AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 21628–21640.
DOI: 10.3934/math.20241051
Received: 21 April 2024
Revised: 11 June 2024
Accepted: 14 June 2024
Published: 05 July 2024

Research article

Hyperbolic Ricci soliton and gradient hyperbolic Ricci soliton on relativistic prefect fluid spacetime

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Abstract: In this research note, we investigated the characteristics of perfect fluid spacetime when coupled with the hyperbolic Ricci soliton. We additionally interacted with the perfect fluid spacetime, with a $\varphi(Q)$-vector field and a bi-conformal vector field that admits the hyperbolic Ricci solitons. Furthermore, we analyze the gradient hyperbolic Ricci soliton in perfect fluid spacetime, employing a scalar concircular field, and discuss about the gradient hyperbolic Ricci soliton's rate of change. In the end, we determined the energy conditions for perfect fluid spacetime in terms of gradient hyperbolic Ricci soliton with a scalar concircular field.

Keywords: perfect fluid spacetime; hyperbolic Ricci solitons; gradient hyperbolic Ricci soliton; Bi-conformal vector field; scalar concircular field; energy condition
Mathematics Subject Classification: 53B30, 53C44, 53C50, 53C80

1. Introduction

Self-similarity and fractals on progressively smaller scales can frequently arise from nonlinear systems supporting solitons when certain requirements are met. Most soliton-supporting systems in nature exhibit these kinds of fractals.

Fractals are more than simply a mathematical example; they are actual physical entities. This universal property of soliton-supporting systems to produce fractals appears to apply to all systems in nature that contain solitons, regardless of the kind of waves (density, electromagnetic, etc.), the medium in which they travel, or the degree of nonlinearity.

The concept of gravity proposed by Einstein is known as the General Theory of Relativity (GTR). According to this theory, the spacetime curvature serves as the gravitational field and gravitational waves, and the energy-momentum tensor $\mathcal{T}$ is its source. Every field theory has its roots in GTR. The Einstein equations, which explain the evolution of spacetime curvature, are specifically the model that all modern particle physics equations are based on. The development of relativistic fluids models and differential geometry in mathematics is the most effective way to comprehend general relativity. The main idea of the GTR holds that spacetime is best represented as a curved manifold [1].

A time-oriented connected four-dimensional Lorentzian manifold, a unique subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature $(-,+,+,+)$, can be used to simulate the spacetime of GTR and cosmology. This manifold has significant implications for GTR. The study of vector nature on the Lorentzian manifold $\mathcal{M}^4$ is the first step toward understanding its geometry. As a result, the Lorentzian manifold $(\mathcal{M}^4, g)$ is the best option for discussing GTR [1].

The energy-momentum tensor $\mathcal{T}$ is a key component of the spacetime's matter content. It is thought that matter is a fluid with density, pressure, and other dynamical and kinematical properties, including shear, expansion, acceleration, and velocity [2].

In traditional cosmological models, the matter component of the universe is believed to behave like a perfect fluid. A perfect fluid is one that has neither viscosity nor heat conduction. A perfect fluid can also be described as an isotropic or star-shaped fluid in its rest frame. Dust is the most basic example of the perfect fluid. According to the GTR for a cosmological model, the matter content of the universe is assumed to behave as a perfect fluid.

If the Ricci tensor has the shape, then a quasi-Einstein Lorentzian manifolds are called perfect fluid spacetime [2, 3],

$$\mathcal{S}_{Ric} = ag + b\eta \otimes \eta, \tag{1.1}$$

wherein scalars a and b are present, and 1-form η is metrically identical to a unit time-like vector field [4]. Additionally, the Lorentzian manifold is a manifold that allows for a vector field that resembles time [5, 6].

The shape of the energy-momentum tensor $\mathcal{T}$ coupled with a perfect fluid is [3]

$$\mathcal{T}(m, n) = pg(m, n) + (\sigma + p)\eta(m)\eta(n), \tag{1.2}$$

for any $m, n \in \chi(\mathcal{M}^4)$, where $\chi(\mathcal{M}^4)$ is a set of C^∞-vector fields defined on $\mathcal{M}^4$, p is the isotropic pressure, σ is the energy-density, g is the Lorentzian metric such that $g(m, \zeta) = \eta(m)$, where η is 1-form, equivalent to the time-like velocity vector ζ of the fluid, and $g(\zeta, \zeta) = -1$.

A radiation fluid is the medium if $\sigma = 3p$. In addition, Eq (1.2) has significant applications in cosmology and stellar structure.

Additional examples of energy-momentum tensors are the scalar field theory and electromagnetic energy-momentum tensors.

Einstein's gravitational equation is the field equation regulating perfect fluid motion [3]

$$\mathcal{S}_{Ric} + \left(\Lambda - \frac{\mathcal{R}_{scal}}{2}\right) g = \kappa \mathcal{T}, \tag{1.3}$$

where $\mathcal{S}_{Ric}$ is the Ricci tensor, $\mathcal{R}_{scal}$ is the scalar curvature of g, Λ is the cosmological constant, and κ is the gravitational constant (which can be interpreted as $8\pi G$, with G being the universal gravitational constant).

Several researchers have investigated in-depth the characteristics of symmetries in the perfect fluid spacetime (PFS) during the past 20 years using a variety of geometric tools, including curvature tensors [7], and most importantly geometric flows. Regarding matter and spacetime geometry, there are numerous symmetries. Metric symmetries are crucial because they make many issues easier to solve. Solitons are mostly used in general relativity to categorize Einstein field equation solutions. Ricci solitons connected to the Ricci flow of spacetime are one of these symmetries. Because it can aid in comprehending the ideas of entropy and energy in GTR, Ricci flow is significant. The places at which the curvatures obey a self-likeness are known as Ricci solitons.

Initially, Ahsan and Ali [8] investigated spacetime symmetries in relation to the Ricci soliton. The geometrical axioms of a PFS were explained by Blaga in [9] using Einstein, Ricci, and their extensions that is, η-Einstein solitons and η-Ricci solitons in a PFS, respectively. Additionally, Ricci solitons were employed by Venkatesha and Kumara [10] to examine the characterization of PFS using the Jacobi and torse-forming vector field. Recently, Siddiqi and Siddiqui [11] discussed the conformal Ricci solitons on PFS. Some properties of PFS with almost Ricci-Bourguignon soliton were investigated by Aliya and Mohammed in [12]. Furthermore, using Ricci-Yamabe solitions, Siddiqi et al. investigated PFS [13] and Static spacetime [14]. Alkhaldi and collaborators [15] have recently explored Ricci-Yamabe solitons on imperfect fluid generalized Robertson Walker spacetime.

The concept of hyperbolic geometric flow was first put forward by Dai et al. [16] in 2010, and then Faraji et al. [17, 18] proposed the ideas of hyperbolic Ricci solitons and gradient hyperbolic Ricci solitons. Blaga and Özgür have recently investigated the idea of hyperbolic Ricci solitons in a number of ways (see [19–21] for additional details).

Thus, inspired by earlier work, in this paper we study relativistic PFS in terms of a hyperbolic Ricci soliton (HRS) and gradient hyperbolic Ricci soliton (GHRS) with various vector fields.

2. Perfect fluid spacetime satisfying Einstein field equation with the cosmological constant $\Lambda > 0$

In GTR, the cosmological constant $\Lambda > 0$, is obtained from Einstein's field equation, which produces a static universe. It is thought to be a potential dark energy contender in contemporary cosmology, which explains why the Universe is expanding faster than ever.

We find from Eqs (1.2) and (1.3),

$$\mathcal{S}_{Ric}(m,n) = -\left(\Lambda - \frac{\mathcal{R}_{scal}}{2} + \kappa p\right) g(m,n) + \kappa(\sigma + p)\eta(m)\eta(n). \tag{2.1}$$

After comparing (6.2) with (2.1), we find the value of pressure and density for PFS $(\mathcal{M}^4, g)$, respectively,

$$p = \frac{1}{\kappa}\left(\frac{\mathcal{R}_{scal}}{2} - \Lambda - a\right), \qquad \sigma = \frac{1}{\kappa}\left(a + b + \Lambda - \frac{\mathcal{R}_{scal}}{2}\right). \tag{2.2}$$

In addition, we gain

$$a = \frac{\mathcal{R}_{scal}}{2} - \kappa p - \Lambda, \qquad b = \kappa(\sigma + p). \tag{2.3}$$

In light of (3.1), one can articulate the following

Theorem 2.1. *If a PFS* $(\mathcal{M}^4, g)$ *is obeying the Einstein field equation with the cosmological constant* Λ*, then density* σ *and the pressure* p *are governed by (2.2).*

Let $(\mathcal{M}^4, g)$ be a PFS fulfilling (2.1). Contracting (2.1) and assuming that $g(\zeta, \zeta) = -1$, we turn up

$$\mathcal{R}_{scal} = 4\Lambda + \kappa(\sigma - 3p). \tag{2.4}$$

Theorem 2.2. *In a PFS $(\mathcal{M}^4, g)$ with pressure p and density σ obeying the Einstein field equation with the cosmological constant Λ, the scalar curvature $\mathcal{R}_{scal}$ is $4\Lambda + \kappa(\sigma - 3p)$.*

Remark 2.3. *After the accelerated expansion of the universe was discovered in 1998 through the detection of supernovas, the energy density associated with dark energy gave rise to a negative pressure p_Λ.*

Remark 2.4. *This cosmological constant Λ is essential to interpreting the accelerated growth of the universe for $\Lambda > 0$. Consequently, the energy density associated with the Λ and speed of light c is defined as [22] and is referred to as the "dark energy density ϵ_Λ" or "vacuum energy density".*

$$\epsilon_\Lambda = \frac{c^4}{\kappa}\Lambda. \tag{2.5}$$

The vacuum energy density's corresponding mass density formula is [22]

$$\sigma_\Lambda = \frac{\epsilon_\Lambda}{c^4}. \tag{2.6}$$

Furthermore, the definition of the equation of state (EoS) for dark energy is [22]

$$p_\Lambda = -\epsilon_\Lambda. \tag{2.7}$$

Next, in view of Remark 2.4 and (2.4), we gain up the following theorem:

Theorem 2.5. *If a PFS $(\mathcal{M}^4, g)$ with pressure p and density σ obeys the Einstein field equation with the cosmological constant Λ, then PFS $(\mathcal{M}^4, g)$ is an accelerating spacetime if, and only if, $\frac{\mathcal{R}_{scal}}{4} > \frac{\kappa(\sigma-3p)}{4}$.*

Corollary 2.6. *If a PFS $(\mathcal{M}^4, g)$ with pressure p and density σ obeys the Einstein field equation with the cosmological constant Λ, then PFS $(\mathcal{M}^4, g)$ is a de-Sitter spacetime if, and only if, $\frac{\mathcal{R}_{scal}}{4} > \frac{\kappa(\sigma-3p)}{4}$.*

Adopting (2.5), (2.6), and (2.4), we get

$$\epsilon_\Lambda = \frac{c^4}{4}[\mathcal{R}_{scal} + (3p - \sigma)], \quad \sigma_\Lambda = \frac{1}{4}[\mathcal{R}_{scal} + (3p - \sigma)]. \tag{2.8}$$

Theorem 2.7. *If a PFS $(\mathcal{M}^4, g)$ obeys the Einstein field equation with the cosmological constant Λ, then the dark energy density ϵ_Λ and vacuum energy density σ_Λ are governed by (2.8).*

Corollary 2.8. *If a radiation fluid spacetime $(\mathcal{M}^4, g)$ obeys the Einstein field equation with the cosmological constant Λ. Then the dark energy density ϵ_Λ and vacuum energy density σ_Λ are evaluated as $\epsilon_\Lambda = c^4\sigma_\Lambda$.*

Furthermore, (2.7) and (2.8) entail the following outcome:

Theorem 2.9. *If a PFS $(\mathcal{M}^4, g)$ obeys the Einstein field equation with the cosmological constant $\Lambda > 0$, then the EoS is $p_\Lambda = -\frac{c^4}{4}[\mathcal{R}_{scal} + (3p - \sigma)]$.*

Corollary 2.10. *If the radiation fluid spacetime $(\mathcal{M}^4, g)$ obeying the Einstein field equation with the cosmological constant $\Lambda > 0$, then the EoS is $p_\Lambda = \frac{c^4\mathcal{R}_{scal}}{4}$.*

3. Hyperbolic Ricci solitons

Hamilton [23] originally introduced the concepts of Ricci flow in 1988. It proves that the soliton of Ricci is the limit of the solutions for the Ricci flow. Additionally, over the past 20 years, a lot of mathematicians have become interested in geometric flow theory, particularly the Ricci flow.

The family of metrics $g(t)$ on a semi-Riemannian M evolves into the Ricci flow [23], if

$$\frac{1}{2}\frac{\partial}{\partial t}g(t) = -\mathcal{S}_{Ric}(t)g(t), \quad g_0 = g(0). \tag{3.1}$$

Definition 3.1. *[23] On the semi-Riemannian manifold, $(\mathcal{M}, g)$, a Ricci soliton is a data (g, ζ, λ) obeying*

$$\frac{1}{2}L_\zeta g + \mathcal{S}_{Ric} + \lambda g = 0, \tag{3.2}$$

where the Ricci tensor is $\mathcal{S}_{Ric}$, and L_X is the Lie-derivative across the vector field ζ. Depending on the constant λ, the manifold (M, g, ζ, λ) is called a hyperbolic Ricci shrinker, expander, or stable soliton, whether $\lambda < 0$, $\lambda > 0$, or $\lambda = 0$.

On the other hand, Kong and Liu studied the hyperbolic Ricci flow [16]. This flow consists of a system of second order nonlinear evolution partial differential equations. The wave characteristics of metrics and manifold curvatures are described by hyperbolic Ricci flow. Consequently inspired by Ricci flow, the hyperbolic Ricci flow is described by the evolution equation that follows:

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}g(t) = -\mathcal{S}_{Ric}(t)g(t), \quad g_0 = g(0), \quad \frac{\partial}{\partial t}g_{ij} = h_{ij}, \tag{3.3}$$

where h_{ij} is a symmetric 2-tensor field. Therefore, an HRS is a self-similar solution of hyperbolic Ricci flow that is characterized as

Definition 3.2. *[17] An HRS is a semi-Riemannian manifold (M^n, g) if, and only if, there is a vector field ζ on M and real scalars μ and λ such that*

$$\frac{1}{2}L_\zeta L_\zeta g + \lambda L_\zeta g + \mathcal{S}_{Ric} = \mu g, \tag{3.4}$$

where S is Ricci curvature of M. In (3.4) λ and μ show the types of solitons and rate of the underlying type, respectively. Moreover, μ represents the rate of change in the solutions and has geometric meaning as well. Depending on the constant μ, the HRS's rate of change can be either shrinking, expanding, or approximately stable, regardless of whether $\mu < 0$, $\mu > 0$, or $\mu = 0$.

If there is a potential function f such that $\zeta = \nabla f$, then an HRS (g, λ, ζ, μ) is referred to as a GHRS [17]. This allows for a rewriting of (3.4) as

$$L_{\nabla f}(Hessf) + 2\lambda Hessf + \mathcal{S}_{Ric} = \mu g. \tag{3.5}$$

Definition 3.3. *[24] If a vector field ζ on a Riemannain manifold $(\mathcal{M}, g)$ satisfies the following relations, it is called a bi-conformal vector field.*

$$(L_\zeta g)(m, n) = \alpha g(m, n) + \beta \mathcal{S}_{Ric}(m, n), \tag{3.6}$$

$$(L_\zeta \mathcal{S}_{Ric})(m, n) = \alpha \mathcal{S}_{Ric}(m, n) + \beta g(m, n), \tag{3.7}$$

where α and β are the arbitrary nonzero smooth functions.

Definition 3.4. *[25] The manifold $(\mathcal{M}, g)$ is said to be an admitted Ricci collineation if*

$$L_\zeta \mathcal{S}_{Ric} = 0, \tag{3.8}$$

where $\mathcal{S}_{Ric}$ is a Ricci tensor.

Now, by the definition of the hyperbolic Ricci soliton

$$\frac{1}{2}(L_\zeta L_X g)(m,n) + \lambda L_\zeta g(m,n) + \mathcal{S}_{Ric}(m,n) = \mu g(m,n). \tag{3.9}$$

Operating the Lie derivative on (3.9), we get

$$\frac{1}{2}L_\zeta(L_\zeta Ł_\zeta g)(m,n) + \lambda(L_\zeta L_\zeta g)(m,n) + L_\zeta \mathcal{S}_{Ric}(m,n) = \mu L_\zeta g(m,n). \tag{3.10}$$

Let ζ be a bi-conformal vector field, and from (3.10), (3.6), and (3.7) we turn up

$$\frac{1}{2}L_\zeta L_\zeta(Ł_\zeta g)(m,n) + \lambda L_\zeta(L_\zeta g)(m,n) + (L_\zeta \mathcal{S}_{Ric})(m,n) = \mu(L_\zeta g)(m,n). \tag{3.11}$$

$$\begin{aligned}&\frac{1}{2}L_\zeta L_\zeta((\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) + \lambda L_\zeta((\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n))\\ &+ (\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)) - \mu(\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) = 0.\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\frac{1}{2}L_\zeta((\alpha(L_\zeta g)(m,n) + \beta(L_\zeta\mathcal{S}_{Ric})(m,n)) + \lambda((\alpha(L_\zeta g)(m,n) + \beta(L_\zeta\mathcal{S}_{Ric})(m,n))\\ &+ (\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)) - \mu(\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) = 0.\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\frac{1}{2}L_\zeta((\alpha(\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) + \alpha\beta(\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)))\\ &+ \lambda((\alpha(\alpha g)(m,n) + \beta\mathcal{S}_{Ric}(m,n)) + \beta(\alpha\mathcal{S}_{Ric})(m,n) + \beta g(m,n))\\ &+ (\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)) - \mu(\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) = 0.\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\frac{1}{2}(\alpha^2(\alpha(L_\zeta g)(m,n)) + \alpha\beta(L_\zeta\mathcal{S}_{Ric})(m,n)) + \alpha\beta(\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)))\\ &+ \lambda((\alpha(\alpha g)(m,n) + \beta\mathcal{S}_{Ric}(m,n)) + \beta(\alpha\mathcal{S}_{Ric})(m,n) + \beta g(m,n))\\ &+ (\alpha\mathcal{S}_{Ric}(m,n) + \beta g(m,n)) - \mu(\alpha g(m,n) + \beta\mathcal{S}_{Ric}(m,n)) = 0.\end{aligned} \tag{3.15}$$

Again, apply the definition of bi-conformal vector field in (3.15), we turn up

$$\mathcal{S}_{Ric}(m,n) = -\frac{[\frac{\alpha^3}{2} + 2\alpha\beta^2 + \lambda(\alpha^2 + \beta^2 + (\beta - \mu\alpha)]}{[\frac{3}{2}\alpha\beta^2 + \frac{\beta^2}{2} + 2\lambda\alpha\beta + (\alpha - \mu\beta)]}g(m,n). \tag{3.16}$$

Thus, we can conclude the following results:

Theorem 3.5. *Let a PFS $(\mathcal{M}^4, g)$ admit the HRS (g, λ, ζ, μ) with a bi-conformal vector field ζ, then the PFS $(\mathcal{M}^4, g)$ is Einstein and the Einstein factor is $-\frac{[\frac{\alpha^3}{2} + 2\alpha\beta^2 + \lambda(\alpha^2 + \beta^2 + (\beta - \mu\alpha)]}{[\frac{3}{2}\alpha\beta^2 + \frac{\beta^2}{2} + 2\lambda\alpha\beta + (\alpha - \mu\beta)]}$.*

In light of (3.16) and (2.1), we gain

$$-\left\{\Lambda - \frac{\mathcal{R}_{scal}}{2} + \kappa p + \frac{[\frac{\alpha^3}{2} + 2\alpha\beta^2 + \lambda(\alpha^2+\beta^2) + (\beta - \mu\alpha)]}{[\frac{3}{2}\alpha\beta^2 + \frac{\beta^2}{2} + 2\lambda\alpha\beta + (\alpha - \mu\beta)]}\right\} g(m,n) + \kappa(\sigma + p)\eta(m)\eta(n) = 0. \quad (3.17)$$

Putting $m = n = \zeta$ in (3.17), we obtain

$$\lambda = \frac{\frac{\alpha^3}{2} + (\beta - \mu\alpha) - [\frac{\kappa}{2}(3\sigma - p) + \Lambda + (\frac{3}{2}\alpha^2\beta + \frac{\beta^2}{2} + (\alpha - \mu\beta)]}{2\alpha\beta[\frac{\kappa}{2}(3\sigma - p) - \Lambda - (\alpha^2+\beta^2)]}. \quad (3.18)$$

Moreover, in the light of the Definition 3.1, we turn up the following outcome.

Theorem 3.6. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a bi-conformal Killing vector ζ field, then the HRS is expanding, steady, or shrinking referred to as*

1) $\frac{\alpha^3}{2} + (\beta - \mu\alpha) > [\frac{\kappa}{2}(3\sigma - p) + \Lambda + (\frac{3}{2}\alpha^2\beta + \frac{\beta^2}{2} + (\alpha - \mu\beta))]$,
2) $\frac{\alpha^3}{2} + (\beta - \mu\alpha) = [\frac{\kappa}{2}(3\sigma - p) + \Lambda + (\frac{3}{2}\alpha^2\beta + \frac{\beta^2}{2} + (\alpha - \mu\beta))]$, *and*
3) $\frac{\alpha^3}{2} + (\beta - \mu\alpha) < [\frac{\kappa}{2}(3\sigma - p) + \Lambda + (\frac{3}{2}\alpha^2\beta + \frac{\beta^2}{2} + (\alpha - \mu\beta))]$, *respectively, provided* $2\alpha\beta[\frac{\kappa}{2}(3\sigma - p) - \Lambda - (\alpha^2+\beta^2)] \neq 0$.

The rate of *HRS* is shrinking, expanding, or stable, depending on the constant μ, whether $\mu < 0$, $\mu > 0$, or $\mu = 0$. Therefore from (3.18) we obtain the nature of the rate of the hyperbolic Ricci flow in a *PFS* $(\mathcal{M}^4, g)$ with a bi-conformal vector field ζ

$$\mu = \frac{\frac{\alpha^2}{2} + \beta + 2\alpha\beta^2 + \lambda(\alpha^2+\beta^2) - [\frac{\kappa}{2}(3\sigma - p) - \Lambda](\frac{3}{2}\alpha^2\beta + \alpha + 2\lambda\alpha\beta))]}{\alpha - [\frac{\kappa}{2}(3\sigma - p) - \Lambda]\beta}. \quad (3.19)$$

Thus, we can articulate the following result:

Theorem 3.7. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a bi-conformal Killing vector field ζ, then the rate of change of the HRS is expanding, steady, or shrinking referred to as*

1) $\frac{\alpha^2}{2} + \beta + 2\alpha\beta^2 + \lambda(\alpha^2+\beta^2) > [\frac{\kappa}{2}(3\sigma - p) - \Lambda](\frac{3}{2}\alpha^2\beta + \alpha + 2\lambda\alpha\beta))$,
2) $\frac{\alpha^2}{2} + \beta + 2\alpha\beta^2 + \lambda(\alpha^2+\beta^2) = [\frac{\kappa}{2}(3\sigma - p) - \Lambda](\frac{3}{2}\alpha^2\beta + \alpha + 2\lambda\alpha\beta))$, *and*
3) $\frac{\alpha^2}{2} + \beta + 2\alpha\beta^2 + \lambda(\alpha^2+\beta^2) < [\frac{\kappa}{2}(3\sigma - p) - \Lambda](\frac{3}{2}\alpha^2\beta + \alpha + 2\lambda\alpha\beta))$, *respectively, provided* $\alpha - [\frac{\kappa}{2}(3\sigma - p) - \Lambda]\beta \neq 0$.

4. Hyberbolic Ricci soliton with a $\varphi(Q)$-vector field on PFS

Definition 4.1. *[26] A vector field φ on a Riemannian manifold M is said to be a $\varphi(Q)$-vector field if it obeys*

$$\nabla_\zeta \varphi = \Omega Q \zeta, \quad (4.1)$$

where ∇, Ω, and Q are the Levi-Civita connection, a constant, and Ricci operator, i.e., $g(Qm, n) = \mathcal{S}_{Ric}(m, n)$, respectively. If $\Omega = 0$ in (4.1), then $\varphi(Q)$ is said to be covariantly constant and φ is a proper $\varphi(Q)$-vector field if $\Omega \neq 0$.

According to (4.1) and the definition of the Lie-derivative, we turn up

$$(L_{\varphi}g)(m,n) = 2\Omega\mathcal{S}_{Ric}(m,n), \tag{4.2}$$

for any $m, n \in \chi(\mathcal{M}^4)$.

Now, in view of (3.9) and (4.2), we find

$$(2\lambda\Omega + 1)\mathcal{S}_{Ric}(m,n) + \Omega L_X\mathcal{S}_{Ric}(m,n) = \mu g(m,n). \tag{4.3}$$

Again adopting that ζ holds the Ricci collineation condition, (4.3) entails that

$$\mathcal{S}_{Ric}(m,n) = \frac{\mu}{(2\lambda\Omega + 1)}g(m,n). \tag{4.4}$$

We can therefore state the following outcome.

Theorem 4.2. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a proper $\varphi(Q)$-vector field ζ and if ζ is a Ricci collineation in the PFS, then the PFS is Einstein and Einstein's factor is $\frac{\mu}{(2\lambda\Omega+1)}$.*

Corollary 4.3. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a covariantly constant $\varphi(Q)$-vector field $(\Omega = 0)$ ζ and if ζ is a Ricci collineation in the PFS, then the PFS is an Einstein.*

Putting $m = n = \zeta$ in (4.4), we obtain

$$\mathcal{S}_{Ric}(\zeta,\zeta) = -\frac{\mu}{(2\lambda\Omega + 1)}. \tag{4.5}$$

Using Eqs (4.5) and (2.1), we turn up

$$\lambda = -\frac{\mu + [\kappa(2p + \sigma) - \frac{\mathcal{R}_{scal}}{2} + \Lambda]}{2\Omega[\kappa(2p + \sigma) - \frac{\mathcal{R}_{scal}}{2} + \Lambda]}. \tag{4.6}$$

Now, Theorem 4.2 and (4.6) entails the following result:

Theorem 4.4. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a proper $\varphi(Q)$-vector field ζ and if ζ is a Ricci collineation in the PFS, then the HRS is shrinking.*

In light of Corollary 4.3 with Eq (4.5) for a covariantly constant $\varphi(\mathcal{R}ic)$-vector field $(\Omega = 0)$, we get

$$\mathcal{S}_{Ric}(\zeta,\zeta) = -\mu. \tag{4.7}$$

Once again with the help of Eqs (2.1) and (4.7), we gain

$$\mu = \frac{\mathcal{R}_{scal}}{2} - [\kappa(2p + \sigma) + \Lambda]. \tag{4.8}$$

Thus, we can state the following theorem:

Theorem 4.5. *If a PFS $(\mathcal{M}^4, g)$ admits the HRS (g, λ, ζ, μ) with a covariantly constant $\varphi(Q)$-vector field $(\Omega = 0)$ ζ and if ζ is a Ricci collineation in the perfect fluid spacetime, then the HRS is steady and the rate of change of the HRS is expanding, steady, or shrinking according to*

1) $\frac{\mathcal{R}_{scal}}{2} > [\kappa(2p + \sigma) + \Lambda]$,
2) $\frac{\mathcal{R}_{scal}}{2} = [\kappa(2p + \sigma) + \Lambda]$, *and*
3) $\frac{\mathcal{R}_{scal}}{2} < [\kappa(2p + \sigma) + \Lambda]$, *respectively, provided steady hyperbolic Ricci solitons.*

5. Gradient hyperbolic Ricci solitons on perfect fluid spacetime

In this section, we determine the GHRS in PFS with scalar concircular field. Thus, we entail the following definition.

Definition 5.1. *[27] If the scalar field $f \in C^{\infty}(\mathcal{M})$ fulfills the equation, it is considered a scalar concircular field.*

$$Hessf = \pi g, \tag{5.1}$$

where the Riemannian metric is g and a scalar field is π. Furthermore, for an arc-length l geodesic, the equation becomes the ordinary differential equation (ODE),

$$\frac{d^2 f}{dl^2} = \pi. \tag{5.2}$$

Now, using Eqs (3.5) with (5.1), we find

$$L_{\nabla f}(Hessf(m,n)) + 2\lambda Hessf(m,n) + \mathcal{S}_{Ric}(m,n) = \mu g(m,n), \tag{5.3}$$

$$L_{\nabla f}(\pi g(m,n)) + 2\lambda \pi g(m,n) + \mathcal{S}_{Ric}(m,n) = \mu g(m,n), \tag{5.4}$$

$$\pi(Hessf(m,n)) + 2\lambda \pi g(m,n) + \mathcal{S}_{Ric}(m,n) = \mu g(m,n), \tag{5.5}$$

$$\mathcal{S}_{Ric}(m,n) = \mu - (\pi^2 + 2\lambda\pi) g(m,n). \tag{5.6}$$

Therefore, we can state the following result.

Theorem 5.2. *If a PFS $(\mathcal{M}^4, g)$ admits the GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with a scalar concircular field f, then the PFS is Einstein.*

Putting $m = n = \zeta$ in (5.6) and using (2.1), we obtain

$$\mu = [\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] - [\kappa(2p + \sigma) + \Lambda]. \tag{5.7}$$

Hence, we articulate the next theorem:

Theorem 5.3. *If a PFS $(\mathcal{M}^4, g)$ admits the GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with a scalar concircular field f, then the rate of change of GHRS is expanding, steady, or shrinking according to*

1) $[\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] > [\kappa(2p + \sigma) + \Lambda]$,
2) $[\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] = [\kappa(2p + \sigma) + \Lambda]$, *and*
3) $[\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] < [\kappa(2p + \sigma) + \Lambda]$, *respectively.*

Moreover, in light of (5.7) and (5.2), one can turn up the following theorem:

Theorem 5.4. *If a PFS $(\mathcal{M}^4, g)$ admits the GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with a scalar concircular field f and with an arc-length l geodesic, then the ODE satisfied by scalar concircular field f is*

$$\left(\frac{d^2 f}{dl^2}\right)^2 + 2\lambda \frac{d^2 f}{dl^2} = \frac{1}{2}(7p + \sigma) + \mu - \Lambda. \tag{5.8}$$

Moreover, we obtain the polynomial solutions of ODE (5.8), which is called the Maclaurin polynomial solution, such as

$$f = K_1\frac{l^2}{2} + K_2 l + K_3, \tag{5.9}$$

where $K_1 = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 2[7p+\sigma)+\mu-\Lambda]}}{2}$.

In a physical context, scalar fields are required to be independent of the choice of reference frame. That is, any two observers using the same units will agree on the value of the scalar field at the same absolute point in spacetime.

In addition, we gain an interesting result.

Theorem 5.5. *If a PFS $(\mathcal{M}^4, g)$ admits the GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with a scalar concircular field f and with an arc-length l geodesic, satisfying the ODE (5.8), then the scalar concircular field f is given by (5.9).*

6. Energy conditions in perfect fluid with gradient hyperbolic Ricci soliton

In this part, referring to [28], we are aware of whether the criterion is satisfied by Ricci tensor $\mathcal{S}_{Ric}$ in the spacetime

$$\mathcal{S}_{Ric}(\zeta, \zeta) > 0, \tag{6.1}$$

for all time-like vector field $\zeta \in \chi(\mathcal{M}^4)$, then Eq (5.5) is referred as the time-like convergence condition (TCC).

From (2.1) and (5.6), it gives

$$\mathcal{S}_{Ric}(\zeta, \zeta) = [\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] - [\kappa(2p+\sigma) + \Lambda + \mu].$$

The PFS in concern holds if it fulfills the TCC, that is, if $\mathcal{S}_{Ric}(\zeta, \zeta) > 0$,

$$[\pi^2 + 2\lambda\pi + \frac{\mathcal{R}_{scal}}{2}] > [\kappa(2p+\sigma) + \Lambda + \mu]. \tag{6.2}$$

Cosmological strong energy condition (SEC) is obeyed by the spacetime [29]. Given the aforementioned information and from (6.2), we can give the following

Theorem 6.1. *If a PFS $(\mathcal{M}^4, g)$ admits a GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with a scalar concircular field f, then the PFS $(\mathcal{M}^4, g)$ satisfies SEC, provided the rate of change of GHRS is expanding.*

Now, we have an interesting remark.

Remark 6.2. *In 1973, Hawking and Ellis [30] showed that $SEC \Rightarrow NEC$ (null energy condition).*

Combining Remark 6.2 with Theorem 6.1 yields the following theorem:

Theorem 6.3. *If a PFS $(\mathcal{M}^4, g)$ admits a GHRS $(g, \lambda, \zeta = \nabla f, \mu)$ with with a scalar concircular field f, then the PFS $(\mathcal{M}^4, g)$ satisfies NEC, if (6.2) holds, provided the rate of change of GHRS is expanding.*

7. Conclusions

This research paper focused on the investigation of various geometric aspects within the framework of a relativistic spacetime attached with perfect fluid. Several key results were obtained and discussed in the context of this study.

To begin, we determined the existence of a hyperbolic Ricci soliton on the relativistic spacetime when combined with perfect fluid matter and a $\varphi(\mathcal{R}ic)$-vector field and bi-conformal vector field. This finding highlights the presence of a specific geometric structure that exhibits soliton-like behavior in the hyperbolic setting. We used a scalar concircular field to study the gradient hyperbolic Ricci soliton in perfect fluid spacetime, examining its rate of change and determining its energy conditions.

Author contributions

Mohd. Danish Siddiqi: Conceptualization, methodology, writing–original draft preparation, writing–review and editing, supervision; Fatemah Mofarreh: Conceptualization, methodology, writing–original draft preparation, writing–review and editing, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, fyalmofarrah@pnu.edu.sa.

Acknowledgments

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions. The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, fyalmofarrah@pnu.edu.sa.

Conflict of interest

The authors declare no conflict of interest.

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