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Research article

Fractional calculus in beam deflection: Analyzing nonlinear systems with Caputo and conformable derivatives

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Abstract: In this paper, our study is divided into two parts. The first part involves analyzing a coupled system of beam deflection type that involves nonlinear equations with sequential Caputo derivatives. The also system incorporates the Caputo derivatives in the initial conditions, which adds a layer of complexity and realism to the problem. We focus on proving the existence of a unique solution for this system, and highlighting the robustness and applicability of fractional derivatives in modeling complex physical phenomena. In the second part of the paper, we employ conformable fractional derivatives, as defined by Khalil, to examine another system consisting of two coupled evolution equations. By the Tanh method, we derive new progressive waves. The connection between these two parts lies in the use of fractional calculus to extend and enhance classical problems.

Keywords: existence of solution; beam deflection; Caputo derivative; conformable fractional derivative; Than method; traveling waves; differential system
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1. Introduction

Ordinary differential equations and partial differential equations are fundamental tools in mathematics, used to model a wide variety of dynamic systems in science and engineering, see papers [2, 6, 16, 17]. Fractional differential equations are a generalization of classical differential equations, where the order of differentiation can be fractional or non-integer. These equations have gained significant attention due to their ability to model complex systems exhibiting memory and hereditary properties, which are not adequately captured by integer-order differential equations and systems [4, 5, 9, 10, 18–20, 23, 24, 35]. The Caputo derivative, one of the commonly used definitions in fractional calculus [13, 22], is particularly useful for initial value problems, making it suitable for physical applications. One important application of fractional differential equations is in the study of beam deflection equations and systems. Beams are structural elements that withstand loads applied laterally to their axis. The classical beam theory, often modeled by fourth-order ordinary differential equations, describes the bending and deflection of beams under various loads [2,3,26,32,34]. However, incorporating fractional derivatives into these models can provide a more accurate representation of materials with viscoelastic properties and non-local behavior, which are common in advanced engineering materials and complex structures. This leads to fractional differential models that can better predict the deflection and dynamic response of beams, offering deeper insights and improved design capabilities in engineering and applied physics.

To provide context, we reference some published papers related to our work. For instance, in [32], Q. Wang and L. Yang studied the existence of positive solutions for the following nonlinear fourth-order system which is used to describe the deformation of an elastic beam:

$$\begin{aligned} u^{(4)}(t) + \beta_1 u''(t) - \alpha_1 u(t) &= f_1(t, u(t), v(t)), t \in (0,1), \\ v^{(4)}(t) + \beta_2 v''(t) - \alpha_2 v(t) &= f_2(t, u(t), v(t)), t \in (0,1), \end{aligned}$$

under the conditions

$$\begin{aligned} u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{aligned}$$

where, $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, and $\beta_i, \alpha_i \in \mathbb{R}$ verify: $\beta_i < 2\pi^2$, $-\beta_i^2/4 \leqslant \alpha_i$, $\alpha_i/\pi^4 + \beta_i/\pi^2 < 1$, $i = 1, 2$.

By giving a cone P in $C([0,1] \times C([0,1]$, the authors proved existence of positive solution results. Then, by constructing over a product cone, they established another positive solution result.

In [31], the authors investigated the existence and uniqueness of solutions for the following system which contains sequential Caputo derivatives:

$$\begin{cases} ({}^cD^{\alpha} + \lambda {}^cD^{\alpha-1})x(t) = f(t, x(t), y(t), I_{0^+}^{P_1} x(t), I_{0^+}^{P_2} y(t)), t \in (0,1), \\ ({}^cD^{\beta} + \mu {}^cD^{\beta-1})y(t) = f(t, x(t), y(t), I_{0^+}^{q_1} x(t), I_{0^+}^{q_2} y(t)), t \in (0,1), \end{cases}$$

with

$$\begin{cases} x(0) = x'(0) = 0, x'(1) = 0, x(1) = \int_0^1 x(s) d\mathfrak{H}_1(s) + \int_0^1 y(s) d\mathfrak{H}_2(s), \\ y(0) = y'(0) = 0, y'(1) = 0, y(1) = \int_0^1 x(s) d\mathfrak{K}_1(s) + \int_0^1 y(s) d\mathfrak{K}_2(s), \end{cases}$$

where, $\alpha, \beta \in (3,4]$, $\lambda, \mu > 0$, $p_1, q_1, p_2, q_2 > 0$, $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions, $I_{0^+}^{J}$ is the Riemann-Liouville integral of order ν, (with $\nu = p_1, q_1, p_2, q_2$), and the Riemann-Stieltjes integrals with given bounded variation functions $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2$. Such systems can be applied in biosciences, see [4] and its references. The authors obtained existence and uniqueness results for solutions of the system.

In [11], the authors were concerned with the existence and uniqueness of solutions for the following coupled system of differential equations with several sequential derivatives:

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}x(t) = H_1(t, x(t), y(t)) + a_1 f_1(x(t)) + b_1 g_1(D^{\alpha_1}D^{\alpha_2}x(t)),\ \ t \in J = [0,1]\,, \\ D^{\beta_1}D^{\beta_2}D^{\beta_3}D^{\beta_4}y(t) = H_2(t, x(t), y(t)) + a_2 f_2(y(t)) + b_2 g_2(D^{\beta_1}D^{\beta_2}y(t)),\ \ t \in J = [0,1]\,, \\ \quad x(0) = x(1) = D^{\alpha_1}D^{\alpha_2}x(1) = D^{\alpha_4}x(0) = 0, \\ \quad y(0) = y(1) = D^{\beta_1}D^{\beta_2}y(1) = D^{\beta_4}y(0) = 0, \end{cases} \tag{1.1}$$

where, $D^{\alpha_1}, D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}, D^{\alpha_4}, D^{\beta_1}, D^{\beta_1}, D^{\beta_2}, D^{\beta_3}, D^{\beta_4}$ are Caputo fractional derivatives. The conditions $0 < \alpha_i \leqslant 1, 0 < \beta_i \leqslant 1,\ i = 1, ..., 4,$ along with the sequential aspect of the derivatives, guarantee that the studied system has a fractional derivative order included in $[3, 4]$. They also supposed that $\alpha_2 + \alpha_1 < \alpha_4, \beta_1 + \beta_2 < \beta_4, f_j : \mathbb{R} \to \mathbb{R}\,, g_j : \mathbb{R} \to \mathbb{R}$, and $H_j : [0,1] \times \mathbb{R}^2 \to \mathbb{R}\,,\ j = 1, 2$ are continuous functions, and

$H_i(t, 0, 0) \neq 0,\ f_i(0) \neq 0,\ g_i(0) \neq 0,\ i = 1, 2.$

In [12], K. Bensassa et al. studied the existence and uniqueness of solutions and stability in the sense of Ulam Hyers of the system

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}u(x) = f_1\,(x, u(x), v(x)) + a_1 g_1(x, u(x)) + b_1 h_1(x, D^{\delta}u(x)), \\ D^{\beta_1}D^{\beta_2}v(x) = f_2\,(x, u(x), v(x)) + a_2 g_2(x, u(x)) + b_2 h_2(x, D^{\delta}u(x)), \end{cases} \tag{1.2}$$

under the conditions

$$\begin{cases} u(0) = u(1) = a, \\ u'(0) = u'(1) = 0, \\ v(0) = v(1) = b, \\ v'(0) = v'(1) = 0, \end{cases} \tag{1.3}$$

where, for $i = 1, 2$, $a_i, b_i \in \mathbb{R}$, $D^{\alpha_i}, D^{\beta_i}, D^{\delta}$ are some fractional derivatives, $0 < \delta \leq 1$ $f_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g_i, h_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Based on the above-cited studies and, in particular, paper [32], in the present paper we shall be concerned with the following problem involving Caputo sequential derivatives:

$$\begin{cases} D^{\alpha_1}D^{\beta_1}u_1(x) = F_1(x, u_1(x), u_2(x)), D^{\delta-1}u_1(x), D^{\delta}u_1(x)), \\ D^{\alpha_2}D^{\beta_2}u_2(x) = F_2(x, u_1(x), u_2(x)), D^{\delta-1}u_2(x), D^{\delta}u_2(x)), \end{cases} \tag{1.4}$$

under the conditions

$$\begin{cases} u_i(0)) = u_i(1) = a_i \in \mathbb{R}, \\ D^{\beta_i}u_i(0) = D^{\beta_i}u_i(1) = 0, \end{cases} \tag{1.5}$$

where, $D^{\alpha_i}, D^{\beta_i}, D^{\delta}$ are Caputo fractional derivatives.

To guarantee the absence of semi-group and commutativity properties on the Caputo derivatives, and to obtain, under different conditions, the above Wang and Yang fourth-order system as a particular case, we also suppose that α_i and β_i satisfy the conditions $2 < \alpha_i \leq 3,\ 0 < \beta_i \leq 1$. We suppose also that $1 < \delta \leq 2$. This condition allows us to obtain the above problem of Wang and Yang [32] as a limiting case of (1.4). For $i = 1, 2$, $F_i : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are some continuous functions.

It is important to note that the problem we are investigating is more significant than the works cited above, and differs from most of them by incorporating Caputo derivatives in the initial conditions. This inclusion adds a layer of complexity and realism to our problem, enhancing its relevance and applicability in physical contexts. These conditions allow for better capturing of memory effects and non-local behaviors of the system while offering compatibility with observable initial conditions and efficient numerical methods.

Furthermore, our system includes two general functions, F_1 and F_2, which extend the applicability of the system beyond previously studied problems, providing a more comprehensive framework for understanding beam deflection dynamics. This broader approach allows for modeling more complex scenarios, thereby offering deeper insights and improved predictive capabilities in engineering and applied physics.

Moreover, our system (1.4) bridges the gap between classical beam deflection theories and modern viscoelastic models. Specifically, if we take the particular case where $F_i(x, u_1(x), u_2(x)), D^{\delta-1}u_1(x), D^{\delta}u_1(x)) = f_i(x, u_1(x), u_2(x)) - a_i u_i'' + b_i u_i', \delta = 2, \alpha_i = 3, \beta_i = 1$, then our system (1.4) can be reduced to the above fourth-order ODEs used to describe the deformation of an elastic beam [32].

Additionally, (1.4) provides a robust tool for studying a wide range of physical phenomena with greater precision and insight. This makes our work both practically relevant and theoretically enriching.

In the second part of our paper, we will use the Tanh method [14, 15] to find new traveling wave solutions for the following coupled beam problem with conformable fractional Khalil derivatives [1, 21]:

$$\begin{cases} T_t^{2\alpha}u + T_x^{4\beta}u + H(u, v, ...) = 0, \\ T_t^{2\alpha}v + T_x^{4\beta}v + L(u, v, ...) = 0, \end{cases} \tag{1.6}$$

where, $0 < \alpha, \beta \leq 1$, and H, L are two given functions.

The connection between these two parts lies in the use of fractional calculus to extend and enhance classical models, demonstrating the versatility and power of fractional derivatives—both Caputo and conformable—in addressing and solving advanced mathematical problems. This unified approach showcases how different fractional derivatives can be effectively applied to study and solve a variety of complex systems.

The choice of the Tanh method is justified by its relative simplicity and straightforward application, often requiring less computational effort compared to other methods. This makes it an efficient tool for finding solutions. Additionally, it provides explicit analytical solutions, which are valuable for understanding the qualitative behavior of the solutions and for validating numerical simulations. By yielding exact traveling wave solutions, the Tanh method helps in gaining insights into the physical phenomena described by the equations, such as wave propagation, solitons, and other localized structures. These advantages make the Tanh method a valuable tool in the study of traveling waves in nonlinear systems. For more details on this method and other important similar techniques, one can refer to papers [25, 27, 33].

The paper is organized as follows: In Section 2, we provide an overview of fractional calculus including Caputo fractional derivatives. In Section 3, we delve into the application of fixed point theory. We introduce the necessary concepts to study our system. Section 4 is devoted to the study of existence and uniqueness of solutions for our system, and an example is presented. In Section 5, we employ conformable fractional derivatives, as defined by Khalil, to examine another system consisting of two coupled evolution equations. Using the Tanh method, we derive new progressive waves. We discuss the advantages of this method and its efficacy in finding explicit analytical solutions. In the final section, we summarize the key findings of our study, emphasizing the connection between the two parts. We highlight how the use of fractional calculus extends and enhances classical models, demonstrating its versatility in addressing complex mathematical problems.

2. Preliminaries

We recall the following notions of fractional calculus that are needed in the proof of our results, see [13, 22].

Definition 1. *The fractional integral of order $\alpha > 0$ of any function $h \in C([\,0,1])$ is defined by*

$$J^{\alpha}h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \tag{2.1}$$

where $\Gamma(\alpha) := \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du.$

Definition 2. *Letting $h \in C^n([\,0,1])$, the Caputo fractional-order derivative of h is defined by*

$$D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds, \tag{2.2}$$

with $n := [\alpha] + 1.$

We also need the following lemmas and remark, see [22].

Lemma 1. *Let $\alpha > 0$, and*

$$D^{\alpha}h(t) = 0, t \in [0,1]. \tag{2.3}$$

Then, $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}$, $c_i \in R$, $i = 0,1,2,.....,n-1$, $n = [\alpha] + 1$.

Lemma 2. *Suppose $\alpha > 0$. Hence,*

$$J^{\alpha}_{a^+}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}, \tag{2.4}$$

such that $c_i \in R$, $i = 0,1,2,...,n-1$.

Remark 1. *(1*) For any $\alpha > 0, \beta > 0, t \in [0,1]$, we have the well known semigroup property: $J^{\alpha}J^{\beta}h(t) = J^{\alpha+\beta}h(t)$.*
(2) If $\beta \geq \alpha > 0$, then by the definition of D^{α} and property (1*), we have $D^{\alpha}J^{\beta}h(t) = J^{\beta-\alpha}h(t)$.*
(3) Taking $\alpha = \beta$ in (2*), we have $D^{\alpha}J^{\alpha}h(t) = h(t)$.*

Now, we prove the following equivalent integral equation.

Lemma 3. *Let $G_1, G_2 \in C([0,1], \mathbb{R})$ and $2 < \alpha_i \leq 3, 0 < \beta_i \leq 1, i = 1,2$. So, the problem*

$$\begin{cases} D^{\alpha_1}D^{\beta_1}u_1(x) = G_1(x), \\ \\ D^{\alpha_2}D^{\beta_2}u_2(x) = G_2(x), \end{cases} \tag{2.5}$$

under the conditions

$$\begin{cases} u_i(0) = u_i(1) = a_i \in \mathbb{R}, \\ D^{\beta_i} u_i(0) = D^{\beta_i} u_i(1) = 0, \end{cases}$$

is equivalent to the following integral equation:

$$\begin{cases} u_1(x) = a_1 + J^{\alpha_1+\beta_1} G_1(x) \\ + \left[\frac{2}{\beta_1 \Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} G_1(s)\, ds - \frac{\Gamma(\beta_1+3)}{\beta_1 \Gamma(\alpha_1+\beta_1)} \int_0^1 (1-s)^{\alpha_1+\beta_1-1} G_1(s)\, ds \right] x^{\beta_1+1} \\ + \left[\frac{\Gamma(\beta_1+3)}{\beta_1 \Gamma(\alpha_1+\beta_1)} \int_0^1 (1-s)^{\alpha_1+\beta_1-1} G_1(s)\, ds - \frac{\beta_1+2}{\beta_1 \Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} G_1(s)\, ds \right] x^{\beta_1+2}, \\ \\ u_2(x) = a_2 + J^{\alpha_2+\beta_2} G_2(x) \\ + \left[\frac{2}{\beta_2 \Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} G_2(s)\, ds - \frac{\Gamma(\beta_2+3)}{\beta_2 \Gamma(\alpha_2+\beta_2)} \int_0^1 (1-s)^{\alpha_2+\beta_2-1} G_2(s)\, ds \right] x^{\beta_2+1} \\ + \left[\frac{\Gamma(\beta_2+3)}{\beta_2 \Gamma(\alpha_2+\beta_2)} \int_0^1 (1-s)^{\alpha_2+\beta_2-1} G_2(s)\, ds - \frac{\beta_2+2}{\beta_2 \Gamma(\alpha 2)} \int_0^1 (1-s)^{\alpha_2-1} G_2(s)\, ds \right] x^{\beta_2+2}. \end{cases} \tag{2.6}$$

Proof. We begin by applying $J^{\alpha_i+\beta_i}$ for (2.5), and then using Lemma 1.2, we can get

$$\begin{aligned} u_1(x) &= J^{\alpha_1+\beta_1} G_1(x) + \frac{c_0 x^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{c_1 x^{\beta_1+1}}{\Gamma(\beta_1+2)} + \frac{2c_2 x^{\beta_1+2}}{\Gamma(\beta_1+3)} + c_3, \\ u_2(x) &= J^{\alpha_2+\beta_2} G_2(x) + \frac{d_0 x^{\beta_2}}{\Gamma(\beta_2+1)} + \frac{d_1 x^{\beta_2+1}}{\Gamma(\beta_2+2)} + \frac{2d_2 x^{\beta_2+2}}{\Gamma(\beta_2+3)} + d_3. \end{aligned} \tag{2.7}$$

Since $u_i(0) = a_i$, we get $c_3 = a_1, d_3 = a_2$.

Now, in the two quantities

$$\begin{aligned} D^{\beta_1} u_1(x) &= \frac{1}{\Gamma(\alpha_1)} \int_0^x (x-s)^{\alpha_1-1} G_1(s)\, ds + c_0 + c_1 x + c_2 x^2, \\ D^{\beta_2} u_2(x) &= \frac{1}{\Gamma(\alpha_2)} \int_0^x (x-s)^{\alpha_2-1} G_2(s)\, ds + d_0 + d_1 x + d_2 x^2, \end{aligned} \tag{2.8}$$

(that have been obtained by differentiation of (2.7)), if we take $x = 0$, then we obtain

$$c_0 = d_0 = 0.$$

Taking $x = 1$ in (2.7) and in (2.8), we have the following four equations:

$$\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1} G_1(s)\,ds + \frac{c_1}{\Gamma(\beta_1+2)} + \frac{2c_2}{\Gamma(\beta_1+3)} = 0,$$

$$\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1} G_1(s)\,ds + c_1 + c_2 = 0,$$

$$\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2-1} G_2(s)\,ds + \frac{d_1}{\Gamma(\beta_2+2)} + \frac{2d_2}{\Gamma(\beta_2+3)} = 0,$$

$$\frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1} G_2(s)\,ds + d_1 + d_2 = 0.$$

Solving these four equations, we obtain

$$c_1 = -\frac{\Gamma(\beta_1+3)}{\beta_1\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1} G_1(s)\,ds + \frac{2}{\beta_1\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1} G_1(s)\,ds,$$

$$c_2 = \frac{\Gamma(\beta_1+3)}{\beta_1\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1} G_1(s)\,ds - \frac{\beta_1+2}{\beta_1\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1} G_1(s)\,ds,$$

$$d_1 = -\frac{\Gamma(\beta_2+3)}{\beta_2\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2-1} G_2(s)\,ds + \frac{2}{\beta_2\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1} G_2(s)\,ds,$$

and

$$d_2 = \frac{\Gamma(\beta_2+3)}{\beta_2\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2-1} G_2(s)\,ds - \frac{\beta_2+2}{\beta_2\Gamma(\alpha2)}\int_0^1 (1-s)^{\alpha_2-1} G_2(s)\,ds.$$

Now, replacing c_i and d_i in (2.7), we have (2.6).

Lemma 3 is thus proved.

3. Application of fixed point theory

We consider the Banach space

$$E = \left\{u \in C([0,1],\mathbb{R}),\ D^{\delta-1}u \in C([0,1],\mathbb{R}), D^{\delta}u \in C([0,1],\mathbb{R})\right\},$$

over which we take the $\infty-$sum norm

$$\|u\|_E = \|u\|_\infty + \left\|D^{\delta-1}u\right\|_\infty + \left\|D^{\delta}u\right\|_\infty,$$

where

$$\|u\|_\infty = \sup_{x\in[0,1]} |u(x)|,\ \left\|D^{\delta-1}u\right\|_\infty = \sup_{x\in[0,1]} \left|D^{\delta-1}u(x)\right|, \left\|D^{\delta}u\right\|_\infty = \sup_{x\in[0,1]} \left|D^{\delta}u(x)\right|.$$

We shall also consider the product space $E \times E$ and the norm

$$\|(u,v)\|_{E\times E} = \|u\|_E + \|v\|_E.$$

Now, to be able to use fixed point theory, we need to introduce an operator $Q := (Q_1, Q_2)$, such that $Q : E \times E \to E \times E$ is defined by the two right-hand sides of (2.6), where the functions $G_1 := F_1, G_2 = F_2$; (F_1 and F_2 are introduced in (1.4)). In other words, we consider

$$Q\left(u\left(x\right), v\left(x\right)\right) := \left(Q_1\left(u\left(x\right), v\left(x\right)\right), Q_2\left(u\left(x\right), v\left(x\right)\right)\right),$$

where, for $i = 1, 2$, we have

$$Q_i\left(u\left(x\right), v\left(x\right)\right) = a_i + J^{\alpha_i+\beta_i} F_i(x, u(x), v(x)), D^{\delta-1}u(x), D^{\delta}v(x))$$

$$+\left[\begin{array}{c} \frac{2}{\beta_i\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha_i-1} F_i(s, u(s), v(s)), D^{\delta-1}u(s), D^{\delta}v(s))ds \\ -\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1 (1-s)^{\alpha_i+\beta_i-1} F_i(s, u(s), v(s)), D^{\delta-1}u(s), D^{\delta}v(s))ds \end{array}\right] x^{\beta_i+1}$$

$$+\left[\begin{array}{c} \frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1 (1-s)^{\alpha_i+\beta_i-1} F_i(s, u(s), v(s)), D^{\delta-1}u(s), D^{\delta}v(s))ds \\ -\frac{\beta_i+2}{\beta_i\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha_1-1} F_i(s, u(s), v(s)), D^{\delta-1}u(s), D^{\delta}v(s))ds \end{array}\right] x^{\beta_i+2}.$$

4. Existence of unique solutions

We suppose the following.

(H2): There is a matrix of positives functions $p_{ji}(x)$, $j = 1, 2..4, i = 1, 2$ such that for all $x \in [0, 1]$ and $(u, v, y, z), (r, s, t, w) \in \mathbb{R}^4$, one has

$$|F_i(x, u, v, y, z) - F_i(x, r, s, t, w)| \leqslant p_{1i}(x)\,|u-r| + p_{2i}(x)\,|v-s|$$

$$+p_{3i}(x)\,|y-t| + p_{4i}(x)\,|z-w|,$$

with, $n_{ji} := \sup\limits_{x\in[0,1]} \left|p_{ji}(x)\right|$, $j = 1, 2, 3, 4, i = 1, 2.$

Then, we can prove the first main result.

Theorem 1. *Suppose the validation of **(H2)**. If $M_1 + M_2 \in [0, 1]$, where*

$$M_i := A_i \max\left\{(n_{1i} + n_{3i}), (n_{2i} + n_{4i})\right\}, i = 1, 2,$$

and

$$\begin{aligned} A_i = &\left(\tfrac{\beta_i+2\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i+1)} + \tfrac{\beta_i+4}{\beta_i\Gamma(\alpha_i+1)}\right) + \tfrac{1}{\Gamma(\alpha_i+\beta_i-\delta+2)} + \tfrac{1}{\Gamma(\alpha_i+\beta_i-\delta+1)} \\ &+\tfrac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+4)}\left(2\left(\beta_i-\delta+3\right)+\left(\beta_i+2\right)^2\right) \\ &+\tfrac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+4)}\left(2\beta_i-\delta+5\right) + \tfrac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+3)}\left(2\beta_i-\delta+4\right) \\ &+\tfrac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+3)}\left(\left(\beta_i+2\right)^2+2\left(\beta_i-\delta+2\right)\right), \end{aligned}$$

then (1.4) and (1.5) admits a unique solution.

Proof. To proceed with the proof, we prove that Q satisfies the Banach contraction principle.

First of all, we note that the stability of the above product space $E \times E$ by the operator Q is trivial, and hence we omit it.

Let us now take two arbitrary elements (u_1, v_1) , $(u_2, v_2) \in E \times E$. So, for all $x \in [0, 1]$ and for $i = 1, 2$, we can write

$$|Q_i(u_1(x), v_1(x)) - Q_i(u_2(x), v_2(x))| \leq$$

$$\frac{1}{\Gamma(\alpha_i+\beta_i)} \left| \int_0^x (x-s)^{\alpha_i+\beta_i-1} \left[\begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right] ds \right|$$

$$+\frac{2}{\beta_i\Gamma(\alpha_i)} \left| \int_0^1 (1-s)^{\alpha_i-1} \left[\begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right] ds \right|$$

$$+\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)} \left| \int_0^1 (1-s)^{\alpha_i+\beta_i-1} \left[\begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right] ds \right|$$

$$+\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)} \left| \int_0^1 (1-s)^{\alpha_i+\beta_i-1} \left[\begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right] ds \right|$$

$$+\frac{\beta_i+2}{\beta_i\Gamma(\alpha_i)} \left| \int_0^1 (1-s)^{\alpha_1-1} \left[\begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right] ds \right|.$$

Therefore,

$$\|Q_i(u_1(x), v_1(x)) - Q_i(u_2(x), v_2(x))\|_\infty \leq$$

$$\frac{\beta_i+2\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)} \int_0^1 (1-s)^{\alpha_i+\beta_i-1} \left| \begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right| ds$$

$$+\frac{\beta_i+4}{\beta_i\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-1} \left| \begin{array}{c} F_i(s, u_1(s), v_1(s)), D^{\delta-1}u_1(s), D^{\delta}v_1(s)) \\ -F_i(s, u_2(s), v_2(s)), D^{\delta-1}u_2(s), D^{\delta}v_2(s)) \end{array} \right| ds.$$

Using (H_2), we obtain

$$\|Q_i(u_1, v_1) - Q_i(u_2, v_2)\|_\infty \leq$$

$$\left(\frac{\beta_i+2\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i+1)} + \frac{\beta_i+4}{\beta_i\Gamma(\alpha_i+1)} \right) \left[\begin{array}{c} n_{1i} \|u_1 - u_2\|_\infty + n_{2i} \|v_1 - v_2\|_\infty \\ +n_{3i} \left\| D^{\delta-1}u_1 - D^{\delta-1}u_2 \right\|_\infty + n_{4i} \left\| D^{\delta}v_1 - D^{\delta}v_2 \right\|_\infty \end{array} \right]. \tag{4.1}$$

Using Caputo derivative, we obtain

$$D^{\delta-1}Q_i\left(u\left(x\right),v\left(x\right)\right)=J^{\alpha_i+\beta_i-\delta+1}F_i(x,u(x),v(x)),D^{\delta-1}u(x),D^{\delta}v(x))$$

$$+\frac{\Gamma(\beta_i+2)}{\Gamma(\beta_i-\delta+3)}x^{\beta_i-\delta+2}\left[\begin{array}{c}\frac{2}{\beta_i\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\\-\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1(1-s)^{\alpha_i+\beta_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\end{array}\right]$$

$$+\frac{\Gamma(\beta_i+3)}{\Gamma(\beta_i-\delta+4)}x^{\beta_i-\delta+3}\left[\begin{array}{c}\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1(1-s)^{\alpha_i+\beta_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\\-\frac{\beta_i+2}{\beta_i\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_1-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\end{array}\right],$$

and

$$D^{\delta}Q_i\left(u\left(x\right),v\left(x\right)\right)=J^{\alpha_i+\beta_i-\delta}F_i(x,u(x),v(x)),D^{\delta-1}u(x),D^{\delta}v(x))$$

$$+\frac{\Gamma(\beta_i+2)}{\Gamma(\beta_i-\delta+2)}x^{\beta_i-\delta+1}\left[\begin{array}{c}\frac{2}{\beta_i\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\\-\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1(1-s)^{\alpha_i+\beta_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\end{array}\right]$$

$$+\frac{\Gamma(\beta_i+3)}{\Gamma(\beta_i-\delta+3)}x^{\beta_i-\delta+2}\left[\begin{array}{c}\frac{\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i)}\int_0^1(1-s)^{\alpha_i+\beta_i-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds\\-\frac{\beta_i+2}{\beta_i\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_1-1}F_i(s,u(s),v(s)),D^{\delta-1}u(s),D^{\delta}v(s))ds.\end{array}\right].$$

Thanks to (H_2), and using the same arguments as in (4.1), we can write

$$\left\|D^{\delta-1}Q_i\left(u_1,v_1\right)-D^{\delta-1}Q_i\left(u_2,v_2\right)\right\|_\infty\leq$$

$$\left[\begin{array}{c}\frac{1}{\Gamma(\alpha_i+\beta_i-\gamma+2)}+\frac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+4)}\left(2\left(\beta_i-\delta+3\right)+\left(\beta_i+2\right)^2\right)\\+\frac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+4)}\left(2\beta_i-\delta+5\right)\end{array}\right]\times\left(\begin{array}{c}n_{1i}\left\|u_1-u_2\right\|_\infty+n_{2i}\left\|v_1-v_2\right\|_\infty\\+n_{3i}\left\|D^{\delta-1}u_1-D^{\delta-1}u_2\right\|_\infty+n_{4i}\left\|D^{\delta}v_1-D^{\delta}v_2\right\|_\infty\end{array}\right). \tag{4.2}$$

Recalculating for this step involves replacing $\delta-1$ with δ everywhere in (2.4), except in the term within the large parentheses. So, we obtain the following estimate:

$$\left\|D^{\delta}Q_i\left(u_1,v_1\right)-D^{\delta}Q_i\left(u_2,v_2\right)\right\|_\infty\leq$$

$$\left[\begin{array}{c}\frac{1}{\Gamma(\alpha_i+\beta_i-\gamma+1)}+\frac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+3)}\left(2\beta_i-\delta+4\right)\\+\frac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+3)}\left(\left(\beta_i+2\right)^2+2\left(\beta_i-\delta+2\right)\right)\end{array}\right]\times\left(\begin{array}{c}n_{1i}\left\|u_1-u_2\right\|_\infty+n_{2i}\left\|v_1-v_2\right\|_\infty\\+n_{3i}\left\|D^{\delta-1}u_1-D^{\delta-1}u_2\right\|_\infty+n_{4i}\left\|D^{\delta}v_1-D^{\delta}v_2\right\|_\infty\end{array}\right). \tag{4.3}$$

Thanks to the definition of the norm over E, and by (4.1)–(4.3), we can write for $i = 1, 2$

$$\|Q_i(u_1(x), v_1(x)) - Q_i(u_2(x), v_2(x))\|_E \leq$$

$$\left[\begin{array}{c} \left(\frac{\beta_i+2\Gamma(\beta_i+3)}{\beta_i\Gamma(\alpha_i+\beta_i+1)} + \frac{\beta_i+4}{\beta_i\Gamma(\alpha_i+1)} \right) \\ + \frac{1}{\Gamma(\alpha_i+\beta_i-\gamma+2)} + \frac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+4)} \left(2(\beta_i - \delta + 3) + (\beta_i + 2)^2 \right) + \frac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+4)} (2\beta_i - \delta + 5) \\ + \frac{1}{\Gamma(\alpha_i+\beta_i-\gamma+1)} + \frac{\Gamma(\beta_i+3)\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+\beta_i+1)\Gamma(\beta_i-\delta+3)} (2\beta_i - \delta + 4) + \frac{\Gamma(\beta_i+2)}{\beta_i\Gamma(\alpha_i+1)\Gamma(\beta_i-\delta+3)} \left((\beta_i + 2)^2 + 2(\beta_i - \delta + 2) \right) \end{array} \right] \tag{4.4}$$
$$\times \left((n_{1i} + n_{3i}) \|u_1 - u_2\|_E + (n_{2i} + n_{4i}) \|v_1 - v_2\|_E \right)$$

$$\leq A_i \Big((n_{1i} + n_{3i}) \|u_1 - u_2\|_E + (n_{2i} + n_{4i}) \|v_1 - v_2\|_E \Big).$$

By (4.4), the definition of M_1 and M_2, and the norm over $E \times E$, we can write

$$\|Q(u_1, v_1) - Q(u_2, v_2)\|_{E\times E} \leq (M_1 + M_2) \|(u_1, v_1) - (u_2, v_2)\|_E .$$

Since $M_1 + M_2 < 1$, then Q is a contraction.

Hence, the Banach fixed point theorem implies that there exists a unique fixed point which is the solution of (1.4) and (1.5).

Example. Consider the following system:

$$\begin{cases} D^{\alpha_1} D^{\beta_1} u_1(x) = F_1(x, u_1(x), u_2(x)), D^{\delta-1} u_1(x), D^{\delta} u_1(x)), \\ D^{\alpha_2} D^{\beta_2} u_2(x) = F_2(x, u_1(x), u_2(x)), D^{\delta-1} u_2(x), D^{\delta} u_2(x)), \\ u_1(0)) = u_1(1) = u_2(0)) = u_2(1) = 0, \\ D^{\beta_1} u_1(0) = D^{\beta_1} u_1(1) = D^{\beta_2} u_2(0) = D^{\beta_2} u_2(1) = 0. \end{cases}$$

We take

$$\alpha_1 = \tfrac{5}{2}, \beta_1 = \tfrac{3}{5}, \alpha_2 = \tfrac{14}{5}, \beta_2 = \tfrac{7}{10}, \delta = \tfrac{3}{2},$$

$$F_1(x, u_1(x), u_2(x)), D^{\delta-1} u_1(x), D^{\delta} u_1(x)) = \tfrac{1}{200e^{2x}} u_1(x) + \left(\tfrac{cos(3+x^2)}{30(10+x^2)}\right) u_2(x) + \left(\tfrac{sin(x)}{10(e^x+24)}\right) D^{\frac{1}{2}} u_1(x)$$
$$+ \left(\tfrac{1}{350e^x}\right) D^{\frac{3}{2}} u_1(x)),$$

$$F_2(x, u_1(x), u_2(x)), D^{\delta-1} u_1(x), D^{\delta} u_1(x)) = \tfrac{1}{23}\left(\tfrac{1}{\pi} + ln(1 + x)\right) u_1(x) + \left(\tfrac{sin(x+1)}{300e^t}\right) u_2(x)$$
$$+ \left(\tfrac{cos(x)}{15(e^x+10)}\right) D^{\frac{1}{2}} u_1(x) + \left(\tfrac{1}{15\pi^2 e^{5t}}\right) D^{\frac{3}{2}} u_1(x)).$$

It is clear that, for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $x \in [0, 1]$, we have

$$|F_1(x, u, v, y, z) - F_1(x, r, s, t, w)| \leqslant \begin{pmatrix} p_{11}(x) |u - r| + p_{21}(x) |v - s| \\ + p_{31}(x) |y - t| + p_{41}(x) |z - w| \end{pmatrix},$$

$$|F_2(x, u, v, y, z) - F_2(x, r, s, t, w)| \leqslant \begin{pmatrix} p_{12}(x) |u - r| + p_{22}(x) |v - s| \\ + p_{32}(x) |y - t| + p_{42}(x) |z - w| \end{pmatrix}.$$

We also have

$$M_1 = \max\{0.2435, 0.1675\} = 0.2435, M_2 = \max\{0.3509, 0.1726\} = 0.3509.$$

The hypotheses of Theorem 1 are valid. Consequently, there exists exactly one solution to this system.

5. Traveling wave for beam systems

In this section, we will employ the Tanh method [7,14,25,28] to discover traveling wave solutions for a coupled system, which incorporates conformable fractional derivatives of the type

$$\begin{cases} \Im_t^{2\alpha} u(t,x) + \Im_x^{4\beta} u(t,x) + H(u,v,\Im_x^{2\beta}(u,v))(t,x) = 0, \\ \Im_t^{2\alpha} v(t,x) + \Im_x^{4\beta} v(t,x) + L(u,v,\Im_x^{2\beta}(u,v))(t,x) = 0, \end{cases} \tag{5.1}$$

where $0 < \alpha,\beta \leq 1$, the two functions H and L are to be specified, and $\Im_t^{\alpha} u(t,x)$ represents the conformable fractional derivative, as defined by Khalil, of the unknown function u with respect to t (see [21,29]); it is expressed by

$$\Im_t^{\alpha} u(t,x) = \frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}} = \lim_{\varepsilon \to 0} \left(\frac{u\left(t+\varepsilon t^{1-\alpha}, x\right) - u(t,x)}{\varepsilon} \right), \quad 0 < \alpha \leq 1.$$

Similarly, we introduce $\Im_x^{\beta} u(t,x)$.

We remark that if $\alpha = \beta = 1$, then the above system can be transformed into the classical coupled beam equations [30,36]:

$$\begin{cases} u_{tt} + u_{xxxx} + H(u,v,(u,v)_{xx}) = 0, \\ v_{tt} + v_{xxxx} + L(u,v,(u,v)_{xx}) = 0. \end{cases}$$

5.1. Tanh Methodology

Now, let us recall the important steps of the Tanh method for the scenario involving Khalil derivatives [15].

(1) We start by considering the coupled equations

$$\begin{cases} \mathbb{F}_1\left(u, v, \Im_t^{\alpha} u, \Im_x^{\beta} u, \Im_t^{\alpha} v, \Im_x^{\beta} v, \Im_t^{2\alpha} u, \Im_t^{\alpha}(\Im_x^{\beta} u), \Im_x^{2\beta} u, \Im_t^{2\alpha} v, \Im_t^{\alpha}(\Im_x^{\beta} v), \Im_x^{2\beta} v, ...\right) = 0, \\ \mathbb{F}_2\left(u, v, \Im_t^{\alpha} u, \Im_x^{\beta} u, \Im_t^{\alpha} v, \Im_x^{\beta} v, \Im_t^{2\alpha} u, \Im_t^{\alpha}(\Im_x^{\beta} u), \Im_x^{2\beta} u, \Im_t^{2\alpha} v, \Im_t^{\alpha}(\Im_x^{\beta} v), \Im_x^{2\beta} v, ...\right) = 0. \end{cases} \tag{5.2}$$

(2) Thanks to

$$\psi = \frac{k}{\alpha} t^{\alpha} + \frac{\omega}{\beta} x^{\beta}, \tag{5.3}$$

our general form can be seen as

$$\begin{cases} \mathbb{G}_1\left(U, V, U', V', U'', V'', U''', V''', ...\right) = 0, \\ \mathbb{G}_2\left(U, V, U', V', U'', V'', U''', V''', ...\right) = 0. \end{cases} \tag{5.4}$$

(3) Now, we use the transformation

$$X = tanh(\psi), \tag{5.5}$$

which allows us to get

$$
\begin{aligned}
\frac{d}{d\psi} &= \left(1-Z^2\right)\frac{d}{dZ},\\
\frac{d^2}{d\psi^2} &= -2Z\left(1-Z^2\right)\frac{d}{dZ}+\left(1-Z^2\right)^2\frac{d^2}{dZ^2},\\
\frac{d^3}{d\psi^3} &= 2\left(1-Z^2\right)\left(3Z^2-1\right)\frac{d}{dZ}-6Z\left(1-Z^2\right)^2\frac{d^2}{dZ^2}+\left(1-Z^2\right)^3\frac{d^3}{dZ^3},\\
\frac{d^4}{d\psi^4} &= -8Z\left(1-X^2\right)\left(3Z^2-2\right)\frac{d}{dZ}+4\left(1-Z^2\right)^2\left(9Z^2-2\right)\frac{d^2}{dZ^2}\\
&\quad -12Z\left(1-Z^2\right)^3\frac{d^3}{dZ^3}+\left(1-Z^2\right)^4\frac{d^4}{dZ^4}.
\end{aligned} \tag{5.6}
$$

(4) Then, we suppose that

$$
\begin{cases}
u(x,t)=U(\psi)=P(Z)=\sum_{i=0}^{m}a_iZ^i,\\
v(x,t)=V(\psi)=Q(Z)=\sum_{i=0}^{n}b_iZ^i.
\end{cases} \tag{5.7}
$$

(5) Finally, employing Wazwaz term-balancing [33], we derive the desired solutions for the constants a_i, b_i.

5.2. Applications

As an application, we propose to find traveling wave solutions for the coupled problem

$$
\begin{cases}
\mathfrak{I}_x^{4\beta}(u)+\mathfrak{I}_t^{2\alpha}(u)+\mathfrak{I}_x^{2\beta}(u)+2b\mathfrak{I}_x^{\beta}((\mathfrak{I}_x^{\beta}v)v)=0,\\
\mathfrak{I}_x^{4\beta}(v)+\mathfrak{I}_t^{2\alpha}(v)+c\mathfrak{I}_x^{2\beta}(uv+ev)=0,
\end{cases} \tag{5.8}
$$

where b, c are two real constants.

We use (5.3) to change (5.8) into the following nonlinear problem:

$$
\begin{cases}
\omega^4U_{\psi\psi\psi\psi}+k^2U_{\psi\psi}+\omega^2(U)_{\psi\psi}+2b\omega^2(V_\psi V)_\psi=0,\\
\omega^4V_{\psi\psi\psi\psi}+k^2V_{\psi\psi}+c\omega^2(UV)_{\psi\psi}+e\omega^2V_{\psi\psi}=0.
\end{cases} \tag{5.9}
$$

Integrating (5.9), we can write

$$
\begin{cases}
\omega^4U_{\psi\psi}+k^2U+\omega^2U+b\omega V^2=0,\\
\omega^4V_{\psi\psi}+k^2V+c\omega^2(UV)+e\omega^2V=0.
\end{cases} \tag{5.10}
$$

Substituting (5.6) and (5.7) into (5.10), the first equation of (5.10) is transformed into the following equation:

$$
\omega^4\Big[-2Z\left(1-Z^2\right)\frac{dP}{dZ}+\left(1-Z^2\right)^2\frac{d^2P}{dZ^2}\Big]+k^2P+b\omega^2Q^2+e\omega^2P=0. \tag{5.11}
$$

The second equation of (5.10) can be transformed into

$$
\omega^4\Big[-2Z\left(1-Z^2\right)\frac{dQ}{dZ}+\left(1-Z^2\right)^2\frac{d^2Q}{dZ^2}\Big]+k^2Q+c\omega^2(PQ)+e\omega^2Q=0. \tag{5.12}
$$

Now, in (5.11), we balance $Z^4\frac{d^2P}{dZ^2}$ with Q^2 to get $2+m=2n$.

Using the same technique with (5.12), we have $2 + n = n + m$.
Therefore, we can write

$$\begin{cases} P(Z) = a_0 + a_1 Z + a_2 Z^2, \\ Q(Z) = b_0 + b_1 Z + b_2 Z^2. \end{cases} \tag{5.13}$$

Substituting (5.13) into (5.11), we observe that

$$\begin{aligned} &-2\omega^4 Z\left(1 - Z^2\right)(a_1 + 2a_2 Z) + 2a_2\omega^4\left(1 - Z^2\right)^2 + k^2(a_0 + a_1 Z + a_2 Z^2) \\ &+ b\omega^2(b_0 + b_1 Z + b_2 Z^2)^2 + \omega^2 e(a_0 + a_1 Z + a_2 Z^2) = 0. \end{aligned} \tag{5.14}$$

Also, substituting (5.13) into (5.12), we get

$$\begin{aligned} &-2\omega^4 Z\left(1 - Z^2\right)(b_1 + 2b_2 Z) + 2b_2\omega^4\left(1 - Z^2\right)^2 + k^2(b_0 + b_1 Z + b_2 Z^2) \\ &+ c\omega^2(a_0 + a_1 Z + a_2 Z^2)(b_0 + b_1 Z + b_2 Z^2) + e\omega^2(b_0 + b_1 Z + b_2 Z^2) = 0. \end{aligned} \tag{5.15}$$

Thus, we obtain the following two sets:
Set 1.

$$\begin{cases} Z^0 : bw^2 b_0^2 - 2w^2 a_1 + 2w^2 a_2 + k^2 a_0 + w^2 a_0 = 0, \\ Z^1 : 2bw^2 b_0 b_1 - 4w^2 a_2 + k^2 a_1 + w^2 a_1 = 0, \\ Z^2 : 2bw^2 b_0 b_2 + bw^2 b_1^2 + 2w^2 a_1 - 4w^2 a_2 + k^2 a_2 + w^2 a_2 = 0, \\ Z^3 : 2bw^2 b_1 b_2 + 4w^2 a_2 = 0, \\ Z^4 : bw^2 b_2^2 + 2w^2 a_2 = 0. \end{cases} \tag{5.16}$$

Set 2.

$$\begin{cases} Z^0 : cw^2 a_0 b_0 - 2w^4 b_1 + 2w^4 b_2 + ew^2 b_0 + k^2 b_0 = 0, \\ Z^1 : cw^2 a_0 b_1 + cw^2 a_1 b_0 - 4w^4 b_2 + ew^2 b_1 + k^2 b_1 = 0, \\ Z^2 : cw^2 a_0 b_2 + cw^2 a_1 b_1 + cw^2 a_2 b_0 + 2w^4 b_1 - 4w^4 b_2 + ewb_2 + k^2 b_2 = 0, \\ Z^3 : cw^2 a_1 b_2 + cw^2 a_2 b_1 + 4w^4 b_2 = 0, \\ Z^4 : cw^2 a_2 b_2 + 2w^4 b_2 = 0. \end{cases} \tag{5.17}$$

Solving (5.16) and (5.17) with the aid of Maple, we obtain the following:
Case 1.

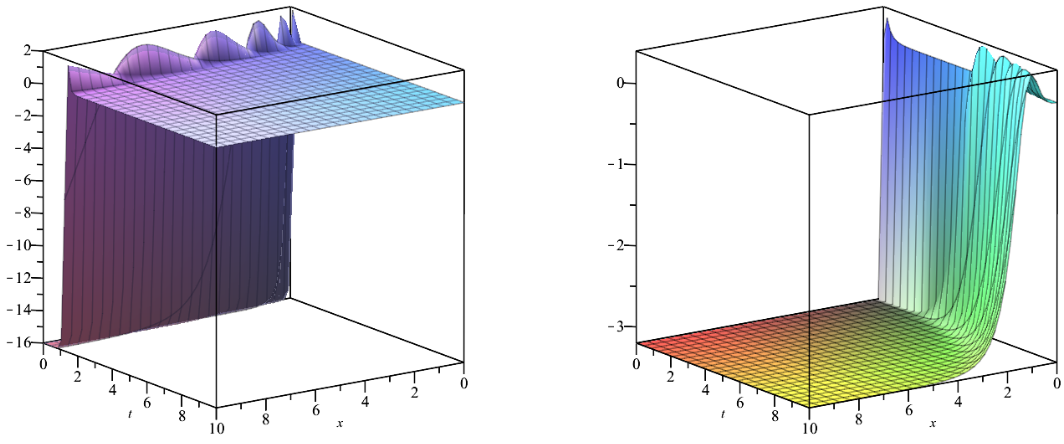
$$\begin{aligned} &\omega = e, \ \ k = \pm e\sqrt{4e^2 - 1}, \ \ a_0 = 0, \ \ a_1 = -\frac{2e^2}{c}, \ \ a_2 = -\frac{2e^2}{c}, \\ &b_0 = 0, \ \ b_1 = \pm 2e^2\sqrt{\frac{1}{bc}}, \ \ b_2 = \pm 2e^2\sqrt{\frac{1}{bc}}. \end{aligned} \tag{5.18}$$

Substituting (5.18) into (5.13), the following traveling wave solution of (5.8) is obtained:

$$u(x,t) = -\frac{2e^2}{c}\tanh(\psi) - \frac{2e^2}{c}\tanh^2(\psi), \tag{5.19}$$

$$v(x,t) = \pm 2e^2\sqrt{\frac{1}{bc}}\tanh(\psi) \pm 2e^2\sqrt{\frac{1}{bc}}\tanh^2(\psi). \tag{5.20}$$

Now, we trace the two components of this traveling wave solution in Figure 1 under specific parameters.



(a) Plots of (5.19) **(b)** Plots of (5.20)

Figure 1. *Plots of (5.19, 5.20), with* $0 \leq x \leq 10$, $0 \leq t \leq 10$ and $b = 5, c = 1, e = 2, \omega = e$, $\alpha = \frac{3}{5}, \beta = \frac{1}{5}$.

Case 2.

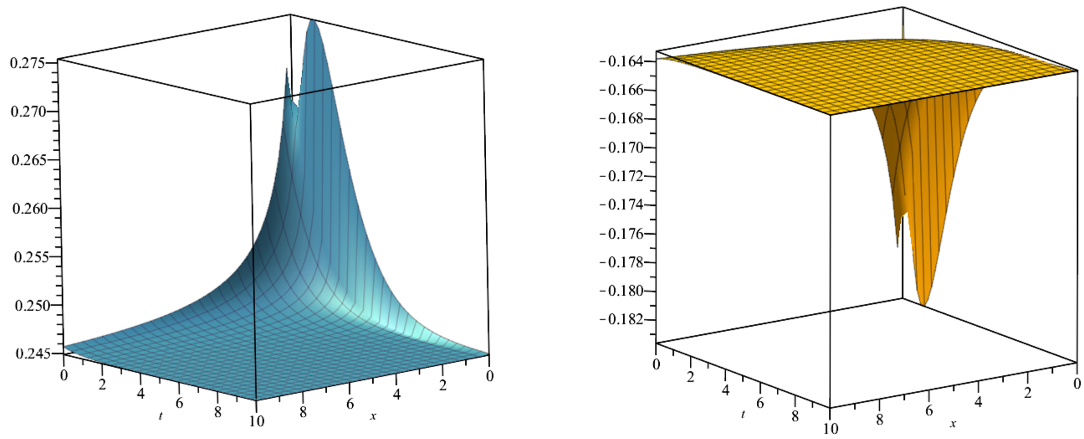
$$\begin{aligned} &\omega = e,\ \ k = k,\ \ a_0 = \frac{4e^2}{c},\ \ a_1 = -\frac{2e^2}{c},\ \ a_2 = -\frac{2e^2}{c}, \\ &b_0 = \pm 4e^2\sqrt{\frac{1}{bc}},\ \ b_1 = \pm 2e^2\sqrt{\frac{1}{bc}},\ \ b_2 = \pm 2e^2\sqrt{\frac{1}{bc}}. \end{aligned} \tag{5.21}$$

Substituting (5.21) into (5.13), the following traveling wave solution of (5.8) is also obtained:

$$u(x,t) = \frac{4e^2}{c} - \frac{2e^2}{c}\tanh(\psi) - \frac{2e^2}{c}\tanh^2(\psi), \tag{5.22}$$

$$v(x,t) = \pm 4e^2\sqrt{\frac{1}{bc}} \pm 2e^2\sqrt{\frac{1}{bc}}\tanh(\psi) \pm 2e^2\sqrt{\frac{1}{bc}}\tanh^2(\psi). \tag{5.23}$$

As above, we trace the two components of this traveling wave solution in Figure 2 under specific parameters.



(a) Plots of solution (5.22)

(b) Plots of solution (5.23)

Figure 2. *Plots of (5.22, 5.23) with* $0 \le x \le 10$, $0 \le t \le 10$ and $b = \frac{3}{2}, c = 3, e = \frac{3}{7}, \omega = e,$ $\alpha = \frac{8}{9}, \beta = \frac{3}{10}$.

6. Conclusions

In the above work, we explored two distinct yet related problems in the field of mathematics. The first part involved analyzing a fractional system with Caputo derivatives, which generalizes a beam deflection type system. We focused on proving the existence of a unique solution for this system. In the second part of our research, we employed conformable fractional derivatives, as defined by Khalil, to examine another system consisting of two coupled evolution equations. By transforming this conformable fractional system, we derived an ordinary differential system characterized by traveling waves. The connection between these two parts lies in the use of fractional calculus to extend and enhance classical models, demonstrating the versatility and power of fractional derivatives—both Caputo and conformable—in addressing and solving advanced mathematical problems. This unified approach shows how different fractional derivatives can be applied to study and solve a variety of complex systems, thereby enriching our understanding and capabilities in mathematical modeling and analysis.

Author contributions

Abdelkader Lamamri: Conceptualization, formal analysis, investigation, methodology and writing-original draft; Iqbal Jebril: Formal analysis, investigation and methodology; Zoubir Dahmani: Supervision and validation; Ahmed Anber: Formal analysis, writing-original numerical draft; Mahdi Rakah: Software and validation; Shawkat Alkhazaleh: Validation and visualisation . All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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