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Research article

Note on normalized solutions to a kind of fractional Schrödinger equation with a critical nonlinearity

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Abstract: In this paper, we study normalized solutions of the fractional Schrödinger equation with a critical nonlinearity

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \mathrm{d}x = a^2, \ u \in H^s(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$, $s \in (0,1)$, $a > 0$, $2 < p < 2_s^* \triangleq \frac{2N}{N-2s}$ and $(-\Delta)^s$ is the fractional Laplace operator. In the purely L^2-subcritical perturbation case $2 < p < 2 + \frac{4s}{N}$, we prove the existence of a second normalized solution under some conditions on a, p, s, and N. This is a continuation of our previous work (*Z. Angew. Math. Phys.*, **73** (2022) 149) where only one solution is obtained.

Keywords: nonlinear fractional Schrödinger equation; normalized solutions; critical nonlinearity
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1. Introduction and main results

In this paper, we study normalized solutions of the fractional Schrödinger equation with a critical nonlinearity of $|u|^{2_s^*-2}u$,

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \mathrm{d}x = a^2, \ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$, $s \in (0,1)$, $a > 0$, and $2 < p < 2_s^* \triangleq \frac{2N}{N-2s}$. The fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = -\frac{C(N,s)}{2} \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \mathrm{d}y$$

$$= \frac{C(N,s)}{2}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y$$

with a positive constant $C(N,s)$, and we normalize the factor $C(N,s)/2 = 1$ for convenience. For problem (1.1), $p = 2 + \frac{4s}{N}$ is the L^2-critical exponent.

The operator $(-\Delta)^s$ arises in physics, chemistry, biology, and finance and can be seen as the infinitesimal generators of the Lévy stable diffusion process (see [1]). Moreover, $(-\Delta + m^2)^{\frac{1}{2}}$ appears in quantum mechanics, where m is the mass of the particle under consideration (see [16]). The study of fractional Laplacian nonlinear equations has attracted much attention from many mathematicians working in different fields. Felmer et al. [11] studied the existence, regularity, and symmetry of positive solutions to the fractional Schrödinger equations in the whole space $\mathbb{R}^N$. Caffarelli et al. investigated a fractional Laplacian with free boundary conditions (see [6, 7]). We also refer the interested readers to the works [5, 9, 10, 19] for more details on the fractional operator and its applications.

Normalized solutions to Schrödinger equations with L^2-supercritical nonlinearity were first studied in the paper [14], where the energy functional was unbounded from below on the L^2-constraint. Recently, Soave in [21] proved several existence (or nonexistence) and stability (or instability) results for the Schrödinger equation with combined nonlinearities as follows:

$$\begin{cases} -\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2\mathrm{d}x = a^2, \ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $2 < q < 2^* \triangleq \frac{2N}{N-2}$. Wei and Wu in [22] extended the results in [21] in three aspects. Firstly, they obtained the existence of a solution of mountain-pass type for $N \geq 3$ and $2 < q < 2 + \frac{4}{N}$. Secondly, the existence and nonexistence of ground states for $2 + \frac{4}{N} \leq q < 2^*$ with $\mu > 0$ large were obtained. Finally, they obtained the precisely asymptotic behaviors of ground states and mountain-pass solutions as $\mu \to 0$. Luo and Zhang in [17] dealt with the existence of normalized ground states for the fractional Schrödinger equation with combined nonlinearities as follows:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2\mathrm{d}x = a^2, \ u \in H^s(\mathbb{R}^N). \end{cases} \tag{1.2}$$

Under different assumptions on $q < p$ and $a > 0$, they proved the existence and nonexistence of normalized solutions in the L^2-subcritical case and L^2-supercritical case, respectively. But they only considered the Sobolev subcritical case $p, q < 2_s^*$. Motivated by the above papers, Zhang and Han in [25] considered problem (1.2) in the Sobolev critical case $q = 2_s^*$, i.e., problem (1.1). They obtained the following results:

(i) Let $N \geq 2$, $s \in (0,1)$, $2 < p < 2 + \frac{4s}{N}$, and assume that $0 < a < \min\{\alpha_1, \alpha_2\}$. Then, $m_a^+ \triangleq \inf_{u\in V(a)^+} E(u) = \inf_{u\in V(a)} E(u) < 0$ and it can be attained by $u_{a,+}$, which is nonnegative and radially decreasing. Moreover, problem (1.1) has a ground state $(u_{a,+}, \lambda_{a,+})$ with $\lambda_{a,+} < 0$;

(ii) Let $N \geq 2$, $s \in (0,1)$, $N^2 > 8s^2$, $p = 2 + \frac{4s}{N}$, and assume that $0 < a < \alpha_3$. Then, $m_a^- \triangleq \inf_{u\in V(a)^-} E(u) \in (0, \frac{s}{N}S_s^{\frac{N}{2s}})$ and it can be attained by $u_{a,-}$ which is nonnegative and radially decreasing. Moreover, problem (1.1) has a ground state $(u_{a,-}, \lambda_{a,-})$ with $\lambda_{a,-} < 0$;

(iii) Let $N \geq 2$, $s \in (0,1)$, $N^2 > 8s^2$, $2 + \frac{4s}{N} < p < 2_s^*$, and assume that $0 < a < \alpha_4$. Then, $m_a^- \triangleq \inf_{u \in V(a)^-} E(u) \in (0, \frac{s}{N} S_s^{\frac{N}{2s}})$ and it can be attained by $u_{a,-}$ which is nonnegative and radially decreasing. Moreover, problem (1.1) has a ground state $(u_{a,-}, \lambda_{a,-})$ with $\lambda_{a,-} < 0$.

The constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that appear in (i)–(iii) are

$$\alpha_1 \triangleq \left[\frac{p(2_s^* - 2)}{2C(s,N,p)(2_s^* - p\gamma_{p,s})} \left(\frac{2_s^* S_s^{\frac{2_s^*}{2}} (2 - p\gamma_{p,s})}{2(2_s^* - p\gamma_{p,s})} \right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}} \right]^{\frac{1}{p(1-\gamma_{p,s})}},$$

$$\alpha_2 \triangleq \left[\frac{22_s^* s}{N\gamma_{p,s} C(s,N,p)(2_s^* - p\gamma_{p,s})} \left(\frac{\gamma_{p,s} S_s^{\frac{N}{2s}}}{2 - p\gamma_{p,s}} \right)^{\frac{2-p\gamma_{p,s}}{2}} \right]^{\frac{1}{p(1-\gamma_{p,s})}},$$

$$\alpha_3 \triangleq \left(\frac{p}{2C(s,N,p)} \right)^{\frac{1}{p-2}}, \quad \alpha_4 \triangleq \gamma_{p,s}^{-\frac{1}{p(1-\gamma_{p,s})}} S_s^{\frac{N}{4s}}$$

where $\gamma_{p,s} \triangleq \frac{N(p-2)}{2ps} < 1$, the constants S_s, $C(s,N,p)$ are defined in (1.3), (1.5) respectively and $V(a)$ is defined in (1.6).

We also refer to the works [2–4, 15, 20, 24, 26] for other related equations.

In order to state our main results, we denote the best constant of the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ by

$$S_s = \inf_{u \in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{\|u\|_{2_s^*}^2}, \tag{1.3}$$

where $D^{s,2}(\mathbb{R}^N)$ denotes the completion of the space $C_c^\infty(\mathbb{R}^N)$ with the norm $\|u\|_{D^{s,2}(\mathbb{R}^N)} = \|(-\Delta)^{\frac{s}{2}} u\|_2$. Solutions to (1.1) can be obtained as the critical points of the associated energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y - \frac{1}{p} \|u(x)\|_p^p - \frac{1}{2_s^*} \|u(x)\|_{2_s^*}^{2_s^*} \tag{1.4}$$

defined on the constraint manifold $S(a) \triangleq \left\{ u \in H^s(\mathbb{R}^N) : \|u\|_2^2 = a^2 \right\}$, where

$$H^s(\mathbb{R}^N) \triangleq \left\{ u \in L^2(\mathbb{R}^N) : \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y < +\infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^s}^2 = \|u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} u\|_2^2.$$

We recall the following fractional Gagliardo–Nirenberg–Sobolev inequality (see [12])

$$\|u\|_p^p \leq C(s,N,p) \|u\|_2^{(1-\gamma_{p,s})p} \|(-\Delta)^{\frac{s}{2}} u\|_2^{\gamma_{p,s} p}, \quad \forall u \in H^s(\mathbb{R}^N). \tag{1.5}$$

Define $H_r^s(\mathbb{R}^N) \triangleq \left\{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \right\}$. It is well known that $H_r^s(\mathbb{R}^N)$ is compactly embedded into $L^p(\mathbb{R}^N)$ for any $p \in (2, 2_s^*)$, and $H_r^s(\mathbb{R}^N)$ is a natural constraint (see [23]).

Lemma 1. (Lemma 2.1 in [25]). *Let* $(u, \lambda) \in S(a) \times \mathbb{R}$ *be a weak solution to problem (1.1). Then* u *belongs to the set*

$$V(a) \triangleq \left\{u \in S(a) : P(u) \triangleq \|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \gamma_{p,s}\|u\|_p^p - \|u\|_{2_s^*}^{2_s^*} = 0\right\}. \tag{1.6}$$

Moreover, $V(a)$ *can be naturally divided into the following three parts:*

$$V(a)^+ \triangleq \left\{u \in V(a) : 2\|(-\Delta)^{\frac{s}{2}}u\|_2^2 > p\gamma_{p,s}^2\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*}\right\},$$
$$V(a)^0 \triangleq \left\{u \in V(a) : 2\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = p\gamma_{p,s}^2\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*}\right\},$$
$$V(a)^- \triangleq \left\{u \in V(a) : 2\|(-\Delta)^{\frac{s}{2}}u\|_2^2 < p\gamma_{p,s}^2\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*}\right\}.$$

In the L^2-subcritical perturbation case $2 < p < 2 + \frac{4s}{N}$, since the functional (1.4) is unbounded below on $S(a)$ as $2_s^* < 2+\frac{4s}{N}$, it will be naturally expected that $E(u)\mid_{S(a)}$ has a second critical point of mountain pass type for problem (1.1). In this paper, we give a complete positive answer to the above expectation (see Theorem 1). Since $H_r^s(\mathbb{R}^N)$ is a natural constraint, we only need to find the critical point for the functional $E(u)$ defined on $H_r^s(\mathbb{R}^N) \cap S(a)$. Define

$$V_r(a)^- \triangleq V(a)^- \cap H_r^s(\mathbb{R}^N).$$

Theorem 1. *Let* $N \geq 2$, $s \in (0, 1)$, $2 < p < 2 + \frac{4s}{N}$, *and assume that* $0 < a < \min\{\alpha_1, \alpha_2\}$. *Then* $m_{a,r}^- \triangleq \inf_{u\in V_r(a)^-} E(u) \in (0, \frac{s}{N}S_s^{\frac{N}{2s}})$ *and it can be attained by* $u_{a,-}$ *which is positive and radially decreasing. Moreover, problem (1.1) has a second solution* $(u_{a,-}, \lambda_{a,-})$ *with some* $\lambda_{a,-} < 0$.

Remark 1. *The method used in this paper can also be applied to the following Sobolev critical fractional Schrödinger equation with a parameter* $\mu > 0$

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu|u|^{p-2}u + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \mathrm{d}x = a^2, \end{cases}$$

which was considered in [27]. We leave the details to the interested readers.

Notations. The notation C in the following context denotes some positive constant that might be changed from line to line and even in the same line. $a \sim b$ means that $Cb \leq a \leq Cb$ and $a \lesssim b$ ($a \gtrsim b$) means that $a \leq Cb$ ($a \geq Cb$) for some positive constant C. The notation $B_z(0)$ denotes the ball in $\mathbb{R}^N$ of center at origin and radius z.

2. Proof of Theorem 1

As in [14], we use the fiber map preserving the L^2-norm $\tau * u = e^{\frac{N\tau}{2}}u(e^\tau x)$ for a.e. $x \in \mathbb{R}^N$. For $u \in S(a)$, define the auxiliary function

$$\Psi_u(\tau) := E(\tau * u) = \frac{1}{2}e^{2s\tau}\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \frac{e^{ps\gamma_{p,s}\tau}}{p}\|u\|_p^p - \frac{e^{2_s^*s\tau}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}, \quad \tau \in \mathbb{R}.$$

Lemma 2. (Lemma 3.3 in [25]). *Let* $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, *and* $0 < a < \alpha_1$. *For every* $u \in S(a)$, *the function* $\Psi_u(\tau)$ *has exactly two critical points* $s_u < t_u \in \mathbb{R}$ *and two zeros* $c_u < d_u \in \mathbb{R}$ *with* $s_u < c_u < t_u < d_u$. *Moreover, we have the following statements:*

(i) $s_u * u \in V(a)^+$ *and* $t_u * u \in V(a)^-$. *If* $\tau * u \in S(a)$, *then either* $\tau = s_u$ *or* $\tau = t_u$.

(ii) *We have*

$$E(t_u * u) = \max\{E(\tau * u) : \tau \in \mathbb{R}\} > 0$$

and $\Psi_u(\tau)$ *is strictly decreasing on* $(t_u, +\infty)$. *In particular, if* $t_u < 0$, *then* $P(u) < 0$.

(iii) *The maps* $u \in V(a) \mapsto s_u \in \mathbb{R}$ *and* $u \in V(a) \mapsto t_u \in \mathbb{R}$ *are of class* C^1.

Proof. Statements (i), (iii), and the first part of (ii) have already been shown in Lemma 3.3 in [25]. From the proof of Lemma 3.3 in [25], we know the functions $\Psi_u(\tau)$ and $\Psi_u''(\tau)$ have exactly two inflection points. In particular, $\Psi_u(\tau)$ is strictly decreasing and concave on $[t_u, +\infty)$. Hence, if $t_u < 0$, then $P(u) = \Psi_u'(0) < 0$. □

Lemma 3. *Let* $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, *and* $0 < a < \min\{\alpha_1, \alpha_2\}$. *Then, we have*

$$m_{a,r}^- \triangleq \inf_{u \in V_r(a)^-} E(u) > 0.$$

Proof. Applying (1.4), (1.5), and $\|u\|_{2_s^*}^2 \leq S_s^{-1}\|(-\Delta)^{\frac{s}{2}}u\|_2^2$, we have

$$\begin{aligned} E(u) &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \frac{1}{p}\|u\|_p^p - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*} \\ &\geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \frac{1}{p}C^p(s,N,p)a^{(1-\gamma_{p,s})p}\|(-\Delta)^{\frac{s}{2}}u\|_2^{\gamma_{p,s}p} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}\|(-\Delta)^{\frac{s}{2}}u\|_2^{2_s^*} \end{aligned}$$

for every $u \in V_r(a)^-$. Define

$$h(t) \triangleq \frac{1}{2}t^2 - \frac{C^p(s,N,p)}{p}a^{(1-\gamma_{p,s})p}t^{p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}t^{2_s^*}.$$

Since $p\gamma_{p,s} < 2 < 2_s^*$, it is easy to see that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$. Let $t_{\max}$ denote the strict maximum of the function $h(t)$, which is at a positive level (see Lemma 3.2 in [25]). For every $u \in V_r(a)^-$, by an easy computation, there exists $\tau_u \in \mathbb{R}$ such that $\|(-\Delta)^{\frac{s}{2}}(\tau_u * u)\|_2 = t_{\max}$. Moreover, by Lemma 2, we see that the value 0 is the unique strict maximum of the function $\Psi_u(\tau)$. Therefore,

$$E(u) = \Psi_u(0) \geq \Psi_u(\tau_u) = E(\tau_u * u) \geq h(\|(-\Delta)^{\frac{s}{2}}(\tau_u * u)\|_2) = h(t_{\max}) > 0.$$

Since $u \in V_r(a)^-$ is arbitrarily chosen, we deduce that

$$m_{a,r}^- \triangleq \inf_{u \in V_r(a)^-} E(u) \geq h(t_{\max}) > 0. \quad \square$$

Lemma 4. *Let* $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, *and* $0 < a < \min\{\alpha_1, \alpha_2\}$. *Then* $m_a^+ \triangleq \inf_{u \in V(a)^+} E(u) = \inf_{u \in V(a)} E(u) < 0$ *and it can be attained by* $u_{a,+}$, *which is positive and radially decreasing. Moreover, problem (1.1) has the ground state solution* $(u_{a,+}, \lambda_{a,+})$ *with* $\lambda_{a,+} < 0$.

Proof. By using a similar method used in Theorem 1.1 in [25], we obtain

$$m_a^+ \triangleq \inf_{u \in V(a)^+} E(u) = \inf_{u \in V(a)} E(u) < 0$$

and it can be attained by $u_{a,+}$, which is nonnegative and radially decreasing. Moreover, problem (1.1) has the ground state $(u_{a,+}, \lambda_{a,+})$ with $\lambda_{a,+} < 0$. Finally, by the strong maximum principle for the fractional Laplacian (see Proposition 2.17 in [19]), we have that $u_{a,+}$ is positive. □

Lemma 5. *Let $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, and $0 < a < \min\{\alpha_1, \alpha_2\}$. Then, we have*

$$m_{a,r}^{-} \triangleq \inf_{u \in V_r(a)^-} E(u) < m_a^{+} + \frac{s}{N} S_s^{\frac{N}{2s}}.$$

Proof. As in [18], the function $U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} u^*(\frac{x}{\varepsilon})$ solves the equation $(-\Delta)^s u = |u|^{2_s^*-2}u$ in $\mathbb{R}^N$, where $u^*(x) = \tilde{u}(x/S_s^{\frac{1}{2s}})/\|\tilde{u}\|_{2_s^*}$ and $\tilde{u}(x) = k(\mu^2 + |x|^2)^{-\frac{N-2s}{2}}$, $x \in \mathbb{R}^N$, with $k > 0$ and $\mu > 0$ are fixed constants. Let $\chi(x) \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function satisfying:

(a) $0 \leq \chi(x) \leq 1$ for any $x \in \mathbb{R}^N$,
(b) $\chi(x) \equiv 1$ in $B_1(0)$,
(c) $\chi(x) \equiv 0$ in $\mathbb{R}^N \setminus \overline{B_2(0)}$.

Define $W_\varepsilon = \chi(x)U_\varepsilon(x)$. According to Propositions 21 and 22 in [18], we know that

$$\|(-\Delta)^{\frac{s}{2}} W_\varepsilon\|_2^2 \leq S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}), \quad \|W_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{N}{2s}} + O(\varepsilon^N), \tag{2.1}$$

$$\|W_\varepsilon\|_p^p = \begin{cases} C\varepsilon^{N-\frac{N-2s}{2}p} + O(\varepsilon^{\frac{N-2s}{2}p}), & N > \frac{p}{p-1}2s, \\ C\varepsilon^{\frac{N}{2}} \log\frac{1}{\varepsilon} + O(\varepsilon^{\frac{N}{2}}), & N = \frac{p}{p-1}2s, \\ C\varepsilon^{\frac{N-2s}{2}p} + O(\varepsilon^{N-\frac{N-2s}{2}p}), & N < \frac{p}{p-1}2s \end{cases} \tag{2.2}$$

and

$$\|W_\varepsilon\|_2^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}), & N > 4s, \\ C\varepsilon^{2s}\log\frac{1}{\varepsilon} + O(\varepsilon^{2s}), & N = 4s, \\ C\varepsilon^{N-2s} + O(\varepsilon^{2s}), & N < 4s. \end{cases} \tag{2.3}$$

Now, we define

$$\widehat{W}_{\varepsilon,t} \triangleq u_{a,+} + tW_\varepsilon \quad \text{and} \quad \overline{W}_{\varepsilon,t} \triangleq \xi^{\frac{N-2s}{2}} \widehat{W}_{\varepsilon,t}(\xi x).$$

Then, it is well known that

$$\|(-\Delta)^{\frac{s}{2}} \overline{W}_{\varepsilon,t}\|_2^2 = \|(-\Delta)^{\frac{s}{2}} \widehat{W}_{\varepsilon,t}\|_2^2, \quad \|\overline{W}_{\varepsilon,t}\|_{2_s^*}^{2_s^*} = \|\widehat{W}_{\varepsilon,t}\|_{2_s^*}^{2_s^*} \tag{2.4}$$

and

$$\|\overline{W}_{\varepsilon,t}\|_2^2 = \xi^{-2s}\|\widehat{W}_{\varepsilon,t}\|_2^2, \quad \|\overline{W}_{\varepsilon,t}\|_p^p = \xi^{(p\gamma_{p,s}-p)s}\|\widehat{W}_{\varepsilon,t}\|_p^p. \tag{2.5}$$

We choose $\xi = \left(\|\widehat{W}_{\varepsilon,t}\|_2/a\right)^{\frac{1}{s}}$, then $\overline{W}_{\varepsilon,t} \in H_r^s(\mathbb{R}^N) \cap S(a)$. By Lemma 2, there exists $\tau_{\varepsilon,t} > 0$ such that $(\overline{W}_{\varepsilon,t})_{\tau_{\varepsilon,t}} \in V_r(a)^-$, where $(\overline{W}_{\varepsilon,t})_{\tau_{\varepsilon,t}} = \tau_{\varepsilon,t}^{\frac{N}{2}} \overline{W}_{\varepsilon,t}(\tau_{\varepsilon,t}x)$. Thus,

$$\|(-\Delta)^{\frac{s}{2}} \overline{W}_{\varepsilon,t}\|_2^2 \tau_{\varepsilon,t}^{s(2-p\gamma_{p,s})} = \gamma_{p,s}\|\overline{W}_{\varepsilon,t}\|_p^p + \|\overline{W}_{\varepsilon,t}\|_{2_s^*}^{2_s^*} \tau_{\varepsilon,t}^{s(2_s^*-p\gamma_{p,s})}. \tag{2.6}$$

Since $u_{a,+} \in V(a)^+$, by Lemma 2, we get $\tau_{\varepsilon,0} > 1$. By (2.1), (2.2), and (2.6), we know that $\tau_{\varepsilon,t} \to 0$ as $t \to +\infty$ uniformly for $\varepsilon > 0$ sufficiently small. Since $\tau_{\varepsilon,t}$ is unique by Lemma 2, it is standard to show that $\tau_{\varepsilon,t}$ is continuous for t, which implies that there exists $t_\varepsilon > 0$ such that $\tau_{\varepsilon,t_\varepsilon} = 1$. It follows that

$$m_{a,r}^- \leq \sup_{t\geq 0} E(\overline{W}_{\varepsilon,t}). \tag{2.7}$$

Recall that $u_{a,+} \in H_r^s(\mathbb{R}^N) \cap S(a)$ and W_ε are positive. By (2.4) and (2.5), we have

$$\begin{aligned} E(\overline{W}_{\varepsilon,t}) &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\widehat{W}_{\varepsilon,t}\|_2^2 - \frac{1}{p}\xi^{(p\gamma_{p,s}-p)s}\|\widehat{W}_{\varepsilon,t}\|_p^p - \frac{1}{2_s^*}\|\widehat{W}_{\varepsilon,t}\|_{2_s^*}^{2_s^*} \\ &= \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}\left(u_{a,+} + tW_\varepsilon\right)\|_2^2 - \frac{1}{p}\xi^{(p\gamma_{p,s}-p)s}\|u_{a,+} + tW_\varepsilon\|_p^p - \frac{1}{2_s^*}\|u_{a,+} + tW_\varepsilon\|_{2_s^*}^{2_s^*}. \end{aligned} \tag{2.8}$$

By the fact that $u_{a,+}$ is a solution to problem (1.1) for some $\lambda_{a,+} < 0$ and the Pohazaev identity satisfied by $u_{a,+}$, we have

$$\lambda_{a,+}a^2 = \lambda_{a,+}\|u_{a,+}\|_2^2 = (\gamma_{p,s} - 1)\|u_{a,+}\|_p^p. \tag{2.9}$$

Then, by (2.8) and (2.9), we deduce that $E(\overline{W}_{\varepsilon,t}) \to m_a^+$ as $t \to 0^+$ and

$$E(\overline{W}_{\varepsilon,t}) < t^2\|(-\Delta)^{\frac{s}{2}}W_\varepsilon\|_2^2 - \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_{a,+}(-\Delta)^{\frac{s}{2}}W_\varepsilon \mathrm{d}x + \frac{\lambda_{a,+}a^2}{p(\gamma_{p,s}-1)} - \frac{1}{2_s^*}t^{2_s^*}\|W_\varepsilon\|_{2_s^*}^{2_s^*} \to -\infty$$

as $t \to +\infty$ uniformly for $\varepsilon > 0$ sufficiently small. Hence, there exists $t_0 > 0$ large enough such that $t_\varepsilon \in (\frac{1}{t_0}, t_0)$ and $E(\overline{W}_{\varepsilon,t}) < 0$ for $t < \frac{1}{t_0}$ and $t > t_0$. Now, we estimate $E(\overline{W}_{\varepsilon,t})$ for $\frac{1}{t_0} \leq t \leq t_0$. Let $u_{a,+}$ be a positive and radially decreasing ground state solution to problem (1.1) (see Lemma 4). Then, we have

$$\begin{aligned} \int_{\mathbb{R}^N} u_{a,+}W_\varepsilon \mathrm{d}x &\sim \int_{B_1(0)} U_\varepsilon(x)\mathrm{d}x \sim \varepsilon^{\frac{N+2s}{2}} \int_0^{\frac{1}{\varepsilon}} \frac{1}{(C+r^2)^{\frac{N-2s}{2}}} r^{N-1}\mathrm{d}r \\ &\sim \varepsilon^{\frac{N+2s}{2}}\left(\frac{1}{\varepsilon}\right)^{2s} \sim \varepsilon^{\frac{N-2s}{2}}. \end{aligned} \tag{2.10}$$

From the definition of $\widehat{W}_{\varepsilon,t}$, we obtain

$$\xi^{2s} = \frac{\|\widehat{W}_{\varepsilon,t}\|_2^2}{a^2} = 1 + \frac{2t}{a^2}\int_{\mathbb{R}^N} u_{a,+}W_\varepsilon \mathrm{d}x + \frac{t^2}{a^2}\|W_\varepsilon\|_2^2 \tag{2.11}$$

for $\frac{1}{t_0} \leq t \leq t_0$. Applying (2.11) and the inequality $(1+\hat{t})^\alpha \geq 1 + \alpha\hat{t}$ for $\hat{t} \geq 0$ and $\alpha < 0$, we obtain that

$$\begin{aligned} \xi^{(p\gamma_{p,s}-p)s} = (\xi^{2s})^{\frac{p\gamma_{p,s}-p}{2}} &= \left(1 + \frac{2t}{a^2}\int_{\mathbb{R}^N} u_{a,+}W_\varepsilon \mathrm{d}x + \frac{t^2}{a^2}\|W_\varepsilon\|_2^2\right)^{\frac{p\gamma_{p,s}-p}{2}} \\ &\geq 1 + \frac{p\gamma_{p,s}-p}{2}\left(\frac{2t}{a^2}\int_{\mathbb{R}^N} u_{a,+}W_\varepsilon \mathrm{d}x + \frac{t^2}{a^2}\|W_\varepsilon\|_2^2\right). \end{aligned} \tag{2.12}$$

Applying (2.8)–(2.10) and (2.12), we have

$$E(\overline{W}_{\varepsilon,t}) \leq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u_{a,+}\|_2^2 + \frac{1}{2}t^2\|(-\Delta)^{\frac{s}{2}}W_\varepsilon\|_2^2 - \frac{1}{p}\|u_{a,+} + tW_\varepsilon\|_p^p$$

$$
\begin{aligned}
&+t\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_{a,+}(-\Delta)^{\frac{s}{2}}W_\varepsilon \mathrm{d}x-\frac{1}{2_s^*}\|u_{a,+}+tW_\varepsilon\|_{2_s^*}^{2_s^*}\\
&-\frac{p\gamma_{p,s}-p}{2p}\left(\frac{2t}{a^2}\int_{\mathbb{R}^N}u_{a,+}W_\varepsilon \mathrm{d}x+\frac{t^2}{a^2}\|W_\varepsilon\|_2^2\right)\|\widehat{W}_{\varepsilon,t}\|_p^p\\
&\le m_a^+ + E(tW_\varepsilon)-\frac{t(\gamma_{p,s}-1)}{a^2}\int_{\mathbb{R}^N}u_{a,+}W_\varepsilon \mathrm{d}x\|\widehat{W}_{\varepsilon,t}\|_p^p\\
&+t\lambda_{a,+}\int_{\mathbb{R}^N}u_{a,+}W_\varepsilon \mathrm{d}x-\frac{t^2(\gamma_{p,s}-1)}{2a^2}\|W_\varepsilon\|_2^2\|\widehat{W}_{\varepsilon,t}\|_p^p\\
&= m_a^+ + E(tW_\varepsilon)+\frac{t^2(1-\gamma_{p,s})}{2a^2}\|W_\varepsilon\|_2^2\|\widehat{W}_{\varepsilon,t}\|_p^p\\
&+\frac{t(1-\gamma_{p,s})}{a^2}\int_{\mathbb{R}^N}u_{a,+}W_\varepsilon \mathrm{d}x\left(\|\widehat{W}_{\varepsilon,t}\|_p^p-\|u_{a,+}\|_p^p\right)
\end{aligned}\tag{2.13}
$$

for $\frac{1}{t_0}\le t\le t_0$. By direct calculation, we have

$$
\begin{aligned}
\|\widehat{W}_{\varepsilon,t}\|_p^p-\|u_{a,+}\|_p^p &= \|u_{a,+}+tW_\varepsilon\|_p^p-\|u_{a,+}\|_p^p\\
&\lesssim \|u_{a,+}\|_p^p+\|tW_\varepsilon\|_p^p-\|u_{a,+}\|_p^p+\int_{\mathbb{R}^N}u_{a,+}^{p-1}tW_\varepsilon \mathrm{d}x.
\end{aligned}\tag{2.14}
$$

Similar to (2.10), we have

$$
\int_{\mathbb{R}^N}u_{a,+}^{p-1}W_\varepsilon \mathrm{d}x\lesssim\int_{B_2(0)}U_\varepsilon(x)\mathrm{d}x\lesssim\varepsilon^{\frac{N-2s}{2}}.\tag{2.15}
$$

By (2.1)–(2.3) and (2.13)–(2.15), we have

$$
\begin{aligned}
E(\overline{W}_{\varepsilon,t}) &\le m_a^+ + \frac{t(1-\gamma_{p,s})}{a^2}\left(O(\varepsilon^{\frac{N-2s}{2}})+\|tW_\varepsilon\|_p^p\right)O(\varepsilon^{\frac{N-2s}{2}})\\
&+E(tW_\varepsilon)+\frac{t^2(1-\gamma_{p,s})}{2a^2}\|W_\varepsilon\|_2^2\|u_{a,+}+tW_\varepsilon\|_p^p\\
&\le m_a^+ + \frac{t^2}{2}\left(S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})\right)-\frac{t^{2_s^*}}{2_s^*}\left(S_s^{\frac{N}{2s}}+O(\varepsilon^N)\right)-O(\|W_\varepsilon\|_p^p)\\
&+O(\varepsilon^{N-2s})+O(\varepsilon^{\frac{N-2s}{2}})O(\|W_\varepsilon\|_p^p)+O(\|W_\varepsilon\|_2^2)+O(\|W_\varepsilon\|_2^2)O(\|W_\varepsilon\|_p^p)\\
&< m_a^+ + \frac{t^2}{2}S_s^{\frac{N}{2s}}-\frac{t^{2_s^*}}{2_s^*}S_s^{\frac{N}{2s}}\le m_a^+ + \frac{s}{N}S_s^{\frac{N}{2s}}
\end{aligned}\tag{2.16}
$$

for $\frac{1}{t_0}\le t\le t_0$ by taking $\varepsilon>0$ sufficiently small. By Lemma 3 and (2.16), we obtain

$$
\begin{aligned}
0<m_{a,r}^- &\triangleq \inf_{u\in V_r(a)^-}E(u)\le E((\overline{W}_{\varepsilon,t_\varepsilon})_{\tau_{\varepsilon,t_\varepsilon}})=E(\overline{W}_{\varepsilon,t_\varepsilon})\\
&\le \sup_{t\in(t_0^{-1},t_0)}E(\overline{W}_{\varepsilon,t})<m_a^+ + \frac{s}{N}S_s^{\frac{N}{2s}}.
\end{aligned}\tag{2.17}
$$

Moreover, by (2.17), for $t<\frac{1}{t_0}$ and $t>t_0$, we have

$$
E(\overline{W}_{\varepsilon,t})<0<m_a^+ + \frac{s}{N}S_s^{\frac{N}{2s}}.\tag{2.18}
$$

It follows from (2.16) and (2.18) that

$$\sup_{t\geq 0} E(\overline{W}_{\varepsilon,t}) < m_a^+ + \frac{s}{N} S_s^{\frac{N}{2s}}.$$

Then, the conclusion follows from (2.7). □

For $0 < c < \min\{\alpha_1, \alpha_2\}$, let $u \in V(c)^{\pm}$. Then $v_b \triangleq \frac{b}{c} u \in S(b)$ for all $b > 0$. By Lemma 2, there exists $\tau_{\pm}(b) > 0$ such that

$$(v_b)_{\tau_{\pm}(b)} = (\tau_{\pm}(b))^{\frac{N}{2}} v_b(\tau_{\pm}(b)x) \in V(b)^{\pm},$$

where $0 < b < \min\{\alpha_1, \alpha_2\}$. Clearly, $\tau_{\pm}(c) = 1$.

Lemma 6. *Let $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, and $0 < c < \min\{\alpha_1, \alpha_2\}$. Then, $(\tau_{\pm}^s(c))'$ exist and*

$$(\tau_{\pm}^s(c))' = \frac{\gamma_{p,s} p \|u\|_p^p + 2_s^* \|u\|_{2_s^*}^{2_s^*} - 2\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{c\left(2\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \gamma_{p,s}^2 p \|u\|_p^p - 2_s^* \|u\|_{2_s^*}^{2_s^*}\right)}. \tag{2.19}$$

Moreover, $E((v_b)_{\tau_{\pm}(b)}) < E(u)$ for all $b > c$ such that $b < \min\{\alpha_1, \alpha_2\}$.

Proof. The proof is mainly inspired by [8, 22]. Since $(v_b)_{\tau_{\pm}(b)} \in V(b)^{\pm}$, we have

$$\left(\frac{b}{c}\tau_{\pm}^s(b)\right)^2 \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \gamma_{p,s}\left(\frac{b}{c}\tau_{\pm}^{\gamma_{p,s}s}(b)\right)^p \|u\|_p^p + \left(\frac{b}{c}\tau_{\pm}^s(b)\right)^{2_s^*} \|u\|_{2_s^*}^{2_s^*}.$$

Next, we define the function

$$\Phi(b, \tau_{\pm}^s) = \left(\frac{b}{c}\tau_{\pm}^s\right)^2 \|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \gamma_{p,s}\left(\frac{b}{c}\tau_{\pm}^{\gamma_{p,s}s}\right)^p \|u\|_p^p - \left(\frac{b}{c}\tau_{\pm}^s\right)^{2_s^*} \|u\|_{2_s^*}^{2_s^*}.$$

Clearly, $\Phi(b, \tau_{\pm}^s(b)) = 0$ for $0 < b < \min\{\alpha_1, \alpha_2\}$. Applying $u \in V(c)^{\pm}$ and Lemma 1, we have

$$\partial_{\tau_{\pm}^s}\Phi(c, 1) = 2\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \gamma_{p,s}^2 p\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*} \neq 0.$$

Applying the implicit function theorem, we have $(\tau_{\pm}^s(c))'$ exists and (2.19) holds. By $u \in V(c)^{\pm}$ once more, we deduce that

$$\begin{aligned} 1 + c(\tau_{\pm}^s(c))' &= 1 + \frac{\gamma_{p,s} p \|u\|_p^p + 2_s^* \|u\|_{2_s^*}^{2_s^*} - 2\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{2\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \gamma_{p,s}^2 p \|u\|_p^p - 2_s^* \|u\|_{2_s^*}^{2_s^*}} \\ &= \frac{p\gamma_{p,s}(1-\gamma_{p,s})\|u\|_p^p}{2\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \gamma_{p,s}^2 p \|u\|_p^p - 2_s^* \|u\|_{2_s^*}^{2_s^*}}. \end{aligned}$$

Since $(v_b)_{\tau_{\pm}(b)} \in V(b)^{\pm}$ and $u \in V(c)^{\pm}$, we obtain

$$\begin{aligned} E((v_b)_{\tau_{\pm}(b)}) &= \left(\frac{1}{2} - \frac{1}{p\gamma_{p,s}}\right)\|(-\Delta)^{\frac{s}{2}}(v_b)_{\tau_{\pm}(b)}\|_2^2 + \left(\frac{1}{p\gamma_{p,s}} - \frac{1}{2_s^*}\right)\|(v_b)_{\tau_{\pm}(b)}\|_{2_s^*}^{2_s^*} \\ &= \left(\frac{b}{c}\tau_{\pm}^s(b)\right)^2\left(\frac{1}{2} - \frac{1}{p\gamma_{p,s}}\right)\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \left(\frac{b}{c}\tau_{\pm}^s(b)\right)^{2_s^*}\left(\frac{1}{p\gamma_{p,s}} - \frac{1}{2_s^*}\right)\|u\|_{2_s^*}^{2_s^*} \end{aligned}$$

$$= \left(\frac{1}{2} - \frac{1}{p\gamma_{p,s}}\right)\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + \left(\frac{1}{p\gamma_{p,s}} - \frac{1}{2_s^*}\right)\|u\|_{2_s^*}^{2_s^*} + o(b-c)$$
$$+ \frac{1 + c(\tau_\pm^s(c))'}{c}\left[2\left(\frac{1}{2} - \frac{1}{p\gamma_{p,s}}\right)\|(-\Delta)^{\frac{s}{2}}u\|_2^2 + 2_s^*\left(\frac{1}{p\gamma_{p,s}} - \frac{1}{2_s^*}\right)\|u\|_{2_s^*}^{2_s^*}\right](b-c)$$
$$= E(u) - \frac{(1-\gamma_{p,s})\|u\|_p^p}{c}(b-c) + o(b-c).$$

Thus, we obtain

$$\frac{dE((v_b)_{\tau_\pm(b)})}{db}\Big|_{b=c} = -\frac{(1-\gamma_{p,s})\|u\|_p^p}{c} < 0.$$

Since $0 < c < \min\{\alpha_1, \alpha_2\}$ is arbitrary and $(v_b)_{\tau_\pm(b)} \in V(b)^\pm$, we have $E((v_b)_{\tau_\pm(b)}) < E(u)$ for all $b > c$ such that $b < \min\{\alpha_1, \alpha_2\}$. □

Lemma 7. *Let $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, and $0 < a < \min\{\alpha_1, \alpha_2\}$. Assume that $u \in V(a)$ is a critical point for $E(u)\mid_{V(a)}$, then u is a critical point for $E(u)\mid_{S(a)}$.*

Proof. By Lemma 3.1 in [25], we have that $V(a)$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^N)$ and $V(a)^0$ is empty. If $u \in V(a)$ is a critical point for $E(u)\mid_{V(a)}$, then by the Lagrange multipliers rule there exist $\lambda, \mu \in \mathbb{R}$ such that

$$E'(u)\varphi - \lambda\int_{\mathbb{R}^N} u\varphi \mathrm{d}x - \mu P'(u)\varphi = 0$$

for every $\varphi \in H^s(\mathbb{R}^N)$. This implies

$$(1-2\mu)(-\Delta)^s u = \lambda u + (1-\mu\gamma_{p,s}p)|u|^{p-2}u + (1-\mu 2_s^*)|u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N.$$

We have to prove that $\mu = 0$. By using the Pohozaev identity for the above equation, we know that

$$(1-2\mu)\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \gamma_{p,s}(1-\gamma_{p,s}p\mu)\|u\|_p^p + (1-\mu 2_s^*)\|u\|_{2_s^*}^{2_s^*}. \tag{2.20}$$

Applying $u \in V(a)$ and (2.20), we deduce that

$$\mu\left(2\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \gamma_{p,s}^2 p\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*}\right) = 0.$$

Since $u \notin V(a)^0$, we have $2\|(-\Delta)^{\frac{s}{2}}u\|_2^2 - \gamma_{p,s}^2 p\|u\|_p^p - 2_s^*\|u\|_{2_s^*}^{2_s^*} \neq 0$. Thus, we have $\mu = 0$. Hence, u is a critical point for $E(u)\mid_{S(a)}$, that is, $V(a)$ is a natural constraint. □

Lemma 8. *Let $N \geq 2$, $2 < p < 2 + \frac{4s}{N}$, and $0 < a < \min\{\alpha_1, \alpha_2\}$. Assume that $m_{a,r}^- < m_a^+ + \frac{s}{N}S_s^{\frac{N}{2s}}$, then $m_{a,r}^-$ can be attained by some $u_{a,-} \in H_r^s(\mathbb{R}^N)$, which is positive and radially decreasing. Furthermore, problem (1.1) has a second solution $u_{a,-}$ with some $\lambda_{a,-} < 0$.*

Proof. Let $\{\bar{u}_n\} \subset V_r(a)^-$ be a minimizing sequence. By Ekeland's variational principle (see [13]), there exists a new minimizing sequence $\{u_n\}$ satisfying

$$\begin{cases} \|\bar{u}_n - u_n\|_{H^s(\mathbb{R}^N)} \to 0, & \text{as } n \to \infty, \\ E(u_n) \to m_{a,r}^-, & \text{as } n \to \infty, \\ P(u_n) \to 0, & \text{as } n \to \infty, \\ E'|_{V_r(a)^-}(u_n) \to 0, & \text{as } n \to \infty. \end{cases} \tag{2.21}$$

Therefore, we obtain

$$E(u_n) - \frac{1}{2}P(u_n) = \frac{1}{p}\left(\frac{p\gamma_{p,s}}{2} - 1\right)\|u_n\|_p^p + \frac{s}{N}\|u_n\|_{2_s^*}^{2_s^*} \to m_{a,r}^-, \quad \text{as } n \to \infty. \tag{2.22}$$

From the third property in (2.21), we have

$$\begin{aligned} E(u_n) &= \frac{s}{N}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 - \frac{1}{p}\left(1 - \frac{\gamma_{p,s}p}{2_s^*}\right)\|u_n\|_p^p + o_n(1) \\ &\geq \frac{s}{N}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 - \frac{1}{p}C^p(s,N,p)\left(1 - \frac{\gamma_{p,s}p}{2_s^*}\right)a^{(1-\gamma_{p,s})p}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{\gamma_{p,s}p} + o_n(1) \end{aligned}$$

by the Gagliardo–Nirenberg–Sobolev inequality (1.5). Then, using that $E(u_n) \leq m_{a,r}^- + 1$ for n large, we deduce that

$$\frac{s}{N}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 \leq \frac{1}{p}C^p(s,N,p)\left(1 - \frac{\gamma_{p,s}p}{2_s^*}\right)a^{(1-\gamma_{p,s})p}\|(-\Delta)^{\frac{s}{2}}u_n\|_2^{\gamma_{p,s}p} + m_{a,r}^- + 1.$$

This implies that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^N)$. Therefore, $u_n \rightharpoonup u_0$ in $H_r^s(\mathbb{R}^N)$ up to a subsequence. By the Sobolev compact embedding theorem $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$, we have $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence. Without loss of generality, we assume that $u_n \rightharpoonup u_0$ weakly in $H_r^s(\mathbb{R}^N)$ and $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ as $n \to \infty$. We claim that $u_0 \neq 0$. Otherwise, $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$. By $P(u_n) \to 0$ as $n \to \infty$, it follows that

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 = \|u_n\|_{2_s^*}^{2_s^*} + o_n(1). \tag{2.23}$$

Applying (2.23) and the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$, we obtain either $u_n \to 0$ strongly in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$ or

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 &= \|u_n\|_{2_s^*}^{2_s^*} + o_n(1) \\ &\geq S_s^{\frac{N}{2s}} + o_n(1). \end{aligned}$$

According to (2.22), either $m_{a,r}^- = 0$ or $m_{a,r}^- \geq \frac{s}{N}S_s^{\frac{N}{2s}}$, which contradicts Lemmas 3 and 5. Therefore, we obtain $u_0 \neq 0$. Let $v_n \triangleq u_n - u_0$. Then, there are the following two cases:

(i) $v_n \to 0$ strongly in $H_r^s(\mathbb{R}^N)$ as $n \to \infty$;
(ii) $\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 + \|v_n\|_2^2 \gtrsim 1$.

In the case (i), $u_0 \in V_r(a)^-$ and $m_{a,r}^-$ is attained by u_0, which is nonnegative and radially decreasing. By Lemma 7, u_0 is a solution to problem (1.1) with $\lambda_0 \in \mathbb{R}$, which appears as a Lagrange multiplier. By multiplying equation (1.1) with u_0 and integrating by parts, it follows from $u_0 \in V_r(a)^-$ that

$$\lambda_0 a^2 = (\gamma_{p,s} - 1)\|u_0\|_p^p < 0.$$

Hence, $\lambda_0 < 0$. By using the strong maximum principle for the fractional Laplacian, we can see that u_0 is positive. It remains to consider the case (ii). Let $\|u_0\|_2^2 = t_0^2$. Then, by Fatou's lemma, we get $0 < t_0 \leq a$. Next, we have the following two subcases:

(a) $\|v_n\|_{2_s^*} \to 0$ as $n \to \infty$ up to a subsequence;
(b) $\|v_n\|_{2_s^*}^{2_s^*} \gtrsim 1$.

In the subcase (a), by Lemma 2, there exists $s_0 > 0$ such that $(u_0)_{s_0} \in V_r(t_0)^-$. Using Lemma 2 again, (2.21) and $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$ as $n \to \infty$ up to a subsequence, we deduce that

$$m_{a,r}^- + o_n(1) = E(u_n) \geq E((u_n)_{s_0}) = E((u_0)_{s_0}) + o_n(1).$$

By Lemma 6, we have $m_{t_0,r}^- \geq m_{a,r}^-$. Therefore, $E((u_0)_{s_0}) = m_{t_0,r}^-$ and $m_{t_0,r}^- = m_{a,r}^-$. If $t_0 < a$, we take $(u_0)_{s_0}$ as the test function in the proof of Lemma 6, and we deduce that $m_{t_0,r}^- > m_{a,r}^-$, which is a contradiction. Thus, in the case of (a), we have $t_0 = a$, and $m_{a,r}^-$ is attained by $(u_0)_{s_0}$, which is nonnegative and radially decreasing. As above, we can see that $(u_0)_{s_0}$ is positive and $(u_0)_{s_0}$ is a solution to problem (1.1) with $\lambda_0' < 0$. Now, it remains to consider the case (b). Let

$$s_n \triangleq \left(\frac{\|(-\Delta)^{\frac{s}{2}} v_n\|_2^2}{\|v_n\|_{2_s^*}^{2_s^*}} \right)^{\frac{1}{(2_s^*-2)s}}.$$

Clearly, in the case (b), $s_n \lesssim 1$, and by the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} (v_n)_{s_n}\|_2^2 &= \|(v_n)_{s_n}\|_{2_s^*}^{2_s^*} \\ &\geq S_s^{\frac{N}{2s}}. \end{aligned}$$

Since $0 < t_0 \leq a$, by Lemma 2, there exists $\tau_0 > 0$ such that $(u_0)_{\tau_0} \in V_r(t_0)^-$. We claim that $s_n \geq \tau_0$ up to a subsequence. If not, suppose the contrary: $s_n < \tau_0$ for all n. Define an auxiliary functional

$$E_0(u) \triangleq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*}.$$

Applying Lemma 2 once more, the Brezis–Lieb lemma (see Lemma 1.32 in [23]), Lemma 6, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ as $n \to \infty$, and the boundedness of $\{s_n\}$, it follows that

$$\begin{aligned} m_{a,r}^- + o_n(1) &= E(u_n) \geq E((u_n)_{s_n}) \\ &= E((u_0)_{s_n}) + E_0((v_n)_{s_n}) + o_n(1) \\ &\geq m_{t_0}^+ + \frac{s}{N} S_s^{\frac{N}{2s}} + o_n(1) \\ &\geq m_a^+ + \frac{s}{N} S_s^{\frac{N}{2s}} + o_n(1), \end{aligned}$$

which is impossible. Therefore, we have $s_n \geq \tau_0$ up to a subsequence. Without loss of generality, we assume that $s_n \geq \tau_0$ for all $n \in \mathbb{N}$. Again, by Lemma 2, the Brezis–Lieb lemma (see Lemma 1.32 in [23]), and the fact that $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ as $n \to \infty$, we obtain

$$\begin{aligned} m_{a,r}^- + o_n(1) &= E(u_n) \geq E((u_n)_{\tau_0}) \\ &= E((u_0)_{\tau_0}) + E_0((v_n)_{\tau_0}) + o_n(1). \end{aligned}$$

According to $s_n \geq \tau_0$, by the proof of Theorem 1.4 in [27] (see Section 8 in [27]), we have

$$E_0((v_n)_{\tau_0}) \geq 0.$$

By Lemma 6, we have $t_0 = a$, and $m^-_{a,r}$ is attained by $(u_0)_{\tau_0}$, which is nonnegative and radially decreasing. By the above analysis, we prove that $(u_0)_{\tau_0}$ is positive and $(u_0)_{\tau_0}$ is a solution for problem (1.1) with some $\lambda''_0 < 0$. Thus, we have proved that $m^-_{a,r}$ can always be attained by $u_{a,-}$, which is positive and radially decreasing. Hence, problem (1.1) has a second solution $(u_{a,-}, \lambda_{a,-})$ with some $\lambda_{a,-} < 0$. □

We are ready to give the proof of Theorem 1.

Proof of Theorem 1. By Lemmas 3 and 5, we have $m^-_{a,r} < m^+_a + \frac{s}{N} S^{\frac{N}{2s}}_s$. Then, Theorem 1 follows from Lemma 8. □

Author contributions

Xizheng Sun: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing-original draft, Writing-review and editing; Zhiqing Han: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

This work does not have any conflicts of interest.

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