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Research article

On the fractional Laplace-Bessel operator

Borhen Halouani* **and Fethi Bouzeffour**

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: halouani@ksu.edu.sa.

Abstract: In this paper, we propose a novel approach to the fractional power of the Laplace-Bessel operator Δ_ν, defined as

$$\Delta_\nu = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu_i}{x_i}\frac{\partial}{\partial x_i}, \quad \nu_i \geq 0.$$

The fractional power of this operator is introduced as a pseudo-differential operator through the multi-dimensional Bessel transform. Our primary contributions encompass a normalized singular integral representation, Bochner subordination, and intertwining relations.

Keywords: Laplace-Bessel operator; fractional calculus; Bessel function
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1. Introduction

Fractional calculus has emerged as both an important and powerful mathematical tool, finding applications across diverse scientific domains where phenomena often display non-local or anomalous behavior [1–4]. In particular, the fractional power of the Laplace operator $\Delta = \sum_{i=1}^{n} \partial_i^2$ is a focal point of investigation in various mathematical and applied contexts [5–8]. Its definition involves the Fourier transform:

$$(-\Delta)^{\alpha/2}\phi = \mathcal{F}^{-1}(\|\xi\|^\alpha \mathcal{F}\phi(\xi)), \quad \text{for } \phi \in S(\mathbb{R}^n),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and its inverse, and $S(\mathbb{R}^n)$ denotes the Schwartz space.

An alternative representation valid for $\alpha \in (0, 2)$ of the fractional Laplacian is provided through a pointwise formula, as articulated in [9]:

$$(-\Delta)^{\alpha/2} f(x) = \frac{1}{\gamma_n(\alpha)} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{f(x) - f(x-y)}{\|y\|^{n+\alpha}}\, dy,$$

where $\gamma_n(\alpha)$ is a constant defined as

$$\gamma_n(\alpha) = \frac{\pi^{n/2}|\Gamma\left(-\frac{\alpha}{2}\right)|}{2^\alpha \Gamma\left(\frac{n+\alpha}{2}\right)},$$

and $B(0,\varepsilon)$ denotes the ball of radius ε centered at the origin.

It is crucial to recognize that the fractional Laplacian exhibits various equivalent definitions throughout $\mathbb{R}^n$, as extensively discussed in [9]. This variability underscores the rich mathematical structure and diverse perspectives associated with this nonlocal differential operator.

Within this expansive framework, our study is dedicated to exploring a specific facet of fractional calculus, with a particular focus on a singular differential operator known as the Laplace-Bessel operator [10, (1.88)], which is given by

$$\Delta_\nu = \sum_{i=1}^{n} B_{\nu_i}, \quad B_{\nu_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\nu_i}{x_i}\frac{\partial}{\partial x_i}, \quad \nu_i \geq 0. \tag{1.1}$$

The Laplace-Bessel operator serves as a mathematical model for phenomena characterized by multi-axial symmetry in various domains. Extensive harmonic analysis related to this operator has been conducted by prominent mathematicians, including B. Muckenhoupt and E. Stein [11], I. Kipriyanov, M. Klyuchantsev [12–14], K. Trimeche [15], L. Lyakhov [16, 17], K. Stempak [18], E. L. Shishkina, and S. M. Sitnik [10], among others.

The fractional Laplace-Bessel operator, denoted as $(-\Delta_\nu)^{\alpha/2}$ with order α, is introduced in our work as a pseudo-differential operator through the multi-dimensional Bessel transform. The investigation into the fractional Laplace-Bessel operator was pioneered by L. Lyakhov [19], and additional insights have been contributed by [16, 20]. For specific insights into the one-dimensional case, refer to [10, 21–23].

Since the fractional Laplace-Bessel operator reduces to the standard Laplacian when the multi-index ν is equal to zero, a challenging question arises: Does the equivalence of the ten definitions of the fractional Laplace operator still hold for the fractional Bessel operator? To address this question, we ground our investigation in a meticulously defined function space, denoted as $\mathscr{H}_\nu^\alpha(\mathbb{R}_+^n)$, capturing functions that exhibit well-behaved behavior under the fractional Laplace-Bessel operator. This space is inspired by the fractional Sobolev spaces introduced by Butzer et al. [24].

To tackle this question, we employ the well-known multi-dimensional Poisson transform $\mathscr{P}_\nu$, as defined in [10, Definition 23], and establish a new intertwining relation between the fractional Laplacian and the fractional Laplace-Bessel operator valid in the Schwartz space $S_*(\mathbb{R}^n)$, given by

$$\mathscr{P}_\nu(-\Delta)^{\alpha/2} = (-\Delta_\nu)^{\alpha/2}\mathscr{P}_\nu. \tag{1.2}$$

This relation is reduced for $\alpha = 2$ to the one obtained in [10, Statement 4, pp. 137]. Since the multi-dimensional Poisson transform keeps the Schwartz space invariant, this particularly partially responds to our starting question.

The structure of the paper is as follows:

In Section 2, we provide an initial overview of foundational concepts. The topics covered encompass the multi-dimensional Bessel transform, generalized translation operator, and generalized convolution, collectively setting the stage for a comprehensive understanding of subsequent content.

Section 3 succinctly presents the primary research outcomes. Here, we summarize the significant findings that have been attained through our investigation.

In Section 4, we furnish a comprehensive proof of the core results. Through meticulous derivation and thorough explanation, we establish the validity of our findings, providing readers with an in-depth grasp of the underlying mathematical foundations.

Finally, in Section 5, we present additional results, including relations such as Bochner subordination and intertwining relations for the fractional Laplace-Bessel.

2. Preliminaries

In the subsequent discussions, we will employ the following notations:

- $S_*(\mathbb{R}^n)$: The space of C^∞ functions, even with respect to each variable, and rapidly decreasing together with their derivatives.
- $L^p_\nu(\mathbb{R}^n_+)$, $1 \leq p \leq \infty$: The space of measurable functions f on $\mathbb{R}^n_+$ such that

$$\|f\|_{\nu,p} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p\, x^\nu dx\right)^{1/p} < \infty, \quad p \in [1, \infty),$$

$$\|f\|_{\nu,\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+} |f(x)| < \infty.$$

Here, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ and $x^\nu = x_1^{\nu_1} \dots x_n^{\nu_n}$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$.

Let $\nu = (\nu_1, \dots, \nu_n)$ be a multi-index with each $\nu_i \geq 0$. The multi-dimensional Bessel transform $\mathscr{F}_\nu\phi$ of a function $\phi \in L^1_\nu(\mathbb{R}^n_+)$ is defined by

$$\mathscr{F}_\nu\phi(\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{R}^n_+} \phi(x)\, \mathscr{J}_\nu(x, \xi)\, x^\nu dx,$$

where the multi-dimensional Bessel function $\mathscr{J}_\nu(x, \xi)$ is defined as

$$\mathscr{J}_\nu(x, \xi) = \prod_{i=1}^{n} j_{\frac{\nu_i - 1}{2}}(x_i\xi_i), \quad \text{with } \mathscr{J}_\nu(0, \xi) = 1.$$

Here, $j_\gamma(t)$ is the normalized Bessel function of the first kind, given by

$$j_\gamma(t) = 2^\gamma\Gamma(\gamma + 1)t^{-\gamma}J_\gamma(t), \gamma \geq -1/2,$$

and $J_\gamma(t)$ denotes the Bessel function of the first kind [25, 26]. We list some well-known basic properties of the multi-dimensional Bessel transform as follows: For the proofs, we refer to [10] and the references therein.

(i) For all $\phi \in L^1_\nu(\mathbb{R}^n_+)$, the function $\mathscr{F}_\nu\phi$ is continuous on $\mathbb{R}^n_+$, and we have

$$\|\mathscr{F}_\nu\phi\|_{\nu,\infty} \leq \|\phi\|_{\nu,1}\,. \tag{2.1}$$

(ii) The multi-dimensional Bessel transform $\mathscr{F}_\nu$ acts as a topological isomorphism on $S_*(\mathbb{R}^n)$, seamlessly extending to an isomorphism on $L^2_\nu(\mathbb{R}^n_+)$, where, for any $\phi \in L^2_\nu(\mathbb{R}^n_+)$, the following Plancherel formula holds:

$$\int_{\mathbb{R}^n_+} |\mathscr{F}_\nu\phi(\xi)|^2 \, \xi^\nu \, d\xi = a_\nu \int_{\mathbb{R}^n_+} |\phi(x)|^2 \, x^\nu \, dx, \tag{2.2}$$

where

$$a_\nu = 2^{|\nu|-n} \prod_{i=1}^{n} \Gamma^2\left(\frac{\nu_i + 1}{2}\right). \tag{2.3}$$

(iii) Inversion formula: Let $\phi \in L^1_\nu(\mathbb{R}^n_+)$ such that $\mathscr{F}_\nu\phi \in L^1_\nu(\mathbb{R}^n_+)$, then we have

$$\mathscr{F}_\nu^{-1}\phi(\xi) = \frac{1}{a_\nu} \int_{\mathbb{R}^n_+} \phi(x) \, \mathscr{J}_\nu(x, \xi) \, x^\nu dx. \tag{2.4}$$

The multi-dimensional generalized translation of a continuous function ϕ on $\mathbb{R}^n$ denoted by $\tau_x\phi$ is defined as follows [10, Definition 26]:

$$\tau_x\phi(y) := \prod_{i=1}^{n} \frac{\Gamma(\nu_i + 1)}{\sqrt{\pi}\Gamma(\nu_i + \frac{1}{2})} \int_0^\pi \cdots \int_0^\pi \phi((x_1, y_1)_{\theta_1}, \ldots, (x_n, y_n)_{\theta_n}) \times \prod_{i=1}^{n} \sin^{2\nu_i} \theta_i \, d\theta_1 \ldots d\theta_n, \tag{2.5}$$

where

$$(x_i, y_i)_{\theta_i} = \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\theta_i}, \quad i = 1, \ldots, n. \tag{2.6}$$

For every $\phi \in L^p_\nu(\mathbb{R}^n_+)$, the function $\tau_x\phi$ belongs to $L^p_\nu(\mathbb{R}^n_+)$, and we have

$$||\tau_x\phi||_{\nu,p} \leq ||\phi||_{\nu,p}. \tag{2.7}$$

The convolution operator determined by τ_x is as follows:

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n_+} \phi(\xi)\tau_x\psi(\xi) \, \xi^\nu d\xi. \tag{2.8}$$

This convolution operation is commutative, associative, and satisfies the following property: For $\phi, \ \psi \in L^1_\nu(\mathbb{R}^n_+)$, the convolution $\phi * \psi \in L^1_\nu(\mathbb{R}^n_+)$, and we have

$$\mathscr{F}_\nu(\phi * \psi) = \mathscr{F}_\nu\phi \cdot \mathscr{F}_\nu\psi.$$

3. Main results

In this section, we outline the central contributions of this paper, beginning with the introduction of the fractional Laplace-Bessel operator $(-\Delta_\nu)^{\alpha/2}$ of order α. This fractional derivative is treated as a pseudo-differential operator using the multi-dimensional Bessel transform. To facilitate our discussions, we operate within a tailored function space defined as follows:

$$\mathscr{H}^\alpha_\nu(\mathbb{R}^n_+) = \left\{\phi \in L^2_\nu(\mathbb{R}^n_+) : \int_{\mathbb{R}^n_+} ||\xi||^{2\alpha} |\mathscr{F}_\nu\phi(\xi)|^2 \xi^\nu \, d\xi < \infty\right\}. \tag{3.1}$$

In accordance with the terminology introduced by Butzer et al. in [24], we commonly denote the space $\mathscr{H}^\alpha_\nu(\mathbb{R})$ as the fractional Bessel space of order α.

Definition 3.1. *Let $0 < \alpha < 2$. For $\phi \in \mathscr{H}_\nu^\alpha(\mathbb{R}_+^n)$, we define the fractional Laplace-Bessel operator $(-\Delta_\nu)^{\alpha/2}\phi$ as*

$$(-\Delta_\nu)^{\alpha/2}\phi = \mathscr{F}_\nu^{-1}(\|\cdot\|^\alpha \mathscr{F}_\nu\phi), \quad in \quad L_\nu^2(\mathbb{R}_+^n). \tag{3.2}$$

Remark 3.1. *If $\|\cdot\|^\alpha \mathscr{F}_\nu\phi \in L_\nu^1(\mathbb{R}_+^n)$, then the integral in* (3.2) *exists as an ordinary Lebesgue integral, that is*

$$(-\Delta_\nu)^{\alpha/2}\phi(\xi) = \frac{1}{a_\nu}\int_{\mathbb{R}_+^n} \|x\|^\alpha \mathscr{F}_\nu\phi(x)\, \mathscr{J}_\nu(x,\xi)\, x^\nu dx, \quad \xi \in \mathbb{R}_+^n.$$

For $\alpha \in (0,2)$ and $\varepsilon > 0$, we set

$$\mathscr{R}_\varepsilon^{(\alpha)}\phi(x) = \frac{1}{\gamma_\nu(\alpha)}\int_{\mathbb{R}_+^n\setminus B(0,\varepsilon)} \frac{\phi(x) - \tau_x\phi(\xi)}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu\, d\xi, \tag{3.3}$$

where the normalized constant $\gamma_\nu(\alpha)$ is given by

$$\gamma_\nu(\alpha) = \frac{|\Gamma(-\frac{\alpha}{2})| \prod_{i=1}^n \Gamma(\frac{\nu_i+1}{2})}{2^{\alpha+n}\Gamma(\frac{|\nu|+\alpha+n}{2})}. \tag{3.4}$$

The following theorem represents the first main result, as it seeks to characterize the fractional Bessel space $\mathscr{H}_\nu^\alpha(\mathbb{R}_+^n)$ defined in (3.1).

Theorem 3.1. *Let $\alpha \in (0,2)$. The following statements are equivalent:*

(i) $\phi \in L_\nu^2(\mathbb{R}_+^n)$ and there exists $\psi \in L_\nu^2(\mathbb{R}_+^n)$ such that:

$$\|\mathscr{R}_\varepsilon^{(\alpha)}\phi - \psi\|_{2,\nu} = o(1) \quad as \quad \varepsilon \downarrow 0;$$

(ii) $\|\mathscr{R}_\varepsilon^{(\alpha)}\phi\|_{2,\nu} = O(1)$ as $\varepsilon \downarrow 0$;

(iii) $\phi \in \mathscr{H}_\nu^\alpha(\mathbb{R}_+^n)$.

In this specific case, the theorem presented below manifests the classical fractional-order derivative, extensively explored by A. Marchaud in 1927 [27]. His work holds fundamental importance in approximation theory and fractional calculus.

Theorem 3.2. *Let $\alpha \in (0,2)$. For a function $\phi \in \mathscr{H}_\alpha(\mathbb{R}_+^n)$, the fractional multi-dimensional Bessel operator $(-\Delta_\nu)^{\alpha/2}\phi(x)$ can be represented as follows:*

$$(-\Delta_\nu)^{\alpha/2}\phi(x) = \frac{1}{\gamma_\nu(\alpha)}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}_+^n\setminus B(0,\varepsilon)} \frac{\phi(x) - \tau_x\phi(\xi)}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu\, d\xi \quad in \quad L_\nu^2(\mathbb{R}_+^n).$$

Building upon Theorem 3.2 with $\nu = (0,\ldots,0)$, we derive the following representation of the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ throughout $\mathbb{R}_+^n$.

Corollary 3.1. *Let $\alpha \in (0,2)$. For a function $\phi \in \mathscr{H}_\alpha(\mathbb{R}_+^n)$, the fractional Laplace operator $(-\Delta)^{\alpha/2}\phi(x)$ on $\mathbb{R}_+^n$ can be represented as follows:*

$$(-\Delta)^{\alpha/2}\phi(x) = \frac{1}{\gamma(\alpha)}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}_+^n\setminus B(0,\varepsilon)} \frac{2\phi(x) - \phi(x+\xi) + \phi(x-\xi)}{\|\xi\|^{n+\alpha}}\,\xi^n\, d\xi,$$

where the normalized constant $\gamma(\alpha)$ is given by

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}}|\Gamma(-\frac{\alpha}{2})|}{2^{n+\alpha}\Gamma(\frac{n+\alpha}{2})}.$$

The corollary that follows, has previously been established in [23]. It is a direct consequence of Theorem 3.2 in the case when $n = 1$ and $\nu \geq 0$.

Corollary 3.2. *For a function $\phi \in \mathscr{H}_\alpha(\mathbb{R}_+)$, the fractional Bessel derivative $(-B_\nu)^{\alpha/2}\phi$, with $\alpha \in (0, 2)$, takes the form:*

$$(-B_\nu)^{\alpha/2}\phi(x) = \frac{2^{\alpha+1}\Gamma(\frac{\nu+\alpha+1}{2})}{\Gamma(\frac{\nu+1}{2})|\Gamma(-\frac{\alpha}{2})|}\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{\phi(x) - \tau^x\phi(\xi)}{\xi^{\alpha+1}}\,d\xi.$$

The next proposition highlights additional properties of the fractional Laplace-Bessel operator, which can be readily discerned by employing (3.2) and the traits of the multi-dimensional Bessel transform. The specific details are intentionally left for the readers.

Proposition 3.1. *Let $\phi, \psi \in \mathscr{H}_\nu(\mathbb{R}^n_+)$.*

(i) Translation invariance: for all $x \in \mathbb{R}^n_+$,

$$\tau_x(-\Delta_\nu)^{\alpha/2}\phi = (-\Delta_\nu)^{\alpha/2}\tau_x\phi, \quad \textit{in} \quad L^2_\nu(\mathbb{R}^n_+).$$

ii) Convolution invariance:

$$(-\Delta_\nu)^{\alpha/2}(\phi * \psi) = ((-\Delta_\nu)^{\alpha/2}\phi) * \psi, \quad \textit{in} \quad L^2_\nu(\mathbb{R}^n_+).$$

iii) Symmetry:

$$\langle(-\Delta_\nu)^{\alpha/2}\phi,\, \psi\rangle_{L^2_\nu(\mathbb{R}^n_+)} = \langle\phi,\, (-\Delta_\nu)^{\alpha/2}\psi\rangle_{L^2_\nu(\mathbb{R}^n_+)}.$$

4. Proof of the main results

Lemma 4.1. *For $\alpha \in (0, 2)$, it holds*

$$\int_{\mathbb{R}^n_+} \frac{1 - \mathscr{T}_\nu(x,\xi)}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu\,d\xi = \gamma_\nu(\alpha)\|x\|^\alpha.$$

Proof. We start by recalling the following formula [25, Ch.12]:

$$\int_0^\infty J_\gamma(ar)r^{\gamma+1}e^{-p^2r^2}dr = \frac{a^\gamma}{(2p^2)^{\gamma+1}}e^{-a^2/4p^2}, \quad \mathrm{Re}(\gamma) > -1. \tag{4.1}$$

Then

$$\frac{1}{2^{|\nu|}t^{\frac{|\nu|+1}{2}}\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})}\int_{\mathbb{R}^n_+} e^{-\|\xi\|^2/4t}\mathscr{J}_\gamma(x,\xi)\xi^\nu d\xi = e^{-t\|x\|^2}. \tag{4.2}$$

In particular, for $x = 0$, and using the fact that $\mathscr{T}_\nu(0,\xi) = 1$, we get

$$\frac{1}{2^{|\nu|}t^{\frac{|\nu|+1}{2}}\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})}\int_{\mathbb{R}^n_+} e^{-\|\xi\|^2/4t}\xi^\nu d\xi = 1. \tag{4.3}$$

Combining (4.2) and (4.3) to get

$$\int_{\mathbb{R}^n_+} \frac{e^{-\|\xi\|^2/4t}}{t^{\frac{1}{2}(|\nu|+n)}}\Big(1 - \mathscr{T}_\nu(x,\xi)\Big)\xi^\nu d\xi = 2^{|\nu|}\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})(1 - e^{-t\|x\|^2}). \tag{4.4}$$

Multiplying both sides of (4.4) by $t^{-1-\alpha/2}$ and integrating over $(0,\infty)$ with respect to the variable t, we obtain

$$\int_{\mathbb{R}_+^n}\int_0^\infty \frac{e^{-\|\xi\|^2/4t}}{t^{1+\frac{1}{2}(|\nu|+\alpha+n)}}\Big(1-\mathscr{T}_\nu(x,\xi)\Big)\xi^\nu dt\,d\xi = 2^{|\nu|}\prod_{i=1}^n \Gamma(\frac{\nu_i+1}{2})\int_0^\infty \frac{1-e^{-t\|\xi\|^2}}{t^{1+\alpha/2}}\,dt. \tag{4.5}$$

A straightforward computation reveals that

$$\int_0^\infty \frac{e^{-\frac{\|\xi\|^2}{4t}}}{t^{1+\frac{1}{2}(|\nu|+\alpha+n)}}\,dt = 2^{|\nu|+n+\alpha}\Gamma((|\nu|+n+\alpha)/2)\|\xi\|^{-(|\nu|+n+\alpha)}.$$

To evaluate the second integral in (4.5), we use the following:

$$\lambda^{\alpha/2} = \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty (1-e^{-t\lambda})\frac{dt}{t^{1+\frac{\alpha}{2}}}, \quad 0<\alpha<2. \tag{4.6}$$

Therefore,

$$\int_{\mathbb{R}_+^n} \frac{1-\mathscr{T}_\nu(x,\xi)}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu\,d\xi = \frac{|\Gamma(-\frac{\alpha}{2})|\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})}{2^{\alpha+n}\Gamma(\frac{|\nu|+\alpha+n}{2})}\|x\|^\alpha.$$

□

We introduce the function

$$\lambda_{\alpha,\varepsilon}(x) = \frac{1}{\gamma_\nu(\alpha)}\int_{\mathbb{R}_+^n\setminus B(0,\varepsilon)} \frac{1-\mathscr{T}_\nu(x,\xi)}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu\,d\xi. \tag{4.7}$$

In the following, we give some elementary properties of $\lambda_{\alpha,\varepsilon}(x)$, and their proofs follows easily from Lemma 4.1.

Lemma 4.2. *For the function $\lambda_{\alpha,\varepsilon}(x)$, the following holds:*

i) $|\lambda_{\alpha,\varepsilon}(x)| \le 1$,
ii) $|\lambda_{\alpha,\varepsilon}(x)| \le \|x\|^\alpha$,
iii) $\lim_{\varepsilon\downarrow 0}\lambda_{\alpha,\varepsilon}(x) = \|x\|^\alpha$.

Proposition 4.1. *For $0<\alpha<2$, the operator $\mathscr{R}_\varepsilon^{(\alpha)}$ is a bounded operator from $L_\nu^2(\mathbb{R}_+^n)$ onto itself, and satisfies*

$$\|\mathscr{R}_\varepsilon^{(\alpha)}f\|_{2,\nu} \le \kappa(\varepsilon)\,\|f\|_{2,\nu}, \tag{4.8}$$

where

$$\kappa(\varepsilon) = \frac{2^{\alpha+1}\Gamma(\frac{|\nu|+\alpha+n}{2})}{\alpha\varepsilon^\alpha|\Gamma(-\frac{\alpha}{2})|\Gamma(\frac{n+|\nu|}{2})}. \tag{4.9}$$

Proof. Applying Holder-Minkowski inequality and using (2.7), we get

$$\|f(x)-\tau_x f(\xi)\|_{2,\nu} \le 2\|f\|_{2,\nu}. \tag{4.10}$$

This leads to

$$\|\mathscr{R}_\varepsilon^{(\alpha)}f\|_{2,\nu} \le \frac{2\|f\|_{2,\nu}}{\gamma_\nu(\alpha)}\int_{^cB(0,\varepsilon)} \frac{1}{\|\xi\|^{|\nu|+\alpha+n}}\xi^\nu d\xi.$$

Utilizing polar coordinates, the integral becomes

$$\int_{^c B(0,\varepsilon)} \frac{1}{\|\xi\|^{|\nu|+\alpha+n}} \xi^\nu d\xi = \int_\varepsilon^\infty \frac{1}{r^{\alpha+1}} \int_{\mathbb{S}_+^{n-1}} \omega^\nu d\sigma(\omega) dr. \tag{4.11}$$

From [28], we have

$$\int_{\mathbb{S}_+^{n-1}} \omega^\nu d\sigma(\omega) = \frac{\prod_{i=1}^n \Gamma(\frac{\nu_i+1}{2})}{2^{n-1}\Gamma(\frac{n+|\nu|}{2})}. \tag{4.12}$$

Therefore,

$$\int_{^c B(0,\varepsilon)} \frac{1}{\|\xi\|^{|\nu|+\alpha+n}} \xi^\nu d\xi = \frac{2^{\alpha+1}\Gamma(\frac{|\nu|+\alpha+n}{2})}{\alpha\varepsilon^\alpha |\Gamma(-\frac{\alpha}{2})| \Gamma(\frac{n+|\nu|}{2})}. \tag{4.13}$$

This completes the proof. □

Proposition 4.2. *Let $\phi \in L^2_\nu(\mathbb{R}^n_+)$. The multi-dimensional Bessel transform of $\mathscr{R}^{(\alpha)}_\varepsilon \phi$ is given by*

$$\mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi) = \lambda_{\alpha,\varepsilon} \mathscr{F}_\kappa \phi, \quad in \quad L^2_\nu(\mathbb{R}^n_+). \tag{4.14}$$

Proof. For $\phi \in L^2_\nu(\mathbb{R}^n_+)$, and since $L^1_\nu(\mathbb{R}^n_+) \cap L^2_\nu(\mathbb{R}^n_+)$ is dense in $L^2_\nu(\mathbb{R}^n_+)$, we choose a sequence $\phi_n \in L^1_\nu(\mathbb{R}^n_+) \cap L^2_\nu(\mathbb{R}^n_+)$ with $\lim_{n\to\infty} \|\phi_n - \phi\|_{2,\nu} = 0$. By Fubini's theorem, we easily obtain

$$\mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi_n)(x) = \lambda_{\alpha,\varepsilon}(x) \mathscr{F}_\nu(\phi_n)(x). \tag{4.15}$$

Then, by Lemma 4.2 and the isometry property of the multi-dimension Bessel transform,

$$\begin{aligned}
&\|\mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi) - \lambda_{\alpha,\varepsilon}(x)\mathscr{F}_\nu \phi\|_{2,\kappa} \\
&\leq \|\mathscr{F}_\kappa(\mathscr{R}^{(\alpha)}_\varepsilon \phi) - \mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi_n)\|_{2,\nu} + \|\lambda_{\alpha,\varepsilon}\{\mathscr{F}_\nu(\phi_n) - \mathscr{F}_\nu(\phi)\}\|_{2,\nu} \\
&\leq C(\varepsilon)\|\phi - \phi_n\|_{2,\nu},
\end{aligned}$$

where $C(\varepsilon) = a_\nu^{1/2}(\kappa(\varepsilon) + 2)$. Thus, this proves the assertion. □

Now, we proceed to the proof of Theorem 3.1.

Proof. We will establish the implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$**:** This implication is direct.

$(ii) \Rightarrow (iii)$**:** Assume that condition (ii) holds. Utilizing Fatou's Lemma, we obtain

$$\begin{aligned}
\|\|.\|^\alpha \mathscr{F}_\nu \phi\|_{2,\nu} &\leq \liminf_{\varepsilon\downarrow 0} \|\lambda_{\alpha,\varepsilon} \mathscr{F}_\nu \phi\|_{2,\nu} \\
&= \liminf_{\varepsilon\downarrow 0} \|\mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi)\|_{2,\nu} \\
&= a_\nu \liminf_{\varepsilon\downarrow 0} \|\mathscr{R}^{(\alpha)}_\varepsilon \phi\|_{2,\nu}.
\end{aligned}$$

Since the last term is finite due to the assumed condition, we establish (iii).

$(iii) \Rightarrow (i)$**:** Assume $\phi \in \mathscr{H}^\alpha_\nu(\mathbb{R}^n_+)$. Given that the multi-dimensional Bessel transform is an isomorphism of $L^2_\nu(\mathbb{R}^n_+)$, there exists $\psi \in L^2_\nu(\mathbb{R}^n_+)$ such that $\mathscr{F}_\nu \psi(x) = \|x\|^\alpha \mathscr{F}_\nu \phi(x)$. Consequently, we have

$$a_\nu^{1/2}\|\mathscr{R}^{(\alpha)}_\varepsilon \phi(x) - \psi(x)\|_{2,\nu} = \|\mathscr{F}_\nu(\mathscr{R}^{(\alpha)}_\varepsilon \phi) - \mathscr{F}_\nu(\psi)\|_{2,\nu} = \|(\lambda_{\alpha,\varepsilon}(x) - \|x\|^\alpha)\mathscr{F}_\nu(\phi)\|_{2,\nu}.$$

Additionally, we find

$$(\lambda_{\alpha,\varepsilon}(x) - \|x\|^{\alpha})^2 |\mathscr{F}_\nu \phi|^2 \leq 4\|x\|^{2\alpha} |\mathscr{F}_\nu \phi|^2 = 4|\mathscr{F}_\nu \psi|^2.$$

Applying the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{\varepsilon \downarrow 0} \|(\lambda_{\alpha,\varepsilon} - \| \cdot \|^{\alpha})^2 \mathscr{F}_\nu \phi\|_{2,\nu}^2 = 0.$$

This completes the proof of (i).

Hence, we have established all three implications, concluding the proof. □

Proof of Theorem 3.2. By applying Proposition 4.1 and utilizing Lemma 4.2 (iii), we have

$$\lim_{\varepsilon \to 0^+} \mathscr{F}_\nu (\mathscr{R}_\varepsilon^{(\alpha)} \phi)(\xi) = \lim_{\varepsilon \to 0^+} \lambda_{\alpha,\varepsilon}(\xi) \mathscr{F}_\nu \phi(\xi) = \|\xi\|^{\alpha} \mathscr{F}_\nu \phi(\xi). \tag{4.16}$$

Furthermore, employing the isometry property of the multi-dimensional Bessel transform,

$$\lim_{\varepsilon \to 0^+} \|\mathscr{F}_\nu (\mathscr{R}_\varepsilon^{(\alpha)} \phi) - \mathscr{F}_\nu (-\Delta_\nu)^{\alpha/2} \phi)\|_{2,\nu} = \lim_{\varepsilon \to 0^+} \|\mathscr{R}_\varepsilon^{(\alpha)} \phi - (-\Delta_\nu)^{\alpha/2} \phi\|_{2,\nu} = 0.$$

Since the pointwise limit must coincide almost everywhere with the strong limit, the assertion follows. □

5. Further results

In what follows, we restrict ourselves to the Schwartz space $S_*(\mathbb{R}^n)$. For $\phi \in S_*(\mathbb{R}^n)$ and $\alpha > 0$, as indicated in Remark 3.1, the fractional Laplace-Bessel operator $(-\Delta_\nu)^{\alpha/2}\phi(\xi)$ is given by

$$(-\Delta_\nu)^{\alpha/2}\phi(\xi) = \frac{1}{a_\nu} \int_{\mathbb{R}^n_+} \|x\|^{\alpha} \mathscr{F}_\nu \phi(x) \mathscr{J}_\nu(x, \xi)\, x^\nu \, dx.$$

Since the multi-dimensional Bessel transform $\mathscr{F}_\nu$ maps the Schwartz space $S_*(\mathbb{R}^n)$ onto itself, it is evident that $(-\Delta_\nu)^{\alpha/2}\phi$ is a C^∞-bounded function on $\mathbb{R}^n$, and it satisfies the relationship

$$\Delta_\nu (-\Delta_\nu)^{\alpha/2} \phi = (-\Delta_\nu)^{\alpha/2} \Delta_\nu \phi.$$

By the dominated convergence theorem, we obtain the following results:

$$\lim_{\alpha \to 0} (-\Delta_\nu)^{\alpha/2} \phi = \phi \quad \text{and} \quad \lim_{\alpha \to 2} (-\Delta_\nu)^{\alpha/2} \phi = -\Delta_\nu \phi.$$

For $\alpha \in (0, 2)$, we have

$$(-\Delta_\nu)^{\alpha/2} \phi(x) = \frac{1}{\gamma_\nu(\alpha)} \int_{\mathbb{R}^n_+} \frac{\phi(x) - \tau^x \phi(\xi)}{\|\xi\|^{|\nu|+n+\alpha}} \xi^\nu d\xi.$$

5.1. Bochner subordination

Before stating our result, let's recall that the Laplace-Bessel operator $-\Delta_\nu$ is the generator of a strongly continuous one-parameter semigroup $\{e^{t\Delta_\nu}\}_{t\geq 0}$, where the operator $e^{t\Delta_\nu}$ is defined as [29]

$$e^{t\Delta_\nu}\phi = \mathscr{F}_\nu^{-1}\left(e^{-t\|\cdot\|^2}\mathscr{F}_\nu\phi\right), \tag{5.1}$$

for all $t \geq 0$ and $\phi \in L^2_\nu(\mathbb{R}^n_+)$. Consequently, $e^{t\Delta_\nu}$ is an integral operator represented by

$$e^{t\Delta_\nu}\phi(x) = (\mathscr{G}^t_\nu * \phi)(x) = \int_{\mathbb{R}^n_+} \tau_x\mathscr{G}^t_\nu(\xi)\phi(\xi)d\mu_\nu(\xi),$$

where

$$\mathscr{G}^t_\nu(x) = \frac{e^{-\frac{\|x\|^2}{4t}}}{2^{|\nu|}t^{\frac{|\nu|+1}{2}}\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})}. \tag{5.2}$$

Of particular interest is the scenario when $p \in [1, 2)$ and $\phi \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n_+)$. In this case, the function $u(t, x) = \mathscr{G}^t_\nu * \phi(x)$ plays a pivotal role as an infinitely smooth solution to the Cauchy problem:

$$\begin{cases}\Delta_\nu u(x,t) = \frac{\partial u(x,t)}{\partial t},\\ u(x,0) = \phi(x).\end{cases}$$

Theorem 5.1. *Let* $0 < \alpha < 2$. *For* $\phi \in S_*(\mathbb{R}^n)$*, we have*

$$(-\Delta_\nu)^{\alpha/2}\phi(x) = \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty (\phi(x) - e^{t\Delta_\nu}\phi(x))\frac{dt}{t^{1+\frac{\alpha}{2}}}. \tag{5.3}$$

Proof. Since $e^{t\Delta_k}\phi \in S_*(\mathbb{R}^n)$, applying the inversion formula for the multi-dimensional Bessel transform and the properties (5.1) of the heat semigroup, we obtain:

$$\phi(x) - e^{t\Delta_\nu}\phi(x) = \frac{1}{a_\nu}\int_{\mathbb{R}^n_+}(1 - e^{-t\|\xi\|^2})\mathscr{F}_\nu\phi(\xi)\mathscr{T}_\nu(x,\xi)\xi^\nu d\xi.$$

This equality, combined with the relation (2.1), implies that

$$\int_0^\infty |\phi(x) - e^{t\Delta_\nu}\phi(x)|\frac{dt}{t^{1+\frac{\alpha}{2}}} = \frac{|\Gamma(-\frac{\alpha}{2})|}{a_\nu}\int_{\mathbb{R}^n_+}\|\xi\|^\alpha|\mathscr{F}_\nu\phi(\xi)|\xi^\nu d\xi < +\infty.$$

Therefore, Fubini's theorem can be applied to obtain

$$\begin{aligned}\frac{1}{\Gamma(-\frac{\alpha}{2})}\int_0^\infty \phi(x) - e^{t\Delta_\nu}\phi(x)\frac{dt}{t^{1+\frac{\alpha}{2}}} &= \frac{1}{a_\nu}\int_{\mathbb{R}^n_+}\|\xi\|^\alpha\mathscr{F}_\nu\phi(\xi)\ \mathscr{T}_\nu(x,\xi)\xi^\nu\\ &= (-\Delta_\nu)^{\alpha/2}\phi(x).\end{aligned}$$

□

5.2. Intertwining relations

Recall the well-known Poisson operator $\mathscr{P}_\nu\phi(x)$. This operator acts on the integrable function $\phi \in L^p_\nu(\mathbb{R}^n_+)$ and is defined by

$$\mathscr{P}_\nu\phi(x) = \frac{c_\nu}{x^{2\nu}} \int_0^{x_1} \cdots \int_0^{x_n} \phi(\xi) \prod_{i=1}^{n} (x_i^2 - \xi_i^2)^{\nu_i - 1/2}\, d\xi_i, \tag{5.4}$$

where the normalized constant c_ν is given by

$$c_\nu = \prod_{i=1}^{n} \frac{2\Gamma(\nu_i + 1)}{\sqrt{\pi}\Gamma(\nu_i + 1/2)}. \tag{5.5}$$

In one dimension, the Poisson transform is also known as the Riemann-Liouville transform. Extending this concept to multiple dimensions, we can refer to the Poisson transform as the multi-dimensional Riemann-Liouville transform. It has been demonstrated in [10, pp. 137] that the Poisson integral transform serves as an intertwining operator between the multi-dimensional Bessel operator and the Laplace operator Δ on $\mathbb{R}^n_+$. More precisely, for $\phi \in S_*(\mathbb{R}^n)$, we have

$$\mathscr{P}_\nu\Delta\phi = \Delta_\nu\mathscr{P}_\nu\phi. \tag{5.6}$$

In the following theorem, we extend the intertwining relation (5.6) to the fractional setting.

Theorem 5.2. *In the Schwartz space $S_*(\mathbb{R}^n)$, the following intertwining relation holds:*

$$\mathscr{P}_\nu(-\Delta)^{\alpha/2} = (-\Delta_\nu)^{\alpha/2}\mathscr{P}_\nu. \tag{5.7}$$

Proof. From [10, formula 3.138], we have

$$\mathscr{T}_\nu(x,\xi) = \mathscr{P}_\nu[e^{-i\langle x,\xi\rangle}]. \tag{5.8}$$

By applying the inversion formula for the standard Fourier transform $\mathcal{F}$ and utilizing Fubini's theorem, we derive the following expression:

$$\begin{aligned}\mathscr{P}_\nu(e^{t\Delta}\phi)(x) &= \frac{1}{a_\nu}\int_{\mathbb{R}^n_+} e^{-t\|\xi\|^2}\mathcal{F}\phi(\xi)\mathscr{P}_\nu[e^{-i\langle x,\xi\rangle}]\xi^\nu\, d\xi\\ &= \frac{1}{a_\nu}\int_{\mathbb{R}^n_+} e^{-t\|\xi\|^2}\mathcal{F}\phi(\xi)\mathscr{T}_\nu(x,\xi)\xi^\nu\, d\xi.\end{aligned}$$

Utilizing this equality, the intertwining relation (5.6), and the differentiation theorem under the integral sign, we observe that

$$\Delta_\nu\mathscr{P}_\nu(e^{t\Delta}\phi) = \mathscr{P}_\nu\Delta(e^{t\Delta}\phi) = \mathscr{P}_\nu\left(\frac{\partial}{\partial t}e^{t\Delta}\phi\right) = \frac{\partial}{\partial t}\mathscr{P}_\nu(e^{t\Delta}\phi).$$

Additionally, the dominated convergence theorem implies that $\lim_{t\to 0}\mathscr{P}_\nu(e^{t\Delta}\phi) = \mathscr{P}_\nu\phi$. Hence, the function $\mathscr{P}_\nu(e^{t\Delta}\phi)$ serves as a solution to the Δ_ν-Cauchy problem:

$$\begin{cases}\Delta_\nu u(x,t) = \frac{\partial u(x,t)}{\partial t},\\ u(x,0) = \mathscr{P}_\nu\phi(x).\end{cases}$$

As $\mathscr{P}_\nu\phi$ is bounded, the solution to this problem is unique, yielding

$$e^{t\Delta_\nu}[\mathscr{P}_\nu(\phi)] = \mathscr{P}_\nu\left(e^{t\Delta}\phi\right).$$

Finally, combining this relation with (5.3) and Fubini's theorem yields the desired intertwining relation. □

6. Conclusions

In conclusion, our study focuses on the domain of fractional calculus by introducing and analyzing the fractional Laplace-Bessel operator as a pseudo-differential operator. We have successfully established a comprehensive framework that not only elucidates the fundamental properties of the fractional Laplace-Bessel operator but also connects it with the classical fractional Laplacian through a novel intertwining relation. This relationship is validated within the Schwartz space. Future research will aim to further explore the practical applications of these theoretical findings and extend the analysis to other related fractional differential operators.

Author contributions

All authors contributed equally and significantly to the study conception and design, material preparation, data collection and analysis. The first draft of the manuscript was written by [F. Bouzeffour] and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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