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Research article

Nonlinear mixed type product $[\mathcal{K},\mathcal{F}]_* \odot \mathcal{D}$ on $*$-algebras

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Abstract: Let $\mathcal{A}$ be a unital $*$-algebra containing a non-trivial projection. In this paper, we prove that if a map $\Omega : \mathcal{A} \to \mathcal{A}$ such that

$$\Omega([\mathcal{K},\mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}),\mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K},\Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K},\mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

where $[\mathcal{K},\mathcal{F}]_* = \mathcal{K}\mathcal{F} - \mathcal{F}\mathcal{K}^*$ and $\mathcal{K} \odot \mathcal{F} = \mathcal{K}^*\mathcal{F} + \mathcal{F}\mathcal{K}^*$ for all $\mathcal{K},\mathcal{F},\mathcal{D} \in \mathcal{A}$, then Ω is an additive $*$-derivation. Furthermore, we extend its results on factor von Neumann algebras, standard operator algebras and prime $*$-algebras. Additionally, we provide an example illustrating the existence of such maps.

Keywords: mixed bi-skew Jordan triple derivation; $*$-derivation; $*$- algebra; involution
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1. Introduction

Consider an algebra $\mathcal{A}$ defined over the complex field $\mathbb{C}$. A map $* : \mathcal{A} \to \mathcal{A}$ is called an involution if the following conditions hold for all $\mathcal{K},\mathcal{F} \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. (i) $(\mathcal{K}+\mathcal{F})^* = \mathcal{K}^* + \mathcal{F}^*$; (ii) $(\alpha\mathcal{K})^* = \bar{\alpha}\mathcal{K}^*$; (iii) $(\mathcal{K}\mathcal{F})^* = (\mathcal{F})^*(\mathcal{K})^*$ and $(\mathcal{K}^*)^* = \mathcal{K}$. An algebra $\mathcal{A}$ with the involution $*$ is called the $*$-algebra. For $\mathcal{K},\mathcal{F} \in \mathcal{A}$, we call $[\mathcal{K},\mathcal{F}]_* = \mathcal{K}\mathcal{F} - \mathcal{F}\mathcal{K}^*$ the skew Lie product, $[\mathcal{K},\mathcal{F}]_\bullet = \mathcal{K}\mathcal{F}^* - \mathcal{F}\mathcal{K}^*$ denotes the bi-skew Lie product and $\mathcal{K} \odot \mathcal{F} = \mathcal{K}^*\mathcal{F} + \mathcal{F}\mathcal{K}^*$ denotes the bi-skew Jordan product. The skew Lie product, the Jordan product, and the bi-skew Jordan product have become increasingly relevant in various research fields, and numerous authors have shown a keen interest in their exploration. This is evident from the numerous studies by authors (see [1–3, 5, 7–10, 13, 15, 16]). Recall that an additive map $\Omega : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\Omega(\mathcal{K}\mathcal{F}) = \Omega(\mathcal{K})\mathcal{F} + \mathcal{K}\Omega(\mathcal{F})$ for all $\mathcal{K}, \mathcal{F} \in \mathcal{A}$. If $\Omega(\mathcal{K}^*) = \Omega(\mathcal{K})^*$ for all $\mathcal{K} \in \mathcal{A}$, then Ω is an additive $*$-derivation. Let $\Omega : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). We say Ω is a nonlinear skew Lie derivation or nonlinear skew Lie triple derivation if

$$\Omega([\mathcal{K}, \mathcal{F}]_*) = [\Omega(\mathcal{K}), \mathcal{F}]_* + [\mathcal{K}, \Omega(\mathcal{F})]_*$$

or

$$\Omega([[\mathcal{K}, \mathcal{F}]_*, \mathcal{D}]_*) = [[\Omega(\mathcal{K}), \mathcal{F}]_*, \mathcal{D}]_* + [[\mathcal{K}, \Omega(\mathcal{F})]_*, \mathcal{D}]_* + [[\mathcal{K}, \mathcal{F}]_*, \Omega(\mathcal{D})]_*$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$. Similarly, a map $\Omega : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear bi-skew Lie derivation or nonlinear bi-skew Lie triple derivation if

$$\Omega([\mathcal{K}, \mathcal{F}]_\bullet = [\Omega(\mathcal{K}), \mathcal{F}]_\bullet + [\mathcal{K}, \Omega(\mathcal{F})]_\bullet$$

or

$$\Omega([[\mathcal{K}, \mathcal{F}]_\bullet, \mathcal{D}]_\bullet) = [[\Omega(\mathcal{K}), \mathcal{F}]_\bullet, \mathcal{D}]_\bullet + [[\mathcal{K}, \Omega(\mathcal{F})]_\bullet, \mathcal{D}]_\bullet + [[\mathcal{K}, \mathcal{F}]_\bullet, \Omega(\mathcal{D})]_\bullet$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$. In 2021, A. Khan [4] established a proof demonstrating that any multiplicative or nonadditive bi-skew Lie triple derivation acting on a factor Von Neumann algebra can be characterized as an additive $*$-derivation.

Numerous authors have recently explored the derivations and isomorphisms corresponding to the novel products created by combining Lie and skew Lie products, skew Lie and skew Jordan products see [6, 11, 12, 14]. As an illustration, Li and Zhang [6] delved into an investigation focused on understanding the arrangement and properties of the nonlinear mixed Jordan triple $*$-derivation within the domain of $*$-algebras. In 2022, Rehman et. al. [12] mixed the concepts of Jordan and Jordan $*$-product and gave the complete characterization of nonlinear mixed Jordan $*$-triple derivation on $*$-algebras. Inspired by the above results, in the present paper, we combined the skew Lie product and bi-skew Jordan product and defined nonlinear mixed bi-skew Jordan triple derivation on $*$-algebras. A map Ω: $\mathcal{A} \to \mathcal{A}$ is called nonlinear mixed bi-skew Jordan triple derivations if

$$\Omega([\mathcal{K}, \mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}), \mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K}, \Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K}, \mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$. Our proof establishes that when Ω represents a nonlinear mixed bi-skew Lie triple derivation acting on $*$-algebras, it necessarily possesses an additive $*$-derivation. In simpler terms, the study demonstrates that specific properties, such as additivity and self-adjointness, can be attributed to the nature of nonlinear mixed bi-skew Jordan triple derivations on $*$-algebras.

2. Main result

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with unity $\mathcal{I}$ containing a non-trivial projection P. Suppose that $\mathcal{A}$ satisfies*

$$\mathcal{X}\mathcal{A}P = 0 \implies \mathcal{X} = 0, \tag{▲}$$

and

$$\mathcal{X}\mathcal{A}(\mathcal{I} - P) = 0 \implies \mathcal{X} = 0. \tag{▼}$$

Define a map $\Omega : \mathcal{A} \to \mathcal{A}$ such that

$$\Omega([\mathcal{K}, \mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}), \mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K}, \Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K}, \mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

then Ω *is an additive* $*$*-derivation.*

Let $P = \mathcal{P}_1$ be a non-trivial projection in $\mathcal{A}$, and $\mathcal{P}_2 = \mathcal{I} - \mathcal{P}_1$, where $\mathcal{I}$ is the unity of this algebra. Then by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{P}_1\mathcal{A}\mathcal{P}_1 \oplus \mathcal{P}_1\mathcal{A}\mathcal{P}_2 \oplus \mathcal{P}_2\mathcal{A}\mathcal{P}_1 \oplus \mathcal{P}_2\mathcal{A}\mathcal{P}_2$ and, denote $\mathcal{A}_{11} = \mathcal{P}_1\mathcal{A}\mathcal{P}_1, \mathcal{A}_{12} = \mathcal{P}_1\mathcal{A}\mathcal{P}_2, \mathcal{A}_{21} = \mathcal{P}_2\mathcal{A}\mathcal{P}_1$ and $\mathcal{A}_{22} = \mathcal{P}_2\mathcal{A}\mathcal{P}_2$. Note that any $\mathcal{K} \in \mathcal{A}$ can be written as $\mathcal{K} = \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}$, where $\mathcal{K}_{ij} \in \mathcal{A}_{ij}$ and $\mathcal{K}_{ij}^* \in \mathcal{A}_{ji}$ for $i, j = 1, 2$.

Several lemmas are used to prove Theorem 2.1.

Lemma 2.1. $\Omega(0) = 0$ *and* $\Omega(\mathcal{I}) = \Omega(\mathcal{I})^*$.

Proof. It is trivial that

$$\Omega(0) = \Omega([0, 0]_* \odot 0) = [\Omega(0), 0]_* \odot 0 + [0, \Omega(0)]_* \odot 0 + [0, 0]_* \odot \Omega(0) = 0.$$

We can easily see that

$$\Omega([\mathcal{I}, i\mathcal{I}]_* \odot \mathcal{I}) = 0.$$

From the other side, we yield

$$\Omega([\mathcal{I}, i\mathcal{I}]_* \odot \mathcal{I}) = [\Omega(\mathcal{I}), i\mathcal{I}]_* \odot \mathcal{I} + [\mathcal{I}, \Omega(i\mathcal{I})]_* \odot \mathcal{I} + [\mathcal{I}, i\mathcal{I}]_* \odot \Omega(\mathcal{I}) = -2i\Omega(\mathcal{I})^* + 2i\Omega(\mathcal{I}).$$

From the equations above, we can deduce

$$\Omega(\mathcal{I}) = \Omega(\mathcal{I})^*.$$

The proof is now concluded. □

Lemma 2.2. *For any* $\mathcal{K}_{12} \in \mathcal{A}_{12}, \mathcal{K}_{21} \in \mathcal{A}_{21}$, *we have*

$$\Omega(\mathcal{K}_{12} + \mathcal{K}_{21}) = \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}).$$

Proof. Let $M = \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}) - \Omega(\mathcal{K}_{12}) - \Omega(\mathcal{K}_{21})$. We have

$$\begin{aligned} \Omega([\mathcal{K}_{12} + \mathcal{K}_{21}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{K}_{12} + \mathcal{K}_{21}), \mathcal{P}_1]_* \odot \mathcal{P}_2 + [\mathcal{K}_{12} + \mathcal{K}_{21}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{12} + \mathcal{K}_{21}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2). \end{aligned}$$

Alternatively, it follows from $[\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$ that

$$\begin{aligned} \Omega([\mathcal{K}_{12} + \mathcal{K}_{21}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= \Omega([\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2) + \Omega([\mathcal{K}_{21}, \mathcal{P}_1]_* \odot \mathcal{P}_2) \\ &= [\Omega(\mathcal{K}_{12}), \mathcal{P}_1]_* \odot \mathcal{P}_2 + [\mathcal{K}_{12}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2) + [\Omega(\mathcal{K}_{21}), \mathcal{P}_1]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{21}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 + [\mathcal{K}_{21}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2). \end{aligned}$$

From the last two expressions, we conclude $[M, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$. That means $\mathcal{P}_1 M^* \mathcal{P}_2 - \mathcal{P}_2 M \mathcal{P}_1 = 0$. By multiplying $\mathcal{P}_2$ from the left, we find $\mathcal{P}_2 M \mathcal{P}_1 = 0$. In similar way, we can easily show that $\mathcal{P}_1 M \mathcal{P}_2 = 0$.

Also, $[i(\mathcal{P}_1 - \mathcal{P}_2), \mathcal{I}]_* \odot \mathcal{K}_{12} = 0$. Thus,

$$\Omega([i(\mathcal{P}_1 - \mathcal{P}_2), \mathcal{I}]_* \odot (\mathcal{K}_{12} + \mathcal{K}_{21}))$$

$$\begin{aligned} &= \Omega([i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot\mathcal{K}_{12})+\Omega([i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot\mathcal{K}_{21})\\ &= [\Omega(i(\mathcal{P}_1-\mathcal{P}_2)),\mathcal{I}]_*\odot\mathcal{K}_{12}+[i(\mathcal{P}_1-\mathcal{P}_2),\Omega(\mathcal{I})]_*\odot\mathcal{K}_{12}\\ &\quad +[i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot\Omega(\mathcal{K}_{12})+[\Omega(i(\mathcal{P}_1-\mathcal{P}_2)),\mathcal{I}]_*\odot\mathcal{K}_{21}\\ &\quad +[i(\mathcal{P}_1-\mathcal{P}_2),\Omega(\mathcal{I})]_*\odot\mathcal{K}_{21}+[i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot\Omega(\mathcal{K}_{21}). \end{aligned}$$

On the other side, we have

$$\begin{aligned} \Omega([i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot(\mathcal{K}_{12}+\mathcal{K}_{21})) &= [\Omega(i(\mathcal{P}_1-\mathcal{P}_2)),\mathcal{I}]_*\odot(\mathcal{K}_{12}+\mathcal{K}_{21})\\ &\quad +[i(\mathcal{P}_1-\mathcal{P}_2),\Omega(\mathcal{I})]_*\odot(\mathcal{K}_{12}+\mathcal{K}_{21})\\ &\quad +[i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot\Omega(\mathcal{K}_{12}+\mathcal{K}_{21}). \end{aligned}$$

From the last two expressions, we obtain $[i(\mathcal{P}_1-\mathcal{P}_2),\mathcal{I}]_*\odot M=0$. That means $-2i\mathcal{P}_1M+2i\mathcal{P}_2M-2iM\mathcal{P}_1+2iM\mathcal{P}_2=0$. By pre and post multiplying by $\mathcal{P}_1$ from both sides, we get $\mathcal{P}_1M\mathcal{P}_1=0$. In the similar way, we can show that $\mathcal{P}_2M\mathcal{P}_2=0$. Hence, $M=0$, i.e.,

$$\Omega(\mathcal{K}_{12}+\mathcal{K}_{21})=\Omega(\mathcal{K}_{12})+\Omega(\mathcal{K}_{21}).$$

The proof is now concluded. □

Lemma 2.3. *For any $\mathcal{K}_{ii}\in\mathcal{A}_{ii},\mathcal{K}_{ij}\in\mathcal{A}_{ij},1\le i,j\le 2$, we have*

$$\Omega(\mathcal{K}_{ii}+\mathcal{K}_{ij}+\mathcal{K}_{ji})=\Omega(\mathcal{K}_{ii})+\Omega(\mathcal{K}_{ij})+\Omega(\mathcal{K}_{ji}).$$

Proof. First, we will demonstrate the case when $i=1$ and $j=2$. Let $M=\Omega(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21})-\Omega(\mathcal{K}_{11})-\Omega(\mathcal{K}_{12})-\Omega(\mathcal{K}_{21})$. Since $[\mathcal{K}_{11},\mathcal{P}_1]_*\odot\mathcal{P}_2=0$ and using Lemma 2.2, we have

$$\begin{aligned} \Omega([(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}),\mathcal{P}_1]_*\odot\mathcal{P}_2) &= \Omega([\mathcal{K}_{11},\mathcal{P}_1]_*\odot\mathcal{P}_2)+\Omega([\mathcal{K}_{12},\mathcal{P}_1]_*\odot\mathcal{P}_2)\\ &\quad +\Omega([\mathcal{K}_{21},\mathcal{P}_1]_*\odot\mathcal{P}_2)\\ &= [\Omega(\mathcal{K}_{11}),\mathcal{P}_1]_*\odot\mathcal{P}_2+[\mathcal{K}_{11},\Omega(\mathcal{P}_1)]_*\odot\mathcal{P}_2\\ &\quad +[\mathcal{K}_{11},\mathcal{P}_1]_*\odot\Omega(\mathcal{P}_2)+[\Omega(\mathcal{K}_{12}),\mathcal{P}_1]_*\odot\mathcal{P}_2\\ &\quad +[\mathcal{K}_{12},\Omega(\mathcal{P}_1)]_*\odot\mathcal{P}_2+[\mathcal{K}_{12},\mathcal{P}_1]_*\odot\Omega(\mathcal{P}_2)\\ &\quad +[\Omega(\mathcal{K}_{21}),\mathcal{P}_1]_*\odot\mathcal{P}_2+[\mathcal{K}_{21},\Omega(\mathcal{P}_1)]_*\odot\mathcal{P}_2\\ &\quad +[\mathcal{K}_{21},\mathcal{P}_1]_*\odot\Omega(\mathcal{P}_2). \end{aligned}$$

On the other side, we have

$$\begin{aligned} \Omega([(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}),\mathcal{P}_1]_*\odot\mathcal{P}_2) &= [\Omega(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}),\mathcal{P}_1]_*\odot\mathcal{P}_2\\ &\quad +[(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}),\Omega(\mathcal{P}_1)]_*\odot\mathcal{P}_2\\ &\quad +[(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}),\mathcal{P}_1]_*\odot\Omega(\mathcal{P}_2). \end{aligned}$$

From the above two expressions, we find $[M,\mathcal{P}_1]_*\odot\mathcal{P}_2=0$, and so, $\mathcal{P}_2M\mathcal{P}_1=0$. Similarly, $\mathcal{P}_1M\mathcal{P}_2=0$. Now, for all $\mathcal{X}_{12}\in\mathcal{A}_{12}$, we have

$$\Omega([\mathcal{X}_{12},(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21})]_*\odot\mathcal{P}_2) = [\Omega(\mathcal{X}_{12}),(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21})]_*\odot\mathcal{P}_2$$

$$+[\mathcal{X}_{12}, \Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21})]_* \odot \mathcal{P}_2$$
$$+[\mathcal{X}_{12}, (\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21})]_* \odot \Omega(\mathcal{P}_2).$$

Also, $[\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \mathcal{P}_2 = 0$ and using Lemma 2.2, we get

$$\begin{aligned}\Omega([\mathcal{X}_{12}, (\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21})]_* \odot \mathcal{P}_2) &= \Omega([\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \mathcal{P}_2) + \Omega([\mathcal{X}_{12}, \mathcal{K}_{12}]_* \odot \mathcal{P}_2)\\ &\quad +\Omega([\mathcal{X}_{12}, \mathcal{K}_{21}]_* \odot \mathcal{P}_2)\\ &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{11}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{11})]_* \odot \mathcal{P}_2\\ &\quad +[\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \Omega(\mathcal{P}_2) + [\Omega(\mathcal{X}_{12}), \mathcal{K}_{12}]_* \odot \mathcal{P}_2\\ &\quad +[\mathcal{X}_{12}, \Omega(\mathcal{K}_{12})]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \mathcal{K}_{12}]_* \odot \Omega(\mathcal{P}_2)\\ &\quad +[\Omega(\mathcal{X}_{12}), \mathcal{K}_{21}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{21})]_* \odot \mathcal{P}_2\\ &\quad +[\mathcal{X}_{12}, \mathcal{K}_{21}]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

From the above two relations, we get $[\mathcal{X}_{12}, M]_* \odot \mathcal{P}_2 = 0$. That means $-\mathcal{X}_{12}M^*\mathcal{P}_2 + \mathcal{P}_2M^*\mathcal{X}_{12}^* = 0$. By post-multiplying by $\mathcal{P}_2$ on both sides, we get $-\mathcal{X}_{12}M^*\mathcal{P}_2 = 0$. Therefore, by using (▲) and (▼), we get $\mathcal{P}_2M\mathcal{P}_2 = 0$. Similarly, $\mathcal{P}_1M\mathcal{P}_1 = 0$. Hence, $M = 0$. i.e.,

$$\Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21}) = \Omega(\mathcal{K}_{11}) + \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}).$$

By using the same technique, we can also show for $i = 2, j = 1$. The proof is now concluded. □

Lemma 2.4. *For any* $\mathcal{K}_{ij} \in \mathcal{A}_{ij}, 1 \le i, j \le 2$, *we have*

$$\Omega(\sum_{i,j=1}^{2} \mathcal{K}_{ij}) = \sum_{i,j=1}^{2} \Omega(\mathcal{K}_{ij}).$$

Proof. Let $M = \Omega(\mathcal{K}_{11}+\mathcal{K}_{12}+\mathcal{K}_{21}+\mathcal{K}_{22})-\Omega(\mathcal{K}_{11})-\Omega(\mathcal{K}_{12})-\Omega(\mathcal{K}_{21})-\Omega(\mathcal{K}_{22})$. Since, $[\mathcal{K}_{11}, \mathcal{P}_1]_*\odot\mathcal{P}_2 = 0$ and using Lemma 2.3 that

$$\begin{aligned}&\Omega([(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}), \mathcal{P}_1]_* \odot \mathcal{P}_2)\\ &= \Omega([\mathcal{K}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2) + \Omega([\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2)\\ &\quad +\Omega([\mathcal{K}_{21}, \mathcal{P}_1]_* \odot \mathcal{P}_2) + \Omega([\mathcal{K}_{22}, \mathcal{P}_1]_* \odot \mathcal{P}_2)\\ &= [\Omega(\mathcal{K}_{11}) + \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{K}_{22}), \mathcal{P}_1]_* \odot \mathcal{P}_2\\ &\quad +[\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2\\ &\quad +[\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

Alternatively, we have

$$\begin{aligned}\Omega([(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}), \mathcal{P}_1]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}), \mathcal{P}_1]_* \odot \mathcal{P}_2\\ &\quad +[(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}), \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2\\ &\quad +[(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}), \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

From the last two relations, we get $[M, \mathcal{P}_1]_*\odot\mathcal{P}_2 = 0$. Thus, $\mathcal{P}_1M^*\mathcal{P}_2-\mathcal{P}_2M\mathcal{P}_1 = 0$. Hence, $\mathcal{P}_2M\mathcal{P}_1 = 0$. Similarly, $\mathcal{P}_1M\mathcal{P}_2 = 0$.

Now, for any $\mathcal{X}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}\Omega([\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22})]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

Also, $[\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \mathcal{P}_2 = 0$, and using Lemma 2.3, we find

$$\begin{aligned}&\Omega([\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \mathcal{P}_2) \\ &= \Omega([\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \mathcal{P}_2) + \Omega([\mathcal{X}_{12}, \mathcal{K}_{12}]_* \odot \mathcal{P}_2) \\ &\quad + \Omega([\mathcal{X}_{12}, \mathcal{K}_{21}]_* \odot \mathcal{P}_2) + \Omega([\mathcal{X}_{12}, \mathcal{K}_{22}]_* \odot \mathcal{P}_2) \\ &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{11}) + \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{K}_{22})]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

Upon comparing the aforementioned two equations, we observe that $[\mathcal{X}_{12}, M]_* \odot \mathcal{P}_2 = 0$. On solving, we get $\mathcal{P}_2 M \mathcal{P}_2 = 0$. Similarly, we can show that $\mathcal{P}_1 M \mathcal{P}_1 = 0$. Hence, $M = 0$, i.e.,

$$\Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}) = \Omega(\mathcal{K}_{11}) + \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{K}_{22}).$$

This ends the proof. □

Lemma 2.5. *For each $\mathcal{K}_{12}, \mathcal{F}_{12} \in \mathcal{A}_{12}$ and $\mathcal{K}_{21}, \mathcal{F}_{21} \in \mathcal{A}_{21}$, we have*

(1) $\Omega(\mathcal{K}_{12} + \mathcal{F}_{12}) = \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{F}_{12})$.
(2) $\Omega(\mathcal{K}_{21} + \mathcal{F}_{21}) = \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{F}_{21})$.

Proof. (1) Let $M = \Omega(\mathcal{K}_{12} + \mathcal{F}_{12}) - \Omega(\mathcal{K}_{12}) - \Omega(\mathcal{F}_{12})$. We have,

$$\begin{aligned}\Omega([\mathcal{K}_{12} + \mathcal{F}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{K}_{12} + \mathcal{F}_{12}), \mathcal{P}_1)]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{12} + \mathcal{F}_{12}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{12} + \mathcal{F}_{12}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

On the other hand, it follows from $[\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$ that

$$\begin{aligned}\Omega([\mathcal{K}_{12} + \mathcal{F}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= \Omega([\mathcal{K}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2) + \Omega([\mathcal{F}_{12}, \mathcal{P}_1]_* \odot \mathcal{P}_2) \\ &= [\Omega(\mathcal{K}_{12}) + \Omega(\mathcal{F}_{12}), \mathcal{P}_1]_* \odot \mathcal{P}_2 + [\mathcal{K}_{12} + \mathcal{F}_{12}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{K}_{12} + \mathcal{F}_{12}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

From the last two relations, we get $[M, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$. This means that $\mathcal{P}_1 M^* \mathcal{P}_2 - \mathcal{P}_2 M \mathcal{P}_1 = 0$. By pre-multiplying $\mathcal{P}_2$ on both sides, we get $\mathcal{P}_2 M \mathcal{P}_1 = 0$. Similarly, we can show that $\mathcal{P}_1 M \mathcal{P}_2 = 0$. Now, for any $\mathcal{X}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}\Omega([\mathcal{X}_{12}, \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{12} + \mathcal{F}_{12})]_* \odot \mathcal{P}_2 \\ &\quad + [\mathcal{X}_{12}, \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \Omega(\mathcal{P}_2).\end{aligned}$$

On the other hand, it follows from $[\mathcal{X}_{12}, \mathcal{K}_{12}]_* \odot \mathcal{P}_2 = 0$ that

$$\begin{aligned}
&\Omega([\mathcal{X}_{12}, \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \mathcal{P}_2) \\
&\quad = \Omega([\mathcal{X}_{12}, \mathcal{K}_{12}]_* \odot \mathcal{P}_2) + \Omega([\mathcal{X}_{12}, \mathcal{F}_{12}]_* \odot \mathcal{P}_2) \\
&\quad = [[\Omega(\mathcal{X}_{12}), \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{F}_{12})]_* \odot \mathcal{P}_2 \\
&\qquad + [\mathcal{X}_{12}, \mathcal{K}_{12} + \mathcal{F}_{12}]_* \odot \Omega(\mathcal{P}_2).
\end{aligned}$$

On comparing the above two relations, we get $[\mathcal{X}_{12}, M]_* \odot \mathcal{P}_2 = 0$. On solving, we get $\mathcal{P}_2 M \mathcal{P}_2 = 0$. Similarly, we can show that $\mathcal{P}_1 M \mathcal{P}_1 = 0$. Hence, $M = 0$, i.e.,

$$\Omega(\mathcal{K}_{12} + \mathcal{F}_{12}) = \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{F}_{12}).$$

(2) By using the same technique, we can show that

$$\Omega(\mathcal{K}_{21} + \mathcal{F}_{21}) = \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{F}_{21}).$$

The proof is now concluded. □

Lemma 2.6. *For each $\mathcal{K}_{ii}, \mathcal{F}_{ii} \in \mathcal{A}_{ii}$ such that $1 \le i \le 2$, we have*

$$\Omega(\mathcal{K}_{ii} + \mathcal{F}_{ii}) = \Omega(\mathcal{K}_{ii}) + \Omega(\mathcal{F}_{ii}).$$

Proof. First, it is prove for $i = 1$. Let $M = \Omega(\mathcal{K}_{11} + \mathcal{F}_{11}) - \Omega(\mathcal{K}_{11}) - \Omega(\mathcal{F}_{11})$. Since, $[\mathcal{K}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$, we have

$$\begin{aligned}
\Omega([\mathcal{K}_{11} + \mathcal{F}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= \Omega([\mathcal{K}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2) + \Omega([\mathcal{F}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2) \\
&= [\Omega(\mathcal{K}_{11}) + \Omega(\mathcal{F}_{11}), \mathcal{P}_1]_* \odot \mathcal{P}_2 + [\mathcal{K}_{11} + \mathcal{F}_{11}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\
&\quad + [\mathcal{K}_{11} + \mathcal{F}_{11}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Omega([\mathcal{K}_{11} + \mathcal{F}_{11}, \mathcal{P}_1]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{K}_{11} + \mathcal{F}_{11}), \mathcal{P}_1]_* \odot \mathcal{P}_2 + [\mathcal{K}_{11} + \mathcal{F}_{11}, \Omega(\mathcal{P}_1)]_* \odot \mathcal{P}_2 \\
&\quad + [\mathcal{K}_{11} + \mathcal{F}_{11}, \mathcal{P}_1]_* \odot \Omega(\mathcal{P}_2).
\end{aligned}$$

Upon comparing the aforementioned two equations, we observe that $[M, \mathcal{P}_1]_* \odot \mathcal{P}_2 = 0$. On solving, we get $\mathcal{P}_1 M^* \mathcal{P}_2 - \mathcal{P}_2 M \mathcal{P}_1 = 0$. By pre-multiplying by $\mathcal{P}_2$ on both sides, we get $\mathcal{P}_2 M \mathcal{P}_1 = 0$. Similarly, $\mathcal{P}_1 M \mathcal{P}_2 = 0$. Now, for any $\mathcal{X}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}
\Omega([\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{11} + \mathcal{F}_{11})]_* \odot \mathcal{P}_2 \\
&\quad + [\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \Omega(\mathcal{P}_2).
\end{aligned}$$

It follows from $[\mathcal{X}_{12}, \mathcal{K}_{11}]_* \odot \mathcal{P}_1 = 0$ that

$$\begin{aligned}
\Omega([\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \mathcal{P}_2) &= [\Omega(\mathcal{X}_{12}), \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \mathcal{P}_2 + [\mathcal{X}_{12}, \Omega(\mathcal{K}_{11}) + \Omega(\mathcal{F}_{11})]_* \odot \mathcal{P}_2 \\
&\quad + [\mathcal{X}_{12}, \mathcal{K}_{11} + \mathcal{F}_{11}]_* \odot \Omega(\mathcal{P}_2).
\end{aligned}$$

By comparing, we get $[\mathcal{X}_{12}, M]_* \odot \mathcal{P}_2 = 0$. That means $-\mathcal{X}_{12} M^* P_2 + P_2 M^* \mathcal{X}_{12}^* = 0$. By pre-multiplying P_2 on both sides, we get $P_2 M^* \mathcal{X}_{12}^* = 0$. Thus, by using (▲) and (▼), we get $P_2 M P_2 = 0$. Similarly, $P_1 M P_1 = 0$. Hence, $M = 0$. This completes the proof. □

Lemma 2.7. Ω *is an additive map.*

Proof. For any $\mathcal{K}, \mathcal{F} \in \mathcal{A}$, we write $\mathcal{K} = \mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22}$ and $\mathcal{F} = \mathcal{F}_{11} + \mathcal{F}_{12} + \mathcal{F}_{21} + \mathcal{F}_{22}$. By using Lemmas 2.4–2.6, we get

$$\begin{aligned}\Omega(\mathcal{K} + \mathcal{F}) &= \Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{K}_{22} + \mathcal{F}_{11} + \mathcal{F}_{12} + \mathcal{F}_{21} + \mathcal{F}_{22}) \\ &= \Omega(\mathcal{K}_{11} + \mathcal{F}_{11}) + \Omega(\mathcal{K}_{12} + \mathcal{F}_{12}) + \Omega(\mathcal{K}_{21} + \mathcal{F}_{21}) + \Omega(\mathcal{K}_{22} + \mathcal{F}_{22}) \\ &= \Omega(\mathcal{K}_{11}) + \Omega(\mathcal{F}_{11}) + \Omega(\mathcal{K}_{12}) + \Omega(\mathcal{F}_{12}) \\ &\quad + \Omega(\mathcal{K}_{21}) + \Omega(\mathcal{F}_{21}) + \Omega(\mathcal{K}_{22}) + \Omega(\mathcal{F}_{22}) \\ &= \Omega(\mathcal{K}_{11} + \mathcal{K}_{12} + \mathcal{K}_{21} + \mathcal{F}_{22}) + \Omega(\mathcal{F}_{11} + \mathcal{F}_{12} + \mathcal{F}_{21} + \mathcal{F}_{22}) \\ &= \Omega(\mathcal{K}) + \Omega(\mathcal{F}).\end{aligned}$$

Hence, Ω is additive. □

Lemma 2.8. *The following conditions holds:*

(i) $\Omega(i\mathfrak{I})^* = \Omega(i\mathfrak{I}) = 0$.
(ii) $\Omega(\mathfrak{I}) = 0$.

Proof. (i) It follows from Lemma 2.7 that

$$\Omega([i\mathfrak{I}, i\mathfrak{I}]_* \odot \mathfrak{I}) = \Omega(-4\mathfrak{I}) = -4\Omega(\mathfrak{I})$$

and

$$\begin{aligned}\Omega([i\mathfrak{I}, i\mathfrak{I}]_* \odot \mathfrak{I}) &= [\Omega(i\mathfrak{I}), i\mathfrak{I}]_* \odot \mathfrak{I} + [i\mathfrak{I}, \Omega(i\mathfrak{I})]_* \odot \mathfrak{I} + [i\mathfrak{I}, i\mathfrak{I}]_* \odot \Omega(\mathfrak{I}) \\ &= (i\Omega(i\mathfrak{I}) - i\Omega(i\mathfrak{I})^*) \odot \mathfrak{I} + 2i\Omega(i\mathfrak{I}) \odot \mathfrak{I} - 2\mathfrak{I} \odot \Omega(\mathfrak{I}) \\ &= -6i\Omega(i\mathfrak{I})^* + 2i\Omega(i\mathfrak{I}) - 4\Omega(\mathfrak{I}).\end{aligned}$$

From the last two expressions, we get

$$-3\Omega(i\mathfrak{I})^* + \Omega(i\mathfrak{I}) = 0. \tag{2.1}$$

Also, we can evaluate

$$\Omega([i\mathfrak{I}, \mathfrak{I}]_* \odot i\mathfrak{I}) = \Omega(2i\mathfrak{I} \odot i\mathfrak{I}) = 4\Omega(\mathfrak{I}).$$

Alternatively, we can write

$$\begin{aligned}\Omega([i\mathfrak{I}, \mathfrak{I}]_* \odot i\mathfrak{I}) &= [\Omega(i\mathfrak{I}), \mathfrak{I}]_* \odot i\mathfrak{I} + [i\mathfrak{I}, \Omega(\mathfrak{I})]_* \odot i\mathfrak{I} + [i\mathfrak{I}, \mathfrak{I}]_* \odot \Omega(i\mathfrak{I}) \\ &= 2i\Omega(i\mathfrak{I})^* - 6i\Omega(i\mathfrak{I}) + 4\Omega(\mathfrak{I})^*.\end{aligned}$$

By comparing above two equations, and also using Lemma 2.1, we find

$$\Omega(i\mathfrak{I})^* - 3\Omega(i\mathfrak{I}) = 0. \tag{2.2}$$

By using Eqs (2.1) and (2.2), we have

$$\Omega(i\mathfrak{I})^* = \Omega(i\mathfrak{I}) = 0.$$

(ii) In the similar way, we can show that $\Omega(\mathfrak{I}) = 0$. □

Lemma 2.9. *Ω preserves star, i.e., $\Omega(\mathcal{K}^*) = \Omega(\mathcal{K})^*$ for all $\mathcal{K} \in \mathcal{A}$.*

Proof. From Lemma 2.7, we have

$$\Omega([\mathcal{K}, i\mathcal{I}]_* \odot i\mathcal{I}) = \Omega(i\mathcal{K} - i\mathcal{K}^*) \odot i\mathcal{I}) = 2\Omega(\mathcal{K}^*) - 2\Omega(\mathcal{K}).$$

Alternatively, it follows from Lemma 2.8 that

$$\Omega([\mathcal{K}, i\mathcal{I}]_* \odot i\mathcal{I}) = [\Omega(\mathcal{K}), i\mathcal{I}]_* \odot i\mathcal{I} = (i\Omega(\mathcal{K}) - i\Omega(\mathcal{K})^*) \odot i\mathcal{I} = 2\Omega(\mathcal{K})^* - 2\Omega(\mathcal{K}).$$

From the above two equations, we obtain

$$\Omega(\mathcal{K}^*) = \Omega(\mathcal{K})^*$$

for all $\mathcal{K} \in \mathcal{A}$. This completes the proof. □

Lemma 2.10. *We prove that $\Omega(i\mathcal{K}) = i\Omega(\mathcal{K})$ for all $\mathcal{K} \in \mathcal{A}$.*

Proof. For any $\mathcal{K} \in \mathcal{A}$, we have

$$\Omega([i\mathcal{I}, \mathcal{I}]_* \odot \mathcal{K}) = \Omega(2i\mathcal{I} \odot \mathcal{K}) = -4\Omega(i\mathcal{K}).$$

Alternatively, it follows from Lemma 2.8 that

$$\Omega([i\mathcal{I}, \mathcal{I}]_* \odot \mathcal{K}) = [i\mathcal{I}, \mathcal{I}]_* \odot \Omega(\mathcal{K}) = (2i\mathcal{I}) \odot \mathcal{K} = -4i\Omega(\mathcal{K}).$$

From the above two expressions, we obtain

$$\Omega(i\mathcal{K}) = i\Omega(\mathcal{K}).$$

□

Proof of Theorem 2.1. For any $\mathcal{K}, \mathcal{F} \in \mathcal{A}$, it follows from Lemmas 2.7 that

$$\Omega(\mathcal{K} + \mathcal{F}) = \Omega(\mathcal{K}) + \Omega(\mathcal{F}). \tag{2.3}$$

Also, by using Lemma 2.9 that

$$\Omega(\mathcal{K}^*) = \Omega(\mathcal{K})^* \tag{2.4}$$

for all $\mathcal{K} \in \mathcal{A}$. Now, we only have to show that Ω is an derivation.

Now, for any $\mathcal{K}, \mathcal{F} \in \mathcal{A}$, and using Lemma 2.7, we have

$$\Omega([\mathcal{K}, \mathcal{I}]_* \odot \mathcal{F}) = \Omega((\mathcal{K} - \mathcal{K}^*) \odot \mathcal{F}) = \Omega(\mathcal{K}^*\mathcal{F}) - \Omega(\mathcal{K}\mathcal{F}) + \Omega(\mathcal{F}\mathcal{K}^*) - \Omega(\mathcal{F}\mathcal{K}).$$

Also, using Lemma 2.8 that

$$\begin{aligned} \Omega([\mathcal{K}, \mathcal{I}]_* \odot \mathcal{F}) &= [\Omega(\mathcal{K}), \mathcal{I}]_* \odot \mathcal{F} + [\mathcal{K}, \mathcal{I}]_* \odot \Omega(\mathcal{F}) \\ &= \Omega(\mathcal{K})^*\mathcal{F} - \Omega(\mathcal{K})\mathcal{F} + \mathcal{F}\Omega(\mathcal{K})^* - \mathcal{F}\Omega(\mathcal{K}) \\ &\quad + \mathcal{K}^*\Omega(\mathcal{F}) - \mathcal{K}\Omega(\mathcal{F}) + \Omega(\mathcal{F})\mathcal{K}^* - \Omega(\mathcal{F})\mathcal{K}. \end{aligned}$$

By comparing the two equations above, we obtain

$$\begin{aligned}\Omega(\mathcal{K}^*\mathcal{F})-\Omega(\mathcal{K}\mathcal{F})+\Omega(\mathcal{F}\mathcal{K}^*)-\Omega(\mathcal{F}\mathcal{K}) &= \Omega(\mathcal{K})^*\mathcal{F}-\Omega(\mathcal{K})\mathcal{F}+\mathcal{F}\Omega(\mathcal{K})^*-\mathcal{F}\Omega(\mathcal{K})\\ &\quad +\mathcal{K}^*\Omega(\mathcal{F})-\mathcal{K}\Omega(\mathcal{F})+\Omega(\mathcal{F})\mathcal{K}^*-\Omega(\mathcal{F})\mathcal{K}. \quad (2.5)\end{aligned}$$

On the other hand, according to Lemma 2.7, we can infer that

$$\begin{aligned}\Omega([i\mathcal{K},\mathcal{I}]_*\odot i\mathcal{F}) &= \Omega((i\mathcal{K}+i\mathcal{K}^*)\odot i\mathcal{F})\\ &= \Omega(\mathcal{K}^*\mathcal{F})+\Omega(\mathcal{K}\mathcal{F})+\Omega(\mathcal{F}\mathcal{K}^*)+\Omega(\mathcal{F}\mathcal{K}).\end{aligned}$$

Alternatively, by using Lemma 2.8, we find

$$\begin{aligned}\Omega([i\mathcal{K},\mathcal{I}]_*\odot i\mathcal{F}) &= [\Omega(i\mathcal{K}),\mathcal{I}]_*\odot i\mathcal{F}+[i\mathcal{K},\mathcal{I}]_*\odot\Omega(i\mathcal{F})\\ &= \Omega(\mathcal{K})^*\mathcal{F}+\Omega(\mathcal{K})\mathcal{F}+\mathcal{F}\Omega(\mathcal{K})^*+\mathcal{F}\Omega(\mathcal{K})\\ &\quad +\mathcal{K}^*\Omega(\mathcal{F})+\mathcal{K}\Omega(\mathcal{F})+\Omega(\mathcal{F})\mathcal{K}^*+\Omega(\mathcal{F})\mathcal{K}.\end{aligned}$$

From the above two expressions, we find

$$\begin{aligned}\Omega(\mathcal{K}^*\mathcal{F})+\Omega(\mathcal{K}\mathcal{F})+\Omega(\mathcal{F}\mathcal{K}^*)+\Omega(\mathcal{F}\mathcal{K}) &= \Omega(\mathcal{K})^*\mathcal{F}+\Omega(\mathcal{K})\mathcal{F}+\mathcal{F}\Omega(\mathcal{K})^*+\mathcal{F}\Omega(\mathcal{K})\\ &\quad +\mathcal{K}^*\Omega(\mathcal{F})+\mathcal{K}\Omega(\mathcal{F})+\Omega(\mathcal{F})\mathcal{K}^*+\Omega(\mathcal{F})\mathcal{K}. \quad (2.6)\end{aligned}$$

Subtracting Eq (2.5) to Eq (2.6), we get

$$\Omega(\mathcal{K}\mathcal{F}+\mathcal{F}\mathcal{K})=\Omega(\mathcal{K})\mathcal{F}+\mathcal{K}\Omega(\mathcal{F})+\mathcal{F}\Omega(\mathcal{K})+\Omega(\mathcal{F})\mathcal{K}. \quad (2.7)$$

By using Lemma 2.10 and the above equation, we find

$$\begin{aligned}\Omega(\mathcal{K}\mathcal{F}-\mathcal{F}\mathcal{K}) &= i\Omega((-i\mathcal{K})(\mathcal{F})+(i\mathcal{F})\mathcal{K})\\ &= \Omega(\mathcal{K})\mathcal{F}+\mathcal{K}\Omega(\mathcal{F})-\mathcal{F}\Omega(\mathcal{K})-\Omega(\mathcal{F})\mathcal{K} \quad (2.8)\end{aligned}$$

Adding Eqs (2.7) and (2.8), we get

$$\Omega(\mathcal{K}\mathcal{F})=\Omega(\mathcal{K})\mathcal{F}+\mathcal{K}\Omega(\mathcal{F}). \quad (2.9)$$

From Eqs (2.3), (2.4) and (2.9), we get Ω is an additive $*$-derivation. This completes the proof.

Now, we provide an example to demonstrate the necessity of the conditions (▲) and (▼) in Theorem 2.1.

Example 2.1. *Consider* $\mathcal{A}=\{\begin{pmatrix} a & 0\\ c & d\end{pmatrix}\}$, *the algebra of all lower triangular matrix of order* 2 *over the field of complex numbers* $\mathbb{C}$ *and* $\mathcal{I}=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$ *be the unity of* $\mathcal{A}$*. The map* $*:\mathcal{A}\to\mathcal{A}$ *given by* $*(\mathcal{K})=\mathcal{K}^\theta$*, where* $\mathcal{K}^\theta$ *denotes the conjugate transpose of matrix A, is an involution. Hence,* $\mathcal{A}$ *is a unital* $*$*-algebra with unity* $\mathcal{I}$*. Now, define a map* $\Omega:\mathcal{A}\to\mathcal{A}$ *such that* $\Omega\begin{pmatrix} a & 0\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & 0\\ -ic & 0\end{pmatrix}$*. Note that* Ω *is a derivation on* $\mathcal{A}$*. So, it also satisfies*

$$\Omega([[\mathcal{K},\mathcal{F}]_\odot,\mathcal{D}]_*)=[[\Omega(\mathcal{K}),\mathcal{F}]_\odot,\mathcal{D}]_*+[[\mathcal{K},\Omega(\mathcal{F})]_\odot,\mathcal{D}]_*+[[\mathcal{K},\mathcal{F}]_\odot,\Omega(\mathcal{D})]_*$$

for all $\mathcal{K},\mathcal{F},\mathcal{D}\in\mathcal{A}$*. Let* $P=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}$ *is a non-trivial projection, so* $P^2=P$ *and* $P^*=P$*. For* $W=\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}\neq 0\in\mathcal{A}$ *and hence* $W\mathcal{A}P=(0)$ *but* $0\neq W\in\mathcal{A}$*. However,* Ω *is not an additive* $*$*-derivation because* $\Omega(\mathcal{K}^*)\neq(\Omega(\mathcal{K}))^*$ *for some* $\mathcal{K}\in\mathcal{A}$*.*

3. Corollaries

The following corollaries arise directly from Theorem 2.1: The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F}$ of real or complex numbers. The dimension of an operator's range is known as its rank. An operator with a finite dimensional range is therefore said to have a finite rank. $\mathcal{F}(\mathcal{H})$ is the subalgebra of all bounded linear operators of finite rank on $\mathcal{H}$.

Let $\mathcal{H}$ be a Banach space over the real or complex number field $\mathbb{F}$. In the case of $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, a subalgebra $\mathcal{K}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ is referred to as a standard operator algebra.

Corollary 3.1. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing an identity operator $\mathcal{I}$. Suppose that $\mathcal{A}$ is closed under adjoint operation. Define $\Omega : \mathcal{A} \to \mathcal{A}$ such that*

$$\Omega([\mathcal{K}, \mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}), \mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K}, \Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K}, \mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$, then Ω is an additive $*$-derivation.

Proof. Every standard operator algebra $\mathcal{A}$ being a prime algebra is a direct consequence of the Hahn-Banach theorem. As a prime algebra, $\mathcal{A}$ naturally fulfills the conditions specified in (▲) and (▼). Consequently, according to Theorem 2.1, it follows that the map Ω described earlier is an additive $*$-derivation. □

A von Neumann algebra is defined as a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ that includes the identity operator, where $\mathcal{B}(\mathcal{H})$ is the space of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In other words, a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ that satisfies the double commutant property, that is, $\mathcal{M}'' = \mathcal{M}$, is considered a von Neumann algebra. In this context, a factor von Neumann algebra is one with a trivial center, which is equal to the intersection of $\mathcal{M}$ and its double commutant, $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathcal{I}$. Additionally, an abelian von Neumann algebra is one where the center is equal to the algebra itself, that is, $\mathcal{Z}(\mathcal{M}) = \mathcal{M}$.

Corollary 3.2. *Let $\mathcal{M}$ ba a factor von Neumann algebra with dim$\mathcal{M} \geq 2$. Define $\Omega : \mathcal{M} \to \mathcal{M}$ such that*

$$\Omega([\mathcal{K}, \mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}), \mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K}, \Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K}, \mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$, then Ω is an $*$-derivation.

Proof. By using [16, Lemma 2.2], it is established that every factor von Neumann algebra $\mathcal{M}$ satisfies the conditions outlined in (▲) and (▼). Therefore, applying Theorem 2.1, we conclude that the map Ω described earlier is an additive $*$-derivation within the context of factor von Neumann algebras. □

An algebra $\mathcal{A}$ is called prime algebra, if $\mathcal{K}\mathcal{A}\mathcal{K} = \{0\}$ for $\mathcal{K}, \mathcal{F} \in \mathcal{A}$ implies either $\mathcal{K} = 0$ or $\mathcal{F} = 0$.

Corollary 3.3. *Let $\mathcal{A}$ be a prime $*$-algebra with unit $\mathcal{I}$ containing non-trivial projection P. A map $\Omega : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Omega([\mathcal{K}, \mathcal{F}]_* \odot \mathcal{D}) = [\Omega(\mathcal{K}), \mathcal{F}]_* \odot \mathcal{D} + [\mathcal{K}, \Omega(\mathcal{F})]_* \odot \mathcal{D} + [\mathcal{K}, \mathcal{F}]_* \odot \Omega(\mathcal{D}),$$

for all $\mathcal{K}, \mathcal{F}, \mathcal{D} \in \mathcal{A}$, then Ω is an additive $*$-derivation.

Proof. By the definition of primeness of $\mathcal{A}$, it is straightforward to observe that $\mathcal{A}$ also satisfies (▲) and (▼). Therefore, by Theorem 2.1, we conclude that Ω is an additive $*$-derivation. □

Author contributions

All authors are contributed equally.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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