AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 21433–21454.
DOI: 10.3934/math.20241041
Received: 27 March 2024
Revised: 13 June 2024
Accepted: 20 June 2024
Published: 04 July 2024

Research article

On general Kirchhoff type equations with steep potential well and critical growth in $\mathbb{R}^2$

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Abstract: In this paper, we study the following Kirchhoff-type equation:

$$M\left(\int_{\mathbb{R}^2}(|\nabla u|^2+u^2)\mathrm{d}x\right)(-\Delta u+u)+\mu V(x)u=K(x)f(u)\ \text{ in }\ \mathbb{R}^2,$$

where $M\in C(\mathbb{R}^+,\mathbb{R}^+)$ is a general function, $V\geq 0$ and its zero set may have several disjoint connected components, $\mu>0$ is a parameter, K is permitted to be unbounded above, and f has exponential critical growth. By using the truncation technique and developing some approaches to deal with Kirchhoff-type equations with critical growth in the whole space, we get the existence and concentration behavior of solutions. The results are new even for the case $M\equiv 1$.

Keywords: Kirchhoff-type equation; steep potential well; critical growth; dimension two; variational method

Mathematics Subject Classification: 35A01, 35A15, 35J20

1. Introduction

The Kirchhoff-type problem appears as a model of several physical phenomena. For example, it is related to the stationary analog of the equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.1}$$

where u is the lateral displacement at x and t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length, and P_0 is the initial axial tension. For more background, see [1, 20] and the references therein. In this paper, we study the following Kirchhoff-type equation with steep potential well and exponential critical nonlinearity:

$$M\left(\int_{\mathbb{R}^2}(|\nabla u|^2+u^2)\mathrm{d}x\right)(-\Delta u+u)+\mu V(x)u=K(x)f(u) \ \text{ in } \ \mathbb{R}^2, \tag{1.2}$$

where $M\in C(\mathbb{R}^+,\mathbb{R}^+)$, $V\in C(\mathbb{R}^2,\mathbb{R}^+)$ with $\Omega=\mathrm{int}(V^{-1}(0))$ having k connected components, $\mu>0$ is a parameter. Because of the presence of the nonlocal term $M\left(\int_{\mathbb{R}^2}(|\nabla u|^2+u^2)\mathrm{d}x\right)$, Eq (1.2) is no longer a pointwise identity, which causes additional mathematical difficulties. The motivation of the present paper arises from results for Schrödinger equations with steep potential well. In [6], Bartsch and Wang studied the following equation with steep potential well:

$$-\Delta u+(1+\mu V(x))u=u^{p-1} \ \text{ in } \ \mathbb{R}^N, \tag{1.3}$$

where $N\geq 3$ and $2<p<2^*=\frac{2N}{N-2}$. Under appropriate conditions on V, the authors obtained the existence of positive ground state solutions for large μ and the concentration behavior of solutions as $\mu\to+\infty$. If p is close to 2^*-1, the authors also obtained multiple positive solutions. In [13], Ding and Tanaka constructed multi-bump positive solutions to Schrödinger equations with steep potential well. In [23], Sato and Tanaka obtained multiple positive and sign-changing solutions. For the critical case, Clapp and Ding [11] considered the following equation with steep potential well:

$$-\Delta u+\mu V(x)u=\lambda u+u^{2^*-1} \ \text{ in } \ \mathbb{R}^N. \tag{1.4}$$

When $N\geq 4$, $\lambda>0$ is small and $\mu>0$ is large, the authors obtained the existence and multiplicity of positive solutions. In [17, 18], Guo and Tang constructed multi-bump solutions of (1.4) in the case that the potential is definite and indefinite. For other related results, see [4, 5, 12, 24–26] and the references therein.

There are relatively few results about Kirchhoff-type equations with steep potential well. In [19], Jia studied the ground-state solutions of the following equation with sign-changing potential well:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+\lambda V(x)u=|u|^{p-2}u \ \text{ in } \ \mathbb{R}^3, \tag{1.5}$$

where $3<p<6$. When $V\geq 0$ and $2<p<6$, Zhang and Du [27] used the truncation technique to obtain the existence of solutions of (1.5). For the critical case, we [29] obtained the existence, multiplicity and concentration behavior of solutions to the following equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+\mu V(x)u=\lambda f(u)+(u^+)^5 \ \text{ in } \ \mathbb{R}^3. \tag{1.6}$$

To the best of our knowledge, there are no results about the existence and concentration behavior of Kirchhoff-type equations with steep potential wells and exponential critical growth nonlinearity in dimension two, especially when the zero set of the steep potential well admits more than one isolated connected component. This is the main motivation of the present paper. Here we say the nonlinearity f has exponential subcritical growth if for any $\alpha>0$,

$$\lim_{u\to+\infty}f(u)e^{-\alpha u^2}=0 \tag{1.7}$$

and the nonlinearity f has exponential critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{u\to+\infty}\frac{f(u)}{e^{\alpha u^2}}=\begin{cases}0, & \forall\alpha>\alpha_0,\\ +\infty, & \forall\alpha<\alpha_0.\end{cases} \tag{1.8}$$

In this paper, we study (1.2) and prove the existence of solutions trapped on one connected component of the potential well.

To study the existence and concentration behavior of solutions, the main difficulty lies in the exponential critical growth of nonlinearity. The Trudinger–Moser inequality plays an important role in dealing with critical nonlinearity. When using this inequality, it is crucial to control the uniform H^1-norm of the sequence. Compared with the classical Schrödinger equation, the nonlocal term of the Kirchhoff type equation prevents us from using the upper bound on energy and the Ambrosetti–Rabinowitz type condition to deduce the desired H^1 norm estimate. If we use the Pohozaev identity, we must impose additional restrictions on V and K. In [3, 22], the authors studied nonlinear scalar field equations in dimension two. We notice that the compactness lemma of Strauss in [7] plays an important role and cannot be used in a non-radial setting. In [10, 16], the authors studied Kirchhoff-type equations with exponential critical growth in a bounded domain. To deal with the critical nonlinearity, a compactness lemma (Lemma 2.1 in [14]) was used. However, this lemma cannot be applied to study a non-radial problem in the whole space. In this paper, we give a compactness lemma restricted to a bounded domain (Lemma 2.5 in Section 2), which is motivated by Lemma 2.1 in [14]. Because this lemma cannot be applied to deal with the non-radial problem in the whole space and the coefficient of the nonlinearity may be unbounded above, we study the problem by penalizing the nonlinearity.

When $N = 2$, to deal with the exponential critical nonlinearity, we need to estimate an upper bound on the energy. In [3], the authors used the following condition:

(f') There exist $\lambda > 0$ and $q > 2$ such that

$$f(u)\ge\lambda u^{q-1},\ \ \forall\ u\ge 0.$$

When $\lambda > 0$ is large, the upper bound on the energy can be controlled. In [14], the authors considered the following Dirichlet problem:

$$-\Delta u = f(x,u)\ \text{ in }\ \Omega,\ \ u=0 \text{ on } \partial\Omega,$$

and introduced the following more natural condition:

(f'') There exists $\beta > \frac{4}{3\alpha_0 d^2}$ such that

$$\lim_{u\to+\infty}\frac{f(x,u)u}{e^{\alpha_0 u^2}}\ge\beta,$$

where d is the radius of the largest open ball in Ω.

By using the Moser sequence of functions, the authors deduced the desired upper bound. Related results can be found in [22,28] for nonlinear scalar field equations and in [10,16,28] for Kirchhoff type equations. Motivated by the above results, we use a direct argument to get the desired upper bound on the energy.

Now we state our results. We assume the following conditions:

(M_1) $M \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\inf_{\mathbb{R}^+} M := M_0 > 0$, and $M(t)$ is strictly increasing for $t \in \mathbb{R}^+$.
(M_2) There exist θ, $\varepsilon_0 > 0$ such that $\frac{M(t)-\varepsilon_0}{t^\theta}$ is decreasing for $t > 0$.
(M_3) There exists $\varepsilon_0' > 0$ such that $\hat{M}(t) - \frac{1}{\theta+1}M(t)t - \varepsilon_0' t$ is increasing for $t \in \mathbb{R}^+$, where $\hat{M}(t) = \int_0^t M(s)\mathrm{d}s$.
(V_1) $V \in C(\mathbb{R}^2, \mathbb{R}^+)$.
(V_2) $\Omega = \mathrm{int}(V^{-1}(0))$ is non-empty with smooth boundary and $\bar{\Omega} = V^{-1}(0)$.
(V_3) Ω consists of k connected components: $\Omega = \cup_{i=1}^k \Omega_i$ and $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for all $i \neq j$.
(V_4) There exists $V_0 > 0$ such that $\left|\{x \in \mathbb{R}^2 : V(x) \leq V_0\}\right| < \infty$.
(K_1) $K \in C(\mathbb{R}^2, \mathbb{R}^+)$ and $k_0 := \inf_{\mathbb{R}^2} K > 0$.
(K_2) There exist $k_1, \alpha > 0$ such that $K(x) \leq k_1 e^{\alpha|x|}$ for $x \in \mathbb{R}^2$.
(f_1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists $l > 1$ such that $\lim_{u\to 0+} \frac{f(u)}{u^l} < +\infty$.
(f_2) There exists $\alpha_0 > 0$ such that

$$\lim_{u\to+\infty} \frac{f(u)}{e^{\alpha u^2} - 1} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

(f_3) There exists $\beta > 0$ such that

$$\beta \leq \lim_{u\to+\infty} \frac{f(u)u}{e^{\alpha_0 u^2}} < +\infty.$$

(f_4) There exists $\sigma > 2(\theta + 1)$ such that $\frac{f(u)}{u^{\sigma-1}}$ is increasing for $u \in \mathbb{R}^+ \setminus \{0\}$.
(f_5) There exist $u_0, L_0 > 0$ such that $F(u) \leq L_0 f(u)$ for $u \geq u_0$, where $F(u) = \int_0^u f(s)\mathrm{d}s$.

Theorem 1.1. *Assume that (M_1)–(M_3), (V_1)–(V_4), (K_1)–(K_2) and (f_1)–(f_5) hold. Let $i_0 \in \{1, 2, \ldots, k\}$. If $\beta > \frac{2M(\frac{4\pi}{\alpha_0})}{k_0 r^2 \alpha_0} e^{\frac{r^2}{2}-1}$, where r is the radius of an open ball contained in Ω_{i_0}, then there exists $\mu_0 > 0$ such that for $\mu > \mu_0$, Eq (1.2) has a positive solution u_μ. Moreover, there exist r_0, c_1, $c_2 > 0$ independent of $\mu > 0$ large such that $\Omega_{i_0}^d \subset B_{r_0}(0)$ and*

$$u_\mu(x) \leq c_2 e^{-c_1\sqrt{\mu}(|x|-r_0)}, \quad \forall\ |x| \geq r_0. \tag{1.9}$$

Besides, for any sequence $\mu_n \to +\infty$, there exists $u_0 \in H_0^1(\Omega_{i_0})$ such that $u_{\mu_n} \to u_0$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$, where $u_0 \in H_0^1(\Omega_{i_0})$ is a positive solution to the limiting problem:

$$M\left(\int_{\Omega_{i_0}} (|\nabla u|^2 + u^2)\mathrm{d}x\right)(-\Delta u + u) = K(x)f(u) \ \text{ in } \ \Omega_{i_0}. \tag{1.10}$$

Remark 1.1. *If $\lim_{u\to+\infty} \frac{f(u)u}{e^{\alpha_0 u^2}} = A \in (0, +\infty)$, then there exists $R > 0$ such that*

$$\frac{A}{2}u^{-1}e^{\alpha_0 u^2} \leq f(u) \leq \frac{3A}{2}u^{-1}e^{\alpha_0 u^2}, \quad \forall\ u \geq R.$$

Moreover,

$$\lim_{u\to+\infty} \frac{F(u)}{f(u)} \leq \lim_{u\to+\infty} \frac{\int_0^R f(s)\mathrm{d}s + \frac{3A}{2}\int_R^u s^{-1}e^{\alpha_0 s^2}\mathrm{d}s}{\frac{A}{2}u^{-1}e^{\alpha_0 u^2}} = 0,$$

from which we get f satisfies (f_5). If $A = \infty$, one can prove it by the L'Hospital rule and the definition of $\epsilon - N$.

Remark 1.2. *Let* $f_1(u) = \frac{\beta(\alpha_0 u^2-1)e^{\alpha_0 u^2}}{\alpha_0 u^3}$*, where* $u > 0$*. Then there exists* $u_1 > 0$ *such that* $f_1(u_1) = u_1^{\sigma-1}$*. Define* $f(u) = u^{\sigma-1}$ *for* $u \in [0, u_1]$ *and* $f(u) = f_1(u)$ *for* $u > u_1$*. Obviously,* f *satisfies* (f_1)*–*(f_3)*. We note that*

$$\left(\frac{f_1(u)}{u^{\sigma-1}}\right)' = \frac{\beta e^{\alpha_0 u^2}}{\alpha_0 u^{3+\sigma}}\left[2\alpha_0^2 u^4 - (\sigma+2)\alpha_0 u^2 + \sigma + 2\right].$$

If $\sigma \leq 6$*, then* $\frac{f_1(u)}{u^{\sigma-1}}$ *is increasing for* $u \geq u_1$*. Moreover,* f *satisfies* (f_4)*. By Remark* 1.1*, we get* f *satisfies* (f_5)*.*

The outline of this paper is as follows: In Section 2, we study the truncated problem; in Section 3, we turn to the original problem and prove Theorem 1.1.

2. Preliminary lemmas

We give some definitions. Denote C as universal positive constant (possibly different). Define $\|u\|_s := \left(\int_{\mathbb{R}^2} |u(x)|^s \mathrm{d}x\right)^{\frac{1}{s}}$, where $s \in [1, \infty)$. Define $H^1(\mathbb{R}^2)$ the Hilbert space with the norm $\|u\|_{H^1} := \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{\frac{1}{2}}$. It is well known that the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^t(\mathbb{R}^2)$ is continuous for all $t \geq 2$. Let $\mu > 0$. Define

$$X_\mu := \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \mathrm{d}x < \infty\right\}$$

the Hilbert space equipped with the norm

$$\|u\|_\mu := \left(\|\nabla u\|_2^2 + \int_{\mathbb{R}^2} (1 + \mu V(x))u^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Obviously, the embedding $X_\mu \hookrightarrow H^1(\mathbb{R}^2)$ is continuous. We give the following Trudinger–Moser inequality:

Lemma 2.1. *([15, 21, 22]) If* $u \in H^1(\mathbb{R}^2)$ *and* $\alpha > 0$*, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty.$$

Moreover, for any fixed $\tau > 0$*, there exists a constant* $C > 0$ *such that*

$$\sup_{u \in H^1(\mathbb{R}^2): \|\nabla u\|_2^2 + \tau\|u\|_2^2 \leq 1} \int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right) \mathrm{d}x \leq C.$$

Since we look for positive solutions, we assume that $f(u) = 0$ for $u \leq 0$. For any $d > 0$, define $\Omega^d := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Omega) < d\}$. By (V_3), we can choose $d > 0$ small such that $\Omega_i^{2d} \cap \Omega_j^{2d} = \emptyset$ for all $i \neq j$. Let $i_0 \in \{1, 2, \ldots, k\}$. Define

$$\chi(x) = \begin{cases} 1, & x \in \Omega_{i_0}^d, \\ 0, & x \in \mathbb{R}^2 \setminus \Omega_{i_0}^d. \end{cases}$$

By (V_4), we know that $\Omega_{i_0}^d$ is bounded. Let $\tau \in (0,1)$. For any $x \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$, define

$$\hat{f}(x,u) = \min\{K(x)f(u), \kappa u^+\},$$

where $u^+ = \max\{u, 0\}$ and $\kappa \in \left(0, \min\left\{\varepsilon_0, \frac{(\theta+1)\varepsilon_0'}{\theta}, M_0(1-\tau)\right\}\right)$. Define

$$g(x,u) = \chi(x)K(x)f(u) + (1-\chi(x))\hat{f}(x,u). \tag{2.1}$$

Then

$$G(x,u) = \int_0^u g(x,s)\mathrm{d}s = \chi(x)K(x)F(u) + (1-\chi(x))\hat{F}(x,u),$$

where $\hat{F}(x,u) = \int_0^u \hat{f}(x,s)\mathrm{d}s$. By (f_4) and the structure of $\hat{f}$, we derive that for all $(x,u) \in \mathbb{R}^2 \times \mathbb{R}$,

$$K(x)f(u)u - \sigma K(x)F(u) \geq 0, \quad \hat{f}(x,u)u - 2\hat{F}(x,u) \geq 0. \tag{2.2}$$

Instead of studying (1.2), we consider the following truncated problem:

$$M\left(\|u\|_{H^1}^2\right)(-\Delta u + u) + \mu V(x)u = g(x,u) \text{ in } \mathbb{R}^2. \tag{2.3}$$

The functional associated with (2.3) is

$$\hat{I}_\mu(u) = \frac{1}{2}\hat{M}\left(\|u\|_{H^1}^2\right) + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} G(x,u)\mathrm{d}x, \quad u \in X_\mu. \tag{2.4}$$

Obviously, $\hat{I}_\mu \in C^1(X_\mu, \mathbb{R})$, and the critical points of $\hat{I}_\mu$ are weak solutions of (2.3).

Lemma 2.2. *Let $l(t) = \hat{I}_\mu(tu)$, where $t \geq 0$ and $u \in X_\mu$ with $\left|\mathrm{supp}u \cap \Omega_{i_0}^d\right| > 0$. Then there exists a unique $t_0 > 0$ such that $l'(t_0) = 0$, $l'(t) > 0$ for $t \in (0, t_0)$, and $l'(t) < 0$ for $t > t_0$.*

Proof. Obviously, $l(0) = 0$. Let $\alpha > \alpha_0$ and $q > 2$. By (K_1) and (f_1)-(f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(x,u)| \leq (\varepsilon + \kappa)|u| + C_\varepsilon |u|^{q-1}(e^{\alpha u^2} - 1), \quad \forall\, (x,u) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.5}$$

Then

$$|G(x,u)| \leq \frac{\varepsilon + \kappa}{2}|u|^2 + \frac{C_\varepsilon}{q}|u|^q(e^{\alpha u^2} - 1), \quad \forall\, (x,u) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.6}$$

By (2.6) and Lemma 2.1, we can choose $\rho > 0$ small such that for $\|u\|_\mu \leq \rho$,

$$\begin{aligned}\left|\int_{\mathbb{R}^2} G(x,u)\mathrm{d}x\right| &\leq \frac{\varepsilon+\kappa}{2}\|u\|_2^2 + \frac{C_\varepsilon}{q}\|u\|_{2q}^q\left[\int_{\mathbb{R}^2}(e^{2\alpha u^2} - 1)\mathrm{d}x\right]^{\frac{1}{2}} \\ &\leq \frac{\varepsilon+\kappa}{2}\|u\|_2^2 + C\|u\|_{2q}^q. \end{aligned} \tag{2.7}$$

By (M_1), we get $\hat{M}(s) \geq M_0 s$ for $s \in \mathbb{R}^+$. Together with (2.7), the choice of κ and the Sobolev embedding theorem, we derive that $l(t) > 0$ for $t > 0$ small. Let $s_0 > 0$. By (M_1)-(M_2), there exists $C_1 > 0$ such that

$$M(s) \leq C_1 + \frac{M(s_0)}{s_0^\theta} s^\theta, \quad s \in \mathbb{R}^+. \tag{2.8}$$

Let $p > 2\theta + 1$. By (f_1)-(f_2), there exist $c_1, c_2 > 0$ such that

$$f(u) \geq c_1 u^p - c_2 u, \quad \forall\, u \in \mathbb{R}. \tag{2.9}$$

By (2.8)-(2.9), we get $l(t) < 0$ for $t > 0$ large. Thus, $\max_{t\geq 0} l(t)$ is attained at $t_0 > 0$ and $l'(t_0) = 0$. Let

$$\begin{aligned} y(t) = &\left[\varepsilon_0 \|u\|_2^2 + \mu \int_{\mathbb{R}^2} V(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^2 \setminus \Omega_{i_0}^d} \frac{\hat{f}(x, tu)u}{t} \mathrm{d}x\right] \\ &+ \left[\left(M(t^2\|u\|_{H^1}^2) - \varepsilon_0\right)\|u\|_2^2 + M(t^2\|u\|_{H^1}^2)\|\nabla u\|_2^2 - t^{2\theta} \int_{\Omega_{i_0}^d} \frac{K(x) f(tu) u}{t^{2\theta+1}} \mathrm{d}x\right]. \end{aligned}$$

Then $y(t_0) = 0$. Moreover, from the structure of g, we derive that for $t > 0$,

$$\begin{aligned} &\varepsilon_0 \|u\|_2^2 + \mu \int_{\mathbb{R}^2} V(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^2 \setminus \Omega_{i_0}^d} \frac{\hat{f}(x, tu)u}{t} \mathrm{d}x > 0, \\ &\left(M(t^2\|u\|_{H^1}^2) - \varepsilon_0\right)\|u\|_2^2 + M(t^2\|u\|_{H^1}^2)\|\nabla u\|_2^2 - t^{2\theta} \int_{\Omega_{i_0}^d} \frac{K(x) f(tu) u}{t^{2\theta+1}} \mathrm{d}x < 0. \end{aligned}$$

By (M_2), we know $\frac{\left(M(t^2\|u\|_{H^1}^2) - \varepsilon_0\right)\|u\|_2^2 + M(t^2\|u\|_{H^1}^2)\|\nabla u\|_2^2}{t^{2\theta}}$ is decreasing for $t > 0$. By (f_4), we know $\int_{\mathbb{R}^2 \setminus \Omega_{i_0}^d} \frac{\hat{f}(x,tu)u}{t} \mathrm{d}x$ is increasing for $t > 0$ and $\int_{\Omega_{i_0}^d} \frac{K(x)f(tu)u}{t^{2\theta+1}} \mathrm{d}x$ is strictly increasing for $t > 0$. Then $y(t) > 0$ for $t < t_0$ and $y(t) < 0$ for $t > t_0$. Moreover, $l'(t) > 0$ for $t \in (0, t_0)$ and $l'(t) < 0$ for $t > t_0$. □

We consider the Moser sequence of functions

$$\bar{\omega}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}}, & 0 \leq |x| \leq \frac{1}{n}, \\ \dfrac{\log \frac{1}{|x|}}{(\log n)^{\frac{1}{2}}}, & \frac{1}{n} \leq |x| \leq 1, \\ 0, & |x| \geq 1. \end{cases}$$

It is well known that $\|\nabla \bar{\omega}_n\|_2^2 = 1$ and $\|\bar{\omega}_n\|_2^2 = \frac{1}{4 \log n} + o(\frac{1}{\log n})$. Choose $x_0 \in \Omega_{i_0}$ and $r > 0$ such that $B_r(x_0) \subset \Omega_{i_0}$, where r is the radius of an open ball contained in Ω_{i_0}. Define the functions $\omega_n(x) = \bar{\omega}_n(\frac{x - x_0}{r})$. Then, $\|\nabla \omega_n\|_2^2 = 1$. Define the functional I_0 as follows:

$$I_0(u) = \frac{1}{2} \hat{M}\left(\int_{\Omega_{i_0}} |\nabla u|^2 + u^2 \mathrm{d}x\right) - \int_{\Omega_{i_0}} K(x) F(u) \mathrm{d}x, \quad u \in H_0^1(\Omega_{i_0}).$$

Lemma 2.3. $\max_{t\geq 0}\hat{I}_\mu(t\omega_n)=\max_{t\geq 0}I_0(t\omega_n)<\frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right)$ *for n large.*

Proof. Obviously, we have $\max_{t\geq 0}\hat{I}_\mu(t\omega_n) = \max_{t\geq 0}I_0(t\omega_n)$. By Lemma 2.2, we derive that $\max_{t\geq 0}\hat{I}_\mu(t\omega_n)$ is attained at a $t_n>0$. By $\left(\hat{I}'_\mu(t\omega_n), t_n\omega_n\right)=0$ and (K_1),

$$\begin{aligned}M(t_n^2+t_n^2\|\omega_n\|_2^2)(t_n^2+t_n^2\|\omega_n\|_2^2)=&\int_\Omega K(x)f(t_n\omega_n)t_n\omega_n\mathrm{d}x\\ \geq& k_0r^2\int_{B_1(0)}f(t_n\bar{\omega}_n)t_n\bar{\omega}_n\mathrm{d}x. \end{aligned}\tag{2.10}$$

If $\lim_{n\to\infty}t_n=0$, then $\lim_{n\to\infty}\hat{I}_\mu(t_n\omega_n)=0$. So we assume that $\lim_{n\to\infty}t_n=l\in(0,+\infty]$. By a direct calculation, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{F(t)}{t^{-2}e^{\alpha_0t^2}}=&\lim_{t\to+\infty}\frac{f(t)}{2\alpha_0t^{-1}e^{\alpha_0t^2}\left(1-\alpha_0^{-1}t^{-2}\right)}\\ =&\lim_{t\to+\infty}\frac{f(t)}{2\alpha_0t^{-1}e^{\alpha_0t^2}}.\end{aligned}$$

So by (f_3), for any $\delta>0$, there exists $t_\delta>0$ such that for $t\geq t_\delta$,

$$f(t)t\geq(\beta-\delta)e^{\alpha_0t^2},\quad F(t)t^2\geq\frac{\beta-\delta}{2\alpha_0}e^{\alpha_0t^2}.\tag{2.11}$$

Since $\lim_{n\to\infty}\frac{t_n}{\sqrt{2\pi}}(\log n)^{\frac{1}{2}}=+\infty$, by (2.10)-(2.11), we derive that

$$\begin{aligned}&M\left(t_n^2+r^2t_n^2\left(\frac{1}{4\log n}+o\left(\frac{1}{\log n}\right)\right)\right)\left(t_n^2+r^2t_n^2\left(\frac{1}{4\log n}+o\left(\frac{1}{\log n}\right)\right)\right)\\ &\geq k_0(\beta-\delta)r^2\pi n^{-2}e^{\frac{\alpha_0}{2\pi}t_n^2\log n}=k_0(\beta-\delta)r^2\pi e^{\left(\frac{\alpha_0}{2\pi}t_n^2-2\right)\log n}.\end{aligned}$$

If $\lim_{n\to\infty}t_n=+\infty$, by (M_2), we get a contradiction. So $\lim_{n\to\infty}t_n=l\in(0,+\infty)$. Moreover, $l\in\left(0,\sqrt{\frac{4\pi}{\alpha_0}}\right]$. If $l\in\left(0,\sqrt{\frac{4\pi}{\alpha_0}}\right)$, then

$$\lim_{n\to\infty}\hat{I}_\mu(t_n\omega_n)\leq\frac{1}{2}\lim_{n\to\infty}\hat{M}(t_n^2\|\omega_n\|_{H^1}^2)<\frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right).\tag{2.12}$$

Now we assume $\lim_{n\to\infty}t_n=\sqrt{\frac{4\pi}{\alpha_0}}$. Let

$$A_n:=\{x\in B_r(x_0):t_n\omega_n(x)\geq t_\delta\}.$$

By (K_1) and (2.11), we have

$$\int_\Omega K(x)F(t_n\omega_n)\mathrm{d}x\geq\frac{(\beta-\delta)k_0}{2\alpha_0}\int_{A_n}t_n^{-2}\omega_n^{-2}e^{\alpha_0t_n^2\omega_n^2}\mathrm{d}x.$$

Let $s\in(0,\frac{1}{2})$. Then, for n large, we have

$$t_n\omega_n(x)\geq t_\delta,\quad\forall\,|x-x_0|\leq\frac{r}{n^s}.$$

Moreover,

$$\int_\Omega K(x)F(t_n\omega_n)\mathrm{d}x \geq \frac{(\beta-\delta)k_0r^2}{2\alpha_0}\int_{B_{\frac{1}{n^s}}(0)} t_n^{-2}\bar{\omega}_n^{-2}e^{\alpha_0 t_n^2\bar{\omega}_n^2}\mathrm{d}x. \tag{2.13}$$

By direct calculation, we obtain

$$\begin{aligned}
&\int_{B_{\frac{1}{n^s}}(0)} t_n^{-2}\bar{\omega}_n^{-2}e^{\alpha_0 t_n^2\bar{\omega}_n^2}\mathrm{d}x\\
=&\int_{|x|\leq\frac{1}{n}}\frac{2\pi n^{\frac{\alpha_0 t_n^2}{2\pi}}}{t_n^2\log n}\mathrm{d}x+\int_{\frac{1}{n}\leq|x|\leq\frac{1}{n^s}}\frac{2\pi\log n e^{\frac{\alpha_0 t_n^2}{2\pi\log n}\log^2|x|}}{t_n^2\log^2|x|}\mathrm{d}x\\
=&\frac{2\pi^2}{t_n^2}\frac{n^{\frac{\alpha_0 t_n^2}{2\pi}-2}}{\log n}+\frac{4\pi^2\log n}{t_n^2}\int_{\frac{1}{n}}^{\frac{1}{n^s}}\frac{xe^{\frac{\alpha_0 t_n^2}{2\pi\log n}\log^2 x}}{\log^2 x}\mathrm{d}x.
\end{aligned} \tag{2.14}$$

Let $C_n=\frac{\alpha_0 t_n^2}{2\pi}$. Then

$$\begin{aligned}
\int_{\frac{1}{n}}^{\frac{1}{n^s}}\frac{xe^{\frac{\alpha_0 t_n^2}{2\pi\log n}\log^2 x}}{\log^2 x}\mathrm{d}x=&\frac{C_n}{\log n}\int_{sC_n}^{C_n} n^{-\frac{2x}{C_n}+\frac{x^2}{C_n}}x^{-2}\mathrm{d}x\\
\geq&\frac{1}{\log n}\int_s^1 n^{-2x+C_nx^2}\mathrm{d}x.
\end{aligned} \tag{2.15}$$

Here

$$\begin{aligned}
\int_s^1 n^{-2x+C_nx^2}\mathrm{d}x \geq& \int_{\frac{2\pi}{\alpha_0 t_n^2}}^1 n^{\left(\frac{\alpha_0 t_n^2}{\pi}-2\right)x-\frac{\alpha_0 t_n^2}{2\pi}}\mathrm{d}x+\int_s^{\frac{2\pi}{\alpha_0 t_n^2}} n^{-2x}\mathrm{d}x\\
=&\frac{n^{-\frac{\alpha_0 t_n^2}{2\pi}}}{\left(\frac{\alpha_0 t_n^2}{\pi}-2\right)\log n}\left(n^{\frac{\alpha_0 t_n^2}{\pi}-2}-n^{2-\frac{4\pi}{\alpha_0 t_n^2}}\right)\\
&+\frac{1}{2\log n}\left(n^{-2s}-n^{-\frac{4\pi}{\alpha_0 t_n^2}}\right).
\end{aligned} \tag{2.16}$$

By (2.13)–(2.16), we derive that there exists $C'>0$ such that

$$\begin{aligned}
\int_\Omega K(x)F(t_n\omega_n)\mathrm{d}x \geq& \frac{(\beta-\delta)k_0\pi^2r^2}{\alpha_0 t_n^2}\frac{n^{\frac{\alpha_0 t_n^2}{2\pi}-2}}{\log n}+\frac{(\beta-\delta)k_0\pi^2r^2}{\alpha_0 t_n^2}\frac{1}{\log n}\left(n^{-2s}-n^{-\frac{4\pi}{\alpha_0 t_n^2}}\right)\\
&+\frac{2(\beta-\delta)k_0\pi^2r^2}{\alpha_0 t_n^2}\frac{n^{-\frac{\alpha_0 t_n^2}{2\pi}}}{\left(\frac{\alpha_0 t_n^2}{\pi}-2\right)\log n}\left(n^{\frac{\alpha_0 t_n^2}{\pi}-2}-n^{2-\frac{4\pi}{\alpha_0 t_n^2}}\right)\\
\geq&\frac{(\beta-\delta)k_0\pi^2r^2}{\alpha_0 t_n^2-2\pi}\frac{n^{\frac{\alpha_0 t_n^2}{2\pi}-2}}{\log n}+\frac{C'n^{-2s}}{\log n}.
\end{aligned} \tag{2.17}$$

Together with (M_1), we have

$$\hat{I}_\mu(t_n\omega_n) \leq \frac{1}{2}\hat{M}\left(t_n^2 + \frac{r^2 t_n^2}{4\log n}\right) + o\left(\frac{1}{\log n}\right) - \frac{(\beta-\delta)k_0\pi^2 r^2}{\alpha_0 t_n^2 - 2\pi}\frac{n^{\frac{\alpha_0 t_n^2}{2\pi}-2}}{\log n} - \frac{C' n^{-2s}}{\log n}. \tag{2.18}$$

By $\lim_{n\to\infty} t_n = \sqrt{\frac{4\pi}{\alpha_0}}$, we obtain that for any $\varepsilon > 0$, there exists N_1 such that $\alpha_0 t_n^2 \leq 4\pi + \varepsilon$ for $n > N_1$. Let

$$l_n(t) = \frac{1}{2}\hat{M}\left(t^2 + \frac{r^2 t^2}{4\log n}\right) - \frac{(\beta-\delta)k_0\pi^2 r^2}{2\pi+\varepsilon}\frac{n^{\frac{\alpha_0 t^2}{2\pi}-2}}{\log n}.$$

Then

$$\hat{I}_\mu(t_n\omega_n) \leq \sup_{t\geq 0} l_n(t) + o\left(\frac{1}{\log n}\right). \tag{2.19}$$

Obviously, there exists $t_n' > 0$ such that $\sup_{t\geq 0} l_n(t) = l_n(t_n')$. Then $(l_n'(t_n'), t_n') = 0$, from which we get

$$M\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right)\left(1 + \frac{r^2}{4\log n}\right) = \frac{(\beta-\delta)k_0\pi r^2\alpha_0}{2\pi+\varepsilon} n^{\frac{\alpha_0(t_n')^2}{2\pi}-2}. \tag{2.20}$$

By (2.19)-(2.20), we have

$$\hat{I}_\mu(t_n\omega_n) \leq \frac{1}{2}\hat{M}\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right) + o\left(\frac{1}{\log n}\right) - \frac{\pi}{\alpha_0\log n}M\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right)\left(1 + \frac{r^2}{4\log n}\right). \tag{2.21}$$

By (2.20) and (M_1), we get $\lim_{n\to\infty}\alpha_0(t_n')^2 = 4\pi$. Moreover,

$$(t_n')^2 = \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0}\frac{\log\frac{(2\pi+\varepsilon)M\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right)\left(1+\frac{r^2}{4\log n}\right)}{(\beta-\delta)k_0\pi r^2\alpha_0}}{\log n} := \frac{4\pi}{\alpha_0} + A_n, \tag{2.22}$$

where $A_n = O(\frac{1}{\log n})$. If $A_n + \frac{r^2(t_n')^2}{4\log n} \geq 0$, by (2.22) and (M_2), we have

$$\begin{aligned}
&\hat{M}\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right) \\
&= \hat{M}\left(\frac{4\pi}{\alpha_0}\right) + \int_{\frac{4\pi}{\alpha_0}}^{(t_n')^2 + \frac{r^2(t_n')^2}{4\log n}} M(s)\mathrm{d}s \\
&\leq \hat{M}\left(\frac{4\pi}{\alpha_0}\right) + \frac{1}{\theta+1}\frac{M(\frac{4\pi}{\alpha_0})}{(\frac{4\pi}{\alpha_0})^\theta}\left[\left(\frac{4\pi}{\alpha_0} + A_n + \frac{r^2(t_n')^2}{4\log n}\right)^{\theta+1} - \left(\frac{4\pi}{\alpha_0}\right)^{\theta+1}\right].
\end{aligned} \tag{2.23}$$

If $A_n + \frac{r^2(t_n')^2}{4\log n} < 0$, by (2.22) and (M_1), we have

$$\hat{M}\left((t_n')^2 + \frac{r^2(t_n')^2}{4\log n}\right) \leq \hat{M}\left(\frac{4\pi}{\alpha_0}\right). \tag{2.24}$$

By (2.21)–(2.24), we obtain that

$$\begin{aligned}\hat{I}_\mu(t_n\omega_n) \leq &\frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right) + o\left(\frac{1}{\log n}\right) + \frac{1}{2}M\left(\frac{4\pi}{\alpha_0}\right)\left(A_n + \frac{\pi r^2}{\alpha_0 \log n}\right) \\ &- \frac{\pi}{\alpha_0 \log n}M\left(\frac{4\pi}{\alpha_0} + A_n + \frac{r^2(t_n')^2}{4\log n}\right).\end{aligned} \tag{2.25}$$

Since $\beta > \frac{2M(\frac{4\pi}{\alpha_0})}{k_0 r^2 \alpha_0} e^{\frac{r^2}{2}-1}$, by choosing δ, ε small and n large, we can derive from (2.25) that $\hat{I}_\mu(t_n\omega_n) < \frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right)$. □

Lemma 2.4. *(Mountain pass geometry) There exist ρ, $\eta > 0$ independent of μ such that $\hat{I}_\mu(u) \geq \eta$ for $\|u\|_\mu = \rho$. Also, there exists a non-negative function $v \in X_\mu$ with $\|v\|_\mu > \rho$ such that $\hat{I}_\mu(v) < 0$.*

Proof. By (M_1), we get $\hat{M}(s) \geq M_0 s$ for $s \in \mathbb{R}^+$. Thus, by choosing $\varepsilon > 0$ small, we can derive from (2.7) and the Sobolev embedding theorem that $\hat{I}_\mu(u) \geq \eta$ for $\|u\|_\mu = \rho$. By (2.8)-(2.9), we get $\lim_{t\to+\infty} \hat{I}_\mu(tv) = -\infty$. □

Define

$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \hat{I}_\mu(\gamma(t)),$$

where $\Gamma := \left\{\gamma \in C([0,1], X_\mu) : \gamma(0) = 0, I_\mu(\gamma(1)) < 0\right\}$. By Lemmas 2.3-2.4 and the mountain pass lemma in [2], there exist $\{u_n\} \subset X_\mu$ and n_0 such that

$$\lim_{n\to\infty} \hat{I}_\mu(u_n) = c_\mu \in [\eta, \max_{t\geq 0} I_0(t\omega_{n_0})], \quad \lim_{n\to\infty} \hat{I}_\mu'(u_n) = 0. \tag{2.26}$$

Moreover,

$$\max_{t\geq 0} I_0(t\omega_{n_0}) < \frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right). \tag{2.27}$$

Now we give a compactness result.

Lemma 2.5. *Suppose Ω is a bounded domain in $\mathbb{R}^2$. Assume that h satisfies the following conditions:*

(h_1) *$h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $\lim_{u\to 0} \frac{h(x,u)}{u} = 0$ uniformly in $x \in \Omega$.*

(h_2) *There exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, $\lim_{u\to+\infty} \frac{h(x,u)}{e^{\alpha u^2}-1} = 0$ uniformly in $x \in \Omega$.*

If $\|u_n\|_{H^1(\Omega)}$, $\int_\Omega |h(x,u_n)u_n| \mathrm{d}x$ are bounded and $u_n(x) \to u(x)$ a.e. $x \in \Omega$, then $\lim_{n\to\infty} \int_\Omega |h(x,u_n) - h(x,u)| \mathrm{d}x = 0$.

Proof. Let $\alpha > \alpha_0$ and $q > 2$. By (h_1)-(h_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h(x,u)| \le \varepsilon|u| + C_\varepsilon |u|^{q-1}(e^{\alpha u^2} - 1), \quad \forall\, (x,u) \in \mathbb{R}^2 \times \mathbb{R}.$$

Then

$$\begin{aligned}\int_\Omega |h(x,u)|^2 \mathrm{d}x \le& C\int_\Omega |u|^2 \mathrm{d}x + C\int_\Omega |u|^{2(q-1)}(e^{2\alpha u^2} - 1)\mathrm{d}x \\ \le& C\int_\Omega |u|^2 \mathrm{d}x + C\left(\int_\Omega |u|^{4(q-1)}\mathrm{d}x\right)^{\frac{1}{2}} \left[\int_\Omega \left(e^{4\alpha u^2} - 1\right)\mathrm{d}x\right]^{\frac{1}{2}}.\end{aligned}$$

Together with Lemma 2.1, we get $h(x,u) \in L^2(\Omega)$. Since $\|u_n\|_{H^1(\Omega)}$ is bounded, we get $\int_\Omega u_n^2 \mathrm{d}x$ is bounded. Let $M > 0$. Then

$$\begin{aligned}&\int_{\{|u_n|\ge M\}\cap\Omega} |h(x,u_n) - h(x,u)|\mathrm{d}x \\ &\le \frac{1}{M}\int_{\{|u_n|\ge M\}\cap\Omega} |h(x,u_n)u_n - h(x,u)u_n|\mathrm{d}x \le \frac{C}{M}.\end{aligned} \tag{2.28}$$

Since $\|u_n\|_{H^1(\Omega)}$ is bounded and $u_n(x) \to u(x)$ a.e. $x \in \Omega$, we get $u_n \to u$ in $L^p(\Omega)$ for any $p > 2$. Thus, by the generalized Lebesgue- dominated convergence theorem, we derive that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{|u_n|\le M\}\cap\Omega} |h(x,u_n) - h(x,u)|\mathrm{d}x \\ &= \lim_{n\to\infty}\int_\Omega |h(x,u_n) - h(x,u)|\chi_{\{|u_n|\le M\}}(x)\mathrm{d}x = 0.\end{aligned} \tag{2.29}$$

By (2.28)-(2.29), we obtain the result. □

Corollary 2.1. *If, $\|u_n\|_{H^1(\Omega^d_{i_0})}$, $\int_{\Omega^d_{i_0}} |K(x)f(u_n)u_n|\mathrm{d}x$ are bounded and $u_n(x) \to u(x)$ a.e. $x \in \Omega^d_{i_0}$, then* $\lim_{n\to\infty}\int_{\Omega^d_{i_0}} |K(x)f(u_n) - K(x)f(u)|\mathrm{d}x = 0.$

Proof. Let $h(x,u) = K(x)f(u)$, where $(x,u) \in \overline{\Omega^d_{i_0}} \times \mathbb{R}$. By (K_1) and (f_1), we get $h \in C(\overline{\Omega^d_{i_0}} \times \mathbb{R}, \mathbb{R})$ and $\lim_{u\to 0}\frac{h(x,u)}{u} = 0$ uniformly in $x \in \Omega^d_{i_0}$. By (K_1) and (f_2), we get $\lim_{u\to+\infty}\frac{h(x,u)}{e^{\alpha u^2}-1} = 0$ uniformly in $x \in \Omega^d_{i_0}$. Then, by Lemma 2.5, we get the result. □

Lemma 2.6. *Let $\mu > 0$. If $\{u_n\} \subset X_\mu$ is a sequence such that $\hat{I}_\mu(u_n) \to c_\mu \in \left(0, \frac{1}{2}\hat{M}\left(\frac{4\pi}{\alpha_0}\right)\right)$ and $\hat{I}'_\mu(u_n) \to 0$, then $\{u_n\}$ converges strongly in X_μ up to a subsequence.*

Proof. By (2.2) and the structure of g, we have

$$\begin{aligned}c_\mu + o_n(1) + o_n(1)\|u_n\|_\mu =& \hat{I}_\mu(u_n) - \frac{1}{2(\theta+1)}(\hat{I}'_\mu(u_n), u_n) \\ \ge& \frac{1}{2}\hat{M}(\|u_n\|^2_{H^1}) - \frac{1}{2(\theta+1)}M(\|u_n\|^2_{H^1})\|u_n\|^2_{H^1} \\ &+ \frac{\theta}{2(\theta+1)}\int_{\mathbb{R}^2}\mu V(x)u_n^2\mathrm{d}x - \frac{\theta\kappa}{2(\theta+1)}\int_{\mathbb{R}^2\backslash\Omega^d_{i_0}} u_n^2\mathrm{d}x\end{aligned}$$

$$+\left(\frac{1}{2(\theta+1)}-\frac{1}{\sigma}\right)\int_{\Omega_{i_0}^d} K(x)f(u_n)u_n\mathrm{d}x. \tag{2.30}$$

Since $\kappa < \frac{(\theta+1)\varepsilon_0'}{\theta}$, by (M_3), we get $\|u_n\|_\mu$ is bounded. Assume that $u_n \rightharpoonup u_\mu$ weakly in X_μ.

We consider two cases.

Case 1. $u_n \rightharpoonup 0$ weakly in X_μ.

By (2.30), we get $\int_{\Omega_{i_0}^d} K(x)f(u_n)u_n\mathrm{d}x$ is bounded. So by Corollary 2.1, we have $\lim_{n\to\infty}\int_{\Omega_{i_0}^d} K(x)f(u_n)\mathrm{d}x = 0$. Together with (K_1), (f_5), and the generalized Lebesgue-dominated convergence theorem, we obtain that

$$\lim_{n\to\infty}\int_{\Omega_{i_0}^d} K(x)F(u_n)\mathrm{d}x = 0.$$

By (M_1), we get

$$\hat{M}(t+s) \geq \hat{M}(t) + M_0 s, \quad \forall\, t, s \geq 0.$$

Thus,

$$\begin{aligned}
c_\mu \geq& \frac{1}{2}\hat{M}\left(\lim_{n\to\infty}\|\nabla u_n\|_2^2 + \tau\lim_{n\to\infty}\|u_n\|_2^2\right) + \frac{M_0(1-\tau)}{2}\lim_{n\to\infty}\|u_n\|_2^2 \\
&- \frac{\kappa}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2\setminus\Omega_{i_0}^d} u_n^2\mathrm{d}x \\
\geq& \frac{1}{2}\hat{M}\left(\lim_{n\to\infty}\|\nabla u_n\|_2^2 + \tau\lim_{n\to\infty}\|u_n\|_2^2\right).
\end{aligned}$$

By (M_1), we have

$$\lim_{n\to\infty}(\|\nabla u_n\|_2^2 + \tau\|u_n\|_2^2) < \frac{4\pi}{\alpha_0}. \tag{2.31}$$

Define $\psi \in C_0^\infty([0,\infty))$ such that $\psi(r) = 1$ on $[1,\infty)$, $\psi(r) = 0$ on $[0,\frac{1}{2}]$ and $0 \leq \psi(r) \leq 1$ on $[0,\infty)$. Define $\psi_R(x) := \psi\left(\frac{|x|}{R}\right)$, where $\Omega_{i_0}^d \subset B_{\frac{R}{2}}(0)$. By $(\hat{I}_\mu'(u_n), \psi_R^2 u_n) = o_n(1)$, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left[M(\|u_n\|_{H^1}^2)\left(|\nabla u_n|^2\psi_R^2 + 2\nabla u_n\nabla\psi_R u_n\psi_R + u_n^2\psi_R^2\right) + \mu V(x)u_n^2\psi_R^2\right]\mathrm{d}x \\
=& \int_{\mathbb{R}^2} g(x,u_n)u_n\psi_R^2\mathrm{d}x + o_n(1) \leq \kappa\int_{\mathbb{R}^2}|u_n\psi_R|^2\mathrm{d}x + o_n(1).
\end{aligned}$$

We note that

$$\int_{\mathbb{R}^2}|u_n|^2|\nabla\psi_R|^2\mathrm{d}x \leq \|\nabla\psi_R\|_{L^\infty(\mathbb{R}^2)}^2\int_{\mathbb{R}^2}|u_n|^2\mathrm{d}x \leq \frac{C}{R^2}.$$

Together with (M_1), we obtain that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{|x|\geq R}\left[|\nabla(u_n\psi_R)|^2 + (1+\mu V(x))|u_n\psi_R|^2\right]\mathrm{d}x = 0. \tag{2.32}$$

Let $A = \lim_{n\to\infty} M(\|u_n\|^2_{H^1})$. Define the functional

$$J_\mu(u) = \frac{A}{2}\|u\|^2_{H^1} + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} G(x,u)\mathrm{d}x, \quad u \in X_\mu.$$

Then $J'_\mu(u_n) = o_n(1)$. Let $P(x,t) = g(x,t)t$ and $Q(t) = t\left(e^{\alpha t^2} - 1\right)$, where $\alpha > \alpha_0$. By (K_1) and (f_2), we have

$$\lim_{t\to\infty} \frac{P(x,t)}{Q(t)} = 0 \text{ uniformly in } x \in \mathbb{R}^2. \tag{2.33}$$

Also,

$$\lim_{n\to\infty} P(x, u_n(x)) = P(x, u_\mu(x)) \; a.e. \; x \in \mathbb{R}^2. \tag{2.34}$$

By (2.31), we can choose $q > 1$(close to 1) and $\alpha > \alpha_0$(close to α_0) such that $q\alpha(\|\nabla u_n\|_2^2 + \tau\|u_n\|_2^2) < 4\pi$ for n large. Let $q' = \frac{q}{q-1}$. By Lemma 2.1, we derive that for n large,

$$\int_{\mathbb{R}^2} Q(u_n)\mathrm{d}x \le \|u_n\|_{q'} \left[\int_{\mathbb{R}^2} (e^{q\alpha u_n^2} - 1)\mathrm{d}x\right]^{\frac{1}{q}} \le C. \tag{2.35}$$

By (2.33)–(2.35) and Lemma 1.2 in [9], we have $\lim_{n\to\infty} \int_{B_R(0)} g(x,u_n)u_n\mathrm{d}x = 0$. Together with (2.32), we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} g(x,u_n)u_n\mathrm{d}x = 0. \tag{2.36}$$

Since $(J'_\mu(u_n), u_n) = o_n(1)$, by (2.36) and (M_1), we get $u_n \to 0$ in X_μ, a contradiction with $c_\mu > 0$.

Case 2. $u_n \rightharpoonup u_\mu \neq 0$ weakly in X_μ.

By $\hat{I}'_\mu(u_n) = o_n(1)$, we get $J'_\mu(u_n) = o_n(1)$. Then $J'_\mu(u_\mu) = 0$. We claim that $\lim_{n\to\infty} \|u_n\|^2_{H^1} = \|u_\mu\|^2_{H^1}$. Otherwise, $\|u_\mu\|^2_{H^1} < \lim_{n\to\infty} \|u_n\|^2_{H^1}$. By (M_1), we get $(\hat{I}'_\mu(u_\mu), u_\mu) < 0$. Since $u_\mu \neq 0$, we get $\left|\mathrm{supp} u_\mu \cap \Omega^d_{i_0}\right| > 0$. By Lemma 2.2, there exists a unique $t_\mu > 0$ such that $(\hat{I}'_\mu(t_\mu u_\mu), t_\mu u_\mu) = 0$. Moreover, $t_\mu \in (0,1)$. By the structure of g, for $x \in \mathbb{R}^2 \setminus \Omega^d_{i_0}$,

$$\frac{\varepsilon'_0}{2}u_n^2 + \left[\frac{1}{2(\theta+1)}\hat{f}(x,u_n)u_n - \hat{F}(x,u_n)\right] \ge 0. \tag{2.37}$$

By (2.2), (2.37), (M_3), and Fatou's lemma, we derive that

$$\begin{aligned} c_\mu &= \hat{I}_\mu(u_n) - \frac{1}{2(\theta+1)}(\hat{I}'_\mu(u_n), u_n) + o_n(1) \\ &\ge \frac{1}{2}\hat{M}(\|u_\mu\|^2_{H^1}) - \frac{1}{2(\theta+1)}M(\|u_\mu\|^2_{H^1})\|u_\mu\|^2_{H^1} \\ &\quad + \frac{\mu\theta}{2(\theta+1)}\int_{\mathbb{R}^2} V(x)u_\mu^2\mathrm{d}x + \int_{\mathbb{R}^2\setminus\Omega^d_{i_0}} \left[\frac{1}{2(\theta+1)}\hat{f}(x,u_\mu)u_\mu - \hat{F}(x,u_\mu)\right]\mathrm{d}x \\ &\quad + \int_{\Omega^d_{i_0}} \left[\frac{1}{2(\theta+1)}K(x)f(u_\mu)u_\mu - K(x)F(u_\mu)\right]\mathrm{d}x + o_n(1). \end{aligned} \tag{2.38}$$

By (f_4), we get $\frac{f(u)}{u^{2\theta+1}}$ is strictly increasing for $u \geq 0$. Then for any $x \in \Omega_{i_0}^d$ and $u > v \geq 0$,

$$\begin{aligned}&\frac{1}{2(\theta+1)}K(x)f(u)u - K(x)F(u)\\&> \frac{1}{2(\theta+1)}K(x)f(v)v - K(x)F(v).\end{aligned} \tag{2.39}$$

By (f_4), we get $\frac{f(u)}{u}$ is strictly increasing for $u \geq 0$. Together with (K_1) and (f_1)-(f_2), we derive that for any $x \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$, there exists a unique $u_x > 0$ such that $K(x)f(u) = \kappa u$ for $u = u_x$, $K(x)f(u) < \kappa u$ for $u < u_x$ and $K(x)f(u) > \kappa u$ for $u > u_x$. Then, for any $x \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$ and $u > v \geq 0$,

$$\begin{aligned}&\frac{\varepsilon_0'}{2}u^2 + \frac{1}{2(\theta+1)}\hat{f}(x,u)u - \hat{F}(x,u)\\&> \frac{\varepsilon_0'}{2}v^2 + \frac{1}{2(\theta+1)}\hat{f}(x,v)v - \hat{F}(x,v).\end{aligned} \tag{2.40}$$

By (2.38)–(2.40), (M_3), Lemma 2.2, and the definition of c_μ, we have

$$\begin{aligned}c_\mu >&\frac{1}{2}\hat{M}(t_\mu^2\|u_\mu\|_{H^1}^2) - \frac{1}{2(\theta+1)}M(t_\mu^2\|u_\mu\|_{H^1}^2)t_\mu^2\|u_\mu\|_{H^1}^2 + \frac{\mu\theta}{2(\theta+1)}\int_{\mathbb{R}^2} V(x)t_\mu^2 u_\mu^2 \mathrm{d}x\\&+ \int_{\mathbb{R}^2\setminus\Omega_{i_0}^d}\left[\frac{1}{2(\theta+1)}\hat{f}(x,t_\mu u_\mu)t_\mu u_\mu - \hat{F}(x,t_\mu u_\mu)\right]\mathrm{d}x\\&+ \int_{\Omega_{i_0}^d}\left[\frac{1}{2(\theta+1)}K(x)f(t_\mu u_\mu)t_\mu u_\mu - K(x)F(t_\mu u_\mu)\right]\mathrm{d}x\\=&\hat{I}_\mu(t_\mu u_\mu) = \max_{t\geq 0}\hat{I}_\mu(tu_\mu) \geq c_\mu,\end{aligned} \tag{2.41}$$

a contradiction. So $\lim_{n\to\infty}\|u_n\|_{H^1}^2 = \|u_\mu\|_{H^1}^2$. Moreover, $\hat{I}_\mu'(u_\mu) = 0$, from which we derive that

$$\begin{aligned}c_\mu &= \lim_{n\to\infty}\hat{I}_\mu(u_n) - \frac{1}{2(\theta+1)}\lim_{n\to\infty}(\hat{I}_\mu'(u_n), u_n)\\&\geq \hat{I}_\mu(u_\mu) - \frac{1}{2(\theta+1)}(I_\mu'(u_\mu), u_\mu) = \hat{I}_\mu(u_\mu) = \max_{t\geq 0}\hat{I}_\mu(tu_\mu) \geq c_\mu.\end{aligned} \tag{2.42}$$

By (2.42), we get $\lim_{n\to\infty}\int_{\mathbb{R}^2} V(x)|u_n - u_\mu|^2\mathrm{d}x = 0$. Then $\lim_{n\to\infty}\|u_n - u_\mu\|_\mu = 0$. □

By (2.26)-(2.27) and Lemma 2.6, we get the following result:

Lemma 2.7. *There exists $u_\mu \in X_\mu$ such that $\hat{I}_\mu(u_\mu) = c_\mu \in [\eta, \max_{t\geq 0} I_0(t\omega_{n_0})]$ and $\hat{I}_\mu'(u_\mu) = 0$, where $\eta > 0$ is independent of μ.*

3. Proof of Theorem 1.1

Define the functional J on $H_0^1(\Omega_{i_0})$ by

$$J(u) = \frac{1}{2}\hat{M}\left(\int_{\Omega_{i_0}}(|\nabla u|^2 + |u|^2)\mathrm{d}x\right) - \int_{\Omega_{i_0}} K(x)F(u)\mathrm{d}x.$$

Lemma 3.1. *For any sequence $\{\mu_n\}$ with $\mu_n \to \infty$ as $n \to \infty$, if $\hat{I}_{\mu_n}(u_{\mu_n}) = c_{\mu_n} \in [\eta, \max_{t\geq 0} I_0(t\omega_{n_0})]$ and $\hat{I}'_{\mu_n}(u_{\mu_n}) = 0$, then $u_{\mu_n} \to u_0$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$, where $u_0 \in H_0^1(\Omega_{i_0})$ is a positive solution of the equation*

$$M\left(\int_{\Omega_{i_0}} (|\nabla u|^2 + u^2)\mathrm{d}x\right)(-\Delta u + u) = K(x)f(u) \ \text{ in } \Omega_{i_0}. \tag{3.1}$$

Proof. Similar to (2.30), we derive that $\|u_{\mu_n}\|_{H^1}$ is bounded. Assume that $u_{\mu_n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^2)$. By Fatou's lemma, we get $\int_{\mathbb{R}^2} V(x)u_0^2\mathrm{d}x = 0$. Moreover, $\int_{\mathbb{R}^2\setminus\Omega} u_0^2\mathrm{d}x = 0$. Then $u_0(x) = 0$ a.e. $x \in \mathbb{R}^2 \setminus \Omega$. By $u_0 \in H^1(\mathbb{R}^2)$, $u_0(x) = 0$ a.e. $x \in \mathbb{R}^2 \setminus \Omega$ with Ω having a smooth boundary and Proposition 9.18 in [8], we get $u_0 \in H_0^1(\Omega)$.

Let $E = \lim_{n\to\infty} M(\|u_{\mu_n}\|_{H^1}^2)$. Define the functional $\tilde{I}_\mu$ on X_μ by

$$\tilde{I}_\mu(u) = \frac{E}{2}\|u\|_{H^1}^2 + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} G(x,u)\mathrm{d}x.$$

Then $\tilde{I}'_{\mu_n}(u_{\mu_n}) = o_n(1)$. For all $\varphi_j \in H_0^1(\Omega_j)$ with $j \neq i_0$, we get

$$E\int_{\Omega_j} (\nabla u_0 \nabla\varphi_j + u_0\varphi_j)\mathrm{d}x = \int_{\Omega_j} g(x,u_0)\varphi_j\mathrm{d}x.$$

Since $u_0 \in H_0^1(\Omega)$, we have $u_0|_{\Omega_j} \in H_0^1(\Omega_j)$. Then

$$E\int_{\Omega_j} (|\nabla u_0|^2 + |u_0|^2)\mathrm{d}x = \int_{\Omega_j} g(x,u_0)u_0\mathrm{d}x. \tag{3.2}$$

By the structure of g, we get $u_0|_{\Omega_j} = 0$. Then $u_0 \in H_0^1(\Omega_{i_0})$.

We claim that $\lim_{n\to\infty} \|u_{\mu_n}\|_{H^1}^2 > 0$. Otherwise, $u_{\mu_n} \to 0$ in $H^1(\mathbb{R}^2)$. Choose $q > 1$(close to 1) and $\alpha > \alpha_0$(close to α_0) such that $q\alpha\|u_{\mu_n}\|_{H^1}^2 < 4\pi$ for n large. Let $t > 2$. By (f_1)-(f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{\Omega_{i_0}^d} f(u_{\mu_n})u_{\mu_n}\mathrm{d}x \leq \varepsilon\|u_{\mu_n}\|_2^2 + C_\varepsilon\int_{\Omega_{i_0}^d} |u_{\mu_n}|^t(e^{\alpha u_{\mu_n}^2} - 1)\mathrm{d}x. \tag{3.3}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega_{i_0}^d} |u_{\mu_n}|^t(e^{\alpha u_{\mu_n}^2} - 1)\mathrm{d}x \\
&\leq \lim_{n\to\infty}\left(\int_{\Omega_{i_0}^d} |u_{\mu_n}|^{\frac{tq}{q-1}}\mathrm{d}x\right)^{\frac{q-1}{q}}\left[\int_{\Omega_{i_0}^d} (e^{q\alpha u_{\mu_n}^2} - 1)\mathrm{d}x\right]^{\frac{1}{q}} \\
&\leq C\lim_{n\to\infty}\left(\int_{\Omega_{i_0}^d} |u_{\mu_n}|^{\frac{tq}{q-1}}\mathrm{d}x\right)^{\frac{q-1}{q}} = 0.
\end{aligned} \tag{3.4}$$

Since $(\hat{I}'_\mu(u_{\mu_n}), u_{\mu_n}) = 0$, by (3.3)-(3.4) and (M_1), we get $\lim_{n\to\infty} \|u_{\mu_n}\|_{\mu_n} = 0$. So $\lim_{n\to\infty} c_{\mu_n} \le 0$, a contradiction. Let $D = \lim_{n\to\infty} \frac{\hat{M}(\|u_{\mu_n}\|^2_{H^1})}{\|u_{\mu_n}\|^2_{H^1}}$. Define the functional $\bar{I}_\mu$ on X_μ by

$$\bar{I}_\mu(u) = \frac{D}{2}\|u\|^2_{H^1} + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^2} G(x,u)\mathrm{d}x.$$

Define the functionals $\bar{J}$ and $\tilde{J}$ on $H^1_0(\Omega_{i_0})$ by

$$\bar{J}(u) = \frac{D}{2}\int_{\Omega_{i_0}} (|\nabla u|^2 + |u|^2)\mathrm{d}x - \int_{\Omega_{i_0}} K(x)F(u)\mathrm{d}x,$$
$$\tilde{J}(u) = \frac{E}{2}\int_{\Omega_{i_0}} (|\nabla u|^2 + |u|^2)\mathrm{d}x - \int_{\Omega_{i_0}} K(x)F(u)\mathrm{d}x.$$

Then $\tilde{J}'(u_0) = 0$. By (M_3), we have $\bar{J}(u_0) \ge 0$. Let $w_{\mu_n} = u_{\mu_n} - u_0$. Then $w_{\mu_n} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^2)$ and

$$c_{\mu_n} = \bar{J}(u_0) + \bar{I}_{\mu_n}(w_{\mu_n}) + o_n(1), \ \left(\tilde{I}'_{\mu_n}(w_{\mu_n}), w_{\mu_n}\right) = o_n(1). \tag{3.5}$$

Similar to the argument in (2.30), we get $\int_{\Omega^d_{i_0}} K(x)f(w_{\mu_n})w_{\mu_n}\mathrm{d}x$ is bounded. Together with Corollary 2.1 and the generalized Lebesgue-dominated convergence theorem, we derive that

$$\lim_{n\to\infty}\int_{\Omega^d_{i_0}} K(x)F(w_{\mu_n})\mathrm{d}x = 0. \tag{3.6}$$

By (3.5)-(3.6), the structure of g and $\hat{M}(t+s) \ge \hat{M}(t) + M_0 s$ for all t, $s \ge 0$, we have

$$\max_{t\ge 0} I_0(t\omega_{n_0}) \ge \lim_{n\to\infty} c_{\mu_n} \ge \frac{1}{2}\lim_{n\to\infty} \hat{M}(\|\nabla w_{\mu_n}\|^2_2 + \tau\|w_{\mu_n}\|^2_2).$$

Together with (2.27), we get $\lim_{n\to\infty}(\|\nabla w_{\mu_n}\|^2_2 + \tau\|w_{\mu_n}\|^2_2) < \frac{4\pi}{\alpha_0}$. By (3.5) and (M_1), we have

$$M_0\|w_{\mu_n}\|^2_{\mu_n} \le \int_{\Omega^d_{i_0}} K(x)f(w_{\mu_n})w_{\mu_n}\mathrm{d}x + \kappa\int_{\mathbb{R}^2\setminus\Omega^d_{i_0}} w^2_{\mu_n}\mathrm{d}x + o_n(1). \tag{3.7}$$

Choose $q > 1$(close to 1) and $\alpha > \alpha_0$(close to α_0) such that $q\alpha(\|\nabla w_{\mu_n}\|^2_2 + \tau\|w_{\mu_n}\|^2_2) < 4\pi$ for n large. By (K_1), (f_1)-(f_2) and Lemma 2.1, we have

$$\lim_{n\to\infty}\int_{\Omega^d_{i_0}} K(x)f(w_{\mu_n})w_{\mu_n}\mathrm{d}x = 0.$$

Together with (3.7), we get $\lim_{n\to\infty}\|w_{\mu_n}\|_{\mu_n} = 0$. So $J'(u_0) = 0$. Since $\lim_{n\to\infty} c_{\mu_n} \ge \eta$, we have $u_0 \ne 0$. The maximum principle shows that u_0 is positive. □

Lemma 3.2. *There exists $\mu' > 0$ such that for $\mu > \mu'$,*

$$\|u_\mu\|_{L^\infty(\mathbb{R}^2\setminus\Omega^d_{i_0})} \le C_0\|u_\mu\|_{H^1(\mathbb{R}^2\setminus\Omega_{i_0})}, \tag{3.8}$$

where $C_0 > 0$ is a constant independent of μ.

Proof. For $i \geq 2$, let $r_i = \frac{2+2^{-i}}{4} r_1$, where $r_1 \in (0, \min\{d, 1\})$. For $y \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$, define $\eta_i \in C_0^\infty(B_{r_i}(y))$ such that $\eta_i(x) = 1$ for $x \in B_{r_{i+1}}(y)$, $0 \leq \eta_i(x) \leq 1$ for $x \in \mathbb{R}^2$, and $|\nabla \eta_i| \leq \frac{2}{r_i - r_{i+1}}$ for $x \in \mathbb{R}^2$. Let $u_\mu^l = \min\{u_\mu, l\}$ and $\beta_i > 1$. By $(I_\mu'(u_\mu), \eta_i^2 |u_\mu^l|^{2(\beta_i-1)} u_\mu) = 0$ and (M_1), we get

$$\begin{aligned}
&M_0 \int_{\mathbb{R}^2} [|\nabla u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 + 2(\beta_i - 1)|\nabla u_\mu^l|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2] \mathrm{d}x \\
&+ M_0 \int_{\mathbb{R}^2} |u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^2} g(x, u_\mu) u_\mu |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x + C \int_{\mathbb{R}^2} |\nabla u_\mu| |\nabla \eta_i| |\eta_i| |u_\mu^l|^{2(\beta_i-1)} |u_\mu| \mathrm{d}x. \qquad (3.9)
\end{aligned}$$

Let $t \geq 2$. By (2.5), (3.9), and Young's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} [|\nabla u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 + 2(\beta_i - 1)|\nabla u_\mu^l|^2 |u_\mu^l|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2] \mathrm{d}x \\
&+ \int_{\mathbb{R}^2} |u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^2} |\nabla \eta_i|^2 |u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \mathrm{d}x + C \int_{\mathbb{R}^2} |u_\mu|^t (e^{\alpha u_\mu^2} - 1) |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x. \qquad (3.10)
\end{aligned}$$

We note that

$$\int_{\mathbb{R}^2} |u_\mu|^t (e^{\alpha u_\mu^2} - 1) |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x = \int_{\mathbb{R}^2} |u_\mu|^t (e^{\alpha \eta_1^2 u_\mu^2} - 1) |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x. \qquad (3.11)$$

By a direct calculation,

$$\|\eta_1 u_\mu\|_{H^1}^2 \leq 2 \int_{B_{r_1}(y)} |\nabla u_\mu|^2 \mathrm{d}x + \left(1 + 2\|\nabla \eta_1\|_{L^\infty(\mathbb{R}^2)}^2\right) \int_{B_{r_1}(y)} |u_\mu|^2 \mathrm{d}x. \qquad (3.12)$$

By (3.12) and Lemma 3.1, we can choose $\mu' > 0$ large such that $\|\eta_1 u_\mu\|_{H^1}^2 < \frac{4\pi}{\alpha_0}$ for $\mu > \mu'$. Choose $q > 1$(close to 1) and $\alpha > \alpha_0$(close to α_0) such that $q\alpha \|\eta_1 u_\mu\|_{H^1}^2 < 4\pi$. Then, by Lemma 2.1, there exists $C > 0$ independent of μ such that

$$\int_{\mathbb{R}^2} (e^{\alpha \eta_1^2 u_\mu^2} - 1)^q \mathrm{d}x \leq \int_{\mathbb{R}^2} (e^{q\alpha \eta_1^2 u_\mu^2} - 1) \mathrm{d}x \leq C. \qquad (3.13)$$

Let $t = 2$ and $p > 2q'$ with $q' = \frac{q}{q-1}$. By (3.10)-(3.11), (3.13), and the Sobolev embedding theorem, we obtain that there exists $C_p > 0$ such that

$$\begin{aligned}
&\|\eta_i u_\mu (u_\mu^l)^{\beta_i - 1}\|_p^2 \\
&\leq C_p \int_{\mathbb{R}^2} \left[|\nabla [\eta_i u_\mu (u_\mu^l)^{\beta_i-1}]|^2 + |\eta_i u_\mu (u_\mu^l)^{\beta_i-1}|^2 \right] \mathrm{d}x \\
&\leq 2C_p \int_{\mathbb{R}^2} [|\nabla u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 + (\beta_i - 1)^2 |\nabla u_\mu^l|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2] \mathrm{d}x \\
&\quad + 2C_p \int_{\mathbb{R}^2} |\nabla \eta_i|^2 |u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \mathrm{d}x + C_p \int_{\mathbb{R}^2} |u_\mu|^2 |u_\mu^l|^{2(\beta_i-1)} \eta_i^2 \mathrm{d}x
\end{aligned}$$

$$\leq C\beta_i^2\int_{\mathbb{R}^2}|\nabla\eta_i|^2|u_\mu|^2|u_\mu^l|^{2(\beta_i-1)}\mathrm{d}x + C\beta_i^2\|\eta_i u_\mu(u_\mu^l)^{\beta_i-1}\|_{2q'}^2. \tag{3.14}$$

By direct calculation, we obtain

$$\frac{1}{r_i - r_{i+1}} = \frac{4}{r_1}2^{i+1} > 1. \tag{3.15}$$

Let $\delta_0 = \frac{2q'}{p}$ and $\beta_i = \delta_0^{-i}$. Then, by (3.14)-(3.15), we have

$$\|u_\mu(u_\mu^l)^{\beta_i-1}\|_{L^p(B_{r_{i+1}}(y))} \leq \frac{C\beta_i}{r_i - r_{i+1}}\|u_\mu(u_\mu^l)^{\beta_i-1}\|_{L^{p\delta_0}(B_{r_i}(y))}. \tag{3.16}$$

Let $l \to \infty$, we obtain

$$\|u_\mu\|_{L^{p\beta_i}(B_{r_{i+1}}(y))} \leq \left(\frac{C\beta_i}{r_i - r_{i+1}}\right)^{\frac{1}{\beta_i}}\|u_\mu\|_{L^{p\beta_{i-1}}(B_{r_i}(y))}. \tag{3.17}$$

By (3.17), we derive that

$$\begin{aligned}\|u_\mu\|_{L^{p\beta_i}(B_{r_{i+1}}(y))} &\leq \prod_{j=2}^{i}\left(\frac{C\beta_j}{r_j - r_{j+1}}\right)^{\frac{1}{\beta_j}}\|u_\mu\|_{L^{p\beta_1}(B_{r_2}(y))}\\ &= \prod_{j=2}^{i}\left[\frac{8C}{r_1}\left(\frac{2}{\delta_0}\right)^j\right]^{\delta_0^j}\|u_\mu\|_{L^{p\beta_1}(B_{r_2}(y))}.\end{aligned}$$

Let $i \to \infty$, we have

$$\|u_\mu\|_{L^\infty(B_{\frac{1}{2}r_1}(y))} \leq C\|u_\mu\|_{L^{p\beta_1}(B_{r_2}(y))} \leq C_0\|u_\mu\|_{H^1(\mathbb{R}^2\setminus\Omega_{i_0})}. \tag{3.18}$$

Since $y \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$ is arbitrary, we finish the proof. □

Lemma 3.3. *There exist r_0, c_1, c_2, $\mu'' > 0$ such that $\Omega_{i_0}^d \subset B_{r_0}(0)$ and for all $\mu > \mu''$,*

$$u_\mu(x) \leq c_2 e^{-c_1\sqrt{\mu}(|x|-r_0)}, \quad \forall\, |x| \geq r_0, \tag{3.19}$$

where r_0, c_1, c_2 are independent of μ.

Proof. By (M_1) and the structure of g, we obtain that for any $x \in \mathbb{R}^2 \setminus \Omega_{i_0}^d$,

$$-M(\|u_\mu\|_{H^1}^2)\Delta u_\mu + \mu V(x)u_\mu + (M_0 - \kappa)u_\mu \leq 0.$$

Similar to (2.30), we can derive from Lemma 2.7 to obtain that $\|u_\mu\|_{H^1}$ is bounded. By (V_4), there exist r_0, $c_0 > 0$ independent of μ such that $\Omega_{i_0}^d \subset B_{r_0}(0)$ and

$$-\Delta u_\mu + c_0\mu u_\mu \leq 0, \quad \forall\, |x| \geq r_0. \tag{3.20}$$

By Lemma 3.2, there exists $c_2 > 0$ such that $u_\mu(x) \leq c_2$ for $|x| = r_0$, where $c_2 > 0$ is independent of $\mu > \mu'$. Let $v_\mu(x) = c_2 e^{-c_1\sqrt{\mu}(|x|-r_0)}$. By choosing $c_1 > 0$ as small, we obtain

$$-\Delta v_\mu + c_0\mu v_\mu \geq 0, \quad \forall\, |x| \geq r_0. \tag{3.21}$$

By (3.20)-(3.21) and the comparison principle, we obtain that $u_\mu(x) \leq v_\mu(x)$ for $|x| \geq r_0$. □

Proof of Theorem 1.1. By Lemma 2.7, there exists $u_\mu \in X_\mu$ such that $\hat{I}_\mu(u_\mu) = c_\mu \in [\eta, \max_{t\geq 0} I_0(t\omega_{n_0})]$ and $\hat{I}'_\mu(u_\mu) = 0$. Let $q > 2$. By (K_2) and (f_1)-(f_2), there exists $C > 0$ such that

$$\frac{K(x)f(u_\mu)}{u_\mu} \leq Ce^{\alpha|x|}\left[u_\mu^{l-1} + |u_\mu|^{q-2}(e^{\alpha u_\mu^2} - 1)\right]. \tag{3.22}$$

By (3.22) and Lemma 3.3, we derive that there exists $\mu'' > 0$ such that for $\mu \geq \mu''$,

$$\frac{K(x)f(u_\mu)}{u_\mu} \leq \kappa, \quad \forall\, |x| \geq 2r_0. \tag{3.23}$$

By (3.22) and Lemmas 3.1-3.2, we derive that there exists $\mu''' > 0$ such that for $\mu \geq \mu'''$,

$$\frac{K(x)f(u_\mu)}{u_\mu} \leq \kappa, \quad \forall\, x \in B_{2r_0}(0) \setminus \Omega_{i_0}^d. \tag{3.24}$$

By (3.23)-(3.24), we know that u_μ is the nonnegative solution of (1.2). The maximum principle shows that u_μ is positive. Together with Lemma 3.1, we obtain the result. □

4. Conclusions

In this paper, we study the Kirchhoff type of elliptic equation, and we assume the nonlinear terms as $K(x)f(u)$, where K is permitted to be unbounded above and f has exponential critical growth. By using the truncation technique and developing some approaches to deal with Kirchhoff-type equations with critical growth in the whole space, we get the existence and concentration behavior of solutions, where the solution satisfies the mountain pass geometry. The results are new even for the case $M \equiv 1$.

Author contributions

Prof. Zhang firstly have the idea of this paper and complete the part of introduction, he also provided the main references. Dr. Lou performed the calculation, and revised the final format of the paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by NSFC (No.12101192) and NSF of Shandong province (No.ZR2023MA037). The authors would like to thank the editors and referees for their useful suggestions and comments.

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. M. Al-Gwaiz, V. Benci, F. Gazzola, Bending and stretching energies in a rectangular plate modeling suspension bridges, *Nonlinear Anal.*, **106** (2014), 18–34. http://dx.doi.org/10.1016/j.na.2014.04.011
2. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381.http://dx.doi.org/10.1016/0022-1236(73)90051-7
3. C. O. Alves, M. A. S. Souto, M. Montenegro, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, *Calc. Var. Partial Differential Equations.*, **43** (2012), 537–554. http://dx.doi.org/10.1007/s00526-011-0422-y
4. T. Bartsch, A. Pankov, Z. Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.*, **3** (2001), 549–569. http://dx.doi.org/10.1142/S0219199701000494
5. T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations.*, **20** (1995), 1725–1741. http://dx.doi.org/10.1080/03605309508821149
6. T. Bartsch, Z. Q. Wang, Multiple positive solutions for a nonlinear Schrödinger equation, *Z. Angew. Math. Phys.*, **51** (2000), 366–384. http://dx.doi.org/10.1007/PL00001511
7. H. Berestycki, P. L. Lions, Nonlinear scalar field equations I. Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345. http://dx.doi.org/10.1007/BF00250555
8. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011. http://dx.doi.org/10.1007/978-0-387-70914-7
9. J. Chabrowski, *Weak convergence methods for semilinear elliptic equations*, Sigapore: World Scientific, 1999. http://dx.doi.org/10.1142/4225
10. S. T. Chen, X. H. Tang, J. Y. Wei, Improved results on planar Kirchhoff-type elliptic problems with critical exponential growth, *Z. Angew. Math. Phys.*, **72** (2021), 38. http://dx.doi.org/10.1007/s00033-020-01455-w
11. M. Clapp, Y. H. Ding, Positive solutions of a Schrödinger equation with critical nonlinearity, *Z. Angew. Math. Phys.*, **55** (2004), 592–605. http://dx.doi.org/10.1007/s00033-004-1084-9
12. Y. H. Ding, A. Szulkin, Bound states for semilinear Schrödinger equations with sign-changing potential, *Calc. Var. Partial Differential Equations.*, **29** (2007), 397–419. http://dx.doi.org/10.1007/s00526-006-0071-8
13. Y. H. Ding, K. Tanaka, Multiplicity of positive solutions of a nonlinear Schrödinger equation, *Manuscripta Math.*, **112** (2003), 109–135. http://dx.doi.org/10.1007/s00229-003-0397-x
14. D. G. de Figueiredo, O. H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var. Partial Differential Equations.*, **3** (1995), 139–153. http://dx.doi.org/10.1007/BF01205003
15. J. M. B. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, *Abstr. Appl. Anal.*, **2** (1997), 301–315.
16. G. M. Figueiredo, U. B. Severo, Ground state solution for a Kirchhoff problem with exponential critical growth, *Milan J. Math.*, **84** (2016), 23–39. http://dx.doi.org/10.1007/s00032-015-0248-8

17. Y. X. Guo, Z. W. Tang, Multi-bump solutions for Schrödinger equation involving critical growth and potential wells, *Discrete Contin. Dyn. Syst.*, **35** (2015), 3393–3415. http://dx.doi.org/10.3934/dcds.2015.35.3393

18. Y. X. Guo, Z. W. Tang, Sign changing bump solutions for Schrödinger equations involving critical growth and indefinite potential wells, *J. Differential Equations.*, **259** (2015), 6038–6071. http://dx.doi.org/10.1016/j.jde.2015.07.015

19. H. F. Jia, Ground state solutions for the nonlinear Kirchhoff type equations with lower term, *J. Math. Phys.*, **61** (2020), 111506. http://dx.doi.org/10.1063/5.0015454

20. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

21. B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$, *J. Funct. Anal.*, **219** (2005), 340–367. http://dx.doi.org/10.1016/j.jfa.2004.06.013

22. B. Ruf, F. Sani, *Geometric properties for parabolic and elliptic PDE's*, New York: Springer, 2013, 251–268. http://dx.doi.org/10.1007/978-88-470-2841-8

23. Y. Sato, K. Tanaka, Sign-changing multi-bump solutions for nonlinear Schrödinger equations with steep potential wells, *Trans. Am. Math. Soc.*, **361** (2009), 6205–6253. http://dx.doi.org/10.1090/S0002-9947-09-04565-6

24. C. A. Stuart, H. S. Zhou, Global branch of solutions for non-linear Schrödinger equations with deepening potential well, *Proc. London Math. Soc.*, **92** (2006), 655–681. http://dx.doi.org/10.1017/S0024611505015637

25. F. A. van Heerden, Z. Q. Wang, Schrödinger type equations with asymptotically linear nonlinearities, *Differ. Integral Equ.*, **16** (2003), 257–280. http://dx.doi.org/10.57262/die/1356060671

26. F. A. van Heerden, Multiple solutions for a Schrödinger type equation with an asymptotically linear term, *Nonlinear Anal.*, **55** (2003), 739–758. http://dx.doi.org/10.1016/j.na.2003.08.008

27. F. B. Zhang, M. Du, Existence and asymptotic behavior of positive solutions for Kirchhoff type problems with steep potential well, *J. Differential Equations.*, **269** (2020), 10085–10106. http://dx.doi.org/10.1016/j.jde.2020.07.013

28. J. J. Zhang, D. G. Costa, J. Marcos do Ó, Existence and concentration of positive solutions for nonlinear Kirchhoff-type problems with a general critical nonlinearity, *Proc. Edinburgh Math. Soc.*, **61** (2018), 1023–1040. http://dx.doi.org/10.1017/s0013091518000056

29. J. Zhang, Z. L. Lou, Existence and concentration behavior of solutions to Kirchhoff type equation with steep potential well and critical growth, *J. Math. Phys.*, **62** (2021), 011506. http://dx.doi.org/10.1063/5.0028510