AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 21686–21702.
DOI: 10.3934/math.20241055
Received: 27 April 2024
Revised: 14 June 2024
Accepted: 24 June 2024
Published: 08 July 2024

Research article

Nonexistence for fractional differential inequalities and systems in the sense of Erdélyi-Kober

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Abstract: Nonexistence theorems constitute an important part of the theory of differential and partial differential equations. Motivated by the numerous applications of fractional differential equations in diverse fields, in this paper, we studied sufficient conditions for the nonexistence of solutions (or, equivalently, necessary conditions for the existence of solutions) for nonlinear fractional differential inequalities and systems in the sense of Erdélyi-Kober. Our approach is based on nonlinear capacity estimates specifically adapted to the Erdélyi-Kober fractional operators and some integral inequalities.

Keywords: fractional differential inequalities; systems; Erdélyi-Kober fractional operators; nonexistence
Mathematics Subject Classification: 26A33, 34A08, 34A12

1. Introduction

Fractional differential equations arise in the mathematical modeling of various problems such as biology, fluid mechanics, electrochemistry, finance, and many other areas of applications; see, e.g., [1–3]. This fact motivated the study of fractional differential equations in various directions: existence of solutions [4, 5], comparison principles [6, 7], inverse problems [8, 9], inequalities [10], etc.

One of the most important topics of the theory of differential and partial differential equations is the issue of nonexistence of solutions, which was initiated by the famous Liouville theorem for harmonic functions (see, e.g., [11]). Nonexistence theorems have several applications, in particular, in the study of blow-up of solutions (see, e.g., [12]). The study of nonexistence of solutions to fractional differential equations and inequalities was initiated by Kirane and his collaborators. Next, this topic was developed by many authors. For instance, Kirane and Malik [13] investigated the profile of the blowing-up solutions to the nonlinear system of fractional differential equations

$$\begin{cases} u'(t) + {}^C D_0^\alpha u(t) = |v(t)|^q, & t > 0, \\ v'(t) + {}^C D_0^\beta v(t) = |u(t)|^p, & t > 0, \\ u(0) = u_0, \ v(0) = v_0, \end{cases}$$

where $0 < \alpha, \beta < 1$, ${}^C D_0^\alpha$ (resp. ${}^C D_0^\beta$) is the Caputo fractional derivative of order α (resp. β), $p, q > 1$, and $u_0, v_0 \in \mathbb{R}$. Laskri and Tatar [14] considered nonlinear fractional differential inequalities of the form

$$\begin{cases} D_0^\alpha u(t) \geq t^\gamma |u(t)|^m, & t > 0, \\ \lim_{t \to 0^+} (I_0^{1-\alpha} u)(t) = b, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, D_0^α is the Riemann-Liouville fractional derivative of order α, $b \in \mathbb{R}$, $m > 1$, and $\gamma \in \mathbb{R}$. Using the test function method (see [12]), the authors obtained sufficient conditions under which (1.1) admits no global solution. Kassim, Furati, and Tatar [15] studied fractional differential inequalities of the form

$$\begin{cases} {}^C D_0^\alpha u(t) + {}^C D_0^\beta u(t) \geq t^\gamma |u(t)|^m, & t > 0, \\ u^{(i)}(0) = b_i, & i = 0, 1, \cdots n - 1, \end{cases}$$

where $0 < \beta \leq \alpha$, $n = -[-\alpha]$, $\gamma \in \mathbb{R}$, $m > 1$, and $b_i \in \mathbb{R}$, $i = 0, 1, \cdots, n - 1$. More recent works can be found in [16–19] (see also the references therein).

In all the above cited works, the fractional derivatives were considered in the sense of Riemann-Liouville or Caputo (see, e.g., [20] for the definitions of these operators). Very recently, in [21], the authors studied fractional differential inequalities of the form

$$\begin{cases} D_{a;\sigma,\eta}^\alpha u(t) \geq V(t)|u(t)|^p, & t > a, \\ \lim_{t \to a^+} \left(I_{a;\sigma,\eta+\alpha}^{1-\alpha} u \right)(t) = u_a, \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$, $a > 0$, $\sigma > 0$, $\eta \in \mathbb{R}$, $p > 1$, $u_a > 0$, and V is a measurable positive function. Here, $D_{a;\sigma,\eta}^\alpha$ denotes the Erdélyi-Kober fractional derivative of order α and parameters σ and η, and $I_{a;\sigma,\eta+\alpha}^{1-\alpha}$ denotes the left-sided Erdélyi-Kober fractional integral of order $1 - \alpha$ and parameters σ and $\eta + \alpha$. It was shown that, if

$$\liminf_{T \to \infty} T^{\frac{-\sigma\alpha p}{p-1}} \int_a^T V^{\frac{-1}{p-1}}(t) t^{\frac{p\alpha\sigma}{p-1} + \sigma(\eta+1) - 1} \, dt = 0, \tag{1.3}$$

then (1.2) admits no weak solution. In particular, when

$$V(t) \geq C_V (t^\sigma - a^\sigma)^\gamma,$$

where $C_V > 0$ is a constant, it was proved that, if one of the following conditions:

$$(\mathrm{C}_1): \quad p(1 - \alpha) - 1 < \gamma < p - 1, \quad (\alpha + \eta)p \leq \eta,$$

$$(\mathrm{C}_2): \quad (p - 1)(1 + \eta) < \gamma < p - 1, \quad (\alpha + \eta)p > \eta,$$

holds, then (1.2) admits no weak solution.

The aim of the first part of this work is to obtain sufficient conditions for the nonexistence of weak solutions to the inhomogeneous version of (1.2) (with $V \equiv 1$), namely,

$$D^{\alpha}_{a;\sigma,\eta}u(t) \geq |u(t)|^p + f(t), \quad t > a \tag{1.4}$$

subject to the initial condition

$$\lim_{t\to a^+} \left(I^{1-\alpha}_{a;\sigma,\eta+\alpha}u\right)(t) = u_a, \tag{1.5}$$

where $a > 0$, $\sigma > 0$, $\eta \in \mathbb{R}$, $0 < \alpha < 1$, $p > 1$, $u_a \in \mathbb{R}$, and $f \in L^1_{\text{loc}}([a,\infty))$. Our motivation for considering problems of type (1.4) is to study the influence of the inhomogeneous term f on the large-time behavior of solutions to (1.2) with $V \equiv 1$. We show that, if $f(t) \geq C_f t^{-\sigma\eta}(t^\sigma - a^\sigma)^\gamma$, $t > a$, where $C_f > 0$ is a constant, and $\gamma > \max\{\eta, -1\}$, then for all $p > 1$, (1.4) and (1.5) admit no weak solution.

We next extend our study to systems of fractional differential inequalities of the form

$$\begin{cases} D^{\alpha}_{a;\sigma,\eta}u(t) \geq g(t)|v(t)|^p, & t > a, \\ D^{\beta}_{a;\sigma,\eta}v(t) \geq h(t)|u(t)|^q, & t > a \end{cases} \tag{1.6}$$

subject to the initial conditions

$$\lim_{t\to a^+} \left(I^{1-\alpha}_{a;\sigma,\eta+\alpha}u\right)(t) = u_a, \ \lim_{t\to a^+} \left(I^{1-\beta}_{a;\sigma,\eta+\alpha}v\right)(t) = v_a, \tag{1.7}$$

where $0 < \alpha,\beta < 1$, $p,q > 1$, g,h are positive measurable functions, and $u_a, v_a \in \mathbb{R}$. Our motivation for considering systems of the form (1.6) is to extend the obtained results in [13] from the Caputo sense to the Erdélyi-Kober sense.

We finally mention that some existence and nonexistence results for a class of nonlinear Erdélyi-Kober type fractional differential equations on unbounded domains were established in [22], making use of some tools from fixed point theory. The approach that we use in this paper is based on nonlinear capacity estimates specifically adapted to Erdélyi-Kober fractional derivatives.

The organization of the rest of the paper is as follows. In Section 2, some notions and properties related to Erdélyi-Kober fractional operators are recalled. The definitions of weak solutions to the considered problems as well as the obtained results are presented in Section 3. Some important lemmas are established in Section 4. Finally, the proofs of our obtained results are given in Section 5.

2. Preliminaries and notation

In this section, we recall briefly some basic notions and properties related to Erdélyi-Kober fractional operators, and fix some notation. For more details, we refer to [20].

Let $a, T \in \mathbb{R}$ be fixed such that $0 < a < T$. We first recall the Riemann-Liouville fractional integral operators.

The left-sided and right-sided Riemann-Liouville fractional integrals of order $\kappa > 0$ of a function $f \in L^1([a,T])$ are defined respectively by

$$(I^\kappa_a f)(t) = \frac{1}{\Gamma(\kappa)} \int_a^t (t-s)^{\kappa-1} f(s)\, ds$$

and

$$(I_T^\kappa f)(t) = \frac{1}{\Gamma(\kappa)} \int_t^T (s-t)^{\kappa-1} f(s)\, ds,$$

for almost everywhere $t \in [a, T]$, where Γ denotes the gamma function.

The left-sided and right-sided Erdélyi-Kober fractional integrals of order $\alpha > 0$ and parameters $\sigma > 0$ and $\eta \in \mathbb{R}$ of a function $f \in L^1([a, T])$, are defined respectively by

$$(I_{a;\sigma,\eta}^\alpha f)(t) = \frac{\sigma t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t \frac{s^{\sigma\eta+\sigma-1} f(s)}{(t^\sigma - s^\sigma)^{1-\alpha}}\, ds$$

and

$$(I_{T;\sigma,\eta}^\alpha f)(t) = \frac{\sigma t^{\sigma\eta}}{\Gamma(\alpha)} \int_t^T \frac{s^{\sigma(1-\alpha-\eta)-1} f(s)}{(s^\sigma - t^\sigma)^{1-\alpha}}\, ds,$$

for almost everywhere $t \in [a, T]$.

Some relations between the Riemann-Liouville and Erdélyi-Kober fractional integrals can be easily obtained. Using the change of variable $z = s^\sigma$, for $a < t < T$, we obtain

$$\begin{aligned} (I_{a;\sigma,\eta}^\alpha f)(t) &= \frac{t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_{a^\sigma}^{t^\sigma} \frac{z^\eta f(z^{\frac{1}{\sigma}})}{(t^\sigma - z)^{1-\alpha}}\, dz \\ &= t^{-\sigma(\alpha+\eta)} \left(I_{a^\sigma}^\alpha \widetilde{f}\right)(t^\sigma), \end{aligned}$$

where

$$\widetilde{f}(z) = z^\eta f(z^{\frac{1}{\sigma}}), \quad a^\sigma < z < T^\sigma.$$

Using the same change of variable, we get

$$\begin{aligned} (I_{T;\sigma,\eta}^\alpha g)(t) &= \frac{t^{\sigma\eta}}{\Gamma(\alpha)} \int_{t^\sigma}^{T^\sigma} \frac{z^{-(\alpha+\eta)} g(z^{\frac{1}{\sigma}})}{(z - t^\sigma)^{1-\alpha}}\, ds \\ &= t^{\sigma\eta} (I_{T^\sigma}^\alpha \widetilde{g})(t^\sigma), \end{aligned}$$

where

$$\widetilde{g}(z) = z^{-(\alpha+\eta)} g(z^{\frac{1}{\sigma}}), \quad a^\sigma < z < T^\sigma.$$

We have the following integration by parts rule (see [21]).

Lemma 2.1. *Let $\mu, \sigma > 0$ and $\eta \in \mathbb{R}$. Let $k, m \geq 1$ and $\frac{1}{k} + \frac{1}{m} \leq 1 + \mu$ ($k \neq 1$ and $m \neq 1$ if $\frac{1}{k} + \frac{1}{m} = 1 + \mu$). If $f \in L^k([a, T])$ and $g \in L^m([a, T])$, then*

$$\int_a^T t^{\sigma-1} (I_{a;\sigma,\eta}^\mu f)(t) g(t)\, dt = \int_a^T t^{\sigma-1} f(t) (I_{T;\sigma,\eta}^\mu g)(t)\, dt.$$

The proof of the following result can be found in [21].

Lemma 2.2. *Let $0 < \mu < 1$, $\sigma > 0$, and $\eta \in \mathbb{R}$. For $\lambda \gg 1$ (λ is sufficiently large), let*

$$\varphi(t) = (T^\sigma - a^\sigma)^{-\lambda} (T^\sigma - t^\sigma)^\lambda, \quad a \leq t \leq T. \tag{2.1}$$

For all $a < t < T$, we have

$$I_{T;\sigma,\eta+1-\mu}^\mu \left(t^{\sigma\eta+1} \varphi'\right)(t) = -\frac{\Gamma(\lambda+1)\sigma}{\Gamma(\lambda+\mu)} (T^\sigma - a^\sigma)^{-\lambda} (T^\sigma - t^\sigma)^{\mu+\lambda-1} t^{\sigma(\eta+1-\mu)}. \tag{2.2}$$

Let $0 < \alpha < 1$, $\sigma > 0$, $\eta \in \mathbb{R}$ and f be a function such that

$$t^{\sigma(\eta+1)} I^{1-\alpha}_{a;\sigma,\eta+\alpha} f \in AC([a,T]),$$

where $AC([a,T])$ denotes the space of absolutely continuous functions on $[a,T]$. The (left-sided) Erdélyi-Kober fractional derivative of order α and parameters σ and η of f is defined by (see, e.g., [20])

$$D^{\alpha}_{a;\sigma,\eta} f(t) = t^{-\sigma\eta} \left(\frac{1}{\sigma t^{\sigma-1}} \frac{d}{dt}\right) \left(t^{\sigma(\eta+1)} I^{1-\alpha}_{a;\sigma,\eta+\alpha} f\right)(t),$$

for almost everywhere $t \in [a,T]$.

Throughout this paper, we shall use the following notations. By C, we mean a positive constant independent of T and the solutions u and v. Its value is not necessarily the same from one line to another. By $\lambda \gg 1$, where $\lambda > 0$, we mean that λ is sufficiently large.

3. The results

In this section, we state our obtained results for problems (1.4)–(1.7).

Let us define weak solutions to (1.4) and (1.5). For all $T > a$, we introduce the set of functions

$$\Psi_T = \left\{\psi \in C^2([a,T]) : \psi \geq 0,\ \psi(T) = 0\right\}.$$

Definition 3.1. We say that u is a weak solution to (1.4) and (1.5), if $u \in L^p_{\mathrm{loc}}([a,\infty))$ and

$$\int_a^T |u(t)|^p \psi(t)\, dt + \int_a^T f(t)\psi(t)\, dt + \frac{a}{\sigma}\psi(a)u_a \leq -\frac{1}{\sigma}\int_a^T t^{\sigma-1} u(t) \left(I^{1-\alpha}_{T;\sigma,\eta+\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)'\right)(t)\, dt, \quad (3.1)$$

for all $T > a$ and $\psi \in \Psi_T$.

Notice that, if u satisfies (1.4) and (1.5), then for all $T > a$, multiplying (1.4) by $\psi \in \Psi_T$, integrating by parts over (a,T), using Lemma 2.1 and (1.5), we obtain (3.1).

Our main result for problems (1.4) and (1.5) is stated in the following theorem.

Theorem 3.2. *Let $a > 0$, $\sigma > 0$, $\alpha \in (0,1)$, and $p > 1$. Let $f \in L^1_{\mathrm{loc}}([a,\infty))$ be such that*

$$f(t) \geq C_f t^{-\sigma\eta} (t^\sigma - a^\sigma)^\gamma, \quad (3.2)$$

for almost everywhere $t > a$, where $C_f > 0$ is a constant. If

$$\gamma > \max\{\eta, -1\}, \quad (3.3)$$

then (1.4) and (1.5) admit no weak solution.

Remark 3.3. From Theorem 3.2, we show that the value of the parameter α has no effect on the nonexistence result.

Remark 3.4. In the homogeneous case ($f \equiv 0$), problem (1.4) under the initial condition (1.5), reduces to problem (1.2) with $V \equiv 1$. In this case, from the obtained result in [21], if one of the conditions:

$$\text{(i)}: \quad p(1-\alpha) < 1, \quad (\alpha+\eta)p \leq \eta,$$

$$\text{(ii)}: \quad \eta < -1, \quad (\alpha+\eta)p > \eta,$$

holds, then we have no weak solution. From Theorem 3.2, we show that under conditions (3.2) and (3.3), the effect of the inhomogeneous term on the large-time behavior of solutions is considerable. Namely, in this case, for every $p > 1$, the inhomogeneous problems (1.4) and (1.5) admit no weak solution.

We now define weak solutions to (1.6) and (1.7).

Definition 3.5. We say that the pair of functions (u, v) is a weak solution to (1.6) and (1.7), if $u \in L^q_{\text{loc}}([a,\infty), h(t)\,dt) \cap L^1_{\text{loc}}([a,\infty))$, $v \in L^p_{\text{loc}}([a,\infty), g(t)\,dt) \cap L^1_{\text{loc}}([a,\infty))$ and

$$\int_a^T |v(t)|^p g(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)u_a \leq -\frac{1}{\sigma}\int_a^T t^{\sigma-1}u(t)\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\,dt, \tag{3.4}$$

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \leq -\frac{1}{\sigma}\int_a^T t^{\sigma-1}v(t)\left(I^{1-\beta}_{T;\sigma,\eta+\beta}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\,dt, \tag{3.5}$$

for all $T > a$ and $\psi \in \Psi_T$.

Notice that, if (u, v) satisfies (1.6) and (1.7), then for all $T > a$, multiplying the first inequality in (1.6) by $\psi \in \Psi_T$, integrating by parts over (a, T), using Lemma 2.1 and (1.7), we obtain (3.4). Similarly, multiplying the second inequality in (1.6) by ψ and integrating by parts over (a, T), we get (3.5).

Our main result for (1.6) and (1.7) is stated in the following theorem.

Theorem 3.6. *Let $a > 0$, $\sigma > 0$, $\eta \in \mathbb{R}$, $0 < \alpha, \beta < 1$, and $p, q > 1$. Assume that $g^{\frac{-1}{p-1}}, h^{\frac{-1}{q-1}} \in L^1_{\text{loc}}([a,\infty))$ and $u_a, v_a \geq 0$. If one of the following conditions:*

(i) $v_a > 0$ and

$$\liminf_{T\to\infty} T^{-\sigma q(\alpha+\beta p)}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}h^{\frac{-1}{q-1}}(t)\,dt\right)^{q-1}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}}g^{\frac{-1}{p-1}}(t)\,dt\right)^{(p-1)q} = 0, \tag{3.6}$$

(ii) $u_a > 0$ and

$$\liminf_{T\to\infty} T^{-\sigma p(\beta+\alpha p)}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}}g^{\frac{-1}{p-1}}(t)\,dt\right)^{p-1}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}h^{\frac{-1}{q-1}}(t)\,dt\right)^{(q-1)p} = 0, \tag{3.7}$$

is satisfied, then (1.6) *and* (1.7) *admit no weak solution.*

Remark 3.7. If $\alpha = \beta$, $p = q$, $g = h = V$, $u_a = v_a$, and $u = v$, then system (1.6) under the initial conditions (1.7) reduces to (1.2). In this case, (3.6) and (3.7) reduce to (1.3). Then, we recover the nonexistence result obtained in [21] for (1.2).

We now consider singular weight functions of the forms

$$g(t) = (t-a)^{-\gamma}, \ \ h(t) = (t-a)^{-\rho}, \tag{3.8}$$

where $\gamma, \rho \geq 0$ are constants. It can be easily seen that $g^{\frac{-1}{p-1}}, h^{\frac{-1}{q-1}} \in L^1_{\text{loc}}([a,\infty))$. From Theorem 3.6, we deduce the following result.

Corollary 3.8. Let $a > 0$, $\sigma > 0$, $\eta < -1$, $0 < \alpha, \beta < 1$, $p, q > 1$, and $u_a, v_a \geq 0$. Let g and h be the functions defined by (3.8), where $\gamma, \rho \geq 0$. If one of the conditions:

(i) $v_a > 0$ and

$$\frac{\gamma q+\rho}{\sigma} < \min\left\{q\left(\alpha+\beta p\right), q\left[\alpha-(\eta+1)(p-1)\right], q\left(\beta p-\eta-1\right)+\eta+1, -(\eta+1)(pq-1)\right\}, \tag{3.9}$$

(ii) $u_a > 0$ and

$$\frac{\rho p+\gamma}{\sigma} < \min\left\{p\left(\beta+\alpha q\right), p\left[\beta-(\eta+1)(q-1)\right], p\left(\alpha q-\eta-1\right)+\eta+1, -(\eta+1)(pq-1)\right\},$$

holds, then (1.6) and (1.7) admit no weak solution.

We provide below an example to illustrate the above result.

Example 3.9. Consider the system of fractional differential inequalities

$$\begin{cases} D^{1/2}_{a;\sigma,-2}u(t) \geq (t-a)^{-\gamma}|v(t)|^2, & t > a, \\ D^{1/4}_{a;\sigma,-2}v(t) \geq (t-a)^{-\rho}|u(t)|^3, & t > a, \end{cases} \tag{3.10}$$

where $a > 0$, $\sigma > 0$, $\gamma, \rho \geq 0$, subject to the initial conditions (1.7) with $u_a, v_a > 0$. System (3.10) is a special case of (1.6), where g, h are defined by (3.8), $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\eta = -2$, $p = 2$, and $q = 3$. From Corollary 3.8, if

$$\frac{3\gamma+\rho}{\sigma} < 3, \tag{3.11}$$

then system (3.10) under the initial conditions (1.7) admits no weak solution. In this case, we have

$$\frac{\gamma q+\rho}{\sigma} = \frac{3\gamma+\rho}{\sigma},$$

and

$$q\left(\alpha+\beta p\right) = 3, \ \ q\left[\alpha-(\eta+1)(p-1)\right] = 3+\frac{3}{2}, q\left(\beta p-\eta-1\right)+\eta+1 = 3+\frac{1}{2}, -(\eta+1)(pq-1) = 5,$$

which shows that (3.9) is equivalent to (3.11).

4. Auxiliary results

In this section, we establish some important lemmas that will be used later in the proofs of our main results.

For all $T > a$ and $\psi \in \Psi_T$, let

$$J(\psi) = \int_a^T t^{\frac{(\sigma-1)p}{p-1}} \psi^{\frac{-1}{p-1}}(t) \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right|^{\frac{p}{p-1}} dt \tag{4.1}$$

and

$$K_1(\psi) = \int_a^T t^{\frac{(\sigma-1)q}{q-1}} h^{\frac{-1}{q-1}}(t) \psi^{\frac{-1}{q-1}}(t) \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right|^{\frac{q}{q-1}} dt, \tag{4.2}$$

$$K_2(\psi) = \int_a^T t^{\frac{(\sigma-1)p}{p-1}} g^{\frac{-1}{p-1}}(t) \psi^{\frac{-1}{p-1}}(t) \left| \left(I_{T;\sigma,\eta+\beta}^{1-\beta} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right|^{\frac{p}{p-1}} dt. \tag{4.3}$$

We have the following a priori estimate for problems (1.4) and (1.5) .

Lemma 4.1. *If u is a weak solution to* (1.4), (1.5), *and $u_a \geq 0$, then*

$$\int_a^T f(t)\psi(t)\, dt \leq C J(\psi), \tag{4.4}$$

for all $T > a$ and $\psi \in \Psi_T$, provided $J(\psi) < \infty$.

Proof. Let u be a weak solution to (1.4), (1.5), and $u_a \geq 0$. Let $T > a$ and $\psi \in \Psi_T$, where $J(\psi) < \infty$. By (3.1), we have

$$\int_a^T |u(t)|^p \psi(t)\, dt + \int_a^T f(t)\psi(t)\, dt + \frac{a}{\sigma}\psi(a) u_a \leq \frac{1}{\sigma} \int_a^T t^{\sigma-1} |u(t)| \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right| dt.$$

Since $u_a \geq 0$, the above inequality yields

$$\int_a^T |u(t)|^p \psi(t)\, dt + \int_a^T f(t)\psi(t)\, dt \leq \frac{1}{\sigma} \int_a^T t^{\sigma-1} |u(t)| \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right| dt. \tag{4.5}$$

On the other hand, we have by Young's inequality that

$$\begin{aligned} &\frac{1}{\sigma} \int_a^T t^{\sigma-1} |u(t)| \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right| dt \\ =& \int_a^T \left(|u(t)| \psi^{\frac{1}{p}}(t) \right) \left(\frac{1}{\sigma} t^{\sigma-1} \psi^{\frac{-1}{p}}(t) \left| \left(I_{T;\sigma,\eta+\alpha}^{1-\alpha} t^{\sigma\eta+1} (t^{1-\sigma(\eta+1)}\psi)' \right)(t) \right| \right) dt \\ \leq& \frac{1}{p} \int_a^T |u(t)|^p \psi(t)\, dt + C J(\psi), \end{aligned}$$

which implies by (4.5) that

$$\left(1 - \frac{1}{p}\right) \int_a^T |u(t)|^p \psi(t)\, dt + \int_a^T f(t)\psi(t)\, dt \leq C J(\psi).$$

Since $1 - \frac{1}{p} > 0$, the above inequality yields (4.4). □

We also have the following a priori estimate for problems (1.6) and (1.7).

Lemma 4.2. *If* (u, v) *is a weak solution to* (1.6)*,* (1.7)*, and* $u_a, v_a \geq 0$*, then*

$$(\psi(a)v_a)^{pq-1} \leq C\,[K_1(\psi)]^{q-1}\,[K_2(\psi)]^{(p-1)q} \tag{4.6}$$

and

$$(\psi(a)u_a)^{pq-1} \leq C\,[K_2(\psi)]^{p-1}\,[K_1(\psi)]^{(q-1)p}\,, \tag{4.7}$$

for all $T > a$ *and* $\psi \in \Psi_T$*, provided* $K_i(\psi) < \infty$*,* $i = 1, 2$*.*

Proof. Let (u, v) be a weak solution to (1.6) and (1.7). Let $T > a$ and $\psi \in \Psi_T$ be such that $K_i(\psi) < \infty$, $i = 1, 2$. By (3.4), we have

$$\int_a^T |v(t)|^p g(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)u_a \leq \frac{1}{\sigma}\int_a^T t^{\sigma-1}|u(t)|\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha} t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|\,dt. \tag{4.8}$$

On the other hand, by Hölder's inequality, we get

$$\begin{aligned}
&\int_a^T t^{\sigma-1}|u(t)|\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha} t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|\,dt\\
=&\int_a^T \left(|u(t)|h^{\frac{1}{q}}(t)\psi^{\frac{1}{q}}(t)\right)\left(t^{\sigma-1}h^{\frac{-1}{q}}(t)\psi^{\frac{-1}{q}}(t)\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha} t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|\right)dt\\
\leq&\left(\int_a^T |u(t)|^q h(t)\psi(t)\,dt\right)^{\frac{1}{q}}[K_1(\psi)]^{\frac{q-1}{q}}\,,
\end{aligned}$$

which implies by (4.8) that

$$\int_a^T |v(t)|^p g(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)u_a \leq \frac{1}{\sigma}\left(\int_a^T |u(t)|^q h(t)\psi(t)\,dt\right)^{\frac{1}{q}}[K_1(\psi)]^{\frac{q-1}{q}}\,. \tag{4.9}$$

Similarly, by (3.5), we have

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \leq \frac{1}{\sigma}\int_a^T t^{\sigma-1}|v(t)|\left|\left(I^{1-\beta}_{T;\sigma,\eta+\beta} t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|\,dt. \tag{4.10}$$

Making use of Hölder's inequality, we obtain

$$\int_a^T t^{\sigma-1}|v(t)|\left|\left(I^{1-\beta}_{T;\sigma,\eta+\beta} t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|\,dt \leq \left(\int_a^T |v(t)|^p g(t)\psi(t)\,dt\right)^{\frac{1}{p}}[K_2(\psi)]^{\frac{p-1}{p}}\,,$$

which implies by (4.10) that

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \leq \frac{1}{\sigma}\left(\int_a^T |v(t)|^p g(t)\psi(t)\,dt\right)^{\frac{1}{p}}[K_2(\psi)]^{\frac{p-1}{p}}\,. \tag{4.11}$$

Since $u_a \geq 0$, it follows from (4.9) that

$$\int_a^T |v(t)|^p g(t)\psi(t)\,dt \leq \frac{1}{\sigma}\left(\int_a^T |u(t)|^q h(t)\psi(t)\,dt\right)^{\frac{1}{q}}[K_1(\psi)]^{\frac{q-1}{q}}\,.$$

The above estimate together with (4.11) implies that

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \le \left(\int_a^T |u(t)|^q h(t)\psi(t)\,dt\right)^{\frac{1}{pq}} \sigma^{\frac{-(p+1)}{p}} \left[K_1(\psi)\right]^{\frac{q-1}{pq}} \left[K_2(\psi)\right]^{\frac{p-1}{p}}.$$

We now use Young's inequality to get

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \le \frac{1}{pq}\int_a^T |u(t)|^q h(t)\psi(t)\,dt + C\left(\left[K_1(\psi)\right]^{\frac{q-1}{pq}} \left[K_2(\psi)\right]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}},$$

that is,

$$\left(1-\frac{1}{pq}\right)\int_a^T |u(t)|^q h(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)v_a \le C\left(\left[K_1(\psi)\right]^{q-1} \left[K_2(\psi)\right]^{(p-1)q}\right)^{\frac{1}{pq-1}},$$

which yields (4.6). Similarly, since $v_a \ge 0$, it follows from (4.11) that

$$\int_a^T |u(t)|^q h(t)\psi(t)\,dt \le \frac{1}{\sigma}\left(\int_a^T |v(t)|^p g(t)\psi(t)\,dt\right)^{\frac{1}{p}} \left[K_2(\psi)\right]^{\frac{p-1}{p}}.$$

The above estimate together with (4.9) gives us that

$$\int_a^T |v(t)|^p g(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)u_a \le \left(\int_a^T |v(t)|^p g(t)\psi(t)\,dt\right)^{\frac{1}{pq}} \sigma^{\frac{-(q+1)}{q}} \left[K_2(\psi)\right]^{\frac{p-1}{pq}} \left[K_1(\psi)\right]^{\frac{q-1}{q}},$$

which implies by Young's inequality that

$$\left(1-\frac{1}{pq}\right)\int_a^T |v(t)|^p g(t)\psi(t)\,dt + \frac{a}{\sigma}\psi(a)u_a \le C\left(\left[K_2(\psi)\right]^{p-1} \left[K_1(\psi)\right]^{(q-1)p}\right)^{\frac{1}{pq-1}}$$

and (4.7) follows. □

For $T > a$ with $T \gg 1$ and $\lambda \gg 1$, let us consider test functions of the form

$$\psi(t) = t^{\sigma(\eta+1)-1}\varphi(t), \quad a \le t \le T, \tag{4.12}$$

where φ is the function defined by (2.1).

Lemma 4.3. *The function ψ defined by* (4.12) *belongs to ψ_T.*

Proof. The result follows immediately from (2.1) and (4.12). □

Let us now estimate the integral terms $J(\psi)$ and $K_i(\psi)$, $i = 1, 2$.

Lemma 4.4. *We have*

$$J(\psi) \le CT^{\frac{-\sigma\alpha p}{p-1}}\left(\ln T + T^{\sigma\left(\eta+1+\frac{p\alpha}{p-1}\right)}\right). \tag{4.13}$$

Proof. By (4.12), for all $a < t < T$, we have

$$t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'(t) = t^{\sigma\eta+1}\varphi'(t),$$

which implies by Lemma 2.2 with $\mu = 1-\alpha$ that

$$\begin{aligned}\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t) &= I^{1-\alpha}_{T;\sigma,\eta+\alpha}\left(t^{\sigma\eta+1}\varphi'\right)(t)\\ &= -\frac{\Gamma(\lambda+1)\sigma}{\Gamma(\lambda+\mu)}(T^\sigma-a^\sigma)^{-\lambda}(T^\sigma-t^\sigma)^{\lambda-\alpha}t^{\sigma(\eta+\alpha)}.\end{aligned} \tag{4.14}$$

Then, by (2.1), it holds that

$$\begin{aligned}&t^{\frac{(\sigma-1)p}{p-1}}\psi^{\frac{-1}{p-1}}(t)\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|^{\frac{p}{p-1}}\\ =&Ct^{\frac{(\sigma-1)p}{p-1}}t^{\frac{1-\sigma(\eta+1)}{p-1}}\varphi^{\frac{-1}{p-1}}(T^\sigma-a^\sigma)^{\frac{-\lambda p}{p-1}}(T^\sigma-t^\sigma)^{\frac{(\lambda-\alpha)p}{p-1}}t^{\frac{\sigma(\eta+\alpha)p}{p-1}}\\ =&Ct^{\frac{(\sigma-1)p}{p-1}+\frac{1-\sigma(\eta+1)}{p-1}+\frac{\sigma(\eta+\alpha)p}{p-1}}(T^\sigma-a^\sigma)^{\frac{\lambda}{p-1}}(T^\sigma-t^\sigma)^{\frac{-\lambda}{p-1}}(T^\sigma-a^\sigma)^{\frac{-\lambda p}{p-1}}(T^\sigma-t^\sigma)^{\frac{(\lambda-\alpha)p}{p-1}}\\ =&Ct^{\sigma(\eta+1)-1+\frac{\sigma p\alpha}{p-1}}(T^\sigma-a^\sigma)^{-\lambda}(T^\sigma-t^\sigma)^{\lambda-\frac{\alpha p}{p-1}}.\end{aligned}$$

Using (4.1) and integrating over (a, T), we obtain

$$\begin{aligned}J(\psi) &= \int_a^T t^{\frac{(\sigma-1)p}{p-1}}\psi^{\frac{-1}{p-1}}(t)\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|^{\frac{p}{p-1}}\,dt\\ &= C(T^\sigma-a^\sigma)^{-\lambda}\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\alpha}{p-1}}(T^\sigma-t^\sigma)^{\lambda-\frac{\alpha p}{p-1}}\,dt\\ &\le C(T^\sigma-a^\sigma)^{\frac{-\alpha p}{p-1}}\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\alpha}{p-1}}\,dt\\ &\le CT^{\frac{-\sigma\alpha p}{p-1}}\left(\ln T+T^{\sigma\left(\eta+1+\frac{p\alpha}{p-1}\right)}\right),\end{aligned}$$

which proves (4.13). ◻

Lemma 4.5. *Assume that* $h^{\frac{-1}{q-1}}\in L^1_{\text{loc}}([a,\infty))$. *We have*

$$K_1(\psi)\le C(T^\sigma-a^\sigma)^{\frac{-\alpha q}{q-1}}\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}h^{\frac{-1}{q-1}}(t)\,dt. \tag{4.15}$$

Proof. By (4.14), it holds that

$$\begin{aligned}&t^{\frac{(\sigma-1)q}{q-1}}h^{\frac{-1}{q-1}}(t)\psi^{\frac{-1}{q-1}}(t)\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|^{\frac{q}{q-1}}\\ =&Ch^{\frac{-1}{q-1}}(t)\left[t^{\frac{(\sigma-1)q}{q-1}}t^{\frac{1-\sigma(\eta+1)}{q-1}}\varphi^{\frac{-1}{q-1}}(T^\sigma-a^\sigma)^{\frac{-\lambda q}{q-1}}(T^\sigma-t^\sigma)^{\frac{(\lambda-\alpha)q}{q-1}}t^{\frac{\sigma(\eta+\alpha)q}{q-1}}\right]\\ =&Ch^{\frac{-1}{q-1}}(t)\left[t^{\frac{(\sigma-1)q}{q-1}+\frac{1-\sigma(\eta+1)}{q-1}+\frac{\sigma(\eta+\alpha)q}{q-1}}(T^\sigma-a^\sigma)^{\frac{\lambda}{q-1}}(T^\sigma-t^\sigma)^{\frac{-\lambda}{q-1}}(T^\sigma-a^\sigma)^{\frac{-\lambda q}{q-1}}(T^\sigma-t^\sigma)^{\frac{(\lambda-\alpha)q}{q-1}}\right]\\ =&Ch^{\frac{-1}{q-1}}(t)t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}(T^\sigma-a^\sigma)^{-\lambda}(T^\sigma-t^\sigma)^{\lambda-\frac{\alpha q}{q-1}}.\end{aligned}$$

Using (4.2) and integrating over (a, T), we obtain

$$\begin{aligned}K_1(\psi) &= \int_a^T t^{\frac{(\sigma-1)q}{q-1}}h^{\frac{-1}{q-1}}(t)\psi^{\frac{-1}{q-1}}(t)\left|\left(I^{1-\alpha}_{T;\sigma,\eta+\alpha}t^{\sigma\eta+1}(t^{1-\sigma(\eta+1)}\psi)'\right)(t)\right|^{\frac{q}{q-1}}\,dt\\ &= C(T^\sigma-a^\sigma)^{-\lambda}\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}h^{\frac{-1}{q-1}}(t)(T^\sigma-t^\sigma)^{\lambda-\frac{\alpha q}{q-1}}\,dt\\ &\le C(T^\sigma-a^\sigma)^{\frac{-\alpha q}{q-1}}\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}h^{\frac{-1}{q-1}}(t)\,dt,\end{aligned}$$

which proves (4.15). □

Similarly, by (2.1), (4.3), and (4.12), we obtain the following estimate of $K_2(\psi)$.

Lemma 4.6. *Assume that* $g^{\frac{-1}{p-1}} \in L^1_{\text{loc}}([a,\infty))$. *We have*

$$K_2(\psi) \le C(T^\sigma - a^\sigma)^{\frac{-\beta p}{p-1}} \int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt.$$

The following estimates follow immediately from Lemmas 4.5 and 4.6.

Lemma 4.7. *Assume that* $h^{\frac{-1}{q-1}} \in L^1_{\text{loc}}([a,\infty))$ *and* $g^{\frac{-1}{p-1}} \in L^1_{\text{loc}}([a,\infty))$. *We have*

$$[K_1(\psi)]^{q-1}\,[K_2(\psi)]^{(p-1)q} \le CT^{-\sigma q(\alpha+\beta p)} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{q-1} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{(p-1)q}$$

and

$$[K_2(\psi)]^{p-1}\,[K_1(\psi))^{(q-1)p} \le CT^{-\sigma p(\beta+\alpha p)} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{p-1} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{(q-1)p}.$$

5. Proofs of the obtained results

This section is devoted to the proofs of Theorems 3.2 and 3.6, and Corollary 3.8.

Proof of Theorem 3.2. We use a contradiction argument. Namely, suppose that u is a weak solution to (1.4) and (1.5). By Lemmas 4.1 and 4.5, we have

$$\int_a^T f(t)\psi(t)\,dt \le CJ(\psi), \tag{5.1}$$

where for $T, \lambda \gg 1$, the function ψ is defined by (4.12). On the other hand, by (2.1), (3.2), and (4.12), we have

$$\begin{aligned}\int_a^T f(t)\psi(t)\,dt &= (T^\sigma - a^\sigma)^{-\lambda} \int_a^T f(t) t^{\sigma(\eta+1)-1}(T^\sigma - t^\sigma)^\lambda\,dt \\ &\ge C_f (T^\sigma - a^\sigma)^{-\lambda} \int_a^T t^{-\sigma\eta}(t^\sigma - a^\sigma)^\gamma t^{\sigma(\eta+1)-1}(T^\sigma - t^\sigma)^\lambda\,dt \\ &= C_f (T^\sigma - a^\sigma)^{-\lambda} \int_a^T (t^\sigma - a^\sigma)^\gamma (T^\sigma - t^\sigma)^\lambda t^{\sigma-1}\,dt.\end{aligned} \tag{5.2}$$

Furthermore, we have

$$\begin{aligned}&(T^\sigma - a^\sigma)^{-\lambda} \int_a^T (t^\sigma - a^\sigma)^\gamma (T^\sigma - t^\sigma)^\lambda t^{\sigma-1}\,dt \\ =&(T^\sigma - a^\sigma)^{-\lambda} \int_a^T (t^\sigma - a^\sigma)^\gamma (T^\sigma - t^\sigma)^\lambda t^{\sigma-1}\,dt \\ =&(T^\sigma - a^\sigma)^{-\lambda} \int_a^T (t^\sigma - a^\sigma)^\gamma \left[(T^\sigma - a^\sigma) - (t^\sigma - a^\sigma)\right]^\lambda t^{\sigma-1}\,dt \\ =&\int_a^T (t^\sigma - a^\sigma)^\gamma \left(1 - \frac{t^\sigma - a^\sigma}{T^\sigma - a^\sigma}\right)^\lambda t^{\sigma-1}\,dt.\end{aligned}$$

Making the change of variable $s = \dfrac{t^\sigma - a^\sigma}{T^\sigma - a^\sigma}$ and using that $\gamma > -1$ (by (3.3)), we obtain

$$\begin{aligned}&(T^\sigma - a^\sigma)^{-\lambda} \int_a^T (t^\sigma - a^\sigma)^\gamma (T^\sigma - t^\sigma)^\lambda t^{\sigma-1}\, dt\\ =&\frac{1}{\sigma}(T^\sigma - a^\sigma)^{\gamma+1} \int_0^1 s^{(\gamma+1)-1}(1-s)^{(\lambda+1)-1}\\ =&\frac{1}{\sigma}(T^\sigma - a^\sigma)^{\gamma+1} B(\gamma+1, \lambda+1),\end{aligned}$$

where B is the beta function. Hence, by (5.2), we have

$$\begin{aligned}\int_a^T f(t)\psi(t)\, dt &\geq C(T^\sigma - a^\sigma)^{\gamma+1}\\ &\geq CT^{\sigma(\gamma+1)}.\end{aligned} \tag{5.3}$$

We now use Lemma 4.4, (5.1), and (5.3) to get

$$T^{\sigma(\gamma+1)} \leq CT^{\frac{-\sigma\alpha p}{p-1}} \left(\ln T + T^{\sigma\left(\eta+1+\frac{p\alpha}{p-1}\right)}\right),$$

that is,

$$1 \leq C\left(T^{\tau_1} \ln T + T^{\tau_2}\right), \tag{5.4}$$

where

$$\tau_1 = -\sigma\left(\frac{\alpha p}{p-1} + (\gamma+1)\right)$$

and

$$\tau_2 = -\sigma(\gamma - \eta).$$

Note that due to (3.3), we have $\tau_i < 0$, $i = 1, 2$. Hence, passing to the limit as $T \to \infty$ in (5.4) , we reach a contradiction. This completes the proof of Theorem 3.2. □

Proof of Theorem 3.6. We also use a contradiction argument. Namely, suppose that (u, v) is a weak solution to (1.6) and (1.7).

We first consider the case (i). By Lemma 4.2, for all $T > a$ and $\psi \in \Psi_T$, we have

$$(\psi(a)v_a)^{pq-1} \leq C\left[K_1(\psi)\right]^{q-1}\left[K_2(\psi)\right]^{(p-1)q}, \tag{5.5}$$

provided $K_i(\psi) < \infty$, $i = 1, 2$. In particular, since $g^{\frac{-1}{p-1}}, h^{\frac{-1}{q-1}} \in L^1_{\text{loc}}([a, \infty))$, then by Lemmas 4.3, 4.5, and 4.6, (5.5) holds for the function ψ defined by (4.12). Since $\psi(a) = a^{\sigma(\eta+1)-1} > 0$, then (5.5) reduces to

$$v_a^{pq-1} \leq C\left[K_1(\psi)\right]^{q-1}\left[K_2(\psi)\right]^{(p-1)q},$$

which implies by the first estimate in Lemma 4.7 that

$$v_a^{pq-1} \leq CT^{-\sigma q(\alpha+\beta p)} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\, dt\right)^{q-1} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\, dt\right)^{(p-1)q}.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality and using (3.6), we obtain (recall that $v_a \geq 0$)

$$v_a^{pq-1} = 0,$$

which contradicts the fact that $v_a > 0$.

Consider now the case (ii). Similarly to the previous case, we obtain by Lemmas 4.2, 4.3, 4.5, and 4.6 that

$$u_a^{pq-1} \leq C\,[K_2(\psi)]^{p-1}\,[K_1(\psi))^{(q-1)p},$$

where ψ is defined by (4.12). Then, from the second estimate in Lemma 4.7, we deduce that

$$u_a^{pq-1} \leq CT^{-\sigma p(\beta+\alpha p)} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{p-1} \left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{(q-1)p}.$$

Passing to the infimum limit as $T \to \infty$ in the above inequality and using (3.7), we obtain (recall that $u_a \geq 0$)

$$u_a^{pq-1} = 0,$$

which contradicts the fact that $u_a > 0$.

Hence, in both cases (i) and (ii), we reach a contradiction. This completes the proof of Theorem 3.6. □

Proof of Corollary 3.8. We only give the proof of the case (i). The proof of the case (ii) follows using a similar argument. By the definition of h, for $T \gg 1$, we have

$$\begin{aligned}
\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt &= \int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} (t-a)^{\frac{\rho}{q-1}}(t)\,dt \\
&\leq (T-a)^{\frac{\rho}{q-1}} \int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}}\,dt \\
&\leq CT^{\frac{\rho}{q-1}}\left(\ln T + T^{\sigma\left(\eta+1+\frac{q\alpha}{q-1}\right)}\right),
\end{aligned}$$

which implies that

$$\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{q-1} \leq C\left(T^{\rho}(\ln T)^{q-1} + T^{\rho+\sigma((\eta+1)(q-1)+q\alpha)}\right). \tag{5.6}$$

Similarly, by the definition of g, we have

$$\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{(p-1)q} \leq C\left(T^{\gamma q}(\ln T)^{(p-1)q} + T^{\gamma q+\sigma((\eta+1)(p-1)q+pq\beta)}\right). \tag{5.7}$$

Then, it follows from (5.6) and (5.7) that

$$\begin{aligned}
&T^{-\sigma q(\alpha+\beta p)}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{q-1}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{(p-1)q} \\
\leq{}& C\left(T^{\eta_1}(\ln T)^{pq-1} + T^{\eta_2}(\ln T)^{q-1} + T^{\eta_3}(\ln T)^{(p-1)q} + T^{\eta_4}\right),
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}\eta_1 &= \gamma q+\rho-\sigma q(\alpha+\beta p),\\ \eta_2 &= \gamma q+\rho+\sigma q\left[(\eta+1)(p-1)-\alpha\right],\\ \eta_3 &= \gamma q+\rho+\sigma\left[q(\eta+1-\beta p)-\eta-1\right],\\ \eta_4 &= \gamma q+\rho+\sigma(\eta+1)(pq-1).\end{aligned}$$

Observe that from (3.9), we have $\eta_i<0$, $i=1,2,3,4$. Hence, from (5.8), we deduce that

$$\lim_{T\to\infty} T^{-\sigma q(\alpha+\beta p)}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma q\alpha}{q-1}} h^{\frac{-1}{q-1}}(t)\,dt\right)^{q-1}\left(\int_a^T t^{\sigma(\eta+1)-1+\frac{\sigma p\beta}{p-1}} g^{\frac{-1}{p-1}}(t)\,dt\right)^{(p-1)q}=0,$$

which shows that (3.6) is satisfied. Then, Theorem 3.6 applies. □

6. Conclusions

Using nonlinear capacity estimates, sufficient conditions for the nonexistence of weak solutions were obtained for the inhomogeneous Erdélyi-Kober fractional differential inequality (1.4) subject to the initial condition (1.5) (see Theorem 3.2) and the system of Erdélyi-Kober fractional differential inequalities (1.6) under the initial conditions (1.7) (see Theorem 3.6). By comparing Theorem 3.2 with the recent result obtained in [21] for the homogeneous problem (1.2) with $V\equiv 1$, we observe that, if the inhomogeneous term f satisfies (3.2) and (3.3), then the nonexistence holds for every $p>1$. However, in the homogeneous case, the nonexistence holds for a certain range of p. Furthermore, Theorem 3.6 recovers the nonexistence result established in [21] (See Remark 3.7).

In this paper, we only studied the nonexistence of solutions to the considered problems. It would be interesting to extend this study in order to get sufficient conditions for the existence of solutions. We hope that in a future work, this question will be solved.

Author contributions

Mohamed Jleli: Conceptualization, methodology, investigation, formal analysis; Bessem Samet: Conceptualization, methodology, validation, investigation, writing review and editing. All authors have read and approved the final draft of the paper for publication.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author is supported by Researchers Supporting Project number (RSP2024R57), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no competing interests.

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