} \end{cases}.$$

Thus, we can choose the expression of $\tilde{\boldsymbol{N}}_2(s)$ as

$$\tilde{\boldsymbol{N}}_2(s) = \frac{1}{\sqrt{2}}(\boldsymbol{N}(s) + \boldsymbol{B}_2(s)).$$

Based on the above conditions, we can obtain

$$\begin{aligned}\tilde{\boldsymbol{B}}_2(s) &= \tilde{\boldsymbol{\gamma}}(s) \wedge \tilde{\boldsymbol{T}}_2(s) \wedge \tilde{\boldsymbol{N}}_2(s)\\ &= \begin{vmatrix} \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}_1(s) & \mathbf{B}_2(s) \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}\\ &= -\frac{1}{\sqrt{2}}\boldsymbol{T}(s) + \frac{1}{2}\boldsymbol{B}_1.\end{aligned} \tag{4.14}$$

By taking the derivative of Eqs (4.2) and (4.14),

$$\tilde{\boldsymbol{T}}_2'(s) = (\frac{1}{\sqrt{2}}\kappa - \frac{1}{2})\boldsymbol{N}(s) - \frac{1}{2}\tau \boldsymbol{B}_2(s),$$

$$\tilde{\boldsymbol{N}}_2'(s) = \frac{1}{\sqrt{2}}[(\kappa - \tau)\boldsymbol{T}(s) - \boldsymbol{B}_1(s)].$$

Hence, we can receive the following Frenet formulas of $\tilde{\boldsymbol{\gamma}}(s)$

$$\begin{cases} \tilde{\boldsymbol{\gamma}}'(s) = \tilde{\boldsymbol{T}}(s) \\ \tilde{\boldsymbol{T}}_2'(s) = -\tilde{\boldsymbol{\gamma}}(s) + \tilde{\kappa}_2(s)\tilde{\boldsymbol{N}}_2 \\ \tilde{\boldsymbol{N}}_2'(s) = -\tilde{\kappa}_2(s)\tilde{\boldsymbol{T}}_2(s) + \tilde{\tau}_2(s)\tilde{\boldsymbol{B}}_2(s) \\ \tilde{\boldsymbol{B}}_2'(s) = \tilde{\tau}_2(s)\tilde{\boldsymbol{N}}_2(s) \end{cases},$$

where

$$\begin{aligned}\tilde{\kappa}_2(s) &= \langle \tilde{\boldsymbol{T}}_2'(s), \tilde{\boldsymbol{N}}_2(s)\rangle\\ &= \frac{\kappa - \tau}{2} - \frac{1}{2\sqrt{2}}\\ &= \kappa,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\tilde{\tau}_2(s) &= -\langle \tilde{\boldsymbol{N}}_2'(s), \tilde{\boldsymbol{B}}_2(s)\rangle\\ &= \frac{\tau - \kappa}{2\sqrt{2}} - \frac{1}{2}\\ &= -\frac{1}{\sqrt{2}}\kappa - \frac{3}{4}.\end{aligned} \tag{4.16}$$

If $\tilde{\kappa}_2(s) = 0$, then $\tilde{\boldsymbol{\gamma}}(s)$ is a straight line, and the form is too simple. We will not consider this situation here, but only consider the case of $\tilde{\kappa}_2(s) \neq 0$.

Let $\tilde{\boldsymbol{V}}$ be a slope axis of a k-type slant helix $\tilde{\boldsymbol{\gamma}}(\tilde{s})$. Then $\tilde{\boldsymbol{V}}$ can be decomposed as

$$\tilde{\boldsymbol{U}} = \tilde{u}_1\tilde{\boldsymbol{\gamma}}(\tilde{s}) + \tilde{u}_2\tilde{\boldsymbol{T}}_2(\tilde{s}) + \tilde{u}_3\tilde{\boldsymbol{N}}_2(\tilde{s}) + \tilde{u}_4\tilde{\boldsymbol{B}}_2(\tilde{s}), \tag{4.17}$$

where $\{\tilde{\boldsymbol{\gamma}}(\tilde{s}), \tilde{\boldsymbol{T}}_2(\tilde{s}), \tilde{\boldsymbol{N}}_2(\tilde{s}), \tilde{\boldsymbol{B}}_2(\tilde{s})\}$ is the Frenet frame of $\tilde{\boldsymbol{\gamma}}(\tilde{s})$, and $\tilde{v}_i = \tilde{v}_i(\tilde{s})(i = 1, 2, 3, 4)$ are differentiable functions for arc-length $\tilde{s}$. Thus

$$\tilde{v}_1 = \langle \tilde{\boldsymbol{\gamma}}(\tilde{s}), \tilde{\boldsymbol{V}} \rangle, \quad \tilde{v}_2 = \langle \tilde{\boldsymbol{T}}_2(\tilde{s}), \tilde{\boldsymbol{V}} \rangle, \quad \tilde{v}_3 = \langle \tilde{\boldsymbol{N}}_2(\tilde{s}), \tilde{\boldsymbol{V}} \rangle, \quad \tilde{v}_4 = \langle \tilde{\boldsymbol{B}}_2(\tilde{s}), \tilde{\boldsymbol{V}} \rangle.$$

By taking derivative on both sides of Eq (4.17), we get

$$(\tilde{v}_1' - \tilde{v}_2)\tilde{\boldsymbol{\gamma}}(\tilde{s}) + (\tilde{v}_2' + \tilde{v}_1 - \tilde{v}_3\tilde{\kappa}_2)\tilde{\boldsymbol{T}}_2(\tilde{s}) + (\tilde{v}_3' + \tilde{v}_2\tilde{\kappa}_2 + \tilde{u}_4\tilde{\tau}_2)\tilde{\boldsymbol{N}}_2(\tilde{s}) + (\tilde{v}_4' + \tilde{v}_3\tilde{\tau}_2)\tilde{\boldsymbol{B}}_2(\tilde{s}) = 0. \tag{4.18}$$

Theorem 4.7. *Let $\boldsymbol{\gamma}(s) : I \longrightarrow \mathbb{R}^4_1$ be a null curve. The Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}(s))$ of $\boldsymbol{\gamma}(s)$ is a 0-type helix if and only if*

$$s\tilde{\kappa}_2'\tilde{\tau}_2 + \tilde{\kappa}_2^2\tilde{\tau}_2' \int s\frac{\tilde{\tau}_2}{\tilde{\kappa}_2} ds - \tilde{\kappa}_2\tilde{\tau}_2(1 + \tilde{\kappa}_2^2) = 0. \tag{4.19}$$

Proof. Based on the definition of null helix, we have

$$\tilde{v}_2 = \langle \tilde{\boldsymbol{V}}, \tilde{\boldsymbol{T}}_2(s) \rangle = \tilde{D}_0,$$

where $\tilde{D}_0$ is a nonzero constant. Substituting it into Eq (4.18), we get

$$\begin{cases} \tilde{v}_1' = \tilde{D}_0 \\ \tilde{v}_1 - \tilde{v}_3\tilde{\kappa}_2 = 0 \\ \tilde{v}_3' + \tilde{D}_0\tilde{\kappa}_2 + \tilde{v}_4\tilde{\tau}_2 = 0 \\ \tilde{v}_4' + \tilde{v}_3\tilde{\tau}_2 = 0 \end{cases},$$

from which we have

$$\begin{cases} \tilde{v}_1 = \tilde{D}_0 s \\ \tilde{v}_2 = \tilde{D}_0 \\ \tilde{v}_3 = \dfrac{\tilde{v}_1}{\tilde{\kappa}_2} \\ \tilde{v}_4 = -\int \tilde{v}_3\tilde{\tau}_2 = -\dfrac{\tilde{v}_3' + \tilde{D}_0\tilde{\kappa}_2}{\tilde{\tau}_2} \end{cases}. \tag{4.20}$$

By the fourth equation in (4.20), we get

$$s\tilde{\kappa}_2'\tilde{\tau}_2 + \tilde{\kappa}_2^2\tilde{\tau}_2' \int s\frac{\tilde{\tau}_2}{\tilde{\kappa}_2} ds - \tilde{\kappa}_2\tilde{\tau}_2(1 + \tilde{\kappa}_2^2) = 0.$$

Conversely, assume Eq (4.19) holds for the Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{V}}$ as follows:

$$\tilde{\boldsymbol{V}} = \tilde{D}_0 s\tilde{\boldsymbol{\gamma}}(s) + \tilde{D}_0\tilde{\boldsymbol{T}}_2(s) + \tilde{D}_0\frac{s}{\tilde{\kappa}_2}\tilde{\boldsymbol{N}}_2(s) - \tilde{D}_0 \int s\frac{\tilde{\tau}_2}{\tilde{\kappa}_2}\tilde{\boldsymbol{B}}_2(s).$$

Then, we have $\tilde{\boldsymbol{V}}' = 0$ and $\langle \boldsymbol{V}, \tilde{\boldsymbol{T}}_2(s) \rangle = \tilde{D}_0 \in \mathbb{R} \setminus \{0\}$. □

As the same method as Theorem 4.7, we can obtain the following corollaries.

Corollary 4.8. *Let $\boldsymbol{\gamma}(s) : I \longrightarrow \mathbb{R}^4_1$ be a null curve. The Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(s)$ is a 1-type slant helix if and only if*

$$[(\frac{\tilde{\tau}_2}{\tilde{\kappa}_2})'' + \frac{\tilde{\tau}_2}{\tilde{\kappa}_2}] \int \tilde{\tau}_2 ds + \frac{\tilde{\tau}_2}{\tilde{\kappa}_2}\tilde{\tau}_2' + 2(\frac{\tilde{\tau}_2}{\tilde{\kappa}_2})'\tilde{\tau}_2 + \tilde{\kappa}_2 = 0. \tag{4.21}$$

Proof. From the definition of 1-type slant helix, we have

$$\tilde{v}_3 = \langle \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{N}}_2(s) \rangle = \tilde{D}_1,$$

where $\tilde{D}_1$ is a nonzero constant. Substituting it into Eq (4.18), we get

$$\begin{cases} \tilde{v}_1' - \tilde{v}_2 = 0 \\ \tilde{v}_2' + \tilde{v}_1 - \tilde{D}_1\tilde{\kappa}_2 = 0 \\ \tilde{v}_2\tilde{\kappa}_2 + \tilde{v}_4\tilde{\tau}_2 = 0 \\ \tilde{v}_4' + \tilde{D}_1\tilde{\tau}_2 = 0 \end{cases}. \tag{4.22}$$

By taking the derivative of the second formula of (4.22), we can receive

$$\tilde{v}_2'' + \tilde{v}_1' - \tilde{D}_1\tilde{\kappa}_2' = 0. \tag{4.23}$$

Then, substituting the first and third formulas of (4.22) into (4.23),

$$[(\frac{\tilde{\tau}_2}{\tilde{\kappa}_2})'' + \frac{\tilde{\tau}_2}{\tilde{\kappa}_2}]\int \tilde{\tau}_2 ds + \frac{\tilde{\tau}_2}{\tilde{\kappa}_2}\tilde{\tau}_2' + 2(\frac{\tilde{\tau}_2}{\tilde{\kappa}_2})'\tilde{\tau}_2 + \tilde{\kappa}_2 = 0$$

can be easily obtained.

Conversely, assume Eq (4.21) holds for the Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{V}}$ as follows:

$$\tilde{\boldsymbol{V}} = \tilde{D}_1[\tilde{\kappa}_2 - (\frac{\tilde{\tau}_2}{\kappa_2}\int \tilde{\tau}_2)']\tilde{\boldsymbol{\gamma}}(s) + \tilde{D}_1\frac{\tilde{\tau}_2}{\kappa_2}\int \tilde{\tau}_2\tilde{\boldsymbol{T}}_2(s) + \tilde{D}_1\tilde{\boldsymbol{N}}_2(s) - \tilde{D}_1\int \tilde{\tau}_2 ds\tilde{\boldsymbol{B}}_2(s).$$

Then, we have $\tilde{\boldsymbol{V}}' = 0$ and $\langle \tilde{\boldsymbol{V}}, \tilde{\boldsymbol{N}}_2(s)\rangle = \tilde{D}_1 \setminus \{0\}$. □

Corollary 4.9. *The Smarandache $\boldsymbol{NB}_2$ curve is a 2-type slant helix if and only if*

$$\frac{\tilde{\tau}_2}{\tilde{\kappa}_2} = \frac{1}{\tilde{D}_2}(d_1\cos s + d_2\sin s). \tag{4.24}$$

Proof. From the definition of 2-type slant helix, we have

$$\tilde{v}_4 = -\langle \tilde{\boldsymbol{V}}, \tilde{\boldsymbol{B}}_2(s)\rangle = \tilde{D}_2,$$

where $\tilde{D}_2$ is a nonzero constant. Substituting it into Eq (4.18), we get

$$\begin{cases} \tilde{v}_1' - \tilde{v}_2 = 0 \\ \tilde{v}_2' + \tilde{v}_1 - \tilde{v}_3\tilde{\kappa}_2 = 0 \\ \tilde{v}_3' + \tilde{v}_2\tilde{\kappa}_2 + \tilde{D}_2\tilde{\tau}_2 = 0 \\ \tilde{v}_3\tilde{\tau}_2 = 0 \end{cases}.$$

Because $\tilde{\tau}_2 \neq 0$, we have

$$\begin{cases} \tilde{v}_1' = \tilde{v}_2 \\ \tilde{v}_2' + \tilde{v}_1 = 0 \\ \tilde{v}_2\tilde{\kappa}_2 + \tilde{D}_2\tilde{\tau}_2 = 0 \end{cases}.$$

Therefore, we can easily obtain

$$\frac{\tilde{\tau}_2}{\tilde{\kappa}_2} = \frac{1}{\tilde{D}_2}(d_1\cos s + d_2\sin s).$$

Conversely, assume Eq (4.24) holds for Smarandache $\boldsymbol{NB}_2$ curve. We can define a vector field $\tilde{\boldsymbol{V}}$ as follows:

$$\tilde{\boldsymbol{V}} = (d_2 \cos s - d_1 \sin s)\tilde{\boldsymbol{\gamma}}(s) + (d_1 \cos s + d_2 \sin s)\tilde{\boldsymbol{T}}_2(s) + \tilde{D}_2\tilde{\boldsymbol{B}}_2(s),$$

where $d_1, d_2 \in \mathbb{R}$. Then, we have $\tilde{\boldsymbol{V}}' = 0$ and $-\langle \tilde{\boldsymbol{V}}, \tilde{\boldsymbol{B}}(s)\rangle = \tilde{D}_2 \in \mathbb{R} \setminus \{0\}$. □

Corollary 4.10. *Let $\boldsymbol{\gamma}(s)$ be a 3-type null slant helix in $\mathbb{R}^4_1$. If the Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \tilde{\boldsymbol{\gamma}}(\varphi(s))$ of $\boldsymbol{\gamma}(s)$ is space-like, then it cannot be any type slant helix.*

Case 3. The Smarandache $\boldsymbol{NB}_2$ curve is a time-like curve.

Hence, we can denote $\{\tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_3(s), \tilde{\boldsymbol{N}}_3(s), \tilde{\boldsymbol{B}}_3(s)\}$ as the moving Frenet frame along the curve $\tilde{\boldsymbol{\gamma}}(s)$, where

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{\gamma}}(s)\rangle = -\langle \tilde{\boldsymbol{T}}_3(s), \tilde{\boldsymbol{T}}_3(s)\rangle = \langle \tilde{\boldsymbol{N}}_3(s), \tilde{\boldsymbol{N}}_3(s)\rangle = \langle \tilde{\boldsymbol{B}}_3(s), \tilde{\boldsymbol{B}}_3(s)\rangle = 1,$$

$$\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{T}}_3(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_3(s)\rangle = \langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{B}}_3(s)\rangle = \langle \tilde{\boldsymbol{T}}_3(s), \tilde{\boldsymbol{N}}_3(s)\rangle = \langle \tilde{\boldsymbol{T}}_3(s), \tilde{\boldsymbol{B}}_3(s)\rangle = \langle \tilde{\boldsymbol{N}}_3(s), \tilde{\boldsymbol{B}}_3(s)\rangle = 0.$$

Suppose that $\tilde{\boldsymbol{N}}_3(s)$ satisfies

$$\tilde{\boldsymbol{N}}_3(s) = \tilde{a}_3\boldsymbol{T}(s) + \tilde{b}_3\boldsymbol{N}(s) + \tilde{c}_3\boldsymbol{B}_1(s) + \tilde{d}_3\boldsymbol{B}_2(s),$$

and the conditions $\langle \tilde{\boldsymbol{\gamma}}(s), \tilde{\boldsymbol{N}}_3(s)\rangle = \langle \tilde{\boldsymbol{T}}_3(s), \tilde{\boldsymbol{N}}_3(s)\rangle = 0$, $\langle \tilde{\boldsymbol{N}}_3(s), \tilde{\boldsymbol{N}}_3(s)\rangle = 1$. We can obtain

$$\begin{cases} 2\tilde{a}_3\tilde{c}_3 + \tilde{b}_3^2 + \tilde{d}_3^2 = 1 \\ \tilde{b}_3 - \tilde{d}_3 = 0 \\ \tilde{c}_3(\kappa + \tau) - \tilde{a}_3 = 0 \end{cases}.$$

We take one set of solutions

$$\begin{cases} \tilde{a}_3 = 0 \\ \tilde{b}_3 = \dfrac{1}{\sqrt{2}} \\ \tilde{c}_3 = 0 \\ \tilde{d}_3 = \dfrac{1}{\sqrt{2}} \end{cases}.$$

Thus, we can choose the expression of $\tilde{\boldsymbol{N}}_3(s)$ as

$$\tilde{\boldsymbol{N}}_3(s) = \frac{1}{\sqrt{2}}(\boldsymbol{N}(s) + \boldsymbol{B}_2(s)).$$

Based on the above conditions, we can obtain

$$\begin{aligned} \tilde{\boldsymbol{B}}_3(s) &= \tilde{\boldsymbol{\gamma}}(s) \wedge \tilde{\boldsymbol{T}}_3(s) \wedge \tilde{\boldsymbol{N}}_3(s) \\ &= \begin{vmatrix} \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}_1(s) & \mathbf{B}_2(s) \\ 0 & \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{2} & 0 & -\dfrac{1}{\sqrt{2}} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{vmatrix} \\ &= -\frac{1}{\sqrt{2}}\boldsymbol{T}(s) - \frac{1}{2}\boldsymbol{B}_1. \end{aligned} \tag{4.25}$$

By taking the derivative of Eqs (4.2) and (4.25),

$$\tilde{\boldsymbol{T}}_3'(s) =(\frac{1}{\sqrt{2}}\kappa - \frac{1}{2})\boldsymbol{N}(s) - \frac{1}{2}\tau \boldsymbol{B}_2(s),$$

$$\tilde{\boldsymbol{N}}_3'(s) = \frac{1}{\sqrt{2}}[(\kappa - \tau)\boldsymbol{T}(s) - \boldsymbol{B}_1(s)].$$

Hence, we can receive the following Frenet formulas of $\tilde{\boldsymbol{\gamma}}(s)$

$$\begin{cases} \tilde{\boldsymbol{\gamma}}'(s) = \tilde{\boldsymbol{T}}_3(s) \\ \tilde{\boldsymbol{T}}_3'(s) = \tilde{\boldsymbol{\gamma}}(s) + \tilde{\kappa}_3(s)\tilde{\boldsymbol{N}}_3 \\ \tilde{\boldsymbol{N}}_3'(s) = -\tilde{\kappa}_3(s)\tilde{\boldsymbol{T}}_3(s) + \tilde{\tau}_3(s)\tilde{\boldsymbol{B}}_3(s) \\ \tilde{\boldsymbol{B}}_3'(s) = -\tilde{\tau}_3(s)\tilde{\boldsymbol{N}}_3(s) \end{cases},$$

where

$$\begin{aligned} \tilde{\kappa}_3(s) &= \langle \tilde{\boldsymbol{T}}_3'(s), \tilde{\boldsymbol{N}}_3(s)\rangle \\ &= \frac{\kappa - \tau}{2} - \frac{1}{2\sqrt{2}} \\ &= \tau, \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \tilde{\tau}_2(s) &= \langle \tilde{\boldsymbol{N}}'(s), \tilde{\boldsymbol{B}}(s)\rangle \\ &= -\frac{\kappa - \tau}{2\sqrt{2}} + \frac{1}{2} \\ &= -\frac{1}{\sqrt{2}}\kappa + \frac{3}{4}. \end{aligned} \tag{4.27}$$

The classification of the Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}$ as time-like is similar to that of the Smarandache $\boldsymbol{NB}_2$ as space-like, and we will not repeat it here.

Corollary 4.11. *Let $\boldsymbol{\gamma}(s)$ be a 3-type null slant helix in $\mathbb{R}^4_1$. If the Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \tilde{\boldsymbol{\gamma}}(\varphi(s))$ of $\boldsymbol{\gamma}(s)$ is time-like, then it cannot be any type slant helix.*

5. The Smarandache curves on the hyperbolic 3-space

In this section, we define the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}$ of the null curve $\boldsymbol{\gamma}(s)$, and $\overline{\boldsymbol{\gamma}}$ lies on the hyperbolic 3-space. We denote $\tilde{s}$ as the arc-length parameter of $\overline{\boldsymbol{\gamma}}$, where

$$\overline{s}(s) = \int_{s_0}^{s} |\overline{\boldsymbol{\gamma}}'(s)| ds,\ s_0 \in I(open\ interval).$$

We construct the Frenet frame that is employed to discuss the geometric relationships between $\boldsymbol{\gamma}(s)$ and its Smarandache $\boldsymbol{TB}_1$ curve.

Definition 5.1. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}^4_1$ with $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}_1(s), \boldsymbol{B}_2(s)\}$ as the moving frame. The curve*

$$\overline{\boldsymbol{\gamma}}(s) = \boldsymbol{T}(s) - \frac{1}{2}\boldsymbol{B}_1(s), \tag{5.1}$$

is called the Smarandache $\boldsymbol{TB}_1$ curve of $\boldsymbol{\gamma}(s)$.

Obviously, $\hat{\boldsymbol{\gamma}}(s)$ fully lies on the hyperbolic 3-space. By taking the derivative of Eq (5.1), we have

$$\overline{\boldsymbol{T}}(s) = (1 + \frac{1}{2}\kappa)\boldsymbol{N}(s) - \frac{1}{2}\tau\boldsymbol{B}_2(s), \tag{5.2}$$

and

$$\langle\overline{\boldsymbol{T}}(s), \overline{\boldsymbol{T}}(s)\rangle = (1 + \frac{1}{2}\kappa)^2 + \frac{1}{4}\tau^2 = 1.$$

Proposition 5.2. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{R}^4_1$, and the Smarandache $\boldsymbol{TB}_1$ curve located on the hyperbolic 3-space can only be space-like.*

Hence, we can denote $\{\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{T}}(s), \overline{\boldsymbol{N}}(s), \overline{\boldsymbol{B}}(s)\}$ as the moving Frenet frame along the curve $\overline{\boldsymbol{\gamma}}(s)$, where

$$-\langle\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{\gamma}}(s)\rangle = \langle\overline{\boldsymbol{T}}(s), \overline{\boldsymbol{T}}(s)\rangle = \langle\overline{\boldsymbol{N}}(s), \overline{\boldsymbol{N}}(s)\rangle = \langle\overline{\boldsymbol{B}}(s), \overline{\boldsymbol{B}}(s)\rangle = 1,$$
$$\langle\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{T}}(s)\rangle = \langle\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{N}}(s)\rangle = \langle\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{B}}(s)\rangle = \langle\overline{\boldsymbol{T}}(s), \overline{\boldsymbol{N}}(s)\rangle = \langle\overline{\boldsymbol{T}}(s), \overline{\boldsymbol{B}}(s)\rangle = \langle\overline{\boldsymbol{N}}(s), \overline{\boldsymbol{B}}(s)\rangle = 0.$$

Suppose that $\overline{\boldsymbol{N}}(s)$ satisfies

$$\overline{\boldsymbol{N}}(s) = \overline{a}\boldsymbol{T}(s) + \overline{b}\boldsymbol{N}(s) + \overline{c}\boldsymbol{B}_1(s) + \overline{d}\boldsymbol{B}_2(s),$$

and the conditions $\langle\overline{\boldsymbol{\gamma}}(s), \overline{\boldsymbol{N}}(s)\rangle = \langle\overline{\boldsymbol{T}}(s), \overline{\boldsymbol{N}}(s)\rangle = 0$, $\langle\overline{\boldsymbol{N}}(s), \overline{\boldsymbol{N}}(s)\rangle = 1$. We can obtain

$$\begin{cases} 2\overline{ac} + \overline{b}^2 + \overline{d}^2 = 1 \\ \overline{b} - \overline{d} = 0 \\ \overline{c}(\kappa + \tau) - \overline{a} = 0 \end{cases}.$$

We take one set of solutions

$$\begin{cases} \overline{a} = 0 \\ \overline{b} = \frac{1}{2}\tau \\ \overline{c} = 0 \\ \overline{d} = 1 + \frac{1}{2}\kappa \end{cases}.$$

Thus, we can choose the expression of $\overline{\boldsymbol{N}}(s)$ as

$$\overline{\boldsymbol{N}}(s) = \frac{1}{2}\tau\boldsymbol{N}(s) + (1 + \frac{1}{2}\kappa)\boldsymbol{B}_2(s).$$

Based on the above conditions, we can obtain

$$\begin{aligned} \overline{\boldsymbol{B}}(s) &= \overline{\boldsymbol{\gamma}}(s) \wedge \overline{\boldsymbol{T}}(s) \wedge \overline{\boldsymbol{N}}(s) \\ &= \begin{vmatrix} \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}_1(s) & \mathbf{B}_2(s) \\ 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 + \frac{1}{2}\kappa & 0 & -\frac{1}{2}\tau \\ 0 & \frac{1}{2}\tau & 0 & 1 + \frac{1}{2}\kappa \end{vmatrix} \\ &= \frac{1}{2}\boldsymbol{T}(s) + \boldsymbol{B}_1. \end{aligned} \tag{5.3}$$

By taking the derivative of Eqs (5.2) and (5.3),

$$\overline{\boldsymbol{T}}'(s) =(\kappa + \frac{1}{2}\kappa^2 + \frac{1}{2}\tau^2)\boldsymbol{T}(s) + \frac{1}{2}\kappa'\boldsymbol{N}(s) - (1 + \frac{1}{2}\kappa) - \frac{1}{2}\tau'\boldsymbol{B}_2(s),$$

$$\overline{\boldsymbol{N}}'(s) = -\tau\boldsymbol{T}(s) + \frac{1}{2}\tau'\boldsymbol{N}(s) - \frac{1}{2}\tau\boldsymbol{B}_1(s) + \frac{1}{2}\kappa'\boldsymbol{B}_2(s).$$

Hence, we can receive the following Frenet formulas of $\overline{\boldsymbol{\gamma}}(s)$

$$\begin{cases} \overline{\boldsymbol{\gamma}}'(s) = \overline{\boldsymbol{T}}(s) \\ \overline{\boldsymbol{T}}'(s) = \overline{\boldsymbol{\gamma}}(s) + \overline{\kappa}(s)\overline{\boldsymbol{N}} \\ \overline{\boldsymbol{N}}'(s) = -\overline{\kappa}(s)\overline{\boldsymbol{T}}(s) + \overline{\tau}(s)\overline{\boldsymbol{B}}(s) \\ \overline{\boldsymbol{B}}'(s) = \overline{\tau}(s)\overline{\boldsymbol{N}}(s) \end{cases},$$

where

$$\begin{aligned} \overline{\kappa}(s) &= \langle \overline{\boldsymbol{T}}'(s), \overline{\boldsymbol{N}}(s)\rangle \\ &= \frac{1}{4}(\kappa'\tau - \kappa\tau' - 2\tau'), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \overline{\tau}(s) &= \langle \overline{\boldsymbol{N}}'(s), \overline{\boldsymbol{B}}(s)\rangle \\ &= -\frac{5}{4}\tau. \end{aligned} \tag{5.5}$$

Similarly, the classification of the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}$ is similar to that of the Smarandache $\boldsymbol{NB}_2$ curve $\tilde{\boldsymbol{\gamma}}$ being time-like, and we will not repeat it here.

Corollary 5.3. *Let $\boldsymbol{\gamma}(s)$ be a 3-type null slant helix in $\mathbb{R}^4_1$. The Smarandache $\boldsymbol{TB}_1$ curve $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \tilde{\boldsymbol{\gamma}}(\varphi(s))$ of $\boldsymbol{\gamma}(s)$ is a straight line.*

6. Examples

Example 6.1. *Let $\boldsymbol{\gamma}(s) = \{\frac{1}{4}e^{-2s} - \frac{1}{8}e^{2s} + 1, -\frac{1}{2}s, \frac{1}{4}e^{-2s}, \frac{1}{8}e^{2s}\}$ be a null curve parameterized by pseudo arc-length s, then it is easy to show that*

$$\boldsymbol{T}(s) = \{-\tfrac{1}{2}e^{-2s} - \tfrac{1}{4}e^{2s}, -\tfrac{1}{2}, -\tfrac{1}{2}e^{-2s}, \tfrac{1}{4}e^{2s}\},$$

$$\boldsymbol{N}(s) = \{e^{-2s} - \tfrac{1}{2}e^{2s}, 0, e^{-2s}, \tfrac{1}{2}e^{2s}\},$$

$$\boldsymbol{B}_1(s) = -2\{-\tfrac{1}{2}e^{-2s} - \tfrac{1}{4}e^{2s}, \tfrac{1}{2}, -\tfrac{1}{2}e^{-2s}, \tfrac{1}{4}e^{2s}\},$$

$$\boldsymbol{B}_2(s) = \{1, 0, 1, -1\},$$

$$||\boldsymbol{T}'(s)|| = 1,$$

$$\kappa(s) = 2,$$

$$\tau(s) = 0.$$

The Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\boldsymbol{\gamma}}(s)$ of $\boldsymbol{\gamma}(s)$,

$$\hat{\boldsymbol{\gamma}}(s) = \boldsymbol{T}(s) + \sqrt{2}\boldsymbol{N}(s) - \boldsymbol{B}_1(s)$$
$$= \{(\sqrt{2} - \frac{3}{2})e^{-2s} - \frac{1}{2}(\sqrt{2} + \frac{3}{2})e^{2s}, \frac{1}{2}, (\sqrt{2} - \frac{3}{2})e^{-2s}, \frac{1}{2}(\sqrt{2} + \frac{3}{2})e^{2s}\}.$$

Obviously, $\hat{\boldsymbol{\gamma}}(s)$ is a 2-type slant helix.

Due to the inability of four-dimensional space to be directly created through drawing, we apply the method of projection for intuitive display. We draw the graphics. We draw the graphics (see Figures 1–4) from four different projection angles, such as from projection angle $\boldsymbol{B}_2 = C(\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1\})$ in Figure 1; from projection angle $\boldsymbol{B}_1 = C(\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_2\})$ in Figure 2; from projection angle $\boldsymbol{N} = C(\{\boldsymbol{T}, \boldsymbol{B}_1, \boldsymbol{B}_2\})$ in Figure 3 and from projection angle $\boldsymbol{T} = C(\{\boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\})$ in Figure 4.

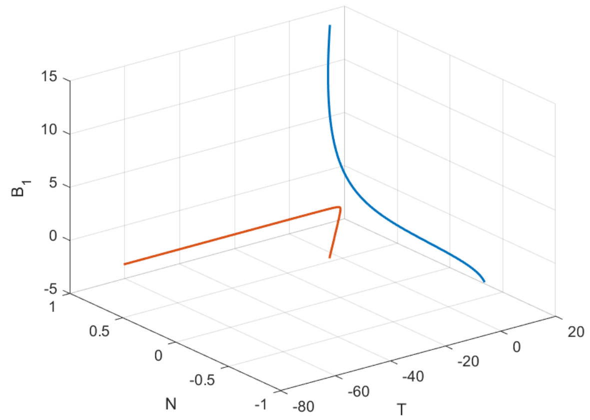

Figure 1. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TNB}_1$ $\tilde{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1\}$.

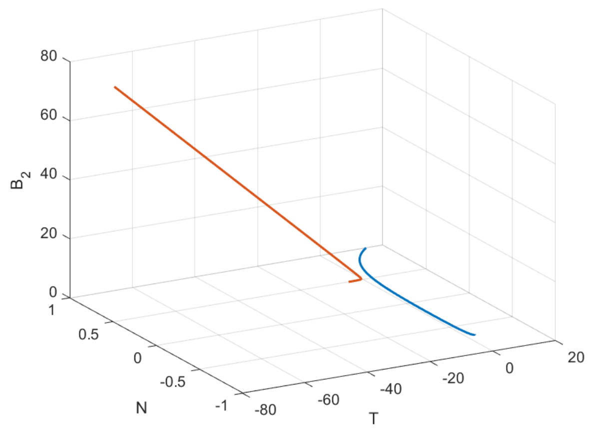

Figure 2. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_2\}$.

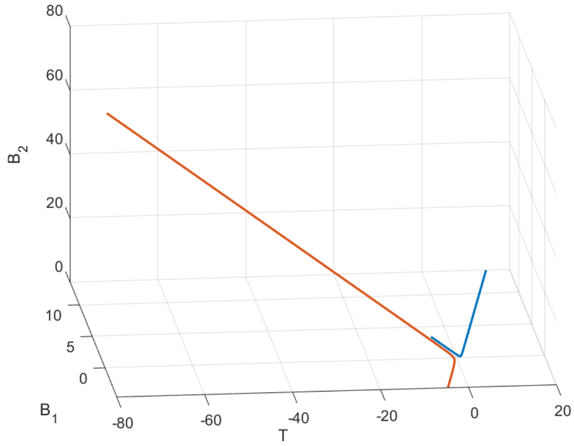

Figure 3. Projection of $\gamma(s)$ (blue) and the Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\gamma}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$.

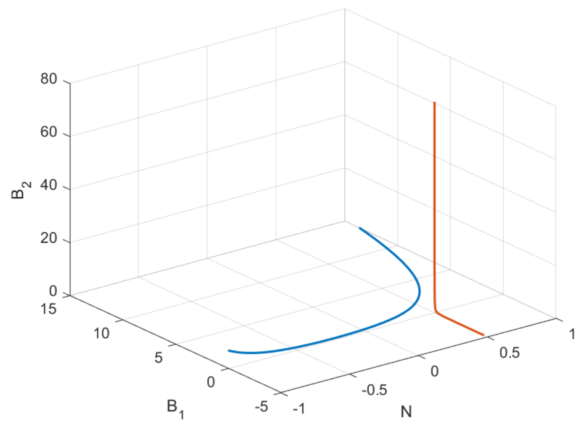

Figure 4. Projection of $\gamma(s)$ (blue) and the Smarandache $\boldsymbol{TNB}_1$ curve $\hat{\gamma}(s)$ (red) on $\{\boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$.

Example 6.2. *Let* $\gamma(s) = \{\frac{1}{6}s^3 + s, s, \frac{1}{2}s^2, \frac{1}{6}s^3\}$ *be a 3-type null slant curve parameterized by pseudo arc-length s, then it is easy to show that*

$$\boldsymbol{T}(s) = \{\frac{1}{2}s^2 + 1, 1, s, \frac{1}{2}s^2\},$$

$$\boldsymbol{N}(s) = \{s, 0, 1, s\},$$

$$\boldsymbol{B}_1(s) = -\{1, 0, 0, 1\},$$

$$\boldsymbol{B}_2(s) = \{1, 0, 1, 1\},$$

$$\|\boldsymbol{T}'(s)\| = 1,$$

$$\kappa(s) = 0,$$

$$\tau(s) = 0.$$

The Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}(s)$ of $\boldsymbol{\gamma}$,

$$\begin{aligned}\overline{\boldsymbol{\gamma}}(s) &= \boldsymbol{T}(s) - \frac{1}{2}\boldsymbol{B}_1(s)\\ &= \{\frac{1}{2}s^2 + \frac{3}{2}, 1, s, \frac{1}{2}s^2 + \frac{1}{2}\}.\end{aligned}$$

Due to the inability of four-dimensional space to be directly created through drawing, we apply the method of projection for intuitive display. We draw the graphics (see Figures 5–8) from four different projection angles, such as from projection angle $\boldsymbol{B}_2 = C(\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1\})$ in Figure 5; from projection angle $\boldsymbol{B}_1 = C(\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_2\})$ in Figure 6; from projection angle $\boldsymbol{N} = C(\{\boldsymbol{T}, \boldsymbol{B}_1, \boldsymbol{B}_2\})$ in Figure 7 and from projection angle $\boldsymbol{T} = C(\{\boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\})$ in Figure 8.

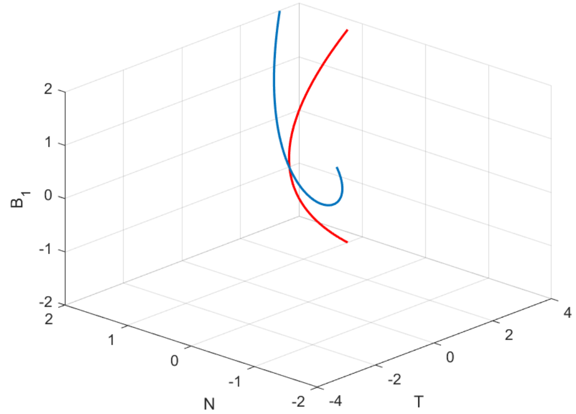

Figure 5. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1\}$.

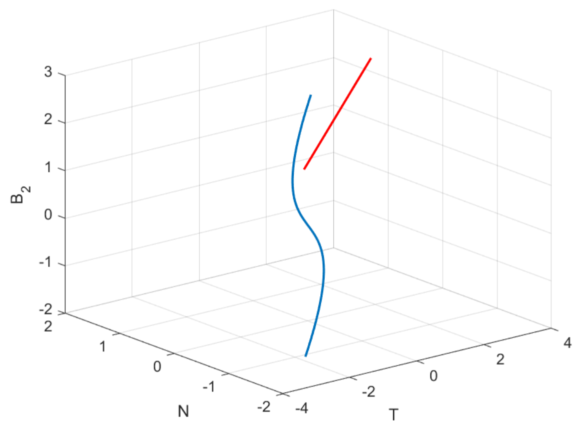

Figure 6. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_2\}$.

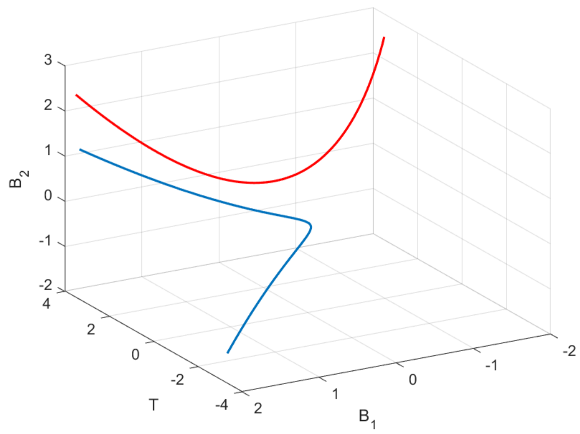

Figure 7. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{T}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$.

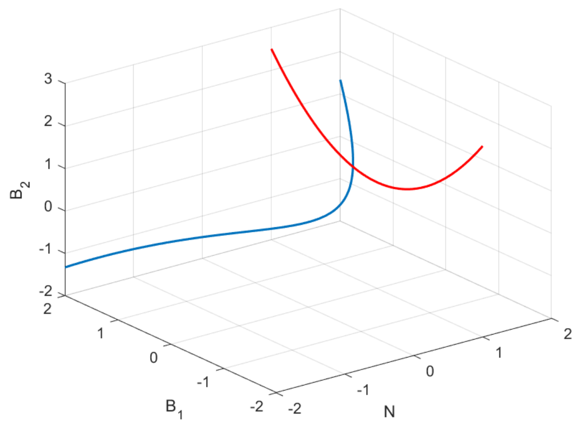

Figure 8. Projection of $\boldsymbol{\gamma}(s)$ (blue) and the Smarandache $\boldsymbol{TB}_1$ curve $\overline{\boldsymbol{\gamma}}(s)$ (red) on $\{\boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$.

7. Conclusions

In this paper, we investigated the Smarandache curves formed by the Frenet vector of the null curve $\boldsymbol{\gamma}(s)$. We establish the relationships of geometric invariants between $\boldsymbol{\gamma}(s)$ and the Smarandache $\boldsymbol{TNB}_1$ curve $\tilde{\boldsymbol{\gamma}}(s)$. We derive the necessary and sufficient conditions for the Smarandache $\boldsymbol{TNB}_1$ curve to be a k-type slant helix on 3-light cone. Based on the relationship between curvature and curvature, we conclude that the Smarandache $\tilde{\boldsymbol{\gamma}}(s)$ cannot be a 1-type slant helix when $\boldsymbol{\gamma}(s)$ is a 3-type null slant helix. The following conclusion can be drawn from the relationship between curvature and curvature. The Smarandache $\boldsymbol{NB}_2$ curve on 3-de Sitter space of $\boldsymbol{\gamma}(s)$ is a null helix if and only if $\tilde{\tau}_1(s) = \tilde{C}(s)\tilde{\kappa}_1(s)$. Furthermore, we find that the Smarandache $\boldsymbol{TB}_1$ curve located on the hyperbolic 3-space can only be a space-like curve.

In our further studies, we plan to address an analogous problem in other spaces, such as Galilean space, among others. Exploring the osculating circles, evolutes, involutes, and other related curves associated with Smarandache curves could provide valuable insights into their geometric behavior. Additionally, investigating special types of Smarandache curves may lead to the discovery of interesting and novel geometric properties.

Author contributions

Huina Zhang: Writing-original draft preparation, Funding acquisition; Yanping Zhao: Writing-review and editing; Jianguo Sun: Writing-original draft preparation, Writing-review and editing, Funding acquisition. All authors have read and agreed to the published version of this manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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