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Research article

Fixed point theorems for enriched Kannan-type mappings and application

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Abstract: The aim of this paper is to establish some fixed point results for enriched Kannan-type mappings in convex metric spaces. We first give an affirmative answer to a recent Berinde and Păcurar's question (Remark 2.3) [*J. Comput. Appl. Math.*, **386** (2021), 113217]. Furthermore, we establish the existence and uniqueness of fixed points for Suzuki-enriched Kannan-type mappings in the setting of convex metric spaces. Finally, we present an application to approximate the solution of the Volterra integral equations to support our results.

Keywords: fixed point; enriched Kannan mapping; suzuki mapping; convex metric space
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1. Introduction and preliminaries

The Banach contraction principle [1], proposed by Banach in 1922, was a fundamental consequence of fixed-point theory. It states that any self-mapping S on a complete metric space (H, d) satisfying

$$d(So, Sz) \leq \alpha d(o, z), 0 \leq \alpha < 1$$

for all $o, z \in H$, then S has a unique fixed point in H. After that, many authors have generalized, improved, and extended this celebrated result by changing either the conditions of the mappings or the construction of the space; see [2–6].

The well-known Nemytzki-Edelstein's results for contractive mappings on compact metric spaces are as follows:

Theorem 1. *[7] Let the self-mapping S on a compact metric space (H, d) satisfy*

$$d(So, Sz) < d(o, z)$$

for any $o, z \in H$ with $o \neq z$. Then S has a unique fixed point in H.

Suzuki [8] established the following version of Edelstein's fixed-point theorem:

Theorem 2. *[8] Let the self-mapping S on a compact metric space (H, d) satisfy*

$$\frac{1}{2}d(o, So) < d(o, z) \qquad \textit{implies} \qquad d(So, Sz) < d(o, z)$$

for all $o, z \in H$ with $o \neq z$. Then S has a unique fixed point in H.

Especially Kannan [9, 10] established the following results, which differ from the Banach contraction principle:

Theorem 3. *[9, 10] Let the self-mapping S on a complete metric space (H, d) satisfying*

$$d(So, Sz) \leq \alpha\,[d(o, So) + d(z, Sz)]$$

for all $o, z \in H$ and $\alpha \in [0, \frac{1}{2})$. Then S has a unique fixed point in H.

It is not difficult to see that contractions are always continuous, while Kannan maps are not necessarily continuous. Another beauty of Kannan mappings is that Kannan's theorem characterizes metric completeness. In 1975, Subrahmanyam [11] presented the following result:

Theorem 4. *[11] A metric space is complete if and only if every Kannan mapping S has a fixed point.*

In addition, Fisher [12] proved the following variant of Theorem 3 for a compact metric space:

Theorem 5. *[12] Let the continuous self-mapping S on a compact metric space (H, d) satisfy*

$$d(So, Sz) < \frac{1}{2}\,[d(o, So) + d(z, Sz)]$$

for all $o, z \in H$ with $o \neq z$. Then S has a unique fixed point in H.

Rencently, Berinde, and Păcurar [13] introduced the notion of enriched Kannan mapping, which is a generalization of that Kannan mapping. A mapping $S : H \to H$ is called an enriched Kannan mapping or a (a, k)-enriched Kannan mapping if there exist $k \in [0, \frac{1}{2})$ and $a \in [0, +\infty)$ such that

$$\|a(o - z) + So - Sz\| \leq k\,(\|o - So\| + \|z - Sz\|)\,. \tag{1.1}$$

We will denote the set of all fixed points of S by $F(S)$. They proved the following:

Theorem 6. *[13] Let $(H, \|\cdot\|)$ be a Banach space and $S : H \to H$ be a $(a, k)-$enriched Kannan mapping. Then the following holds:*

(i) $F(S) = \{o\}$;

(ii) There exists $\lambda \in [0, 1)$, the sequence $\{o_n\}_{n=0}^{+\infty}$ defined by

$$o_{n+1} = \lambda o_n + (1 - \lambda)So_n$$

converges to o in H;

(iii) set $\mu = \frac{k}{1-k}$, for any $n \in \mathbb{N}$, then

$$\|o_{n+i-1} - p\| \leq \frac{\mu^i}{1 - \mu}\,\|o_n - o_{n-1}\|\,.$$

And then, a question was raised following the above theorem in [13].

Question 1. *Does the enriched Kannan mapping fixed point theorem still characterize the metric completeness?*

On the other hand, Takahashi [14] introduces the notion of the convex structure in metric space as follows:

Definition 1. *[14] Let (H, d) be a metric space. Define a function $W : H \times H \times [0, 1] \to H$ is said to be a convex structure in H if*

$$d(v, W(o, z; \lambda)) \leq \lambda d(v, o) + (1 - \lambda)d(v, z)$$

holds for each $z, o, z \in H$ and $\lambda \in [0, 1]$. (H, d, W) is called convex metric space.

Remark 1. *It is worth mentioning that a linear normed space embedded with the natural convex structure*

$$W(o, z; \lambda) = \lambda o + (1 - \lambda)z$$

is a convex metric space, but it is not valid for some metric spaces, see [15, 16].

Lemma 1. *[14–16] Let (H, d, W) be a convex metric space, and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$. For any $o, z \in H$, the following holds:*

(i) $W(o, o; \lambda) = o$; $W(o, z; 0) = z$ *and* $W(o, z; 1) = o$;
(ii) $d(o, z) = d(o, W(o, z; \lambda)) + d(z, W(o, z; \lambda))$;
(iii) $d(o, W(o, z; \lambda)) = (1 - \lambda)d(o, z)$ *and* $d(z, W(o, z; \lambda)) = \lambda d(o, z)$;
(iv) $|\lambda_1 - \lambda_2| d(o, z) \leq d(W(o, z; \lambda_1), W(o, z; \lambda_2))$.

Lemma 2. *[17] Let the self-mapping S on a convex metric space (H, d, W) and $S_\lambda : H \to H$ defined by*

$$S_\lambda o = W(o, S o; \lambda), o \in H.$$

Then, we have $F(S) = F(S_\lambda)$ for any $\lambda \in [0, 1)$.

Berinde and Păcurar [17] gave the concept of enriched Kannan mapping on a convex metric space as below:

Definition 2. *[17] A self-mapping S on a convex metric space (H, d, W) is said to be an enriched Kannan mapping if there exist $k \in [0, \frac{1}{2})$ and $\lambda \in [0, 1)$ satisfying*

$$d(W(o, S o; \lambda), W(z, S z; \lambda)) \leq k \left[d(o, W(o, S o; \lambda)) + d(z, W(z, S z; \lambda))\right], o, z \in H. \tag{1.2}$$

Notice that the continuous Kannan contractive mapping $S : H \to H$ is such that

$$d(S o, S z) < \frac{1}{2} \left[d(o, S o) + d(z, S z)\right]$$

for all $o, z \in H$ with $o \neq z$, in a complete but nocompact metric space may be fixed-point free ([18–20]). Hence, a question about enriched Kannan contractive mapping may arise:

Question 2. *Does there exists a complete but noncompact convex metric space (H, d) and continuous enriched Kannan contractive mapping $S : H \to H$ satisfying*

$$d(W(o, So; \lambda), W(z, Sz; \lambda)) < \frac{1}{2}\left[d(o, W(o, So; \lambda)) + d(z, W(z, Sz; \lambda))\right]$$

and S is fixed-point-free?

In this work, we first give an affirmative answer to Question 1 by proving that every enriched Kannan contractive mapping has a fixed point and characterizes the completeness of the underlying normed space. Furthermore, we provide an example answer to Question 2. Moreover, we present some new fixed point results for Suzuki-enriched Kannan-type mappings in the setting of convex metric spaces. Finally, we apply the fixed point result to approximating the solution of nonlinear Volterra integral equations.

2. Main results

In what follows, the symbol F represents the set of all functions $f : [0, +\infty) \to [0, \frac{1}{2})$ such that

$$f(o_n) \to \frac{1}{2} \quad implies \quad o_n \to 0 \quad as \quad n \to +\infty;$$

the symbol Ψ represents the set of all strictly monotonic, increasing, and continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that

$$\psi(o) = 0 \quad if\ and\ only\ if \quad o = 0.$$

We start with the following theorem, which is an affirmative answer to Question 1.

Theorem 7. *Let $(H, \|\cdot\|)$ be a normed space and $S : H \to H$ be a mapping satisfying*

$$\|a(o - z) + So - Sz\| < \frac{1}{2}\left[\|o - So\| + \|z - Sz\|\right] \tag{2.1}$$

for any $o, z \in H$ with $o \neq z$. If S has a fixed point, then $(H, \|\cdot\|)$ is a Banach space.

Proof. If $a = 0$, the result follows from Theorem 4. Suppose that $a > 0$, we observe that (2.1) can be rewritten as follows:

$$\left\|\frac{a}{a+1}(o - z) + \frac{1}{a+1}(So - Sz)\right\| < \frac{1}{2(a+1)}\left[\|o - So\| + \|z - Sz\|\right].$$

Let $\lambda = \frac{a}{a+1}$, clearly $a = \frac{\lambda}{1-\lambda}$, then (2.1) becomes

$$\|\lambda(o - z) + (1 - \lambda)(So - Sz)\| < \frac{1 - \lambda}{2}\left[\|o - So\| + \|z - Sz\|\right].$$

Set $S_\lambda o = \lambda o + (1 - \lambda)So$, we deduce that

$$\|S_\lambda o - S_\lambda z\| < \frac{1}{2}\left[\|o - S_\lambda o\| + \|z - S_\lambda z\|\right]. \tag{2.2}$$

For the purpose of contradiction, suppose that $\{o_n\} \in H$ is a Cauchy sequence but does not converge. Set $D(o, M) = \inf\{\|o - z\| : z \in M\}$ and $M = \{o_n : n \in \mathbb{N}\}$ be a set of divergent sequences of distinct elements in H. Let $o \in H$, and we have the subsequent cases:

Case 1. If $o \notin H$, as $\{o_n\}$ is a Cauchy sequence, there exists an integer $N_0(o)$ such that

$$\|o_m - o_n\| < \frac{1}{2} D(o, H) \leq \frac{1}{2} \|o - o_l\|, m, n \geq N_0(o),$$

for any $l \in \mathbb{N}$. In particular

$$\left\|o_m - o_{N_0(o)}\right\| < \frac{1}{2} \|o - o_l\|,$$

for any $l \in \mathbb{N}$, $m \geq N_0(o)$.

Case 2. If $o \in H$, then $o = o_{n(o)}$ for some $n(o) \in \mathbb{N}$, and there exists an integer $n_0(o) \in N$ such that

$$\left\|o_m - o_{n_0(o)}\right\| < \frac{1}{2} \left\|o_{n_0(o)} - o_{n(o)}\right\|,$$

for any $m \geq n_0(o) > n(o)$. Define $S : H \to H$ by

$$S o = \begin{cases} \frac{o_{N_0(o)} - \lambda o}{1 - \lambda}, & \text{if} \quad o \notin H, \\ \frac{o_{n_0(o)} - \lambda o}{1 - \lambda}, & \text{if} \quad o = o_{n(o)} \in H. \end{cases}$$

Let $o, z \in H$, and we will show that S is an enriched Kannan contractive mapping. Indeed, we need to consider the following four cases:

Case 1. If $o, z \notin M$, then $S o = \frac{o_{N_0(o)} - \lambda o}{1 - \lambda}$ and $S z = \frac{o_{N_0(z)} - \lambda z}{1 - \lambda}$, which imply that $S_\lambda o = o_{N_0(o)}$ and $S_\lambda z = o_{N_0(z)}$. We can suppose that $o_{N_0(z)} > o_{N_0(o)}$. It follows that

$$\left\|o_{N_0(o)} - o_{N_0(z)}\right\| < \frac{1}{2} \left\|o - o_{N_0(o)}\right\| = \frac{1}{2} \|o - S_\lambda o\|,$$

which shows that $\|S_\lambda o - S_\lambda z\| < \frac{1}{2} [\|o - S_\lambda o\| + \|z - S_\lambda z\|]$.

Case 2. If $o, z \in M$, there exist $n(o), n(z) \in \mathbb{N}$ such that $o = o_{n(o)}, z = o_{n(z)}$. Then $S o = \frac{o_{n_0(o)} - \lambda o}{1 - \lambda}$ and $S z = \frac{o_{n_0(z)} - \lambda z}{1 - \lambda}$, which implies that $S_\lambda o = o_{n_0(o)}$ and $S_\lambda z = o_{n_0(z)}$. Suppose that $n_0(z) > n_0(o)$. We obtain

$$\left\|o_{n_0(z)} - o_{n_0(o)}\right\| < \frac{1}{2} \left\|o_{n_0(o)} - o_{n(o)}\right\| = \frac{1}{2} \|S_\lambda o - o\|,$$

which shows that $\|S_\lambda o - S_\lambda z\| < \frac{1}{2} [\|o - S_\lambda o\| + \|z - S_\lambda z\|]$.

Case 3. If $o \in M$, $z \notin M$, there exists $n(o) \in N$ satisfying $o = o_{n(o)}$, then $S o = \frac{o_{n_0(o)} - \lambda o}{1 - \lambda}$ and $S z = \frac{o_{N_0(z)} - \lambda z}{1 - \lambda}$ which imply that $S_\lambda o = o_{n_0(o)}$ and $S_\lambda z = o_{N_0(z)}$.

Subcase 1. If $n_0(o) \geq N_0(z)$, we obtain

$$\left\|o_{n_0(o)} - o_{N_0(z)}\right\| < \frac{1}{2} \left\|z - o_{N_0(z)}\right\| = \frac{1}{2} \|z - S_\lambda z\|,$$

which shows that $\|S_\lambda o - S_\lambda z\| < \frac{1}{2} [\|o - S_\lambda o\| + \|z - S_\lambda z\|]$.

Subcase 2. If $n_0(o) < N_0(z)$, we obtain

$$\left\|o_{N_0(z)} - o_{n_0(o)}\right\| < \frac{1}{2} \left\|o_{n_0(o)} - o\right\| = \frac{1}{2} \|S_\lambda o - o\|,$$

which shows that $\|S_\lambda o - S_\lambda z\| < \frac{1}{2}[\|o - S_\lambda o\| + \|z - S_\lambda z\|]$.
Case 4. If $o \notin M$ and $z \in M$, in this case, $S_\lambda o = o_{N_0(o)}$ and $S_\lambda z = o_{n_0(z)}$, similarly to Case 3, we can also get that $\|S_\lambda o - S_\lambda z\| < \frac{1}{2}[\|o - S_\lambda o\| + \|z - S_\lambda z\|]$.
Therefore, by summarizing all cases, for any $o, z \in H$ with $o \neq z$, we have

$$\|S_\lambda o - S_\lambda z\| < \frac{1}{2}\|o - S_\lambda o\| + \|z - S_\lambda z\|,$$

which shows that S is an enriched Kannan contractive mapping. Notice that S is fixed point-free, which is a contradiction. Therefore, $(H, \|\cdot\|)$ is a Banach space. □

Now, we give the following example to answer Question 2.

Example 1. *Let $H = \mathbb{N}$ and $W(o, z; \lambda) = \lambda o + (1-\lambda)z$, $\lambda \in [0, 1)$. We define the metric $d : H \times H \to [0, +\infty)$ by*

$$d(o, z) = \begin{cases} 1 + \left|\frac{1}{o} - \frac{1}{z}\right|, & o \neq z, \\ 0, & o = z. \end{cases}$$

Then (H, d, W) is a complete convex metric space, but H is not compact since the sequence (n) has no convergent subsequence in H. A mapping $S : H \to H$ is defined by $So = 7o$ for any $o \in H$. Clearly, S is continuous. Moreover, for $\lambda = \frac{1}{2}$, we have $W(o, z; \frac{1}{2}) = 4o$. For all $o, z \in H$ with $o < z$, we conclude that

$$d\left(W(o, So; \frac{1}{2}), W(z, Sz; \frac{1}{2})\right) = 1 + \left|\frac{1}{4o} - \frac{1}{4z}\right| < 1 + \frac{1}{4o}.$$

Notice that

$$\begin{aligned}\frac{1}{2}\left[d\left(o, W(o, So; \frac{1}{2})\right) + d\left(z, W(z, Sz; \frac{1}{2})\right)\right] &= \frac{1}{2}\left[1 + \left|\frac{1}{o} - \frac{1}{4o}\right| + 1 + \left|\frac{1}{z} - \frac{1}{4z}\right|\right] \\ &= 1 + \frac{3}{8o} + \frac{3}{8z} > 1 + \frac{3}{8o}.\end{aligned}$$

Thus

$$d\left(W(o, So; \frac{1}{2}), W(z, Sz; \frac{1}{2})\right) < \frac{1}{2}\left[d\left(o, W(o, So; \frac{1}{2})\right) + d\left(z, W(z, Sz; \frac{1}{2})\right)\right],$$

for all $o, z \in H$ with $o < z$. Similarly, one can prove it for the case $o, z \in H$ with $o > z$. Therefore, S is an enriched Kannan contractive mapping, but S has no fixed point.

Suzuki enriched the Kannan-type mapping fixed point theorem as follows:

Theorem 8. *Let (H, d, W) be a complete convex metric space and $S : H \to H$ be a mapping. If there exists $\lambda \in [0, 1)$ such that for any $o, z \in H$,*

$$\frac{1-\lambda}{2} d(o, So) < d(o, z)$$

implies

$$\psi(d(W(o, So; \lambda), W(z, Sz; \lambda))) \leq f(d(o, z))[\psi(d(o, W(o, So; \lambda))) + \psi(d(z, W(z, Sz; \lambda)))],$$

where $f \in F, \psi \in \Psi$. Then, S has a unique fixed point $o^* \in H$, and the sequence $o_{n+1} = W(o_n, So_n; \lambda)$ converges to o^*.

Proof. For any $o \in H$, set $S_\lambda o = W(o, So; \lambda)$. In this case, the given assumption becomes

$$\frac{1}{2}d(o, S_\lambda o) < d(o, z) \quad implies \quad \psi(d(S_\lambda o, S_\lambda z)) \leq f(d(o, z))[\psi(d(o, S_\lambda o)) + \psi(d(z, S_\lambda z))]. \tag{2.3}$$

Choose $o_1 \in H$ and construct the Picard iteration associated with S_λ, this is $o_{n+1} = S_\lambda o_n$. Without loss of generality, suppose that $o_n \neq o_{n+1}$ for all $n \in \mathbb{N}$. Indeed, if $o_n = o_{n+1}$ for some $n \in \mathbb{N}$, we have

$$d(o_n, So_n) = d(o_{n+1}, So_n) = d(o_n, W(o_n, So_n; \lambda)) \leq (1 - \lambda)d(o_n, So_n),$$

which means that $d(o_n, So_n) = 0$. Thus, o_n is a fixed point of S. Since

$$\frac{1}{2}d(o_n, S_\lambda o_n) < d(o_n, S_\lambda o_n) = d(o_n, o_{n+1}),$$

by applying the condition (2.3), we have

$$\psi(d(S_\lambda o_n, S_\lambda o_{n+1})) \leq f(d(o_n, o_{n+1}))\left[\psi(d(o_n, S_\lambda o_n)) + \psi(d(o_{n+1}, S_\lambda o_{n+1}))\right]. \tag{2.4}$$

Then we see that

$$\begin{aligned}\psi(d(o_{n+1}, o_{n+2})) &\leq f(d(o_n, o_{n+1}))\left[\psi(d(o_n, o_{n+1})) + \psi(d(o_{n+1}, o_{n+2}))\right]\\ &< \frac{1}{2}\left[\psi(d(o_n, o_{n+1})) + \psi(d(o_{n+1}, o_{n+2}))\right].\end{aligned}$$

Hence $\psi(d(o_{n+1}, o_{n+2})) < \psi(d(o_n, o_{n+1}))$. Since ψ is increasing, we have $d(o_{n+1}, o_{n+2}) < d(o_n, o_{n+1})$. Thus, $\{d(o_n, o_{n+1})\}$ is a decreasing sequence of nonnegative real numbers, and hence it is convergent. Assume that $\lim_{n\to+\infty} d(o_n, o_{n+1}) = r \geq 0$. If $r > 0$. From (2.4), we obtain that

$$\frac{\psi(d(o_{n+1}, o_{n+2}))}{\psi(d(o_n, o_{n+1})) + \psi(d(o_{n+1}, o_{n+2}))} \leq f(d(o_n, o_{n+1}))$$

letting $n \to +\infty$, we get $\frac{1}{2} \leq \lim_{n\to+\infty} f(d(o_n, o_{n+1}))$, a contradiction , thus $r = 0$. Now, we show that $\{o_n\}$ is a Cauchy sequence. If not, then there exist $\varepsilon > 0$ and two sequences $\{q_k\}, \{p_k\}$ of positive integers such that

$$p_k > q_k > k, d(o_{q_k}, o_{p_k}) \geq \varepsilon \quad and \quad d(o_{q_k}, o_{p_k-1}) < \varepsilon.$$

We obtain

$$\varepsilon \leq d(o_{q_k}, o_{p_k}) \leq d(o_{q_k}, o_{p_k-1}) + d(o_{p_k-1}, o_{p_k}) \leq \varepsilon + d(o_{p_k-1}, o_{p_k}).$$

Let $k \to +\infty$, we deduce that $\lim_{k\to+\infty} d(o_{q_k}, o_{p_k}) = \varepsilon$. Further, from

$$d(o_{q_k}, o_{p_k}) \leq d(o_{q_k}, o_{q_k+1}) + d(o_{q_k+1}, o_{p_k+1}) + d(o_{p_k+1}, o_{p_k})$$

and

$$d(o_{q_k+1}, o_{p_k+1}) \leq d(o_{q_k+1}, o_{q_k}) + d(o_{q_k}, o_{p_k}) + d(o_{p_k}, o_{p_k+1}),$$

we obtain $\lim_{k\to+\infty} d(o_{q_k+1}, o_{p_k+1}) = \varepsilon$. Note that $\lim_{n\to+\infty} d(o_n, o_{n+1}) = 0$, there exists $N_0 \in \mathbb{N}$ such that

$$d(o_n, o_{n+1}) < \varepsilon \quad \textit{and} \quad \frac{1}{2}d(o_{q_k}, S_\lambda o_{q_k}) = \frac{1}{2}d(o_{q_k}, o_{q_k+1}) < \varepsilon \leq d(o_{p_k}, o_{q_k}),$$

for any $n > N_0$. By applying the condition (2.3) again, we find

$$\psi\left(d(S_\lambda o_{p_k}, S_\lambda o_{q_k})\right) \leq f(d(o_{p_k}, o_{q_k}))\left[\psi\left(d(o_{p_k}, S_\lambda o_{p_k})\right) + \psi\left(d(o_{q_k}, S_\lambda o_{q_k})\right)\right],$$

for all $n > N_0$. Thus

$$\psi\left(d(o_{p_k+1}, o_{q_k+1})\right) < \frac{1}{2}\left[\psi\left(d(o_{p_k}, o_{p_k+1})\right) + \psi\left(d(o_{q_k}, o_{q_k+1})\right)\right].$$

Letting $k \to +\infty$, we obtain that

$$\psi(\varepsilon) = \lim_{k\to+\infty} \psi\left(d(S_\lambda o_{p_k}, S_\lambda o_{q_k})\right) \leq \frac{1}{2}\left[\psi(0) + \psi(0)\right] = 0,$$

which implies $\varepsilon = 0$ and leads to a contradiction. Therefore, $\{o_n\}$ is a Cauchy sequence and $o_n \to o^* \in H$ as $n \to +\infty$. We assume that there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2}d(o_n, o_{n+1}) \geq d(o_n, o^*) \quad \textit{and} \quad \frac{1}{2}d(o_{n+1}, o_{n+2}) \geq d(o_{n+1}, o^*).$$

Then we obtain

$$d(o_n, o_{n+1}) \leq d(o_n, o^*) + d(o^*, o_{n+1}) \leq \frac{1}{2}\left[d(o_n, o_{n+1}) + d(o_{n+1}, o_{n+2})\right],$$

which implies

$$d(o_n, o_{n+1}) \leq d(o_{n+1}, o_{n+2})$$

a contradiction. Therefore, one of the following conditions holds:

(a) $\frac{1}{2}d(o_n, o_{n+1}) < d(o_n, o^*)$ for any n in some infinite subset E of $\mathbb{N}$;
(b) $\frac{1}{2}d(o_{n+1}, o_{n+2}) < d(o_{n+1}, o^*)$ for any n in some infinite subset U of $\mathbb{N}$.

If (a) holds, then we have

$$\begin{aligned}\psi(d(o_{n+1}, S_\lambda o^*)) = \psi(d(S_\lambda o_n, S_\lambda o^*)) &\leq f(d(o_n, o^*))\left[\psi(d(o_n, S_\lambda o_n)) + \psi(d(o^*, S_\lambda o^*))\right] \\ &< \frac{1}{2}\left[\psi(d(o_n, o_{n+1})) + \psi(d(o^*, o_{n+1}) + d(o_{n+1}, S_\lambda o^*))\right].\end{aligned}$$

Hence, $\lim_{n\in E, n\to+\infty} \psi(d(o_{n+1}, S_\lambda o^*)) = 0$ and $\lim_{n\to+\infty} o_{n+1} = S_\lambda o^*$, that is, $\{o_n\}$ has a subsequence converging to $S_\lambda o^*$. Similarly, if (b) holds, we also obtain that $\{o_n\}$ has a subsequence converging to $S_\lambda o^*$. Since $\{o_n\}$ is converging to o^*, $o^* = S_\lambda o^*$. If z^* is another fixed point of S_λ, that is, $o^* = S_\lambda o^* \neq S_\lambda z^* = z^*$. Since $\frac{1}{2}d(o^*, S_\lambda o^*) = 0 < d(o^*, z^*)$, then, we have

$$\psi(d(o^*, z^*)) = \psi(d(S_\lambda o^*, S_\lambda z^*)) \leq f(d(o^*, z^*))\left[\psi(d(o^*, S_\lambda o^*)) + \psi(d(z^*, S_\lambda z^*))\right] = 0,$$

thus $d(o^*, z^*) = 0$, which is a contradiction. Combining this with Lemma 2, we have that S has a unique fixed point in H. □

Example 2. *Let $H = [0,2]$ and d be a Euclidean metric on H. Set $W(o,z;\lambda) = \lambda o + (1-\lambda)z$ for any $\lambda \in [0,1)$. Then (H,d,W) is a complete convex metric space. Let us define $S : H \to H$ by*

$$So = \begin{cases} \frac{8-o}{7}, & o \in [0,2), \\ \frac{2}{7}, & o = 2. \end{cases}$$

We choose $\lambda = \frac{1}{8}$, we have

$$S_{\frac{1}{8}} = W(o,So;\frac{1}{8}) = \begin{cases} 1, & o \in [0,2), \\ \frac{1}{2}, & o = 2. \end{cases}$$

Clearly, S satisfies the contractive (2.3) ($\psi(z) = t$ and $f(z) = -\frac{z}{12} + \frac{1}{2}$ for all $z \geq 0$). All conditions of Theorem 8, hold and therefore S has a unique fixed point 1 in H.

Next, we give a generalization of Theorem 8 in the setting of compact convex spaces.

Theorem 9. *Let (H,d,W) be a compact convex metric space with continuous convex structure and $S : H \to H$ be a continuous mapping. If there exists $\lambda \in [0,1)$ such that for any $o,z \in H$,*

$$\frac{1-\lambda}{2} d(o,So) < d(o,z)$$

implies

$$\psi(d(W(o,So;\lambda),W(z,Sz;\lambda))) < \frac{1}{2}[\psi(d(o,W(o,So;\lambda))) + \psi(d(z,W(z,Sz;\lambda)))],$$

where $\psi \in \Psi$. Then, S has a unique fixed point $o^* \in H$, and the sequence $o_{n+1} = W(o_n,So_n;\lambda)$ converges to o^*.

Proof. For any $o \in H$, we set $S_\lambda o = W(o,So;\lambda)$. It is clear that S_λ is continuous. In this case, the given assumption becomes

$$\frac{1}{2}d(o,S_\lambda o) < d(o,z) \quad implies \quad \psi(d(S_\lambda o,S_\lambda z)) < \frac{1}{2}[\psi(d(o,S_\lambda o)) + \psi(d(z,S_\lambda z))], \tag{2.5}$$

for any $o,z \in H$ and $\lambda \in [0,1)$. Let $g(o) = d(o,S_\lambda o)$. It is clear that $g(o)$ is continuous. By the fact that H is compact, there exists a point $o^* \in H$ such that $g(o^*) = \inf\{g(o) : o \in H\}$, then $g(o^*) \leq g(o)$ for any $o \in H$. Suppose that $o^* \neq S_\lambda o^*$, thus

$$\frac{1}{2}d(o^*,S_\lambda o^*) < d(o^*,S_\lambda o^*) \leq d(S_\lambda o^*,S_\lambda^2 o^*),$$

then we obtain

$$\psi\left(d(S_\lambda o^*,S_\lambda^2 o^*)\right) < \frac{1}{2}\left[\psi(d(o^*,S_\lambda o^*)) + \psi\left(d(S_\lambda o^*,S_\lambda^2 o^*)\right)\right].$$

This implies that $\psi\left(d(S_\lambda o^*,S_\lambda^2 o^*)\right) < \psi(d(o^*,S_\lambda o^*))$, contradicting $d(o^*,S_\lambda o^*) \leq d(S_\lambda o^*,S_\lambda^2 o^*)$. Hence $o^* = S_\lambda o^*$. Assume that z^* is another fixed point, that is, $o^* = S_\lambda o^* \neq S_\lambda z^* = z^*$. Since $\frac{1}{2}d(o^*,S_\lambda o^*)$ $=0 < d(o^*,z^*)$, we have

$$\psi(d(o^*,z^*)) = \psi(d(S_\lambda o^*,S_\lambda z^*)) < \frac{1}{2}[\psi(d(o^*,S_\lambda o^*)) + \psi(d(z^*,S_\lambda z^*))] = 0.$$

Thus, $d(o^*, z^*) = 0$, i.e. $o^* = z^*$, combining this with Lemma 2, we have that S has a unique fixed point o^* in H. Choose $o_1 \in H$ and the iterative process $\{o_n\}_{n=0}^{+\infty}$ defined by

$$o_{n+1} = S_\lambda o_n = W(o_n, S o_n; \lambda).$$

Suppose that $o_n \neq S_\lambda o_n$ for all $n \in \mathbb{N}$. Since $\frac{1}{2} d(o_n, S_\lambda o_n) < d(o_n, S_\lambda o_n) = d(o_n, o_{n+1})$, we have

$$\psi\left(d(S_\lambda o_n, S_\lambda o_{n+1})\right) < \frac{1}{2}\left[\psi\left(d(o_n, S_\lambda o_n)\right) + \psi\left(d(o_{n+1}, S_\lambda o_{n+1})\right)\right],$$

then $\psi\left(d(o_{n+1}, o_{n+2})\right) < \psi\left(d(o_n, o_{n+1})\right)$. Assume that $\lim_{o\to+\infty} \psi\left(d(o_n, o_{n+1})\right) = r \geq 0$. Suppose that $r > 0$. Since H is compact, there exists a convergent subsequence $\{o_{n_k}\}$ of $\{o_n\}$ such that $o_{n_k} \to o$, as $k \to +\infty$. Thus,

$$0 < r = \lim_{k\to+\infty} \psi\left(d(o_{n_k}, S_\lambda o_{n_k})\right) = d(o, S_\lambda o).$$

Since $\frac{1}{2} d(o, S_\lambda o) < d(o, S_\lambda o)$, we have

$$0 < r = \lim_{k\to+\infty} \psi\left(d(S o_{n_k}, S_\lambda o_{n_k+1})\right) = \psi\left(d(S o, S_\lambda^2 o)\right) < \frac{1}{2}\left[\psi\left(d(o, S_\lambda o)\right) + \psi\left(d(S o, S_\lambda^2 o)\right)\right],$$

it follows that

$$0 < r = \psi\left(d(S o, S_\lambda^2 o)\right) < \psi\left(d(o, S_\lambda o)\right) = r,$$

which is a contradiction, so $r = 0$, which implies $o = o^*$. Since $0 = \frac{1}{2} d(o^*, S_\lambda o^*) < d(o_n, o^*)$, then we have

$$\psi\left(d(o_{n+1}, o^*)\right) = \psi\left(d(S_\lambda o_n, S_\lambda o^*)\right) < \frac{1}{2}\left[\psi\left(d(o_n, S_\lambda o_n)\right) + \psi\left(d(o^*, S_\lambda o^*)\right)\right],$$

thus $\lim_{n\to+\infty} \psi(d(o_{n+1}, o^*)) = 0$, which implies $d(o_{n+1}, o^*) \to 0$ as $n \to +\infty$. Therefore, $o_{n+1} = W(o_n, S o_n; \lambda)$ converges to o^*. □

Remark 2. *Let H, d, W and the map $S : H \to H$ be defined as in Example 2. Then (H, d, W) is a compact convex metric space, and S satisfies the contractive (2.5) for any $\psi(t) = t$ for all $t \geq 0$, S has a unique fixed point 1 in H. However, it is to be noted that S is not continuous, so Theorem 9 is not applicable.*

Question 3. *Does the conclusion of Theorem 9 still hold true if we remove the condition "S is continuous"?*

The following theorem is an answer to the above question:

Theorem 10. *Let (H, d, W) be a compact convex metric space and $S : H \to H$ be a mapping. If there exists $\lambda \in [0, 1)$ such that for any $o, z \in H$,*

$$\frac{1-\lambda}{2} d(o, S o) < d(o, z)$$

implies

$$\psi\left(d(W(o, S o; \lambda), W(z, S z; \lambda))\right) < \frac{1}{2}[\psi\left(d(o, W(o, S o; \lambda))\right) + \psi\left(d(z, W(z, S z; \lambda))\right)],$$

where $\psi \in \Psi$. Then, S has a unique fixed point $o^* \in H$, and the sequence $o_{n+1} = W(o_n, S o_n; \lambda)$ converges to o^*.

Proof. For any $o \in H$, we set $S_\lambda o = W(o, So; \lambda)$, and the given assumption becomes

$$\frac{1}{2}d(o, S_\lambda o) < d(o,z) \quad \textit{implies} \quad \psi(d(S_\lambda o, S_\lambda z)) < \frac{1}{2}[\psi(d(o, S_\lambda o)) + \psi(d(z, S_\lambda z))], \tag{2.6}$$

for all $o, z \in H$ and $\lambda \in [0,1)$. We set $L = \inf\{d(o, S_\lambda o) : o \in H\}$. Then we can find e sequence $\{o_n\} \in H$ such that

$$\lim_{n\to+\infty} d(o_n, S_\lambda o_n) = L.$$

By the compactness property of H, we put $\lim_{n\to+\infty} o_n = e$ and $\lim_{n\to+\infty} S_\lambda o_n = c$ for some $e, c \in H$. Then we deduce

$$L = \lim_{n\to+\infty} d(o_n, S_\lambda o_n) = \lim_{n\to+\infty} d(o_n, c) = \lim_{n\to+\infty} d(e, S_\lambda o_n) = d(e,c).$$

Now we check that $L = 0$. Assume the contrary, i.e., $L > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$d(o_n, c) > \frac{2}{3}L, d(o_n, S_\lambda o_n) < \frac{4}{3}L,$$

for all $n \geq N_0$. Thus

$$\frac{1}{2}d(o_n, S_\lambda o_n) < d(o_n, c),$$

which implies

$$\psi(d(S_\lambda o_n, S_\lambda c)) < \frac{1}{2}\left[\psi(d(o_n, S_\lambda o_n)) + \psi(d(c, S_\lambda c))\right].$$

Let $n \to +\infty$, and we obtain

$$\psi(d(c, S_\lambda c)) \leq \frac{1}{2}\left[\psi(d(e,c)) + \psi(d(c, S_\lambda c))\right],$$

which implies $\psi(d(c, S_\lambda c)) \leq \psi(d(e,c))$. Thus, we obtain $\psi(d(c, S_\lambda c)) \leq \psi(L)$. Moreover, since $\frac{1}{2}d(c, S_\lambda c) < d(c, S_\lambda c)$ then, we have

$$\psi\left(d(S_\lambda c, S_\lambda^2 c)\right) < \frac{1}{2}\left[\psi(d(c, S_\lambda c)) + \psi\left(d(S_\lambda c, S_\lambda^2 c)\right)\right],$$

it follows that $\psi\left(d(S_\lambda c, S_\lambda^2 c)\right) < \psi(d(c, S_\lambda c)) \leq \psi(L)$. Hence $d(S_\lambda c, S_\lambda^2 c) < L$, a contradiction. Therefore, $L = 0$ and $\lim_{n\to+\infty} o_n = \lim_{n\to+\infty} S o_n = e$. We shall show that S_λ has a fixed point in H. Assume on the contrary S_λ does not have a fixed point. Since $\frac{1}{2}d(o_n, S_\lambda o_n) < d(o_n, S_\lambda o_n)$, then we have

$$\psi\left(d(S_\lambda o_n, S_\lambda^2 o_n)\right) < \frac{1}{2}\left[\psi(d(o_n, S_\lambda o_n)) + \psi\left(d(S_\lambda o_n, S_\lambda^2 o_n)\right)\right],$$

which implies that

$$\psi\left(d(S_\lambda o_n, S_\lambda^2 o_n)\right) < \psi(d(o_n, S_\lambda o_n)).$$

Due to ψ is increasing, thus $d(S_\lambda o_n, S_\lambda^2 o_n) < d(o_n, S_\lambda o_n)$. By using triangular inequality, we obtain

$$d(e, S_\lambda^2 o_n) \leq d(e, S_\lambda o_n) + d(S o_n, S_\lambda^2 o_n) < d(e, S_\lambda o_n) + d(o_n, S_\lambda o_n).$$

Hence $\lim_{n\to+\infty} d(e, S_\lambda^2 o_n) = 0$. Assume that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{2}d(o_N, S_\lambda o_N) \geq d(o_N, e) \quad and \quad \frac{1}{2}d(S_\lambda o_N, S_\lambda^2 o_N) \geq d(S_\lambda o_N, e),$$

we deduce that

$$\begin{aligned} d(o_N, S_\lambda o_N) &\leq d(o_N, e) + d(e, S_\lambda o_N) \\ &\leq \frac{1}{2}d(o_N, S_\lambda o_N) + \frac{1}{2}d(S_\lambda o_N, S_\lambda^2 o_N) \\ &< \frac{1}{2}d(o_N, S_\lambda o_N) + \frac{1}{2}d(o_N, S_\lambda o_N) \\ &= d(o_N, S_\lambda o_N) \end{aligned}$$

a contradiction. Thus, either

$$\frac{1}{2}d(o_n, S_\lambda o_n) < d(o_n, e) \quad or \quad \frac{1}{2}d(S_\lambda o_n, S_\lambda^2 o_n) < d(S_\lambda o_n, e)$$

holds for any $n \in \mathbb{N}$. This yields one of the following conditions:
(1) There exists an infinite subset I of $\mathbb{N}$ such that

$$\psi\left(d(S_\lambda o_n, S_\lambda e)\right) < \frac{1}{2}\left[\psi\left(d(o_n, S_\lambda o_n)\right) + \psi\left(d(e, S_\lambda e)\right)\right], \quad for\ any \quad n \in I;$$

(2) There exists an infinite subset J of $\mathbb{N}$ such that

$$\psi\left(d(S_\lambda^2 o_n, Te)\right) < \frac{1}{2}\left[\psi\left(d(S o_n, S_\lambda^2 o_n)\right) + \psi\left(d(e, Te)\right)\right], \quad for\ any \quad n \in J.$$

For the first case, let $n \to +\infty$, we obtain

$$\psi\left(d(e, S_\lambda e)\right) \leq \frac{1}{2}\left[\psi\left(d(e, e)\right) + \psi\left(d(e, S_\lambda e)\right)\right],$$

and consequently, $\psi\left(d(e, S_\lambda e)\right) = 0$. This yields $d(e, S_\lambda e) = 0$. Thus, $e = S_\lambda e$. Also in the second case, let $n \to +\infty$, we obtain that

$$\psi\left(d(e, S_\lambda e)\right) \leq \frac{1}{2}\left[\psi\left(d(e, e)\right) + \psi\left(d(e, S_\lambda e)\right)\right].$$

Similarly, we can conclude that $e = S_\lambda e$. Hence, e is a fixed point of S_λ in both cases, a contradiction. If we assume that z^* is another fixed point of S_λ, that is, $o^* = S_\lambda o^* \neq S_\lambda z^* = z^*$. Since $\frac{1}{2}d(o^*, S_\lambda o^*) = 0 < d(o^*, z^*)$, we have

$$\psi\left(d(o^*, z^*)\right) = \psi\left(d(S_\lambda o^*, S_\lambda z^*)\right) < \frac{1}{2}\left[\psi\left(d(o^*, S_\lambda o^*)\right) + \psi\left(d(z^*, S_\lambda z^*)\right)\right] = 0.$$

Thus $d(o^*, z^*) = 0$, i.e., o^* is the unique fixed point of S_λ. Combining with Lemma 2, we have that S has a unique fixed point in H. □

3. Application

In this part, we establish the existences to solution of nonlinear Volterra integral equations

$$o(x) = g(x) + \int_0^x G(x, r, o(r))dr, x \in [0, l], \tag{3.1}$$

where $l > 0$, $G : [0, l] \times [0, l] \times \mathbb{R} \to \mathbb{R}$, and $g : [0, l] \to \mathbb{R}$.

Theorem 11. *Assume that*

(1) The function G is continuous;
(2) For any $\lambda \in [0, 1)$, there exists $f \in F$ such that

$$\left|\lambda(o(x) - z(x)) + (1 - \lambda) \int_0^x G(x, r, o(r)) - G(x, r, z(r))dr\right|$$
$$\leq f(|o(x) - z(x)|)(1 - \lambda) \left[\left|o(x) - g(x) - \int_0^x G(x, r, o(r))dr\right| + \left|z(x) - g(x) - \int_0^x G(x, r, z(r))dr\right|\right].$$

Then (3.1) has a unique solution. Moreover, the solution is exhibited as follows:

$$z(x) = g(x) + \int_0^x G(x, r, z(r))dr,$$

where $z(x) = \lim_{n\to+\infty} o_n(x)$, $o_0(x) = o_0 \in H$, and

$$o_{n+1}(x) = \lambda o_n(x) + (1 - \lambda) \left[g(x) + \int_0^x G(x, r, o_n(r))dr\right], \lambda \in [0, 1), n \in N.$$

Proof. Let $H = C([0, l], \mathbb{R})$ be the set of all continuous functions on the interval $[0, l]$. Define the metric $d : H \times H \to \mathbb{R}^+$ by

$$d(o, z) = \sup_{x\in[0,l]} |o(x) - z(x)|,$$

and the mapping $W : H \times H \times [0, 1) \to H$ by the formula

$$W(o, z; \lambda) = \lambda o + (1 - \lambda)z.$$

Clearly, (H, d, W) is a complete convex metric space. Consider the mapping

$$(So)(x) = g(x) + \int_0^x G(x, r, o(r))dr, o \in H, x \in [0, l].$$

It is clear that $o(x)$ is a solution of Eq (3.1) if and only if $o(x)$ is a fixed point of S, that is, $So = o$. Obviously, S is well defined. Define $S_\lambda o$ by

$$(S_\lambda o)(x) = \lambda o(x) + (1 - \lambda)(So)(x),$$

and set $o_{n+1}(x) = (S_\lambda o_n)(x) = \lambda o_n(x) + (1 - \lambda)(So_n)(x), n \in \mathbb{N}$. Let $o, z \in H$, we have

$$|(S_\lambda o)(x) - (S_\lambda z)(x)| = |\lambda o(x) + (1 - \lambda)(So)(x) - \lambda z(x) + (1 - \lambda)(Sz)(x)|$$

$$
\begin{aligned}
&= |\lambda(o(x) - z(x)) + (1-\lambda)((S o)(x) - (S z)(x))| \\
&\leq f(|o(x) - z(x)|)\left[\left|o - g(x) - \int_0^x G(x, r, o(r))dr\right| + \left|z - g(x) - \int_0^x G(x, r, z(r))dr\right|\right] \\
&= f(|o(x) - z(x)|)\left[|o - (S_\lambda o)(x)| + |z - (S_\lambda z)(x)|\right].
\end{aligned}
$$

This implies that

$$
d(S_\lambda o, S_\lambda z) \leq f(d(o, z))\left[d(o, S_\lambda o) + d(z, S_\lambda z)\right].
$$

Hence, S is an enriched Kannan-type mapping. Meanwhile, by Theorem 8 ($\psi(o) = o$ for all $o > 0$), S has a unique fixed point $z(x)$, satisfying $z(x) = (S z)(x) = (S_\lambda z)(x)$, which means that $z(x)$ is the solution of (3.1). Now, we will show that $z(x) = f(x) + \int_0^x G(x, r, z(r))dr$. Note that

$$
\begin{aligned}
|z(x) - o_{n+1}(x)| &= |(S_\lambda z)(x) - (S_\lambda o_{n+1})(x)| \\
&= \left|\lambda(z(x) - o_n(x)) + (1-\lambda)\int_0^x [G(x, r, z(r)) - G(x, r, o_n(r))]dr\right| \\
&\leq f(|z(x) - o_n(x)|)(1-\lambda)\left[\left|z(x) - g(x) - \int_0^x G(x, r, z(r))dr\right|\right. \\
&\quad \left. + \left|o_n(x) - g(x) - \int_0^x G(x, r, o_n(r))dr\right|\right] \\
&= f(|z(x) - o_n(x)|)\left[|z(x) - (S_\lambda z)(x)| + |o_n(x) - (S_\lambda o_n)(x)|\right],
\end{aligned}
$$

which implies that $\limsup_{n\to+\infty} |z(x) - o_{n+1}(x)| = 0$. Thus

$$
\begin{aligned}
z(x) = \lim_{n\to+\infty} o_{n+1}(x) &= \lambda \lim_{n\to+\infty} o_n(x) + (1-\lambda)\left[g(x) + \int_0^x G(x, r, \lim_{n\to+\infty} o_n(r))dr\right] \\
&= \lambda z(x) + (1-\lambda)\left[g(x) + \int_0^x G(x, r, z(r))dr\right].
\end{aligned}
$$

Hence, we have

$$
z(x) = g(x) + \int_0^x G(x, r, z(r))dr.
$$

□

4. Conclusions

In this paper, we prove three questions about enriched Kannan-type mapping, including the open question raised by Berinde and Păcurar [13]. We defined and studied Suzuki-enriched Kannan-type mappings in convex metric spaces. Several examples related to theorems are also provided to show the validity of our main results. The solution of an integral equation is also investigated. Suzuki enriched Kannan-type mappings are natural generalizations of enriched Kannan mappings. Our results extend fundamental findings previously established in related research.

Author contributions

Yao Yu, Chaobo Li and Dong Ji: Conceptualization, Methodology, Validation, Writing-original draft and Writing-review & editing. All authors contributed equally to the writing of this article. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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