AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 21672–21685.
DOI: 10.3934/math.20241054
Received: 05 March 2024
Revised: 05 June 2024
Accepted: 20 June 2024
Published: 08 July 2024

Research article

Revisiting the m-weak core inverse

Jinyong Wu[1], Wenjie Shi[2] and Sanzhang Xu[2,*]

[1] School of Mathematics and Physics, Wenzhou University, Wenzhou 325035, China
[2] Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian, 223003, China

* **Correspondence:** Email: xusanzhang5222@126.com.

Abstract: The m-weak core inverse of a complex matrix was introduced by D. E. Ferreyra and Saroj B. Malik in 2024. We have revisited this inverse by using the inverse along two matrices, that is, we have proved that the m-weak core inverse of a complex matrix coincides with the inverse along two complex matrices. Moreover, the necessary and sufficient conditions of the m-weak core inverse of a complex matrix have been obtained. The one-sided m-weak core inverse has been introduced by using the core-EP (EP means Equal Prohection) inverse of A.

Keywords: the inverse along two matrices; index; the m-weak core inverse; core-EP inverse; complex matrix
Mathematics Subject Classification: 15A09

1. Introduction

The symbol $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ matrices over the complex field $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Let $\mathbb{C}_n^{CM} = \{A \in \mathbb{C}^{n\times n} \mid \operatorname{rank}(A^2) = \operatorname{rank} A\}$. The column subspace of A is $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$, and the null subspace of A is $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$. If there exists a smallest positive integer $k \in \mathbb{Z}$ such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ holds, this integer is called the index of $A \in \mathbb{C}^{n\times n}$ with the symbol $\operatorname{ind}(A)$. A complex matrix A is called normal if $AA^* = A^*A$, where $A \in \mathbb{C}^{n\times n}$.

Let $A, X \in \mathbb{C}^{m\times n}$. If $AXA = A$ and $XAX = X$, where AX and XA are Hermitian, we call the matrix X is the Moore-Penrose inverse of A [12, 16] and using the symbol $A^\dagger$ denotes the Moore-Penrose inverse of A. Let $A, X \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The algebraic definition of the Drazin inverse is as follows: If

$$AXA = A,\ XA^{k+1} = A^k, \text{ and } AX = XA,$$

then X is called a Drazin inverse of A. It is unique and denoted by A^D [6]. Note that for a square complex matrix, the algebraic definition of the Drazin inverse is equivalent to the functional definition of the Drazin inverse. If $\mathrm{ind}(A) = 1$, the Drazin inverse is called the group inverse of A and denoted by $A^{\#}$. Let $A \in \mathbb{C}^{n\times n}$. The DMP (Drazin Moore-Penrose) inverse of A was introduced by using the Drazin and the Moore-Penrose inverses of A in [15], and the formula of the DMP inverse of A is $A^{D,\dagger} = A^D AA^\dagger$ [15, Theorem 2.2]. Manjunatha Prasad and Mohana [13] introduced the core-EP inverse of a matrix[13, Definition 3.1]. Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\textcircled{\dagger}}$. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. The m-weak group inverse was introduced by Zhou, Chen, and Zhou in [22]. A matrix $X \in \mathbb{C}^{n\times n}$ is called the m-weak group inverse of A if $XA^{k+1} = A^k$, $AX^2 = X$, $(A^*)^k A^{m+1} X = (A^*)^k A^m$ for $m \in \mathbf{Z}$. In [18, Theorem 2.1], Wang introduced a new matrix decomposition, namely, the core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Given a matrix $A \in \mathbb{C}^{n\times n}$, A can be written as the sum of matrices $A_1 \in \mathbb{C}^{n\times n}$ and $A_2 \in \mathbb{C}^{n\times n}$, that is, $A = A_1 + A_2$, where $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$, and $A_1^* A_2 = A_2 A_1 = 0$. In [18, Theorem 2.3 and Theorem 2.4], Wang proved this matrix decomposition is unique, and there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{1.1}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular, and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent with $\mathrm{rank}(A^k) = r$. In [18, Theorem 2.3], Wang proved that A_1 can be described by using the Moore-Penrose inverse of A^k. The explicit expressions of A_1 can be found in the following lemma.

Lemma 1.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $A = A_1 + A_2$ is the core-EP decomposition of A, then $A_1 = A^k(A^k)^\dagger A$.*

Definition 1.1. *Let $A, B, C \in \mathbb{C}^{n\times n}$. A matrix $Y \in \mathbb{C}^{n\times n}$ is the inverse along B and C of A if we have*

$$YAB = B,\ CAY = C,\ \mathcal{N}(C) \subseteq \mathcal{N}(Y) \text{ and } \mathcal{R}(Y) \subseteq \mathcal{R}(B).$$

If such Y exists, then it is unique (see [1, Definition 4.1] and [17, Definition 1.2]). In [8, Definition 1.2] and [11, Definition 2.1], the authors introduced the one-sided (b, c)-inverse in rings. In [1, Definition 2.7], the authors introduced the one-sided (B, C)-inverse for complex matrices. Let $A, B, C \in \mathbb{C}^{n\times n}$. We say that $X \in \mathbb{C}^{n\times n}$ is a left (B, C)-inverse of A if we have $\mathcal{N}(C) \subseteq \mathcal{N}(X)$ and $XAB = B$. We say that $Y \in \mathbb{C}^{n\times n}$ is a right (B, C)-inverse of A if we have $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$ and $CAY = C$.

The m-weak core inverse was introduced by Ferreyra and Malik in [10], and this inverse can be introduced by using the m-weak group inverse.

2. When the m-weak core inverse is an inverse along two matrices

The relationships of the core inverse, DMP (Drazin Moore-Penrose) inverse, core-EP inverse, WG (weak group) inverse, WC (weak core)inverse, m-weak group inverse, and m-weak core inverse can be explained as in the following picture, Figure 1.

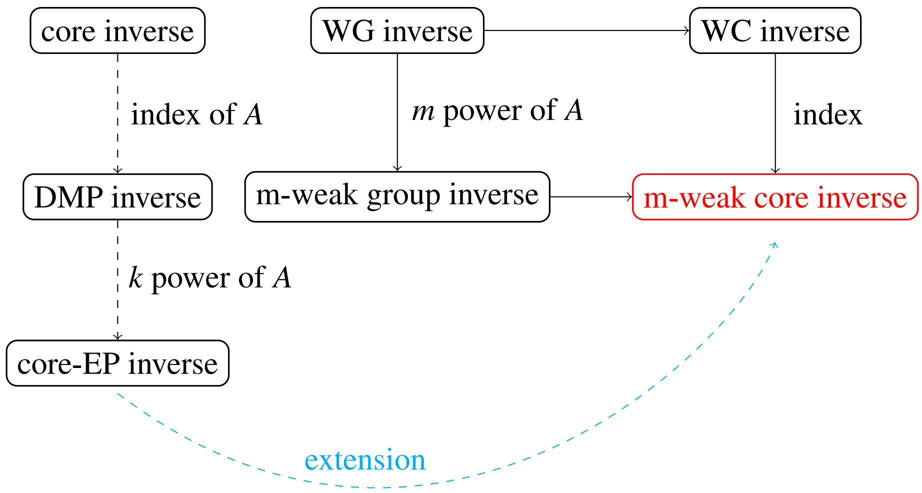

Figure 1. Relationships of several generalized inverses.

The m-weak core inverse coincides with the WC inverse if $m = 1$, and the m-weak core inverse coincides with the core-EP inverse if $m \geqslant k$ by [10, Remark 4.2]. Thus, we assume that $2 \leqslant m < k$.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k, m \in \mathbb{Z}$, and k is the index of A. If $2 \leqslant m < k$, and A^m is normal, then $\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left((A^k)^* A^m\right)$.*

Proof. If $2 \leqslant m < k$ and A^m is normal, then

$$\begin{aligned}
&\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left((A^m A^{k-m})^* A^{2m}(A^m)^\dagger\right)\\
&= \mathcal{N}\left((A^{k-m})^*(A^m)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left((A^{k-m})^*(A^m)^* A^m A^m (A^m)^\dagger\right)\\
&= \mathcal{N}\left((A^{k-m})^*((A^m)^* A^m)^*(A^m(A^m)^\dagger)^*\right)\\
&= \mathcal{N}\left((A^{k-m})^*(A^m(A^m)^\dagger(A^m)^* A^m)^*\right)\\
&= \mathcal{N}\left((A^{k-m})^*(A^m(A^m)^\dagger A^m(A^m)^*)^*\right)\\
&= \mathcal{N}\left((A^{k-m})^*(A^m(A^m)^*)^*\right) = \mathcal{N}\left((A^{k-m})^*((A^m)^* A^m)^*\right)\\
&= \mathcal{N}\left(((A^m)^* A^m A^{k-m})^*\right) = \mathcal{N}\left(((A^m)^* A^k)^*\right) = \mathcal{N}\left((A^k)^* A^m\right).
\end{aligned} \tag{2.1}$$

□

The following counterexample shows that if A^m is not a normal matrix, then $\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left((A^k)^* A^m\right)$ does not hold in general. Note that the precondition is $2 \leqslant m < k$, so we start the following example by using a 4×4 matrix with $\mathrm{ind}(A) = 3$ and $m = 2$.

Example 2.1. *Let* $A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. *It is easy to check that* ind(A) = 3. *Then, we have* $(A^3)^*A^4(A^2)^\dagger = \begin{bmatrix} -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \\ -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \\ -\frac{162}{5} & -\frac{81}{5} & \frac{162}{5} & -\frac{81}{5} \\ -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \end{bmatrix}$ *and* $(A^3)^*A^2 = \begin{bmatrix} -43 & 27 & 11 & 27 \\ -43 & 27 & 11 & 27 \\ -129 & 81 & 33 & 81 \\ -43 & 27 & 11 & 27 \end{bmatrix}$. *So,* $(A^3)^*A^4(A^2)^\dagger \left((A^3)^*A^2\right)^\dagger (A^3)^*A^2 = \begin{bmatrix} -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \\ -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \\ -\frac{162}{5} & -\frac{81}{5} & \frac{162}{5} & -\frac{81}{5} \\ -\frac{54}{5} & -\frac{27}{5} & \frac{54}{5} & -\frac{27}{5} \end{bmatrix}$, *which says that* $(A^3)^*A^4(A^2)^\dagger \left((A^3)^*A^2\right)^\dagger (A^3)^*A^2 \neq (A^3)^*A^4(A^2)$, *that is,* $\mathcal{N}\left((A^3)^*A^4(A^2)^\dagger\right) \supseteq \mathcal{N}\left((A^3)^*A^2\right)$ *does not hold in general. Note that the condition* $\mathcal{N}\left((A^3)^*A^4(A^2)^\dagger\right) \supseteq \mathcal{N}\left((A^3)^*A^2\right)$ *if and only if* $(A^3)^*A^4 (A^2)^\dagger \left((A^3)^*A^2\right)^\dagger (A^3)^*A^2 = (A^3)^*A^4(A^2)^\dagger$.

Lemma 2.2. *Let* $k, m \in \mathbb{Z}$. *Then, the conditions* $2 \leqslant m < k$ *and* $2m > k+1$ *are equivalent to* $m < k < 2m-1$.

Proof. " $\Leftarrow$ " If $m < k < 2m-1$, then $m < 2m-1$, that is, $m > 1$, which implies $m \geqslant 2$ by $m \in \mathbb{Z}$. The opposite is trivial. □

The following lemma will be used several times in the sequel.

Lemma 2.3. *Let* $A \in \mathbb{C}^{n\times n}$, $k, m \in \mathbb{Z}$, *and* k *is the index of* A. *Then,* $\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right)$.

Proof.

$$\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^k(A^k)^\dagger A^{2m}(A^m)^\dagger\right) \tag{2.2}$$

The equality (2.2) is equivalent to the following equality by $A^k(A^k)^\dagger = AA^{\textcircled{\dagger}}$.

$$\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(AA^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) \tag{2.3}$$

As $A^{\textcircled{\dagger}}$ is an outer inverse of A, we have

$$\mathcal{N}\left(AA^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) \tag{2.4}$$

The proof is completed by equality (2.3) and equality (2.4). □

From the proof of the above lemma we have the following lemma.

Lemma 2.4. *Let* $A \in \mathbb{C}^{n\times n}$, $k, m \in \mathbb{Z}$, *and* k *is the index of* A. *Then,* $\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A_1A^{2m-1}(A^m)^\dagger\right)$, *where* A_1 *is the core part of the core-EP decomposition.*

Proof. By the proof of Lemma 2.3, we have

$$\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(AA^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right)$$

Note that $\mathcal{N}\left(AA^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A_1A^{2m-1}(A^m)^\dagger\right)$ by $A_1 = AA^{\textcircled{\dagger}}A$. Thus, $\mathcal{N}\left((A^k)^*A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(AA^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A_1A^{2m-1}(A^m)^\dagger\right)$ by Lemma 1.1. □

Lemma 2.5. *Let $A \in \mathbb{C}^{n\times n}$, $k, m \in \mathbb{Z}$, and k is the index of A. If $m < k < 2m-1$, then $\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right)$.*

Proof. By the proof of Lemma 2.3, we have

$$\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^{\textcircled{\dagger}} A^{2m}(A^m)^\dagger\right) \tag{2.5}$$

Note that $\mathcal{N}\left(A^{\textcircled{\dagger}} A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^{\textcircled{\dagger}} A^{k+1} A^{2m-k-1}(A^m)^\dagger\right)$ by $m < k < 2m-1$. So, by $A^{\textcircled{\dagger}} A^{k+1} = A^k$, we have

$$\mathcal{N}\left(A^{\textcircled{\dagger}} A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^k A^{2m-k-1}(A^m)^\dagger\right) = \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right) \tag{2.6}$$

Thus, the proof is completed by (2.5) and (2.6). $\square$

Lemma 2.6. *Let $A \in \mathbb{C}^{n\times n}$, $k, m \in \mathbb{Z}$, and k is the index of A. If $m < k < 2m-1$, then $\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^k(A^m)^\dagger\right)$.*

Proof. By Lemma 2.5, now, we just need to show the following equation:

$$\mathcal{N}\left(A^k(A^m)^\dagger\right) = \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right) \tag{2.7}$$

For any $u \in \mathcal{N}\left(A^k(A^m)^\dagger\right)$, we have

$$A^{2m-1}(A^m)^\dagger u = A^{2m-k-1}A^k(A^m)^\dagger u = 0,$$

which says that

$$\mathcal{N}\left(A^k(A^m)^\dagger\right) \subseteq \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right). \tag{2.8}$$

For any $v \in \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right)$, we have

$$\begin{aligned} A^k(A^m)^\dagger v &= A^D A^{k+1}(A^m)^\dagger v = A^D A A^D A^{k+1}(A^m)^\dagger v \\ &= (A^D)^2 A^{k+2}(A^m)^\dagger v = A^D(A^D A A^D)A^{k+2}(A^m)^\dagger v \\ &= (A^D)^3 A^{k+3}(A^m)^\dagger v \\ &= \cdots \\ &= (A^D)^{2m-1-k} A^{2m-1}(A^m)^\dagger v \\ &= 0 \end{aligned}$$

which says that

$$\mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right) \subseteq \mathcal{N}\left(A^k(A^m)^\dagger\right). \tag{2.9}$$

Thus, $\mathcal{N}\left(A^k(A^m)^\dagger\right) = \mathcal{N}\left(A^{2m-1}(A^m)^\dagger\right)$ holds by (2.8) and (2.9). $\square$

The following counterexample shows that if $m < k < 2m-1$ does not hold, then $\mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) = \mathcal{N}\left(A^k(A^m)^\dagger\right)$ does not hold in general. Note that the precondition is $2 \leqslant m < k$ and $2m > k+1$, so we start the following example by assuming that $k > 2m-1$ and $2 \leqslant m < k$. For example, let $m = 2$, and then $2m-1 = 3$. Thus, we can choose $k = 4$ and let the related matrix be a 5×5 matrix.

Example 2.2. *Let* $A = \begin{bmatrix} 2 & 2 & -2 & -1 & 1 \\ -1 & 2 & -2 & 0 & 0 \\ -1 & 2 & -2 & -2 & -1 \\ -2 & -2 & 2 & -1 & -2 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{C}^{5\times5}$. *It is easy to check that* $\operatorname{ind}(A) = 4$ *and* $m = 2$.

Then, the following equality is obvious: $(A^4)^*A^4(A^2)^\dagger = \begin{bmatrix} \frac{702}{7} & \frac{702}{7} & -\frac{702}{7} & -\frac{2106}{7} & \frac{2106}{7} \\ \frac{702}{7} & \frac{702}{7} & -\frac{702}{7} & -\frac{2106}{7} & \frac{2106}{7} \\ -\frac{702}{7} & -\frac{702}{7} & \frac{702}{7} & \frac{2106}{7} & -\frac{2106}{7} \\ -\frac{351}{7} & -\frac{351}{7} & \frac{351}{7} & \frac{1053}{7} & -\frac{1053}{7} \\ \frac{2016}{7} & \frac{2016}{7} & -\frac{2016}{7} & -\frac{6318}{7} & \frac{6318}{7} \end{bmatrix}$. *Also we check that* $A^4(A^2)^\dagger = \begin{bmatrix} 216 & 216 & -216 & -108 & 648 \\ -108 & -108 & 108 & 54 & -324 \\ -54 & -54 & 54 & 27 & -162 \\ -162 & -162 & 162 & 81 & -486 \\ 162 & 162 & -162 & -81 & 486 \end{bmatrix}$. *So,*

$$(A^4)^*A^4(A^2)^\dagger \left(A^4(A^2)^\dagger\right)^\dagger A^4(A^2)^\dagger = \begin{bmatrix} \frac{8731}{79} & \frac{8731}{79} & -\frac{8731}{79} & -\frac{10223}{185} & \frac{29177}{88} \\ \frac{8731}{79} & \frac{8731}{79} & -\frac{8731}{79} & -\frac{10223}{185} & \frac{29177}{88} \\ -\frac{8731}{79} & -\frac{8731}{79} & \frac{8731}{79} & \frac{10223}{185} & -\frac{29177}{88} \\ -\frac{10223}{185} & -\frac{10223}{185} & \frac{10223}{185} & \frac{9477}{343} & -\frac{29177}{176} \\ \frac{29177}{88} & \frac{29177}{88} & -\frac{29177}{88} & -\frac{29177}{176} & \frac{84547}{85} \end{bmatrix},$$

which says that $(A^4)^*A^4(A^2)^\dagger \left((A^4)^*A^2\right)^\dagger A^4(A^2)^\dagger \neq (A^4)^*A^4(A^2)^\dagger$, *that is, the implication* $\mathcal{N}\left((A^4)^*A^4(A^2)^\dagger\right) \supseteq \mathcal{N}\left(A^4(A^2)^\dagger\right)$ *does not hold in general.*

Lemma 2.7. [18, Corollary 3.3] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *Then,* $AA^{\textcircled{\dagger}} = A^k(A^k)^\dagger$.

In the following theorem, we will show that the m-weak core inverse can be expressed by using the core-EP inverse. This theorem is one of the main results in this paper.

Theorem 2.8. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$ *and* $m \in \mathbb{Z}$. *Then, X is the m-weak core inverse of A if and only if both* $AX = (A^{\textcircled{\dagger}})^m A^{2m}(A^m)^\dagger$ *and* $X = AA^{\textcircled{\dagger}}X$ *hold.*

Proof. " $\Rightarrow$ " Let X be the m-weak core inverse of A. Then, we have $AX = (A^{\textcircled{\dagger}})^m A^{2m}(A^m)^\dagger$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. So, $X = A^kU$ for some $U \in \mathbb{C}^{n\times n}$ by $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Hence,

$$X = A^kU = AA^{\textcircled{\dagger}}X,$$

by Lemma 2.7.

" $\Leftarrow$ " $X = AA^{\textcircled{\dagger}}X = A^k(A^k)^\dagger X$ implies $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, so, X is the m-weak core inverse of A by [10, Theorem 4.1]. □

Lemma 2.9. [18, Theorem 3.2] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *If* $A = A_1 + A_2$ *is the core-EP decomposition of A with* A_1, A_2 *as in* (1.1), *then,* $A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$, *where U and T same as* (1.1).

Lemma 2.10. [10, Theorem 4.9] *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. Then, $AA^{\textcircled{w}_m} = (A^{\textcircled{\dagger}})^m A^{2m}(A^m)^{\dagger}$.*

The following theorem is also one of the main results in this paper.

Theorem 2.11. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. Then, the m-weak core inverse of A is the inverse along A^k and $(A^k)^*A^{2m}(A^m)^{\dagger}$.*

Proof. Let X be the m-weak core inverse of A. Then, $XAA^k = XA^{k+1} = A^k$ by [10, Theorem 4.7 (d)]. Let $A = A_1 + A_2$ be the core-EP decomposition of A, where $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$, and $A_1^*A_2 = A_2A_1 = 0$. By [18, Theorem 2.3 and Theorem 2.4], there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{2.10}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular, and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent. Then,

$$A^m = U\begin{bmatrix} T^m & \Phi_m \\ 0 & N^m \end{bmatrix}U^*, \tag{2.11}$$

where $\Phi_m = \sum_{j=1}^{m} T^{j-1}SN^{m-j}$. By (3.5) in [19], we have

$$A^k = U\begin{bmatrix} T^k & \Phi_k \\ 0 & 0 \end{bmatrix}U^*, \tag{2.12}$$

where $\Phi_k = \sum_{j=1}^{k} T^{j-1}SN^{k-j}$. By (2.12), we have

$$(A^k)^* = U\begin{bmatrix} (T^k)^* & 0 \\ (\Phi_k)^* & 0 \end{bmatrix}U^*. \tag{2.13}$$

By [10, Remark 3.2 (3.5)], we have

$$A^m(A^m)^{\dagger} = U\begin{bmatrix} E_r & 0 \\ 0 & N^m(N^m)^{\dagger} \end{bmatrix}U^*. \tag{2.14}$$

By (2.11), (2.13), and (2.14), we have

$$\begin{aligned} &(A^k)^*A^{2m}(A^m)^{\dagger} = (A^k)^*A^mA^m(A^m)^{\dagger} \\ &= U\begin{bmatrix} (T^k)^* & 0 \\ (\Phi_k)^* & 0 \end{bmatrix}\begin{bmatrix} T^m & \Phi_m \\ 0 & N^m \end{bmatrix}\begin{bmatrix} E_r & 0 \\ 0 & N^m(N^m)^{\dagger} \end{bmatrix}U^* \\ &= U\begin{bmatrix} (T^k)^*T^m & (T^k)^*\Phi_mN^m(N^m)^{\dagger} \\ (\Phi_k)^*T^m & (\Phi_k)^*\Phi_mN^m(N^m)^{\dagger} \end{bmatrix}U^*. \end{aligned} \tag{2.15}$$

By Lemma 2.9, Lemma 2.10, and (2.15), we have

$$\begin{aligned}
&(A^k)^*A^{2m}(A^m)^{\dagger}AX = (A^k)^*A^{2m}(A^m)^{\dagger}(A^{\textcircled{\dagger}})^mA^{2m}(A^m)^{\dagger}\\
&= U\begin{bmatrix} (T^k)^*T^m & (T^k)^*\Phi_mN^m(N^m)^{\dagger} \\ (\Phi_k)^*T^m & (\Phi_k)^*\Phi_mN^m(N^m)^{\dagger} \end{bmatrix}\begin{bmatrix} T^{-m} & 0 \\ 0 & 0 \end{bmatrix}\\
&\begin{bmatrix} T^m & \Phi_m \\ 0 & N^m \end{bmatrix}\begin{bmatrix} E_r & 0 \\ 0 & N^m(N^m)^{\dagger} \end{bmatrix}U^* \qquad (2.16)\\
&= U\begin{bmatrix} (T^k)^*T^m & (T^k)^*\Phi_mN^m(N^m)^{\dagger} \\ (\Phi_k)^*T^m & (\Phi_k)^*\Phi_mN^m(N^m)^{\dagger} \end{bmatrix}U^*.
\end{aligned}$$

By (2.15) and (2.16), we have

$$(A^k)^*A^{2m}(A^m)^{\dagger}AX = (A^k)^*A^{2m}(A^m)^{\dagger}. \qquad (2.17)$$

By [10, Theorem 4.7], we have $\mathcal{N}((A^k)^*A^{2m}(A^m)^{\dagger}) \subseteq \mathcal{N}(X)$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Thus, the m-weak core inverse of A is the inverse along A^k and $(A^k)^*A^{2m}(A^m)^{\dagger}$ by $XA^{k+1} = A^k$, (2.17), and Definition 1.1. □

Definition 2.1. [20, Definition 3.1] *Let $A \in \mathbb{C}^{n\times n}$, A^D is the Drazin inverse of A, and $i, m \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called an $\langle i, m\rangle$-core inverse of A if it satisfies*

$$X = A^DAX \ \ and \ \ A^mX = A^i(A^i)^{\dagger}. \qquad (2.18)$$

If such an X exists, then it is unique and denoted by $A^{\oplus}_{i,m}$.

In the following lemma, the expression of the $\langle i, m\rangle$-core inverse of A can be found by using the core-EP decomposition of A.

Lemma 2.12. [21, Theorem 2.7] *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If $A = A_1 + A_2$ is the core-EP decomposition of A with A_1, A_2 as in* (1.1)*, then, the expression of the $\langle i, m\rangle$-core inverse of A is $A^{\oplus}_{i,m} = U\begin{bmatrix} T^{-m} & 0 \\ 0 & 0 \end{bmatrix}U^*$ for all $i \geqslant k$.*

In the following theorem, we will show that the m-weak core inverse can be described by using the $\langle i, m\rangle$-core inverse of A.

Theorem 2.13. *Let $A, X \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and $m \in \mathbb{Z}$. Then, the following are equivalent:*

(1) *X is the m-weak core inverse of A;*
(2) *$AX = A^{\oplus}_{i,m}A^{2m}(A^m)^{\dagger}$ and $X = AA^{\textcircled{\dagger}}X$;*
(3) *$AX = A^{\oplus}_{i,m}A^{2m}(A^m)^{\dagger}$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$;*
(4) *$XAX = X$, $AX = A^{\oplus}_{i,m}A^{2m}(A^m)^{\dagger}$ and $XA = A^{\oplus}_{i,m+1}A^{2m}(A^m)^{\dagger}$.*

Proof. (1) ⇔ (2). By Theorem 2.8 and Lemma 2.12.

(1) ⇔ (3). By [10, Theorem 4.11] and Lemma 2.12.

(1) ⇔ (4). By [10, Theorem 4.9] and Lemma 2.12. □

Theorem 2.14. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and $m \in \mathbb{Z}$. If $2 \leqslant m < k$ and A^m is normal, then the m-weak core inverse of A is the inverse along A^k and $(A^k)^*A^m$.*

Proof. Let X be the m-weak core inverse of A. By Lemma 2.1, we have

$$\mathcal{N}\left((A^k)^* A^{2m} (A^m)^\dagger\right) = \mathcal{N}\left((A^k)^* A^m\right). \tag{2.19}$$

By Theorem 2.11, we have

$$\mathcal{N}\left((A^k)^* A^{2m} (A^m)^\dagger\right) \subseteq \mathcal{N}(X). \tag{2.20}$$

The equalities (2.19) and (2.20) give

$$\mathcal{N}\left((A^k)^* A^m\right) \subseteq \mathcal{N}(X). \tag{2.21}$$

By the proof of Theorem 2.11, we have

$$(A^k)^* A^{2m} (A^m)^\dagger AX = (A^k)^* A^{2m} (A^m)^\dagger. \tag{2.22}$$

The equality (2.22) implies that

$$AX - E_n \in \mathcal{N}\left((A^k)^* A^{2m} (A^m)^\dagger\right). \tag{2.23}$$

The equalities (2.19) and (2.23) give

$$AX - E_n \in \mathcal{N}\left((A^k)^* A^{2m} (A^m)^\dagger\right) = \mathcal{N}\left((A^k)^* A^m\right), \tag{2.24}$$

which says that

$$(A^k)^* A^m AX = (A^k)^* A^m. \tag{2.25}$$

Thus, the m-weak core inverse of A is the inverse along A^k and $(A^k)^* A^m$ by (2.20), (2.25), the proof of Theorem 2.11, and Definition 1.1. □

Remark 2.1. *By the proof of Theorem* 2.11*, we have*

$$(A^k)^* A^m = U \begin{bmatrix} (T^k)^* T^m & (T^k)^* \Phi_m \\ (\Phi_k)^* T^m & (\Phi_k)^* \Phi_m \end{bmatrix} U^* \tag{2.26}$$

and

$$AX = U \begin{bmatrix} E_r & T^{-m} \Phi_m N^m (N^m)^\dagger \\ 0 & 0 \end{bmatrix} U^*. \tag{2.27}$$

By (2.26) *and* (2.27)*, we have*

$$\begin{aligned} (A^k)^* A^m AX &= U \begin{bmatrix} (T^k)^* T^m & (T^k)^* \Phi_m \\ (\Phi_k)^* T^m & (\Phi_k)^* \Phi_m \end{bmatrix} \begin{bmatrix} E_r & T^{-m} \Phi_m N^m (N^m)^\dagger \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (T^k)^* T^m & (T^k)^* \Phi_m N^m (N^m)^\dagger \\ (\Phi_k)^* T^m & (\Phi_k)^* \Phi_m N^m (N^m)^\dagger \end{bmatrix} U^* \end{aligned} \tag{2.28}$$

The equalities (2.19) *and* (2.28) *give*

$$\begin{cases} (T^k)^* \Phi_m = (T^k)^* \Phi_m N^m (N^m)^\dagger \\ (\Phi_k)^* \Phi_m = (\Phi_k)^* \Phi_m N^m (N^m)^\dagger \end{cases} \tag{2.29}$$

Thus, by (2.29) *and T being invertible, we have*

$$\Phi_m = \Phi_m N^m (N^m)^\dagger.$$

Theorem 2.15. *Let $A, X \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. Then, the following are equivalent:*

(1) *X is the m-weak core inverse of A;*
(2) *X is the inverse along A^k and $A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}$;*
(3) *X is the inverse along A^k and $A_1A^{2m-1}(A^m)^{\dagger}$, where A_1 is the core part of the core-EP decomposition.*

Proof. (1) ⇒ (2) and (1) ⇒ (3). It is obvious by Lemma 2.3, Lemma 2.4, and Theorem 2.11.

(2) ⇒ (1). By Theorem 2.11 and Definition 1.1, it is enough to show that

$$A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}AX = A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger} \text{ and } \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}\right) \subseteq \mathcal{N}(X). \tag{2.30}$$

By Lemma 2.3, we have $\mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right) = \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}\right)$. Hence, the condition $\mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}\right) \subseteq \mathcal{N}(X)$ holds by Theorem 2.11. By the proof of Theorem 2.11, we have

$$(A^k)^*A^{2m}(A^m)^{\dagger}AX = (A^k)^*A^{2m}(A^m)^{\dagger}. \tag{2.31}$$

The equality (2.31) implies that

$$AX - E_n \in \mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right). \tag{2.32}$$

The property (2.32) and Lemma 2.3 give

$$AX - E_n \in \mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right) = \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}\right),$$

which says that

$$A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}AX = A^{\textcircled{\dagger}}A^{2m}(A^m)^{\dagger}.$$

(3) ⇒ (1). By Theorem 2.11 and Definition 1.1, it is enough to show that

$$A_1A^{2m-1}(A^m)^{\dagger}AX = A_1A^{2m-1}(A^m)^{\dagger} \text{ and } \mathcal{N}\left(A_1A^{2m-1}(A^m)^{\dagger}\right) \subseteq \mathcal{N}(X).$$

One can see that the proof of (3) ⇒ (1) is similar to the proof of (2) ⇒ (1). □

The DMP inverse is the inverse along A^k and $A^kA^{\dagger}$, and in the following theorem, we will show that the m-weak core inverse of A is the inverse along A^k and $A^k(A^m)^{\dagger}$ under the condition $m < k < 2m - 1$.

Theorem 2.16. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. If $m < k < 2m - 1$, then the m-weak core inverse of A is the inverse along A^k and $A^k(A^m)^{\dagger}$.*

Proof. Let X be the m-weak core inverse of A. By Lemma 2.6, we have

$$\mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right) = \mathcal{N}\left(A^k(A^m)^{\dagger}\right). \tag{2.33}$$

By the proof of Theorem 2.14, we have

$$AX - E_n \in \mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right). \tag{2.34}$$

The equalities (2.33) and (2.34) give

$$AX - E_n \in \mathcal{N}\left((A^k)^*A^{2m}(A^m)^{\dagger}\right) = \mathcal{N}\left(A^k(A^m)^{\dagger}\right), \tag{2.35}$$

which says that

$$A^k(A^m)^{\dagger}AX = A^k(A^m)^{\dagger}. \tag{2.36}$$

Thus, the m-weak core inverse of A is the inverse along A^k and $A^k(A^m)^{\dagger}$ by (2.36), the proof of Theorem 2.11, and Definition 1.1. □

3. One sided m-weak core inverse

Motivated by the ideal of the one-sided (B, C)-inverse of A, the one-sided m-weak core inverse was introduced by using the core-EP inverse of A.

Definition 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. We say that $X \in \mathbb{C}^{n\times n}$ is a left m-weak core inverse of A if we have*

$$XA^{k+1} = A^k \text{ and } \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) \subseteq \mathcal{N}(X). \tag{3.1}$$

We say that $Y \in \mathbb{C}^{n\times n}$ is a right m-weak core inverse of A if we have

$$Y = AA^{\textcircled{\dagger}}Y \text{ and } AY - E \in \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right). \tag{3.2}$$

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. Then, $Y \in \mathbb{C}^{n\times n}$ is a right m-weak core inverse of A if and only if $Y = AA^{\textcircled{\dagger}}Y$ and $(A^k)^*A^{2m}(A^m)^\dagger AY = (A^k)^*A^{2m}(A^m)^\dagger$ hold.*

Proof. By Lemma 3.1 and the definition of the right m-weak core inverse, the condition $AY - E \in \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right)$ is equivalent to

$$(A^k)^*A^{2m}(A^m)^\dagger AY = (A^k)^*A^{2m}(A^m)^\dagger. \tag{3.3}$$

Thus,

$$\begin{aligned}
&(A^k)^*A^{2m}(A^m)^\dagger AY = (A^k)^*A^{2m}(A^m)^\dagger \\
\Leftrightarrow\ &(A^k)^*\left(A^{2m}(A^m)^\dagger AY - A^{2m}(A^m)^\dagger\right) = 0 \\
\Leftrightarrow\ &(A^k)^\dagger\left(A^{2m}(A^m)^\dagger AY - A^{2m}(A^m)^\dagger\right) = 0 \\
\Leftrightarrow\ &A^k(A^k)^\dagger\left(A^{2m}(A^m)^\dagger AY - A^{2m}(A^m)^\dagger\right) = 0 \\
\Leftrightarrow\ &AA^{\textcircled{\dagger}}\left(A^{2m}(A^m)^\dagger AY - A^{2m}(A^m)^\dagger\right) = 0 \\
\Leftrightarrow\ &A^{\textcircled{\dagger}}\left(A^{2m}(A^m)^\dagger AY - A^{2m}(A^m)^\dagger\right) = 0.
\end{aligned}$$

□

The following theorem gives the main results for the one sided m-weak core inverse.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $m \in \mathbb{Z}$. If A is both left and right m-weak core invertible, then the left m-weak core inverse of A and the right m-weak core inverse of A are unique. Moreover, the left m-weak core inverse of A coincides with the right m-weak core inverse of A.*

Proof. Let X be a left m-weak core inverse of A and Y be a right m-weak core inverse of A. Then, by Definition 3.1, we have

$$XA^{k+1} = A^k \text{ and } \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) \subseteq \mathcal{N}(X), \tag{3.4}$$

and

$$Y = AA^{\textcircled{\dagger}}Y \text{ and } AY - E \in \mathcal{N}\left(A^{\textcircled{\dagger}}A^{2m}(A^m)^\dagger\right) \tag{3.5}$$

hold. By Lemma 3.1, the equality (3.5) is equivalent to

$$Y = AA^{\textcircled{\dagger}}Y \text{ and } (A^k)^*A^{2m}(A^m)^\dagger AY = (A^k)^*A^{2m}(A^m)^\dagger. \tag{3.6}$$

By Lemma 2.3, the equality (3.5) is equivalent to

$$XA^{k+1} = A^k \text{ and } \mathcal{N}\left((A^k)^* A^{2m}(A^m)^\dagger\right) \subseteq \mathcal{N}(X). \tag{3.7}$$

Thus $X = U(A^k)^* A^{2m}(A^m)^\dagger$ and $Y = AA^{\textcircled{\dagger}}Y = A^k(A^k)^\dagger Y$ for some $U, V \in \mathbb{C}^{n\times n}$ by (3.6) and and (3.7). Therefore,

$$\begin{aligned} X &= U(A^k)^* A^{2m}(A^m)^\dagger = U(A^k)^* A^{2m}(A^m)^\dagger AY = XAY; \\ Y &= A^k(A^k)^\dagger Y = XA^{k+1}(A^k)^\dagger Y = XAA^k(A^k)^\dagger Y = XAY. \end{aligned} \tag{3.8}$$

Hence, $X = Y$ by (3.8).

If Z is a another right m-weak core inverse of A, one can prove $X = Z$ in a similar way. Then $Y = Z$ by $X = Y$ and $X = Z$, which says the right MPCEP-inverse of A is unique. One also can prove the left m-weak core inverse of A is unique by a similar proof of the uniqueness of the right m-weak core inverse of A. By the above proof, we can get that the left m-weak core inverse of A coincides with the right m-weak core inverse of A. □

4. The relationship between different new-typed generalized inverses with same column subspace

The following table (Table 1) shows several generalized inverses that have the same column subspace.

Table 1. New-typed generalized inverses with same column subspace.

New-typed generalized inverses	matrix B	matrix C
Drazin inverse	$B = A^k$	$C = A^k$
core-EP inverse	$B = A^k$	$C = (A^k)^*$
DMP inverse	$B = A^k$	$C = A^k A^\dagger$
WG inverse	$B = A^k$	$C = (A^k)^* A$
m-weak core inverse	$B = A^k$	$C = A^k(A^m)^\dagger$

Note that if A^k is an EP matrix, then the Drazin inverse coincides with the core-EP inverse.

5. Conclusions

The m-weak core inverse of a complex matrix have been revisited this inverse by using the inverse along two matrices. Moreover, the necessary and sufficient conditions of the m-weak core inverse of a complex matrix have been obtained. The one-sided m-weak core inverse has been introduced by using the core-EP inverse of A. We believe that investigation related to the generalized inverses along the core parts of related matrix decompositions will attract attention, and we describe perspectives for further research:

(1). Considering weak generalized inverses based on the core-EP decompositions.

(2). Extending the m-weak core inverse of a complex matrix to an element in rings.

(3). The column space and the null space of a complex matrix are useful tools in the generalized inverses theory of a complex.

Author contributions

Jinyong Wu: Writing-original draft preparation; Wenjie Shi and Sanzhang Xu: writing-review and editing; Sanzhang Xu: methodology; Jinyong Wu and Wenjie Shi: supervision.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The research article is supported by the National Natural Science Foundation of China (No.11701429 and No.12001223), the Qing Lan Project of Jiangsu Province, the Natural Science Foundation of Jiangsu Province of China (No.BK20220702), the "Five-Three-Three" talents of Huai′ an city and College Students Innovation and Entrepreneurship Training Program (No. 202311049024Z).

Conflict of interest

There is no conflict of interest for all authors.

References

1. J. Benítez, E. Boasso, H. W. Jin, On one-sided (B, C)-inverses of arbitrary matrices, *Electron. J. Linear Al.,* **32** (2017), 391–422. https://doi.org/10.48550/arXiv.1701.09054
2. E. Boasso, G. Kantún-Montiel, The (b, c)-inverses in rings and in the Banach context, Mediterr, *Mediterr. J. Math.,* **14** (2017), 21 pages. http://dx.doi.org/ 10.1007/s00009-017-0910-1
3. O. M. Baksalary, G. Trenkler, Core inverse of matrices, *Linear Multilinear A.,* **58** (2010), 681–697. http://dx.doi.org/10.1080/03081080902778222
4. X. F. Cao, Y. Y. Huang, X. Hua, T. Y. Zhao, S. Z. Xu, Matrix inverses along the core parts of three matrix decompositions, *AIMS Math.,* **8** (2023), 30194–30208. http://dx.doi.org/ 10.3934/math.20231543
5. J. L. Chen, D. Mosić, S. Z. Xu, On a new generalized inverse for Hilbert space operators, *Quaest. Math.,* **43** (2020), 1331–1348. http://dx.doi.org/10.2989/16073606.2019. 1619104
6. M. P. Drazin, Pseudo-inverses in associative rings and semigroup, *Am. Math. Mon.,* **65** (1958), 506–514. http://dx.doi.org/10.1080/00029890.1958.11991949
7. M. P. Drazin, A class of outer generalized inverses, *Linear Algebra Appl.,* **43** (2012), 1909–1923. https://doi.org/10.1016/j.laa.2011.09.004

8. M. P. Drazin, Left and right generalized inverses, *Linear Algebra Appl.,* **510** (2016), 64–78. https://dx.doi.org/10.1016/j.laa.2016.08.010
9. D. E. Ferreyra, F. . Levis, N. Thome, Characterizations of k-commutative equalities for some outer generalized inverses, *Linear Multilinear A.,* **68** (2020), 177–192. https://dx.doi.org/10.1080/03081087.2018.1500994
10. D. E. Ferreyra, S. B. Malik, The m-weak core inverse, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM,* **118** (2024), 17 pages. https://dx.doi.org/10.1007/s13398-023-01539-y
11. Y. Y. Ke, J. Višnjić, J. L. Chen, One sided (b, c)-inverse in rings, *Filomat,* **34** (2020), 727–736. https://doi.org/10.2298/FIL2003727K
12. E. H. Moore, On the reciprocal of the general algebraic matrix, *Bull. Amer. Math. Soc.,* **26** (1920), 394–395.
13. K. Manjunatha Prasad, K. S. Mohana, Core-EP inverse, *Linear Multilinear A.,* **62** (2014), 792–804. https://doi.org/10.1080/03081087.2013.791690
14. M. Mehdipour, A. Salemi, On a new generalized inverse of matrices, *Linear Multilinear A.,* **66** (2018), 1046–1053. https://doi.org/10.1080/03081087.2017.1336200
15. S. B. Malik, N. Thome, On a new generalized inverse for matrices of an arbitrary index, *Appl. Math. Comput.,* **226** (2014), 575–580. http://dx.doi.org/10.1016/j.amc.2013.10.060
16. R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.,* **51** (1955), 406–413. http://dx.doi.org/10.1017/S0305004100030401
17. D. S. Rakić, A note on Rao and Mitra's constrained inverse and Drazin's (b,c) inverse, *Linear Algebra Appl.,* **523** (2017), 102–108. https://doi.org/10.1016/j.laa.2017.02.025
18. H. X. Wang, Core-EP decomposition and its applications, *Linear Algebra Appl.,* **508** (2016), 289–300. https://doi.org/10.1016/j.laa.2016.08.008
19. H. X. Wang, J. L. Chen, Weak group inverse, *Open Math.,* **16** (2018), 1218–1232. http://dx.doi.org/10.1515/math-2018-0100
20. S. Z. Xu, J. L. Chen, J. Benítez, Dingguo Wang, Generalized core inverse of matrices, *Miskolc Math. Notes,* **20** (2019), 565–584. http://dx.doi.org/10.18514/MMN.2019.2594
21. S. Z. Xu, H. X. Wang, J. L. Chen, X. F. Chen, T. W. Zhao, Generalized WG inverse, *J. Algebra Appl.,* **20** (2021), 2150072(12 pages). https://doi.org/10.1142/S0219498821500729
22. Y. K. Zhou, J. L. Chen, M. M. Zhou, m-weak group inverses in a ring with involution, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM,* **115** (2021), 13 pages. https://doi.org/10.1007/s13398-020-00932-1