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Research article

Choquard equations with critical exponential nonlinearities in the zero mass case

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Abstract: We investigate Choquard equations in $\mathbb{R}^N$ driven by a weighted N-Laplace operator with polynomial kernel and zero mass. Since the setting is limiting for the Sobolev embedding, we work with nonlinearities which may grow up to the critical exponential. We establish the existence of a positive solution by variational methods, complementing the analysis in [32], where the case of a logarithmic kernel was considered.

Keywords: Choquard equation; zero mass; exponential growth; variational methods; limiting Sobolev embeddings
Mathematics Subject Classification: 35A15, 35J20, 35J60, 35B33

1. Introduction

The aim of this work is to study the weighted Choquard equation with zero mass and polynomial kernel given by

$$-\mathrm{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right) = \left(\frac{1}{|\cdot|^{\mu}} * Q(|\cdot|)F(u)\right)Q(|x|)\,f(u) \quad \text{in } \mathbb{R}^N \tag{Ch$_0$}$$

with $N \geq 2$. Here, A and Q are positive radial weight functions, $\mu \in (0, N)$, and the nonlinearity f is positive. Since the operator is built on the N-Laplacian, one expects that the maximal integrability for the nonlinearity is exponential. This is indeed the framework we are considering, with the additional difficulty of the absence of a mass term.

Choquard-type equations, namely Schrödinger equations with a nonlocal right-hand side, appear in many physics contexts, since they originate from systems where a Schrödinger and a Poisson equation are coupled: those systems, indeed, model, among others, the interaction of two identically charged particles in electromagnetism, and the self-interaction of the wave function with its own gravitational field in quantum mechanics. For the physics background, we refer to [8, 27] and to the references therein. The mathematical interest lies on the fact that the equations of the form

$$-\Delta u + V(x)u = \left(\frac{1}{|\cdot|^{\mu}} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{Ch}$$

where f is a subcritical or critical nonlinearity, can be treated by variational methods. Indeed, if $N \geq 3$ and in the case where the potential $V > 0$, one usually works in the natural Sobolev space $H^1(\mathbb{R}^N)$ and takes advantage of the Hardy-Littlewood-Sobolev inequality (see Lemma 2.2 below) to prove that the functional associated to (Ch) is well-defined, see [16, 17, 29, 30, 34]. The planar case $N = 2$ is more delicate, since this setting is limiting for the Sobolev embedding, and specific techniques need to be developed, see [1, 2]. Note, however, that in order to retrieve the connection with the Schrödinger-Poisson system, the kernel $|\cdot|^{-\mu}$ should be replaced by $-\log|\cdot|$, which is sign-changing and unbounded from above and below, and this makes the analysis even harder: we refer to [12, 15, 19, 28] and to the recent developments in [11, 13, 14].

However, some physics models prescribe that the potential V appearing in the Schrödinger equation is identically zero, e.g., in the study of the Yang-Mills equation in the nonabelian gauge theory of particle physics, see [25]. Such "zero mass case" is mathematically intriguing, since the absence of the mass implies a lack of control of the L^2-part of the norm in $H^1(\mathbb{R}^N)$. Therefore, even if the right-hand side is just local, one is lead to study the equation in the *homogeneous Sobolev space* $D_0^{1,2}(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla \cdot\|_2$. In the higher dimensional case $N \geq 3$, one can still work in this homogeneous space thanks to the critical Sobolev embedding $D_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, see e.g. [4, 6, 9] for Schrödinger equations and [5] for Choquard equations. However, in the Sobolev limiting case $N = 2$, where already the additional difficulties due to exponential nonlinearities appear, the space $D_0^{1,2}(\mathbb{R}^2)$ is not a space of functions anymore; indeed, one cannot distinguish between u and $u + c$ for all $c \in \mathbb{R}$, and no Sobolev embeddings can be proved in this setting. The same problem of course occurs for $D_0^{1,N}(\mathbb{R}^N)$. Nevertheless, we point out that, when dealing with Choquard equations with zero mass and logarithmic kernel, that is originating from Schrödinger-Poisson systems, a sort of mass term may be retrieved from the nonlocal term anyway by a careful splitting of the logarithm, and this enables one to work again in a (possibly inhomogeneous) Sobolev space, see [18, 36] for the linear case $f(u) = u$ and [10] for the delicate extension for a general class of subcritical and critical nonlinearities. This trick however does not work in the case of a polynomial kernel.

In the context of Schrödinger equations with zero mass in $\mathbb{R}^N$, and extending the results of [35] for the non-conformal case, in their recent paper [21], de Albuquerque and Carvalho managed to retrieve a good functional framework by modifying the operator, namely introducing in the standard N-Laplacian $\operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right)$ a positive radial *coercive* weight function A, which satisfies

(A) $A : \mathbb{R}^+ \to \mathbb{R}$ is continuous, $\liminf_{r\to 0^+} A(r) > 0$, and there exist $A_0, \ell > 0$ such that $A(r) \geq A_0 r^\ell$ for all $r > 0$,

and considering the weighted operator $\operatorname{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right)$. Equations driven by weighted p-Laplace operators may be found in several branches of physics such as fluid mechanics, see e.g. [20]. In our case, the functional space which naturally arises is

$$E := \left\{u \in L^N_{loc}(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} A(|x|)|\nabla u|^N \,\mathrm{d}x < +\infty\right\}, \tag{1.1}$$

which is a reflexive* Banach space when endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} A(|x|)|\nabla u|^N \, \mathrm{d}x \right)^{\frac{1}{N}}, \tag{1.2}$$

see [21, Lemma 2.1 and Corollary 1.5]. In particular, in its radial subspace, denoted by E_{rad}, one may recover the Sobolev embeddings, which are necessary not only to enable us to accomplish our estimates, but also to show that E_{rad} is in fact a space of functions. For $p \geq 1$, let us first define the Q-weighted Lebesgue space

$$L^p_Q(\mathbb{R}^N) := \left\{ u \in \mathcal{M}(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} Q(|x|)|u|^p \, \mathrm{d}x < +\infty \right\},$$

where $\mathcal{M}(\mathbb{R}^N)$ stands for the set of all measurable functions on $\mathbb{R}^N$.

Theorem A. *([21], Theorem 1.2) Assume (A) and*

(Q) $Q : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and there exist $b_0, b > -N$ such that

$$\limsup_{r \to 0^+} \frac{Q(r)}{r^{b_0}} < +\infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < +\infty .$$

Then, the embedding $E_{rad} \hookrightarrow L^p_Q(\mathbb{R}^N)$ is continuous for $\gamma \leq p < +\infty$, where

$$\gamma := \max\left\{ N, \frac{(b - \ell + N)(N+1)}{\ell} + N \right\} = \begin{cases} N & \text{if } b < \ell - N, \\ \frac{(b-\ell+N)(N+1)}{\ell} + N & \text{if } b \geq \ell - N. \end{cases} \tag{1.3}$$

Furthermore, the embedding is compact for $\gamma \leq p < +\infty$ when $b < \ell - N$, and for $\gamma < p < +\infty$ when $b \geq \ell - N$.

Note that assumption (Q) allows for weight functions which can be singular at the origin and vanishing at infinity, and has also been used in the study of Choquard equations with vanishing potential, see e.g. [1].

In [21] the authors were also able to prove that in this limiting setting, a sort of Pohožaev-Trudinger-Moser inequality holds. The critical exponential growth is the same as in the unweighted case, namely $t \mapsto \mathrm{e}^{\alpha|t|^{\frac{N}{N-1}}}$, while the influence of the weight functions appears in the Moser exponent. Since we are considering the whole space, one needs to subtract the first terms of the Taylor expansion from the exponential, by introducing the functions

$$\Phi_{\alpha, j_0}(t) := \mathrm{e}^{\alpha|t|^{\frac{N}{N-1}}} - \sum_{j=0}^{j_0 - 1} \frac{\alpha^j}{j!} |t|^{j \frac{N}{N-1}}, \tag{1.4}$$

for $\alpha > 0$ and $j_0 \in \mathbb{N}$.

*The reflexivity of E can be shown in the usual way thanks to the reflexivity of the weighted Lebesgue spaces $L^N(\mathbb{R}^N, A(|\cdot|)\,\mathrm{d}x)$ for $N \geq 2$ see e.g. [23].

Theorem B. *([21], Theorem 1.6) Assume (A) and (Q) hold, and let* $j_0 = \inf\{j \in \mathbb{N} \mid j \geq \frac{\gamma(N-1)}{N}\}$*. Then, for each* $u \in E_{rad}$ *and* $\alpha > 0$*, the function* $\Phi_{\alpha,j_0}(u)$ *belongs to* $L^1_Q(\mathbb{R}^N)$*. Moreover, if*

$$0 < \alpha < \widetilde{\alpha}_N(Q) := \alpha_N \left(1 + \frac{b_0}{N}\right)\left(\inf_{x \in B_1(0)} A(|x|)\right)^{\frac{1}{N-1}}, \tag{1.5}$$

where $\alpha_N := N\omega_{N-1}^{1/(N-1)}$*, with* ω_{N-1} *denoting the measure of the unit sphere in* $\mathbb{R}^N$*, then*

$$\sup_{u \in E_{rad},\, \|u\| \leq 1} \int_{\mathbb{R}^N} Q(|x|)\, \Phi_{\alpha,j_0}(u)\, \mathrm{d}x < +\infty\,.$$

With these tools available in E_{rad}, the existence for the Schrödinger equation with zero mass

$$-\mathrm{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right) = Q(|x|)f(u) \quad \text{in } \mathbb{R}^N$$

was proved in [21], in the case of a positive critical exponential nonlinearity f in the sense of Theorem B, which undergoes a strong growth condition, which is effective in a neighbourhood of zero, namely

$$F(s) \geq \lambda s^\nu \qquad \text{with } \nu > \gamma \text{ and } \lambda \text{ large enough,} \tag{1.6}$$

for all $s \in \mathbb{R}$ and γ defined in (1.3). In this functional framework, a Schrödinger-Poisson system with zero mass, in gradient form and with critical exponential nonlinearities, was recently considered in [32]. After reducing the system to the Choquard equation with logarithmic kernel

$$-\mathrm{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right) = C_N\left(\log\frac{1}{|\cdot|} * Q(|\cdot|)F(u)\right)Q(|x|)f(u) \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

existence is proved using a variational approximating procedure in the spirit of [13, 14, 28]: in fact, the difficulties due to a sign-changing kernel which is unbounded from below and above, are overcome by means of a uniform approximation which exploits suitable kernels having a polynomial behaviour. The global condition (1.6) was also avoided by obtaining a fine upperbound on the mountain-pass level by means of a careful analysis on Moser sequences.

In this paper, we study (Ch_0), which is the counterpart of (1.7) where the logarithm is substituted by the polynomial kernel $|\cdot|^{-\mu}$, $\mu \in (0, N)$, hence we need to face the combined difficulties due to the conformal framework (where one naturally considers exponential nonlinearities), to the absence of mass (where one looses the standard functional setting) and the nonlocality of the right-hand side. As far as we know, Choquard equations with zero mass and polynomial kernels in the conformal case have not been studied before. On the one hand, the analysis will be less involved than the one in [32], since we do not have to face the problem of a sign-changing kernel, and thus we can work directly with the equation without relying on approximation procedures; on the other hand, we would like to avoid the global growth condition (1.6), so a fine analysis on the mountain-pass level should still be performed.

Before stating our results, let us introduce some additional conditions on A and Q:

(Q_μ) $Q : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and there exist $b_0, b > \frac{\mu}{2} - N$ such that

$$\limsup_{r \to 0^+} \frac{Q(r)}{r^{b_0}} < +\infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < +\infty\,;$$

(A') there exist $r_0 > 0$ and $L > 0$ such that $A_0(1 + |x|^{\ell}) \le A(|x|) \le A_0(1 + |x|^{L})$ for all $x \in B_{r_0}(0)$, with A_0, ℓ given by (A);

(Q') $\liminf\limits_{r\to 0^+} \dfrac{Q(r)}{r^{b_0}} = C_Q > 0\,.$

The last two conditions will be needed in estimating the mountain pass level, and can also be found in [21, 32], while (Q_μ) is the adaptation of assumption (Q) to the Choquard case, and is used to prove that the functional assocated to (Ch_0) is well-defined, see Lemma 3.1 below.

Notation: With a little abuse, from now on $A(x) := A(|x|)$ and similarly $Q(x) := Q(|x|)$.

Concerning the nonlinearity f, aiming at modeling both the subcritical and the critical case, we consider the following conditions:

(f_0) $f \in C^1(\mathbb{R})$, $f(t) > 0$ for $t > 0$, and $f(t) = 0$ for $t \le 0$;

(f_1^s) f is *subcritical* in the sense of Trudinger-Moser, namely

$$\lim_{t\to+\infty} \frac{f(t)}{\mathrm{e}^{\alpha t^{\frac{N}{N-1}}}} = 0 \quad \text{for all } \alpha > 0\,;$$

(f_1^c) f is *critical* in the sense of Trudinger-Moser, namely there exists $\alpha_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{\mathrm{e}^{\alpha t^{\frac{N}{N-1}}}} = \begin{cases} 0 & \text{for } \alpha > \alpha_0\,, \\ +\infty & \text{for } \alpha < \alpha_0\,; \end{cases}$$

(f_2) there exists $\widetilde{p} > \left(1 - \frac{\mu}{2N}\right)\gamma$ such that $f(t) = o(t^{\widetilde{p}-1})$ as $t \to 0^+$;

(f_3) there exist $\tau \in \left(1 - \frac{2}{N}, 1\right)$ and $C > 0$ such that

$$\tau \le \frac{F(t)f'(t)}{(f(t))^2} \le C \quad \text{for any } t > 0\,;$$

(f_ξ) there exist $\xi > 0$ and $\nu > \gamma$ such that

$$F(t) \ge \xi t^{\nu} \quad \text{for } t \in (0, 1]\,;$$

(f_4) there exist $t_0, M_0 > 0$ and $\theta \in (0, N-1]$ such that

$$0 < t^{\theta} F(t) \le M_0 f(t) \quad \text{for } t \ge t_0\,;$$

(f_5) there exists $\beta_0 > 0$ such that

$$\liminf_{t\to+\infty} \frac{F(t)}{\mathrm{e}^{\alpha_0 t^{\frac{N}{N-1}}}} \ge \beta_0 > 0\,.$$

Definition 1.1 (Solution of (Ch_0))**.** We say that $u \in E$ is a *weak solution of* (1.7) if

$$\int_{\mathbb{R}^N} A(x)|\nabla u|^{N-2}\nabla u \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^{\mu}} \,\mathrm{d}y \right) Q(x) f(u(x))\, \varphi(x) \,\mathrm{d}x$$

for all $\varphi \in E$.

Theorem 1.1. *Let $\mu \in (0, N)$, under conditions (A), (Q_μ), (f_0), (f_2), and (f_3), assume either that*

S) the problem is subcritical, namely (f_1^s) holds,

or that

C) the problem is critical, namely (f_1^c) holds, and

i) (f_ξ) holds with $\xi > \xi_0$ (depending on ν) given in (3.11)

or, alternatively,

ii) (A'), (Q'), (f_4)-(f_5) are fulfilled.

Then, (1.7) *has a positive radially symmetric weak solution in E_{rad}.*

Remark 1. We stress the fact that our results are new even in the planar case $N = 2$. Moreover, they can be seen as an extension of the corresponding results in [1,2] to the zero mass case, of those in [32] to the case of polynomial kernels, and of those in [21] to the Choquard framework.

Remark 2. Since the weight A is continuous and bounded below by (A), it is clear that for all $\Omega \subset\subset \mathbb{R}^N$ there exist constants $\underline{a}_0, \overline{a}_0 > 0$ such that $\underline{a}_0 < A(x) < \overline{a}_0$ for all $x \in \Omega$. This implies that $E \subset D^{1,N}(\mathbb{R}^N) \subset W^{1,N}_{loc}(\mathbb{R}^N)$, where $D^{1,N}(\mathbb{R}^N)$ is the homogeneous Sobolev space defined in (1.1) with $A \equiv 1$, see [24, Lemma II.6.1]. Therefore, it is sufficient to prove the existence of a nonnegative solution of (1.7) in order to retrieve its positivity by the strong maximum principle for quasilinear equations, see [31, Theorem 11.1].

Remark 3. The coercivity of the weight function A in the operator is striking in order to work in a suitable functional setting. By now, the more natural case $A \equiv 1$ seems still out of reach in both conformal Schrödinger and Choquard frameworks, with the exception of the Schrödinger-Poisson system, see [10, 18].

Notation For $R > 0$ and $x_0 \in \mathbb{R}^N$ we denote by $B_R(x_0)$ the ball of radius R and center x_0. Given a set $\Omega \subset \mathbb{R}^N$, its characteristic function is denoted by χ_Ω and $\Omega^c := \mathbb{R}^N \setminus \Omega$. The space of the infinitely differentiable functions which are compactly supported is $C_0^\infty(\mathbb{R}^N)$, while $L^p(\mathbb{R}^N)$ with $p \in [1, +\infty]$ is the Lebesgue space of p-integrable functions. The norm of $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$. For $q > 0$, we define $\lfloor q \rfloor$ as the largest integer strictly less than q; if $q > 1$, its conjugate Hölder exponent is $q' := \frac{q}{q-1}$. The symbol $\lesssim$ indicates that an inequality holds up to a multiplicative constant depending only on structural constants. Finally, $o_n(1)$ denotes a vanishing real sequence as $n \to +\infty$. Hereafter, the letter C will be used to denote positive constants which are independent of relevant quantities and whose value may change from line to line.

Overview After the short Section 2, in which we discuss some consequences of our assumptions and state some useful results, we prove existence for the Choquard equation (1.7), splitting the proof in Sections 3 and 4, according to the set of assumptions considered in Theorem 1.1.

2. Preliminaries

From now on, we set $\Phi_\alpha := \Phi_{j_0,\alpha}$ with j_0 defined in Theorem B. We start by collecting some comments on our assumptions:

Remark 4. (i) From (f_0)-(f_1^c)-(f_2) and (1.4), it is easy to infer that for fixed $\alpha > \alpha_0$, $p > 1$, and for any $\varepsilon > 0$ one has

$$|f(t)| \le \varepsilon |t|^{\widetilde{p}-1} + C_1(\alpha, p, \varepsilon)|t|^{p-1}\Phi_\alpha(t)\,, \qquad t \in \mathbb{R}\,, \tag{2.1}$$

for some $C_1(\alpha, p, \varepsilon) > 0$, and consequently,

$$|F(t)| \le \varepsilon |t|^{\widetilde{p}} + C_2(\alpha, p, \varepsilon)|t|^{p}\Phi_\alpha(t)\,, \qquad t \in \mathbb{R}\,, \tag{2.2}$$

for some $C_2(\alpha, p, \varepsilon) > 0$. In the case (f_1^s) holds in place of (f_1^c), inequalities (2.1) and (2.2) are valid with $\alpha > 0$ arbitrary.

(ii) Assumption (f_3) implies that f is monotone increasing and

$$F(t) \le (1-\tau)tf(t) \quad \text{for any } \ t \ge 0\,. \tag{2.3}$$

(iii) Although frequent in the literature, see e.g. [1,3,21], assumption (f_ξ) is very strong, not because of the polynomial growth $t \mapsto t^\nu$ with $\nu > \gamma$, which is reasonable since it excludes just exponential decays at 0, but mainly because of the fact that one should prescribe this behaviour in the whole range $[0,1]$ and not just asymptotically. In fact, it is not easy to verify. For instance, the easiest example $F(t) = t^\kappa$ with $\gamma < \kappa < \nu$ verifies this growth condition just in a small right neighborhood of 0 and not in the whole $[0,1]$. This is the reason why we are also considering an alternative proof of our main result which uses assumptions (f_4) and (f_5), although the argument which exploits (f_ξ) is much easier.

(iv) (f_5) is a condition at infinity, compatible with the critical growth (f_1^c) and related to the well-known de Figueiredo-Miyagaki-Ruf condition [22]. It is crucial in order to estimate the mountain pass level and gain compactness, see Lemma 4.1. A similar condition appears also in [2,10,11, 15,32], however, as in [1,18], we do not prescribe β_0 to be large.

(v) Examples of admissible subcritical or critical nonlinearities are $F(t) = t^q e^{t^\alpha}$ with $q > \left(1 - \frac{\mu}{2N}\right)\gamma$ and $\alpha \in \left[0, \frac{N}{N-1}\right]$. The critical case (C-ii) corresponds to the choice of $\alpha = \frac{N}{N-1}$. Regarding the weight functions A and Q, possible examples which fulfill (A)-(A') or (Q_μ)-(Q'), respectively, are:

- $A(r) = A_0(1 + r^\ell)$ or $A(r) = A_0 e^r$, with $A_0, \ell > 0$;
- $Q \equiv 1$, or $Q(r) = r^{b_0}\chi_{\{r\le 1\}} + r^b\chi_{\{r>1\}}$ with $b_0, b > \frac{\mu}{2} - N$, or $Q(r) = re^{-r}$.

The next lemma assures that the function Φ_α introduced in (1.4) has the same properties of the exponential.

Lemma 2.1. *For $\alpha > 0$, $r > 1$, and $\nu \in \mathbb{R}$ it holds that*

$$(\Phi_\alpha(t))^r \le \Phi_{\alpha r}(t) \qquad \text{for all } \ t > 0 \tag{2.4}$$

and

$$\Phi_\alpha(\nu t) = \Phi_{\alpha\nu^{\frac{N}{N-1}}}(t) \qquad \text{for all } \ t > 0\,. \tag{2.5}$$

Proof. For the first inequality, see [37, Lemma 2.1]; the second is just an easy calculation. □

We end this section by recalling the well-known Hardy-Littlewood-Sobolev inequality, see [26, Theorem 4.3], which will be frequently used throughout the paper.

Lemma 2.2. *(Hardy-Littlewood-Sobolev inequality) Let $N \ge 1$, $s, r > 1$, and $\mu \in (0, N)$ with $\frac{1}{s} + \frac{\mu}{N} + \frac{1}{r} = 2$. There exists a constant $C = C(N, \mu, s, r)$ such that for all $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * f\right) h \, dx \le C\|f\|_s \|h\|_r\,.$$

3. Proof of Theorem 1.1: the subcritical case and the critical case (*i*)

We start by proving that the functional J, formally associated to (Ch_0),

$$J(u) := \frac{1}{N}\int_{\mathbb{R}^N} A(x)|\nabla u|^N \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right)Q(x)F(u(x))\,\mathrm{d}x$$

is well-defined in the space E_{rad}, is C^1 with derivative

$$J'(u)[\varphi] = \int_{\mathbb{R}^N} A(x)|\nabla u|^{N-2}\nabla u\nabla\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right)Q(x)f(u(x))\,\varphi(x)\,\mathrm{d}x\,,$$

and possesses a mountain-pass geometry.

Lemma 3.1. *Under assumptions (f_0), (f_2), and either (f_1^c) or (f_1^s), the functional $J : E_{rad} \to \mathbb{R}$ is well-defined and C^1. If f satisfies also (f_3), there exist constants $\rho, \eta > 0$ and $e \in E_{rad}$ such that:*

(i) $J|_{S_\rho} \geq \eta > 0$, *where* $S_\rho = \{u \in E_{rad} \,|\, \|u\| = \rho\}$*;*
(ii) $\|e\| > \rho$ *and* $J(e) < 0$.

Proof. Although the proof is standard, the main tool being the Hardy-Littlewood-Sobolev inequality (Lemma 2.2), we retrace it here, in particular to show the rôle of assumption (Q_μ).

We focus on the second term of J, the first one being already $\|u\|^N$, see (1.2). By Lemma 2.2 with $r = t = \frac{2N}{2N-\mu}$, the estimate (2.2), and Hölder's inequality, one infers

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right)Q(x)F(u(x))\,\mathrm{d}x &\lesssim \left(\int_{\mathbb{R}^N}|QF(u)|^{\frac{2N}{2N-\mu}}\right)^{\frac{2N-\mu}{N}}\\ &\lesssim \left(\int_{\mathbb{R}^N} Q^{\frac{2N}{2N-\mu}}|u|^{\frac{2N\widetilde{p}}{2N-\mu}}\right)^{\frac{2N-\mu}{N}} + \left(\int_{\mathbb{R}^N} Q^{\frac{2N}{2N-\mu}}|u|^{\frac{2Npq'}{2N-\mu}}\right)^{\frac{2N-\mu}{Nq'}}\left(\int_{\mathbb{R}^N} Q^{\frac{2N}{2N-\mu}}|\Phi_\alpha(u)|^{\frac{2Nq}{2N-\mu}}\right)^{\frac{2N-\mu}{Nq}},\end{aligned}$$

for $\alpha > \alpha_0$ in case (f_1^c) holds (resp. $\alpha > 0$ if (f_1^s) holds). In order to use now the Sobolev embedding given by Theorem A, as well as to bound the exponential term by Theorem B, in both cases the weight function $\widetilde{Q} := Q^{\frac{2N}{2N-\mu}}$ must verify assumption (Q), and the exponent of u, namely $\frac{2N\widetilde{p}}{2N-\mu}$, should be greater than γ. However, it is not difficult to show that this is the case under (Q_μ) and (f_2). As a result, using also (2.4), one infers

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right)Q(x)F(u(x))\,\mathrm{d}x \lesssim \|u\|^{2\widetilde{p}} + \|u\|^{2p}\left(\int_{\mathbb{R}^N}\widetilde{Q}\,\Phi_{\frac{2Nq\alpha}{2N-\mu}}(u)\right)^{\frac{2N-\mu}{Nq}} < +\infty\,. \tag{3.1}$$

This shows the well-posedness of J in E_{rad}, while the regularity of J follows by standard arguments. In order to show (*i*), from (3.1) and (2.5) we deduce

$$J(u) \gtrsim \|u\|^N - \|u\|^{2\widetilde{p}} - \|u\|^{2p}\left(\int_{\mathbb{R}^N}\widetilde{Q}\,\Phi_{\frac{2Nq\alpha}{2N-\mu}\|u\|^{\frac{N}{N-1}}}\left(\tfrac{u}{\|u\|}\right)\right)^{\frac{2N-\mu}{Nq}}.$$

Therefore, in order to apply the uniform estimate of Theorem (B), one needs $\frac{2Nq\alpha}{2N-\mu}\|u\|^{\frac{N}{N-1}} < \widetilde{\alpha}_N(\widetilde{Q})$ defined in (1.5), namely to require that $\rho < \left(\frac{2N-\mu}{2Nq\alpha}\,\widetilde{\alpha}_N(\widetilde{Q})\right)^{\frac{N-1}{N}}$. If so,

$$J(u) \gtrsim \|u\|^N - \|u\|^{2\widetilde{p}} - \|u\|^{2p},$$

which implies that 0 is a local minimum by choosing p large enough, since $2\widetilde{p} > (2N-\mu)\frac{\gamma}{N} > N$. Let us now take $0 \le \varphi \in E_{\rm rad}$ and define

$$\psi(t) := \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(t\varphi)\right) Qf(t\varphi)\,\mathrm{d}x\,.$$

Using (2.3), it is then standard to show that $\frac{\psi'(t)}{\psi(t)} \ge \frac{2}{(1-\tau)t}$, which in turn implies $\psi(t) \ge \psi(1)t^{\frac{2}{1-\tau}}$. Hence,

$$J(t\varphi) = \frac{t^N}{N}\|\varphi\|^N - \psi(t) \le \frac{t^N}{N}\|\varphi\|^N - Ct^{\frac{2}{1-\tau}} \to -\infty\,,$$

since $\tau \in \left(1-\frac{2}{N}, 1\right)$ by (f_3). It is then sufficient to take $e := t_0\varphi$ with t_0 large enough, to conclude that (ii) holds. □

As a consequence of this mountain-pass geometry, one infers the existence of a Cerami sequence in $E_{\rm rad}$ at level

$$c_{mp} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))\,,$$

where

$$\Gamma := \{\gamma \in C([0,1], E_{\rm rad}) \,|\, \gamma(0) = 0, \gamma(1) = e\}\,,$$

namely, a sequence $(u_k)_k \subset E_{\rm rad}$ such that

$$J(u_k) \to c_{mp} \quad \text{and} \quad (1+\|u_k\|)J'(u_k) \to 0 \quad \text{in } (E_{\rm rad})' \tag{3.2}$$

as $k \to +\infty$. In details,

$$J(u_k) = \frac{1}{N}\int_{\mathbb{R}^N} A(x)|\nabla u_k|^N\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) QF(u_k) = c_{mp} + o_k(1)\,, \tag{3.3}$$

and for all $\varphi \in E_{\rm rad}$ one has

$$J'(u_k)[\varphi] = \int_{\mathbb{R}^N} A(x)|\nabla u_k|^{N-2}\nabla u_k\nabla\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) Qf(u_k)\varphi = o_k(1)\|\varphi\|\,, \tag{3.4}$$

from which

$$J'(u_k)[u_k] = \int_{\mathbb{R}^N} A(x)|\nabla u_k|^N\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) Qf(u_k)u_k = o_k(1)\|u_k\|\,. \tag{3.5}$$

Lemma 3.2. *Assume that (f_0)–(f_3) hold. Let $(u_k)_k \subset E_{rad}$ be a Cerami sequence of J at level c_{mp}. Then $(u_k)_k$ is bounded in E with*

$$\|u_k\|^N \le c_{mp}\left(\frac{1}{N} - \frac{1-\tau}{2}\right)^{-1} + o_k(1)\,, \tag{3.6}$$

and there exists $u \in E_{rad}$ such that $u_k \rightharpoonup u$ in E_{rad}.

Proof. By (3.2) and (3.5) we obtain

$$
\begin{aligned}
c_{mp} + o_k(1) &= J(u_k) - \frac{1-\tau}{2} J'(u_k)[u_k] \\
&= \left(\frac{1}{N} - \frac{1-\tau}{2}\right) \|\nabla u_k\|^N - \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) Q\left(F(u_k) - (1-\tau) f(u_k) u_k\right) \mathrm{d}x \\
&\geq \left(\frac{1}{N} - \frac{1-\tau}{2}\right) \|\nabla u_k\|^N
\end{aligned}
$$

by (2.3). The weak convergence follows since E_{rad} is a closed subspace of a reflexive Banach space. □

To show that the limit function u is indeed a weak solution of (Ch_0), we may prove that $u_k \to u$ in E_{rad}. This is manageable in the subcritical case. On the other hand, in the critical case, we first need to get a suitable uniform control on the mountain-pass level, see (3.7) below, so that one can use the uniform estimate given by Theorem B in order to prove the convergence of the nonlocal term in the functional. Under assumption (f_ξ), this is relatively easy, since by taking the constant ξ large enough, one can decrease the value of the mountain pass level up to the desired threshold. This is the aim of the last part of this section, which therefore contains the proof of Theorem 1.1 under the first set of assumptions, while we defer its proof under the more verifiable assumptions (f_4) and (f_5) to Section 4.

In the spirit of [1] we then prove the following.

Lemma 3.3. *Under (f_0), (f_1^c), (f_2), there exists $\xi_0 > 0$ explicit such that, if f satisfies (f_ξ) with $\xi > \xi_0$, then*

$$
c_{mp} < \left(\frac{1}{N} - \frac{1-\tau}{2}\right)\left(\frac{2N-\mu}{2N}\,\frac{\widetilde{\alpha}_N(\widetilde{Q})}{\alpha_0}\right)^{N-1} =: c_*\,, \tag{3.7}
$$

from which

$$
\|u_k\|^{\frac{N}{N-1}} < \frac{2N-\mu}{2N\alpha_0}\,\widetilde{\alpha}_N(\widetilde{Q})\,. \tag{3.8}
$$

Proof. Fix a nonnegative radial function $\varphi_0 \in C_0^\infty(B_1(0))$ with values in $[0,1]$ such that $\varphi_0 \equiv 1$ in $B_{\frac{1}{2}}(0)$ and $|\nabla\varphi_0| \leq 2$. Then

$$
\begin{aligned}
J(\varphi_0) &= \frac{1}{N}\int_{B_1(0)\setminus B_{\frac{1}{2}}(0)} A(x)|\nabla\varphi_0|^N \,\mathrm{d}x - \frac{1}{2}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * QF(\varphi_0)\right) QF(\varphi_0) \\
&\leq \frac{\omega_N}{N}\left(2^N - 1\right) \sup_{B_1(0)\setminus B_{\frac{1}{2}}(0)} A - \frac{\xi^2}{2}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * Q\varphi_0^\nu\right) Q\varphi_0^\nu\,.
\end{aligned}
$$

Noting that the right-hand side tends to $-\infty$ as $\xi \to +\infty$, one may take $\xi > \xi_1$, where ξ_1 is chosen such that

$$
\frac{\xi_1^2}{2}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * Q\varphi_0^\nu\right) Q\varphi_0^\nu = \frac{\omega_N}{N}\left(2^N - 1\right) \sup_{B_1(0)\setminus B_{\frac{1}{2}}(0)} A\,,
$$

and get $J(\varphi_0) \leq 0$. As a result, by definition of c_{mp} we can estimate as follows:

$$
\begin{aligned}
c_{mp} &\leq \max_{t\in[0,1]} J(t\varphi_0) \leq \max_{t\in[0,1]}\left(t^N \int_{B_1(0)} A(x)\frac{|\nabla\varphi_0|^N}{N}\,\mathrm{d}x - \frac{\xi^2 t^{2\nu}}{2}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * Q\varphi_0^\nu\right) Q\varphi_0^\nu\right) \\
&\leq \frac{\xi_1^2}{2}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * Q\varphi_0^\nu\right) Q\varphi_0^\nu \max_{t\in[0,1]}\left(t^N - \chi t^{2\nu}\right),
\end{aligned} \tag{3.9}
$$

where $\chi := \left(\frac{\xi}{\xi_1}\right)^2 > 1$. It is standard to prove that the map $h(t) := t^N - \chi t^{2\nu}$ achieves its maximum in $t_0 := \left(\frac{N}{2\nu\chi}\right)^{\frac{1}{2\nu-N}} \in (0,1)$ since $\nu > \gamma > N$. Hence, inserting $h(t_0)$ in (3.9), one gets

$$c_{mp} \leq \frac{\xi_1^{\frac{4\nu}{2\nu-N}}}{\xi^{\frac{2N}{2\nu-N}}}\left(\frac{N}{2\nu}\right)^{\frac{N}{2\nu-N}}\frac{2\nu-N}{4\nu}\int_{B_1(0)}\left(\frac{1}{|\cdot|^\mu} * Q\,\varphi_0^\nu\right)Q\,\varphi_0^\nu =: c_0(\nu.N,\xi_1,Q,\varphi_0)\,\xi^{-\frac{2N}{2\nu-N}}. \tag{3.10}$$

To show (3.7) we then need to choose ξ so that the right-hand side is below the threshold c_*, namely

$$\xi > \xi_0 := \max\{\xi_1, \xi_*\}, \tag{3.11}$$

where ξ_* satisfies the equality in (3.10). At this point, combining the uniform bounds in (3.6) and (3.7), it is immediate to infer a nice uniform control on the norm of $(u_k)_k$ given by (3.8). □

We are now ready to prove Theorem 1.1 under assumptions (f_0)-(f_3) and (f_ξ) with $\xi > \xi_0$ defined in (3.11).

Proof of Theorem 1.1 (S)-(C-i). We aim at proving that

$$T(u_k) := \int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)(u_k-u) \to 0 \tag{3.12}$$

as $n \to +\infty$. Indeed, if so, by (3.4) with $\varphi = u$ and (3.5), one would infer

$$\int_{\mathbb{R}^N} A(x)|\nabla u_k|^{N-2}\nabla u_k \nabla(u_k-u)\,\mathrm{d}x \to 0$$

which, combined with

$$\int_{\mathbb{R}^N} A(x)|\nabla u|^{N-2}\nabla u \nabla(u_k-u)\,\mathrm{d}x \to 0$$

by weak convergence, would guarantee that $u_k \to u$ strongly in E by means of the simple inequality (see [33, inequality (2.2)])

$$(|y_1|^{N-2}y_1 - |y_2|^{N-2}y_2)(y_1-y_2) \geq C(N)|y_1-y_2|^N \quad \text{for all } y_1, y_2 \in \mathbb{R}^N.$$

Since the functional is C^1, the fact that u is a weak solution of (Ch_0) directly follows.

Hence, we can show (3.12). By the Hardy-Littlewood-Sobolev inequality, we obtain

$$|T(u_k)| \leq \|QF(u_k)\|_{\frac{2N}{2N-\mu}}\|Qf(u_k)(u_k-u)\|_{\frac{2N}{2N-\mu}}, \tag{3.13}$$

and we prove that the first term on the right is uniformly bounded, while the second converges to 0. Indeed, similarly to (3.1), we have

$$\|QF(u_k)\|_{\frac{2N}{2N-\mu}} \lesssim \|u_k\|^{\widetilde{p}} + \|u_k\|^p\left(\int_{\mathbb{R}^N} Q^{\frac{2N}{2N-\mu}}\,\Phi_{\frac{2Nq\alpha}{2N-\mu}\|u_k\|^{\frac{N}{N-1}}}\left(\frac{u_k}{\|u_k\|}\right)\right)^{\frac{2N-\mu}{2N}}. \tag{3.14}$$

If (f_1^s) holds, since $\|u_k\|$ is uniformly bounded by Lemma 3.2, then

$$\frac{2Nq\alpha}{2N-\mu}\|u_k\|^{\frac{N}{N-1}} < \widetilde{\alpha}_N(\widetilde{Q}) \tag{3.15}$$

follows by taking a sufficiently small $\alpha > 0$. On the other hand, in the critical case (f_1^c), by (3.8) one may take $q > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0, so that (3.15) holds. In both cases the last term in (3.14) is then bounded uniformly in k. As a result,

$$\|QF(u_k)\|_{\frac{2N}{2N-\mu}} \leq C \tag{3.16}$$

by Lemma 3.2. Similarly, recalling the notation $\widetilde{Q} := Q^{\frac{2N}{2N-\mu}}$, by (2.1) and the Hölder inequality with conjugate exponents $\widetilde{p}, \widetilde{p}' = \frac{\widetilde{p}}{\widetilde{p}-1}$ for the first term, and r, r' and ν, ν' for the second, we get

$$\begin{aligned}\|Qf(u_k)(u_k-u)\|_{\frac{2N}{2N-\mu}}^{\frac{2N}{2N-\mu}} &\lesssim \left(\int_{\mathbb{R}^N} \widetilde{Q}|u_k|^{\frac{2N\widetilde{p}}{2N-\mu}}\right)^{\frac{\widetilde{p}-1}{\widetilde{p}}} \left(\int_{\mathbb{R}^N} \widetilde{Q}|u_k-u|^{\frac{2N\widetilde{p}}{2N-\mu}}\right)^{\frac{1}{\widetilde{p}}} \\ &+ \left(\int_{\mathbb{R}^N} \widetilde{Q}|u_k|^{(p-1)\frac{2Nr'}{2N-\mu}}\right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^N} \widetilde{Q}|u_k-u|^{\frac{2Nr\nu'}{2N-\mu}}\right)^{\frac{1}{r\nu'}} \left(\int_{\mathbb{R}^N} \widetilde{Q}\,\Phi_{\frac{2Nr\nu\alpha}{2N-\mu}\|u_k\|^{\frac{N}{N-1}}}\left(\frac{u_k}{\|u_k\|}\right)\right)^{\frac{1}{r\nu}}.\end{aligned}$$

As before, in the subcritical case, again a choice of α small enough is sufficient to control the exponential term, while in the critical case one needs to choose $r, \nu > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0, and consider the upper bound (3.8). In both cases, we may show the boundedness of the exponential term. Up to a smaller ν and a bigger p, one also has $\frac{2Nr\nu'}{2N-\mu} > \gamma$ and $(p-1)\frac{2Nr'}{2N-\mu} > \gamma$. Hence,

$$\|Qf(u_k)(u_k-u)\|_{\frac{2N}{2N-\mu}} \lesssim \|u_k\|^{\widetilde{p}-1}\|u_k-u\|_{L_{\widetilde{Q}}^{\frac{2N\widetilde{p}}{2N-\mu}}} + \|u_k\|^{p-1}\|u_k-u\|_{L_{\widetilde{Q}}^{\frac{2Nr\nu'}{2N-\mu}}} \to 0 \tag{3.17}$$

by Lemma 3.2 and the compact embedding given by Theorem A. Combining (3.16) and (3.17) with (3.13), (3.12) holds, and the strong convergence $u_k \to u$ follows, which proves that u is a weak solution of (Ch_0). □

4. Proof of Theorem 1.1: the critical case (*ii*)

As we mentioned in the introduction, the global growth assumption (f_ξ), introduced in the critical case, is in fact not easy to verify. In this section, we prove the existence of a weak solution of (Ch_0) in the critical case by using (f_4)-(f_5) instead of (f_ξ); however, we will need to prescribe some control from below of the weight functions A and Q as in (A')-(Q'). The argument, inspired by [1, 2], exploits the concentration behavior of the Moser sequences to infer a suitable uniform bound for $\|u_k\|$, which turns out to depend on all structural constants of the equation.

Let us introduce the Moser sequence as

$$\widetilde{w}_n(x) := \begin{cases} (\log n)^{1-\frac{1}{N}} & \text{if } 0 \leq |x| \leq \frac{\rho}{n}, \\ \dfrac{\log\frac{\rho}{|x|}}{(\log n)^{\frac{1}{N}}} & \text{if } \frac{\rho}{n} < |x| < \rho, \\ 0 & \text{if } |x| \geq \rho, \end{cases}$$

where $\rho \leq r_0$ is given by (A'). Using (A'), we estimate from below its norm in E as

$$\begin{aligned}\int_{\mathbb{R}^N} A(x)|\nabla\widetilde{w}_n|^N \,\mathrm{d}x &= \frac{\omega_{N-1}}{\log n}\int_{\frac{\rho}{n}}^{n} \frac{A(r)}{r}\,\mathrm{d}r \geq \frac{\omega_{N-1}A_0}{\log n}\int_{\frac{\rho}{n}}^{n} \frac{1+r^\ell}{r}\,\mathrm{d}r \\ &= \omega_{N-1}A_0\left(1 + \frac{\rho^\ell}{\ell\log n} + o\left(\frac{1}{\log n}\right)\right),\end{aligned}$$

and analogously from above, hence we can state that

$$\|\widetilde{w}_n\|^N = \omega_{N-1}A_0(1+\delta_n)\,, \qquad \text{with} \quad \frac{\rho^\ell/\ell}{\log n} + o\left(\frac{1}{\log n}\right) \le \delta_n \le \frac{\rho^L/L}{\log n} + o\left(\frac{1}{\log n}\right). \tag{4.1}$$

Hence, defining

$$w_n := \frac{\widetilde{w}_n}{(\omega_{N-1}A_0(1+\delta_n))^{\frac{1}{N}}}\,,$$

one has $\|w_n\| = 1$ for all $n \in \mathbb{N}$.

Lemma 4.1. *Under (A)-(A'), (Q_μ)-(Q'), (f_0)-(f_3), and (f_4)-(f_5), one has*

$$c_{mp} < \frac{\omega_{N-1}A_0}{N}\left(\frac{2\widetilde{b}_0 + 2N - \mu}{2\alpha_0}\right)^{N-1}. \tag{4.2}$$

Proof. We aim at showing that there exist a suitable Б > 0 (to be chosen later) and $n_0 \in \mathbb{N}$ such that

$$\max_{t\ge 0} J(tw_{n_0}) < Б\,. \tag{4.3}$$

Suppose by contradiction that (4.3) does not hold. This means that for all $n \in \mathbb{N}$ there exists $t_n > 0$ such that

$$J(t_n w_n) = \max_{t\ge 0} J(tw_n) \ge Б\,.$$

Since the convolution term is positive and $\|w_n\| = 1$ for all $n \in \mathbb{N}$, this implies

$$t_n^N \ge NБ\,. \tag{4.4}$$

On the other hand, one may suppose that t_n is chosen such that $J(t_n w_n) = \max\{J(tw_n)\,|\,t > 0\}$ by the geometry of the functional on radial functions with compact support obtained in Lemma 3.1. Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_n} J(tw_n) = 0$, from which

$$t_n^N = \int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(t_n w_n)\right) Qf(t_n w_n) t_n w_n\,. \tag{4.5}$$

Using assumptions (f_4) and (f_5), for all $\varepsilon > 0$ fixed, there exists $t_\varepsilon > 0$ such that for $t > \max\{t_0, t_\varepsilon\}$ one has

$$tf(t)F(t) \ge \frac{t^{\theta+1}}{M_0}(F(t))^2 \ge \frac{\beta_0^2 - \varepsilon}{M_0} t^{\theta+1} \mathrm{e}^{2\alpha_0 t^{\frac{N}{N-1}}}. \tag{4.6}$$

Hence, recalling that w_n is constant in $B_{\frac{\varrho}{n}}(0)$, we can estimate the right-hand side of (4.5) from below by (4.6) as

$$\begin{aligned} t_n^N &\ge \int_{B_{\frac{\varrho}{n}}(0)}\left(\int_{B_{\frac{\varrho}{n}}(0)} \frac{Q(y)F(t_n w_n(y))}{|x-y|^\mu}\,\mathrm{d}y\right) Q(x) f(t_n w_n(x)) t_n w_n(x)\,\mathrm{d}x \\ &\ge \frac{(\beta_0^2 - \varepsilon) t_n^{\theta+1} (\log n)^{\left(1-\frac{1}{N}\right)(\theta+1)}}{M_0\left((1+\delta_n)A_0\omega_{N-1}\right)^{\frac{\theta+1}{N}}} \mathrm{e}^{\frac{2\alpha_0 t_n^{\frac{N}{N-1}} \log n}{((1+\delta_n)A_0\omega_{N-1})^{\frac{1}{N-1}}}} \int_{B_{\frac{\varrho}{n}}(0)}\int_{B_{\frac{\varrho}{n}}(0)} \frac{Q(x)Q(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned} \tag{4.7}$$

By (Q'), we have $Q(r) > cr^{b_0}$ in $B_{\frac{\varrho}{n}}(0)$ for n large enough. Hence, using the simple estimate $\frac{1}{|x-y|^\mu} \geq \left(\frac{n}{2\rho}\right)^\mu$ for all $x, y \in B_{\frac{\varrho}{n}}(0)$, we obtain

$$\begin{aligned}\int_{B_{\frac{\varrho}{n}}(0)} \int_{B_{\frac{\varrho}{n}}(0)} \frac{Q(x)Q(y)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y &\geq c^2 \left(\frac{n}{2\rho}\right)^\mu \left(\int_{B_{\frac{\varrho}{n}}(0)} |x|^{b_0}\,\mathrm{d}x\right)^2 \\ &= c^2 \left(\frac{n}{2\rho}\right)^\mu \omega_{N-1}^2 \left(\int_0^{\frac{\varrho}{n}} r^{b_0+N-1}\,\mathrm{d}r\right)^2 = \frac{c^2\omega_{N-1}^2}{2^\mu(b_0+N)^2} \left(\frac{\rho}{n}\right)^{2b_0+2N-\mu}.\end{aligned}$$

Hence, from (4.7) one infers

$$t_n^{N-\theta-1} \geq K \exp\left\{\left(\frac{2\alpha_0\, t_n^{\frac{N}{N-1}}}{(A_0\omega_{N-1}(1+\delta_n))^{\frac{1}{N-1}}} - (2b_0+2N-\mu)\right)\log n + \frac{N-1}{N}(\theta+1)\log\log n\right\}, \tag{4.8}$$

where the constant K is defined as

$$K := \frac{(\beta_0^2-\varepsilon)c^2\omega_{N-1}^2\,\rho^{2b_0+2N-\mu}}{M_0 2^\mu(b_0+N)^2\,(A_0\omega_{N-1}(1+\delta_n))^{\frac{\theta+1}{N}}}.$$

Applying the log on both sides of (4.8) yields

$$\begin{aligned}(N-1-\theta)\frac{(N-1)}{N}t_n^{\frac{N}{N-1}} \geq (N-1-\theta)\log(t_n) &\geq \log K + \frac{N-1}{N}(\theta+1)\log\log n \\ &+ \left(\frac{2\alpha_0\, t_n^{\frac{N}{N-1}}}{(A_0\omega_{N-1}(1+\delta_n))^{\frac{1}{N-1}}} - (2b_0+2N-\mu)\right)\log n.\end{aligned} \tag{4.9}$$

Dividing by $t_n^{\frac{N}{N-1}}$, we obtain

$$(N-1-\theta)\frac{(N-1)}{N} \geq \left(\frac{2\alpha_0}{(A_0\omega_{N-1}(1+\delta_n))^{\frac{1}{N-1}}} - \frac{2b_0+2N-\mu}{t_n^{\frac{N}{N-1}}}\right)\log n.$$

If $t_n \to +\infty$, then one would get a contradiction for large n, since $\theta \in (0, N-1]$, and similarly, if the factor in front of $\log n$ is positive. Hence, we infer that $(t_n)_n$ is bounded with

$$t_n^N \leq A_0\omega_{N-1}(1+\delta_n)\left(\frac{2b_0+2N-\mu}{2\alpha_0}\right)^{N-1}. \tag{4.10}$$

Comparing (4.4) and (4.10), and since $\delta_n = o_n(1)$ as $n \to +\infty$, we see that by choosing

$$\text{Б} := \frac{A_0\omega_{N-1}}{N}\left(\frac{2b_0+2N-\mu}{2\alpha_0}\right)^{N-1}, \tag{4.11}$$

one reaches the claimed contradiction. Namely one gets

$$\exists \lim_{n\to+\infty} t_n = A_0\omega_{N-1}\left(\frac{2b_0+2N-\mu}{2\alpha_0}\right)^{N-1}.$$

Now, combining (4.4), (4.11), and (4.10), from (4.9) we deduce

$$\begin{aligned}C &\geq \left(\frac{2\alpha_0\, t_n^{\frac{N}{N-1}}}{(A_0\omega_{N-1}(1+\delta_n))^{\frac{1}{N-1}}} - (2b_0+2N-\mu)\right)\log n + \frac{N-1}{N}(\theta+1)\log\log n \\ &\geq (2b_0+2N-\mu)\left(\frac{1}{(1+\delta_n)^{\frac{1}{N-1}}} - 1\right)\log n + \frac{N-1}{N}(\theta+1)\log\log n \\ &\geq (2b_0+2N-\mu)\left(\frac{-\delta_n}{N-1} + o(\delta_n)\right)\log n + \frac{N-1}{N}(\theta+1)\log\log n \\ &= o_n(1) + \frac{N-1}{N}(\theta+1)\log\log n\,,\end{aligned}$$

recalling (4.1), which is again a contradiction. Therefore, (4.3) with (4.11) must hold true, which readily implies (4.2). □

With the fine upperbound of the mountain-pass level given by Lemma 4.1 we are in a position to prove the existence of a nontrivial weak solution of (Ch_0). The argument follows the line of [2], see also [1], and we only sketch it, but paying attention to the more delicate points.

Proof of Theorem 1.1(C-ii). First, we prove that

$$\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)\varphi \to \left(\frac{1}{|\cdot|^\mu} * QF(u)\right)Qf(u)\varphi \qquad \text{in } L^1(\mathbb{R}^N) \tag{4.12}$$

for all test functions φ, where u is the limit point of the Cerami sequence $(u_k)_k$. For such φ, it is easy to prove that $w_k := \frac{\varphi}{1+u_k} \in E_{\mathrm{rad}}$. Indeed,

$$\begin{aligned}\|w_n\|^N &\leq \int_{\mathbb{R}^N} A(x)\left(\frac{|\nabla\varphi|^N}{(1+u_k)^N} + \frac{|\varphi|^N|\nabla u_k|^N}{(1+u_k)^{2N}}\right)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} A(x)|\nabla\varphi|^N\,\mathrm{d}x + C(\varphi)\int_{\mathbb{R}^N} A(x)|\nabla u_k|^N\,\mathrm{d}x \lesssim \|\varphi\|^N + \|u_k\|^N \leq C\end{aligned}$$

by Lemma 3.2. This implies that one may test (3.4) with w_k and find

$$\begin{aligned}\int_\Omega\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)\frac{\varphi}{1+u_k}\,\mathrm{d}x &= \int_{\mathbb{R}^N} A(x)|\nabla u_k|^{N-2}\nabla u_k\nabla w_k\,\mathrm{d}x + o_k(1)\|w_k\| \\ &\leq \int_{\mathbb{R}^N} A(x)|\nabla u_k|^N|\varphi|\,\mathrm{d}x + \int_{\mathbb{R}^N} A(x)|\nabla u_k|^{N-1}\frac{|\nabla\varphi|}{1+u_k}\,\mathrm{d}x + o_k(1)\left(\|\varphi\|+\|u_k\|\right) \\ &\leq 2\|u_k\|^N + \|\varphi\|^N + o_k(1) \leq C\,,\end{aligned} \tag{4.13}$$

since $u_k \geq 0$ in the second integral, and having used the Hölder inequality there. Let $\Omega \subset\subset \mathbb{R}^N$ and $\varphi \geq 0$ be a test function such that $\varphi \equiv 1$ on Ω. Then,

$$\begin{aligned}&\int_\Omega\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)\,\mathrm{d}x \\ &\leq 2\int_{\{u_k\leq 1\}\cap\Omega}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)\frac{Qf(u_k)}{1+u_k} + \int_{\{u_k\geq 1\}\cap\Omega}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)u_k \\ &\leq \int_\Omega\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)\frac{\varphi}{1+u_k} + \int_{\mathbb{R}^N}\left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right)Qf(u_k)u_k \leq C\,,\end{aligned}$$

thanks to (4.13), (3.5) and (3.6). As a result, the measure ν_n defined by

$$\nu_n(\Omega) := \int_\Omega \left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) Qf(u_k)\,\mathrm{d}x$$

has uniformly bounded total variation, hence there exists a measure ν such that, up to a subsequence, $\nu_n \overset{*}{\rightharpoonup} \nu$, namely

$$\int_\Omega \left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) Qf(u_k)\,\varphi\,\mathrm{d}x \to \int_\Omega \varphi\,\mathrm{d}\nu$$

for all $\varphi \in C_0^\infty(\Omega)$. As in [2, Lemma 2.4] we may then conclude that ν is absolutely continuous with respect to the Lebesgue measure and it can be identified as $\nu = \left(\frac{1}{|\cdot|^\mu} * QF(u)\right) Qf(u)\,\mathrm{d}x$, which proves (4.12).

Combining (4.12) with the weak convergence $u_k \rightharpoonup u$ in E, we infer that u is a weak solution of (Ch_0). We need now to prove that $u \not\equiv 0$. To this aim, we first show that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) QF(u_k) \to \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * QF(u)\right) QF(u)\,. \tag{4.14}$$

Reasoning as in [2, Lemma 2.4], thanks to (f_4) it is possible to reduce the proof of (4.14) to show

$$\begin{aligned}\int_{\{u_k \le M\}} &\left(\int_{\{u_k \le K\}} \frac{Q(y)F(u_k(y))}{|x-y|^\mu}\,\mathrm{d}y\right) Q(x)F(u_k(x))\,\mathrm{d}x \\ &\to \int_{\{u \le M\}} \left(\int_{\{u \le K\}} \frac{Q(y)F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right) Q(x)F(u(x))\,\mathrm{d}x \end{aligned} \tag{4.15}$$

for all $M, K > 0$ large enough. However, if u_k is pointwisely bounded, by (f_2) one deduces $F(u_k) \le C_{M,K}|u_k|^{\widetilde{p}}$, therefore,

$$\int_{\{u_k \le M\}} \left(\int_{\{u_k \le K\}} \frac{Q(y)F(u_k(y))}{|x-y|^\mu}\,\mathrm{d}y\right) Q(x)F(u_k(x))\,\mathrm{d}x \lesssim \|Q|u_k|^{\widetilde{p}}\|^2_{\frac{2N}{2N-\mu}} \to \|Q|u|^{\widetilde{p}}\|^2_{\frac{2N}{2N-\mu}} \tag{4.16}$$

by the strong convergence given by Theorem A. Hence, by the inverse of the dominated convergence theorem [7, Theorem 1.2.7], the left-hand side of (4.16) is uniformly bounded and we can use the dominated convergence theorem to prove (4.15), and in turn (4.14).

Assuming by contradiction $u \equiv 0$, combining (4.14), $F(0) = 0$, and (3.3) one then infers

$$\begin{aligned} c_{mp} &= J(u_k) + o_k(1) \\ &= \frac{\|u_k\|^N}{N} + \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * QF(u_k)\right) QF(u_k) + o_k(1) = \frac{\|u_k\|^N}{N} + o_k(1)\,, \end{aligned} \tag{4.17}$$

from which, by Lemma 4.1,

$$\begin{aligned} \frac{2N\alpha_0}{2N-\mu}\|u_k\|^{\frac{N}{N-1}} &= \frac{2N\alpha_0}{2N-\mu}(Nc_{mp})^{\frac{1}{N-1}} + o_k(1) \\ &< \frac{2N\alpha_0}{2N-\mu}(\omega_{N-1}A_0)^{\frac{1}{N-1}}\,\frac{2b_0 + 2N - \mu}{2\alpha_0} \\ &= N(\omega_{N-1}A_0)^{\frac{1}{N-1}}\left(1 + \frac{2b_0}{2N-\mu}\right). \end{aligned} \tag{4.18}$$

By (3.5) and the Hardy-Littlewood inequality, we have

$$\|u_k\|^N + o_k(1) \leq \|QF(u_k)\|_{\frac{2N}{2N-\mu}} \|Qf(u_k)u_k\|_{\frac{2N}{2N-\mu}} \tag{4.19}$$

and we estimate the two terms as in (3.14) thanks to (2.2) and (2.1), respectively. The exponential term is then uniformly bounded by Theorem B by (4.18), since

$$\widetilde{\alpha}_N(\widetilde{Q}) = N(\omega_{N-1}A_0)^{\frac{1}{N-1}}\left(1 + \frac{1}{N}b_0\frac{2N}{2N-\mu}\right).$$

Since $u_k \to 0$ in $L^t_{\widetilde{Q}}(\mathbb{R}^N)$ for $t > \gamma$, from (4.19) we conclude that $\|u_k\| \to 0$, which leads to a contradiction with (4.17). We can thus conclude that the weak solution u is nontrivial. □

5. Conclusions

In this paper we studied Choquard equations in $\mathbb{R}^N$ with subcritical or critical exponential nonlinearities and with polynomial kernel, in the framework of zero mass problems. By using variational methods in a suitable functional setting, we proved the existence of a positive solution by means of a careful analysis on the mountain-pass level. Our work extends previous results in [1, 2, 21] to the case of zero mass Choquard equations, and complements the analysis carried out in [32], where the case of a logarithmic kernel was considered.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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