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Research article

Novel generalized inequalities involving a general Hardy operator with multiple variables and general kernels on time scales

M. Zakarya[1], Ghada AlNemer[2], A. I. Saied[3,4] and H. M. Rezk[5,*]

[1] King Khalid University, College of Science, Department of Mathematics, P.O. Box 9004, 61413 Abha, Saudi Arabia

[2] Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia

[3] Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001, Košice, Slovakia

[4] Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt

[5] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt

* **Correspondence:** Email: haythamrezk@azhar.edu.eg.

Abstract: This paper introduced novel multidimensional Hardy-type inequalities with general kernels on time scales, extending existing results in the literature. We established generalized inequalities involving a general Hardy operator with multiple variables and kernels on arbitrary time scales. Our findings not only encompassed known results in the realm of real numbers ($\mathbb{T} = \mathbb{R}$), but also provided refinements and generalizations thereof. The proposed inequalities offered versatile applications in mathematical analysis and beyond, contributing to the ongoing exploration of inequalities on diverse time scales.

Keywords: Hardy type inequality; Hardy's operator; weighted functions; inequalities with kernels; time scales
Mathematics Subject Classification: 26D10, 26D15, 34N05, 42B25, 42C10, 47B38

1. Introduction

Opic and Kufiner [1] proved that if $1 < r \leq \varrho < \infty$, then

$$\left(\int_a^b u(\varkappa)\left(\int_a^{\varkappa} h(\tau)d\tau\right)^{\varrho} d\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_a^b h^r(\varkappa)\upsilon(\varkappa)\,d\varkappa\right)^{\frac{1}{r}}, \tag{1.1}$$

holds for the nonnegative function h, if

$$K := \sup_{a<\varkappa<b}\left(\int_{\varkappa}^{b} u(\tau)d\tau\right)^{\frac{1}{\varrho}}\left(\int_{a}^{\varkappa} \upsilon^{1-\varpi}(\tau)d\tau\right)^{\frac{1}{\varpi}} < \infty,$$

where $-\infty \leq a \leq b \leq \infty$ and u, v are measurable positive functions in (a, b). Furthermore, an estimate for the constant C in (1.1) is given by

$$C \leq \left(1+\frac{\varrho}{\varpi}\right)^{\frac{1}{\varrho}}\left(1+\frac{\varpi}{\varrho}\right)^{\frac{1}{\varpi}} K, \quad \text{where } \varpi = \frac{r}{r-1}.$$

Stepanov [2] proved that if $0 < r \leq 1$, $r \leq \varrho < \infty$ and $k \geq 0$ is a measurable kernel, then

$$\left(\int_0^\infty u(\varkappa)\left(\int_0^\infty k(\varkappa,\eta)h(\eta)d\eta\right)^{\varrho} d\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_0^\infty h^r(\varkappa)\upsilon(\varkappa)\,d\varkappa\right)^{\frac{1}{r}}, \tag{1.2}$$

holds for the nonnegative nondecreasing function h, if

$$L = \sup_{\tau>0}\left(\int_\tau^\infty \upsilon(\varkappa)\,d\varkappa\right)^{-\frac{1}{r}}\left(\int_0^\infty u(\varkappa)\left(\int_\tau^\infty k(\varkappa,\eta)d\eta\right)^{\varrho} d\varkappa\right)^{\frac{1}{\varrho}} < \infty.$$

Furthermore, if C in (1.2) is the smallest feasible, then $L = C$.

Heinig and Maligranda [3] demonstrated that if $0 < r \leq 1$, $r \leq \varrho < \infty$, and $k \geq 0$ is a measurable kernel, then

$$\left(\int_0^\infty u(\varkappa)\left(\int_0^\infty k(\varkappa,\tau)h(\tau)d\tau\right)^{\varrho} d\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_0^\infty h^r(\varkappa)\upsilon(\varkappa)\,d\varkappa\right)^{\frac{1}{r}},$$

holds for the nonnegative nonincreasing function h, if

$$\left(\int_0^\infty u(\varkappa)\left(\int_0^s k(\varkappa,\tau)d\tau\right)^{\varrho} d\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_0^s \upsilon(\varkappa)\,d\varkappa\right)^{\frac{1}{r}},$$

holds for all $s > 0$.

Oguntuase et al. [4] proved that if $1 < r \leq \varrho < \infty$, $0 < b_j \leq \infty$, $s_j \in (1, r)$, $j = 1, 2, ..., m$, ϕ is a nonnegative and convex function on (a, d), $-\infty \leq a < d \leq \infty$. Define $u(\varkappa_1, ..., \varkappa_m)$ and $v(\varkappa_1, ..., \varkappa_m)$ as nonnegative weighted functions such that $v(\varkappa_1, ..., \varkappa_m) = v_1(\varkappa_1)v_2(\varkappa_2)...v_m(\varkappa_m)$, then

$$\begin{aligned}
&\left(\int_0^{b_1}...\int_0^{b_m}\left[\phi\left(A_k h\left(\varkappa_1,...,\varkappa_m\right)\right)\right]^{\varrho} u(\varkappa_1,...,\varkappa_m)\frac{d\varkappa_1...d\varkappa_m}{\varkappa_1...\varkappa_m}\right)^{\frac{1}{\varrho}}\\
\leq\ & C\left(\int_0^{b_1}...\int_0^{b_m}\phi^r(h(\varkappa_1,...,\varkappa_m))v(\varkappa_1,...,\varkappa_m)\frac{d\varkappa_1...d\varkappa_m}{\varkappa_1...\varkappa_m}\right)^{\frac{1}{r}},
\end{aligned} \tag{1.3}$$

holds $\forall h(\varkappa_1, ..., \varkappa_m)$ such that $a < h(\varkappa_1, ..., \varkappa_m) < d$, if

$$A(s_1,...,s_m) = \sup_{0<\eta_1,...,\eta_m<b_1,...,b_m}\left[V_1(\eta_1)\right]^{\frac{s_1-1}{r}}...\left[V_m(\eta_m)\right]^{\frac{s_m-1}{r}}$$

$$\times\left(\begin{array}{c}\int_{\eta_1}^{b_1}\cdots\int_{\eta_m}^{b_m}\left(\frac{k(\varkappa_1,\ldots,\varkappa_m,\eta_1,\ldots,\eta_m)}{K(\varkappa_1,\ldots,\varkappa_m)}\right)^{\varrho}[V_1(\varkappa_1)]^{\frac{\varrho(r-s_1)}{r}}\\ \times[V_m(\varkappa_m)]^{\frac{\varrho(r-s_m)}{r}}\frac{u(\varkappa_1,\ldots,\varkappa_m)}{\varkappa_1\ldots\varkappa_m}d\varkappa_1\ldots d\varkappa_m\end{array}\right)^{\frac{1}{\varrho}}<\infty,$$

holds, where $V_j(\eta_j)=\int_0^{\eta_j}\left[v_j(\tau_j)\right]^{\frac{-1}{r-1}}\left(\tau_j\right)^{\frac{1}{r-1}}d\tau_j$, $j=1,2,....,m$,

$$A_k h(\varkappa_1,\ldots,\varkappa_m)=\frac{1}{K(\varkappa_1,\ldots,\varkappa_m)}\int_0^{\varkappa_1}\cdots\int_0^{\varkappa_m}k(\varkappa_1,\ldots,\varkappa_m,\eta_1,\ldots,\eta_m)h(\eta_1,\ldots,\eta_m)d\eta_1\ldots d\eta_m,$$

and

$$K(\varkappa_1,\ldots,\varkappa_m)=\int_0^{\varkappa_1}\cdots\int_0^{\varkappa_m}k(\varkappa_1,\ldots,\varkappa_m,\tau_1,\ldots,\tau_m)d\tau_1\ldots d\tau_m.$$

Furthermore, if C is the best feasible, then

$$C\le\inf_{1<s_1,\ldots,s_m<r}\left(\frac{r-1}{r-s_1}\right)^{\frac{r-1}{r}}\cdots\left(\frac{r-1}{r-s_m}\right)^{\frac{r-1}{r}}A(s_1,\ldots,s_m).$$

Oguntuase and Durojaye [5] showed that if $1<r\le\varrho<\infty$, $0<b_j\le\infty$, $s_j\in(1,r)$, $j=1,2,\ldots,m$ and ϕ is a nonnegative function on (a,d), $-\infty\le a<d\le\infty$. Let there exist a convex function ψ on (a,d) such that

$$A\psi(\varkappa)\le\phi(\varkappa)\le B\psi(\varkappa),$$

holds for constants $0<A\le B<\infty$ and $u(\varkappa_1,\ldots,\varkappa_m)$, $v(\varkappa_1,\ldots,\varkappa_m)$, which are nonnegative weighted functions such that $v(\varkappa_1,\ldots,\varkappa_m)=v_1(\varkappa_1)v_2(\varkappa_2)\ldots v_m(\varkappa_m)$. Then,

$$\begin{aligned}&\left(\int_0^{b_1}\cdots\int_0^{b_m}\left[\phi\left(A_k h\left(\varkappa_1,\ldots,\varkappa_m\right)\right)\right]^{\varrho}\frac{u(\varkappa_1,\ldots,\varkappa_m)}{\varkappa_1\ldots\varkappa_m}d\varkappa_1\ldots d\varkappa_m\right)^{\frac{1}{\varrho}}\\ \le\;&C\left(\int_0^{b_1}\cdots\int_0^{b_m}\phi^r(h(\varkappa_1,\ldots,\varkappa_m))\frac{v(\varkappa_1,\ldots,\varkappa_m)}{\varkappa_1\ldots\varkappa_m}d\varkappa_1\ldots d\varkappa_m\right)^{\frac{1}{r}},\end{aligned}\tag{1.4}$$

holds $\forall h(\varkappa_1,\ldots,\varkappa_m)$ such that $a<h(\varkappa_1,\ldots,\varkappa_m)<d$ if

$$\begin{aligned}A(s_1,\ldots,s_m)=&\sup_{0<\eta_1,\ldots,\eta_m<b_1,\ldots,b_m}[V_1(\eta_1)]^{\frac{s_1-1}{r}}\cdots[V_m(\eta_m)]^{\frac{s_m-1}{r}}\\ &\times\left(\begin{array}{c}\int_{\eta_1}^{b_1}\cdots\int_{\eta_m}^{b_m}\left(\frac{k(\varkappa_1,\ldots,\varkappa_m,\eta_1,\ldots,\eta_m)}{K(\varkappa_1,\ldots,\varkappa_m)}\right)^{\varrho}[V_1(\varkappa_1)]^{\frac{\varrho(r-s_1)}{r}}[V_m(\varkappa_m)]^{\frac{\varrho(r-s_m)}{r}}\\ .\frac{u(\varkappa_1,\ldots,\varkappa_m)}{\varkappa_1\ldots\varkappa_m}d\varkappa_1\ldots d\varkappa_m\end{array}\right)^{\frac{1}{\varrho}}<\infty,\end{aligned}$$

holds, where $V_j(\eta_j)=\int_0^{\eta_j}\left[v_j(\tau_j)\right]^{\frac{-1}{r-1}}\left(\tau_j\right)^{\frac{1}{r-1}}d\tau_j$, $j=1,2,....,m$,

$$A_k h(\varkappa_1,\ldots,\varkappa_m)=\frac{1}{K(\varkappa_1,\ldots,\varkappa_m)}\int_0^{\varkappa_1}\cdots\int_0^{\varkappa_m}k(\varkappa_1,\ldots,\varkappa_m,\eta_1,\ldots,\eta_m)h(\eta_1,\ldots,\eta_m)d\eta_1\ldots d\eta_m,$$

and

$$K(\varkappa_1,\ldots,\varkappa_m)=\int_0^{\varkappa_1}\cdots\int_0^{\varkappa_m}k(\varkappa_1,\ldots,\varkappa_m,\tau_1,\ldots,\tau_m)d\tau_1\ldots d\tau_m.$$

In addition, if C is the best constant, then

$$C \leq \frac{B}{A} \inf_{1<s_1,...,s_m<r} \left(\frac{r-1}{r-s_1}\right)^{\frac{r-1}{r}} ... \left(\frac{r-1}{r-s_m}\right)^{\frac{r-1}{r}} A(s_1, ..., s_m).$$

In recent years, the study of dynamic inequalities on time scales has received a lot of attention and has become a major field in pure and applied mathematics. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. The case is when the time scale is equal to the reals or to the integers representing the classical theories of continuous and of discrete inequalities. Any inequality that can be proven on time scales should be avoided twice, once in the continuous case and once in the discrete case.

Saker et al. [6] established the time scale version of (1.1) as the following: Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $1 < r \leq \varrho < \infty$, $h \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ be a nonnegative the function, and $f, g \in C_{rd}((a, b)_{\mathbb{T}}, \mathbb{R})$ be positive functions. Then,

$$\left(\int_a^b f(\varkappa)\left(\int_a^{\sigma(\varkappa)} h(\tau)\Delta\tau\right)^{\varrho} \Delta\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_a^b h^r(\varkappa) g(\varkappa)\Delta\varkappa\right)^{\frac{1}{r}}, \tag{1.5}$$

holds, if

$$K = \sup_{a<\varkappa<b}\left(\int_{\varkappa}^b f(\tau)\Delta\tau\right)^{\frac{1}{\varrho}}\left(\int_a^{\sigma(\varkappa)} g^{1-\varpi}(\tau)\Delta\tau\right)^{\frac{1}{\varpi}} < \infty, \text{ where } \varpi = \frac{r}{r-1}.$$

Furthermore, for the constant C in (1.5), the following estimate is satisfied:

$$K \leq C \leq \left(1+\frac{\varrho}{\varpi}\right)^{\frac{1}{\varrho}}\left(1+\frac{\varpi}{\varrho}\right)^{\frac{1}{\varpi}} K.$$

In the same paper [6], the authors proved the dual form for (1.5) in the following: Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $1 < r \leq \varrho < \infty$, $h \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ be a nonnegative the function, and $f, g \in C_{rd}((a, b)_{\mathbb{T}}, \mathbb{R})$ be positive functions. Then,

$$\left(\int_a^b f(\varkappa)\left(\int_x^b h(\tau)\Delta\tau\right)^{\varrho} \Delta\varkappa\right)^{\frac{1}{\varrho}} \leq C\left(\int_a^b h^r(\varkappa) g(\varkappa)\Delta\varkappa\right)^{\frac{1}{r}}, \tag{1.6}$$

holds, if

$$L = \sup_{a<\varkappa<b}\left(\int_a^{\sigma(\varkappa)} f(\tau)\Delta\tau\right)^{\frac{1}{\varrho}}\left(\int_x^b g^{1-\varpi}(\tau)\Delta\tau\right)^{\frac{1}{\varpi}} < \infty, \text{ where } \varpi = \frac{r}{r-1}.$$

Furthermore, for the constant C in (1.6), the following estimate is satisfied:

$$L \leq C \leq \left(1+\frac{\varrho}{\varpi}\right)^{\frac{1}{\varrho}}\left(1+\frac{\varpi}{\varrho}\right)^{\frac{1}{\varpi}} L.$$

For more details about the dynamic inequalities of Hardy-type, we refer the reader to the papers [7–11] and the book by Agarwal et al. [12].

The aim of this paper is to demonstrate multidimensional Hardy-type inequalities with general kernels on time scales. As special cases of our results on time scales, when $\mathbb{T} = \mathbb{R}$, we get the integral inequalities (1.3) and (1.4) proved by Oguntuase et al. [4] and Oguntuase and Durojaye [5], respectively. Also, as special cases of the main reslts, when $\mathbb{T} = \mathbb{N}$, we can obtain other inequalities in the discrete calculus, which are essentially new for the reader.

The following is the structure of this document. Section 2 covers the fundamentals of time scales calculus. In Section 3, we prove our main results, where some classical and modern inequalities are derived.

2. Preliminaries

This section includes definitions and lemmas which are fundamentals of time scales calculus; see [13–15]. Consider the time scale $\mathbb{T}$ and $\tau \in \mathbb{T}$. The forward jump operator is defined by: $\sigma(\tau) = \inf\{v \in \mathbb{T} : v > \tau\}$. A function $\Phi : \mathbb{T} \to \mathbb{R}$, is characterized as rd-continuous when it exhibits continuity at every right-dense point within $\mathbb{T}$ and possesses finite left-sided limits at left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is ushered by $C_{rd}(\mathbb{T}, \mathbb{R})$, and for any function $\Phi : \mathbb{T} \to \mathbb{R}$, the notation $\Phi^{\sigma}(\tau)$ denotes $\Phi(\sigma(\tau))$.

The derivatives of $\Phi\varpi$ and Φ/ϖ (where $\varpi\varpi^{\sigma} \neq 0$) of two differentiable functions Φ and ϖ are given by

$$(\Phi\varpi)^{\Delta} = \Phi^{\Delta}\varpi + \Phi^{\sigma}\varpi^{\Delta} = \Phi\varpi^{\Delta} + \Phi^{\Delta}\varpi^{\sigma}, \ \left(\frac{\Phi}{\varpi}\right)^{\Delta} = \frac{\Phi^{\Delta}\varpi - \Phi\varpi^{\Delta}}{\varpi\varpi^{\sigma}}.$$

If $G^{\Delta}(r) = \varpi(r)$, then the delta integral is predefined as

$$\int_{r_0}^{r} \varpi(t)\Delta t = G(r) - G(r_0).$$

It can be demonstrated that if $\varpi \in C_{rd}(\mathbb{T}, \mathbb{R})$, then the Cauchy integral $G(r) = \int_{r_0}^{r} \varpi(t)\Delta t$ exists, $r_0 \in \mathbb{T}$, and it satisfies $G^{\Delta}(r) = \varpi(r)$. The integration by parts formula is provided by

$$\int_{\upsilon_0}^{\upsilon} \lambda(\tau)\varphi^{\Delta}(\tau)\Delta\tau = [\lambda(\tau)\varphi(\tau)]_{\upsilon_0}^{\upsilon} - \int_{\upsilon_0}^{\upsilon} \lambda^{\Delta}(\tau)\varphi^{\sigma}(\tau)\Delta\tau.$$

The time scale chain rule is stated as follows:

$$(\varphi \circ g)^{\Delta}(\tau) = \varphi^{'}(g(\varkappa))\, g^{\Delta}(\tau), \text{ where } \varkappa \in [\tau, \sigma(\tau)], \tag{2.1}$$

where it is supposed that $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable.

The Hölder inequality is expressed as:

$$\begin{aligned} &\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} |h(\tau)g(\tau)|\Delta\tau_1...\Delta\tau_m \\ \leq \ &\left(\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} |h(\tau)|^{\gamma}\Delta\tau_1...\Delta\tau_m\right)^{\frac{1}{\gamma}} \left(\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} |g(\tau)|^{\nu}\Delta\tau_1...\Delta\tau_m\right)^{\frac{1}{\nu}}, \end{aligned} \tag{2.2}$$

where $a_1, ..., a_m, b_1, ..., b_m \in \mathbb{T}$, $h, g : \mathbb{T}^m \to \mathbb{R}$ such that

$$h(\tau) = h(\tau_1, \tau_2, ..., \tau_m), \ g(\tau) = g(\tau_1, \tau_2, ..., \tau_m),$$

$\gamma > 1$ and $1/\gamma + 1/\nu = 1$.

Theorem 2.1. *(Jensen's inequality) Assume that $a_j, b_j \in \mathbb{T}$, $j = 1, 2, .., m$, and $c, d \in \mathbb{R}$. If $g : \mathbb{T}^m \to (c, d)$ is rd-continuous and $\Phi : (c, d) \to \mathbb{R}$ is continuous and convex, then*

$$\Phi\left(\frac{1}{\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\xi, \tau)\Delta\tau} \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\xi, \tau)g(\tau)\Delta\tau\right)$$
$$\leq \frac{1}{\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\xi, \tau)\Delta\tau} \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\xi, \tau)\Phi(g(\tau))\Delta\tau, \tag{2.3}$$

where

$$\Delta\tau = \Delta\tau_1...\Delta\tau_m,\ h(\xi, \tau) = h(\xi_1, ..., \xi_m, \tau_1, ..., \tau_m) \text{ and } g(\tau) = g(\tau_1, ..., \tau_m).$$

Theorem 2.2. *(Minkowski's inequality) Assume that $a_j, b_j \in \mathbb{T}$, $j = 1, 2, ..., m$, and $\gamma \geq 1$. If $k : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}$, $w, h : \mathbb{T}^m \to \mathbb{R}$ are nonnegative rd-continuous functions, then*

$$\left(\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} w(\xi)\left(\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\tau)k(\xi, \tau)\Delta\tau\right)^{\gamma} \Delta\xi\right)^{\frac{1}{\gamma}}$$
$$\leq \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} h(\tau)\left(\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} w(\xi)k^{\gamma}(\xi, \tau)\Delta\xi\right)^{\frac{1}{\gamma}} \Delta\tau, \tag{2.4}$$

where

$$k(\xi, \tau) = k(\xi_1, ..., \xi_m, \tau_1, ..., \tau_m),\ w(\xi) = w(\xi_1, ..., \xi_m) \text{ and } h(\tau) = h(\tau_1, ..., \tau_m).$$

3. The main findings

We shall assume in this work that the functions are nonnegative rd-continuous functions and the considered integrals exist (and are finite, i.e., convergent). Throughout, we are using the following assumption: Define the nonnegative functions $h : \mathbb{T}^m \to \mathbb{R}$, $k : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}$ as the following:

$$h(\eta) = h(\eta_1, ..., \eta_m) \text{ and } k(\xi, \eta) = k(\xi_1, ..., \xi_m, \eta_1, ..., \eta_m).$$

Also, we define the general Hardy operator A_k as the following:

$$A_k h(\xi_1, ..., \xi_m) = \frac{1}{K(\xi_1, ..., \xi_m)} \int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k(\xi_1, ..., \xi_m, \eta_1, ..., \eta_m)h(\eta_1, ..., \eta_m)\Delta\eta,$$

with

$$K(\xi_1, ..., \xi_m) = \int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k(\xi_1, ..., \xi_m, \tau_1, ..., \tau_m)\Delta\tau,$$

and

$$A(s_1, ..., s_m) = \sup_{a_j<\eta_j<b_j} \left(\int_{\eta_1}^{b_1} ... \int_{\eta_m}^{b_m} \left(\frac{k(\xi, \eta)}{K(\xi)}\right)^{\varrho} [V_1^{\sigma}(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}} ... [V_m^{\sigma}(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}}\right.$$
$$\left.\times \frac{u(\xi)}{(\sigma(\xi_1) - a_1) ... (\sigma(\xi_m) - a_m)}\Delta\xi\right)^{\frac{1}{\varrho}} [V_1^{\sigma}(\eta_1)]^{\frac{s_1-1}{\mu}} ... [V_m^{\sigma}(\eta_m)]^{\frac{s_m-1}{\mu}}, \tag{3.1}$$

where $V_j(\eta_j) = \int_{a_j}^{\eta_j} \left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\tau_j\right) - a_j\right)^{\frac{1}{\mu-1}} \Delta\tau_j$, $j = 1, 2, ..., m$.

Mathematical applications of this work are given in the form of remarks, examples, and corollaries. Now, we start with the time scale version of (1.3).

Theorem 3.1. *Let $a_j, b_j \in \mathbb{T}$, $1 < \mu \leq \varrho < \infty$, $s_j \in (1,\mu)$, $j = 1,2,...,m$, and ψ be a nonnegative and convex function on (a,d), $-\infty \leq a < d \leq \infty$. We define $u(\xi_1,...,\xi_m)$ and $v(\xi_1,...,\xi_m)$ as nonnegative weighted functions such that*

$$v(\xi_1,...,\xi_m) = v_1(\xi_1)v_2(\xi_2)...v_m(\xi_m). \tag{3.2}$$

If (3.1) holds, then

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}[\psi(A_k h(\xi))]^{\varrho}\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\right)^{\frac{1}{\varrho}}\\
\leq\ &\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}}...\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}}A(s_1,...,s_m)\\
&\times\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\psi^{\mu}(h(\eta))\frac{v(\eta)}{(\sigma(\eta_1)-a_1)...(\sigma(\eta_m)-a_m)}\Delta\eta\right)^{\frac{1}{\mu}}.
\end{aligned}\tag{3.3}$$

Proof. By applying (2.3), we see that

$$\begin{aligned}
&\int_{a_1}^{b_1}...\int_{a_m}^{b_m}[\psi(A_k h(\xi))]^{\varrho}\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\\
=\ &\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\left(\psi\left(\frac{1}{K(\xi)}\int_{a_1}^{\sigma(\xi_1)}...\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)h(\eta)\Delta\eta\right)\right)^{\varrho}\\
&\times\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\\
\leq\ &\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\left(\frac{1}{K(\xi)}\int_{a_1}^{\sigma(\xi_1)}...\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)\psi(h(\eta))\Delta\eta\right)^{\varrho}\\
&\times\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\\
=\ &\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\frac{u(\xi)}{K^{\varrho}(\xi)}\frac{J^{\varrho}(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi,
\end{aligned}\tag{3.4}$$

where

$$J(\xi) = \int_{a_1}^{\sigma(\xi_1)}...\int_{am}^{\sigma(\xi_m)}k(\xi,\eta)\psi(h(\eta))\Delta\eta. \tag{3.5}$$

Denote $\psi^{\mu}(h(\eta))\frac{v_1(\eta_1)...v_m(\eta_m)}{(\sigma(\eta_1)-a_1)...(\sigma(\eta_m)-a_m)} = \Psi(\eta)$ and substitute it into (3.5) to obtain that

$$\begin{aligned}
J(\xi) =\ &\int_{a_1}^{\sigma(\xi_1)}...\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)[\Psi(\eta)]^{\frac{1}{\mu}}[v_1(\eta_1)]^{\frac{-1}{\mu}}...[v_m(\eta_m)]^{\frac{-1}{\mu}}\\
&\times[V_1^{\sigma}(\eta_1)]^{\frac{s_1-1}{\mu}}...[V_m^{\sigma}(\eta_m)]^{\frac{sm-1}{\mu}}[V_1^{\sigma}(\eta_1)]^{\frac{1-s_1}{\mu}}...[V_m^{\sigma}(\eta_m)]^{\frac{1-sm}{\mu}}\\
&\times(\sigma(\eta_1)-a_1)^{\frac{1}{\mu}}...(\sigma(\eta_m)-a_m)^{\frac{1}{\mu}}\Delta\eta,
\end{aligned}\tag{3.6}$$

where $V_j(\eta_j) = \int_{a_j}^{\eta_j} \left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\tau_j\right) - a_j\right)^{\frac{1}{\mu-1}} \Delta\tau_j,\ j = 1,2,...,m$. Applying (2.2) with $\mu > 1$ and $\mu/(\mu-1)$ in (3.6), we see that

$$\begin{aligned} J(\xi) \leq & \left(\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k^{\mu}(\xi,\eta)\Psi(\eta) \left[V_1^{\sigma}(\eta_1)\right]^{s_1-1} ... \left[V_m^{\sigma}(\eta_m)\right]^{s_m-1} \Delta\eta\right)^{\frac{1}{\mu}} \\ & \times\Bigg[\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} \left[V_1^{\sigma}(\eta_1)\right]^{\frac{1-s_1}{\mu-1}} ... \left[V_m^{\sigma}(\eta_m)\right]^{\frac{1-s_m}{\mu-1}} \\ & \times \left[v_1(\eta_1)\right]^{\frac{-1}{\mu-1}} ... \left[v_m(\eta_m)\right]^{\frac{-1}{\mu-1}} \left(\sigma(\eta_1) - a_1\right)^{\frac{1}{\mu-1}} ... \left(\sigma(\eta_m) - a_m\right)^{\frac{1}{\mu-1}} \Delta\eta\Bigg]^{\frac{\mu-1}{\mu}}. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), we have

$$\begin{aligned} & \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left[\psi\left(A_k h(\xi)\right)\right]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1) - a_1) ... (\sigma(\xi_m) - a_m)} \Delta\xi \\ \leq & \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \frac{u(\xi)}{(\sigma(\xi_1) - a_1) ... (\sigma(\xi_m) - a_m) K^{\varrho}(\xi)} \\ & \times\left(\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k^{\mu}(\xi,\eta)\Psi(\eta) \left[V_1^{\sigma}(\eta_1)\right]^{s_1-1} ... \left[V_m^{\sigma}(\eta_m)\right]^{s_m-1} \Delta\eta\right)^{\frac{\varrho}{\mu}} \\ & \times\Bigg[\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} \left[V_1^{\sigma}(\eta_1)\right]^{\frac{1-s_1}{\mu-1}} ... \left[V_m^{\sigma}(\eta_m)\right]^{\frac{1-s_m}{\mu-1}} \\ & \times \left[v_1(\eta_1)\right]^{\frac{-1}{\mu-1}} ... \left[v_m(\eta_m)\right]^{\frac{-1}{\mu-1}} \left(\sigma(\eta_1) - a_1\right)^{\frac{1}{\mu-1}} ... \left(\sigma(\eta_m) - a_m\right)^{\frac{1}{\mu-1}} \Delta\eta\Bigg]^{\frac{\varrho(\mu-1)}{\mu}} \Delta\xi. \end{aligned} \tag{3.8}$$

Since

$$V_j(\eta_j) = \int_{a_j}^{\eta_j} \left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\tau_j\right) - a_j\right)^{\frac{1}{\mu-1}} \Delta\tau_j,\ j = 1,2,...,m,$$

then

$$V_j^{\Delta}(\eta_j) = \left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_j\right) - a_j\right)^{\frac{1}{\mu-1}} > 0. \tag{3.9}$$

Therefore, the function V_j is increasing. Applying the chain rule formula (2.1) on $\left[V_j(\eta_j)\right]^{1-(s_j-1/(\mu-1))}$, we obtain

$$\left[\left[V_j(\eta_j)\right]^{1-\frac{s_j-1}{\mu-1}}\right]^{\Delta} = \left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta} = \left(\frac{\mu-s_j}{\mu-1}\right)\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}} V_j^{\Delta}(\eta_j), \tag{3.10}$$

where $\xi_j \in [\eta_j, \sigma(\eta_j)]$, $j = 1,2,...,m$. Thus, by substituting (3.9) into (3.10), we see

$$\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta} = \left(\frac{\mu-s_j}{\mu-1}\right)\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}} \left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_j\right) - a_j\right)^{\frac{1}{\mu-1}}. \tag{3.11}$$

Since $\xi_j \leq \sigma(\eta_j)$ and V_j is increasing, we have

$$V_j\left(\xi_j\right) \leq V_j^{\sigma}(\eta_j).$$

Using the relation $1 < s_j < \mu,\ j = 1, 2, ..., m$, we get

$$\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}} \geq \left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}}. \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta} \geq \left(\frac{\mu - s_j}{\mu - 1}\right)\left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}} \left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_j\right) - a_j\right)^{\frac{1}{\mu-1}},$$

and then

$$\begin{aligned} &\int_{a_j}^{\sigma(\xi_j)} \left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta} \Delta\eta_j \\ \geq\ & \left(\frac{\mu - s_j}{\mu - 1}\right)\int_{a_j}^{\sigma(\xi_j)} \left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}} \left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_j\right) - a_j\right)^{\frac{1}{\mu-1}} \Delta\eta_j. \end{aligned}$$

Thus, we have (note $V_j(a_j) = 0$) that

$$\begin{aligned} &\int_{a_j}^{\sigma(\xi_j)} \left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}} \left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_j\right) - a_j\right)^{\frac{1}{\mu-1}} \Delta\eta_j \\ \leq\ & \left(\frac{\mu - 1}{\mu - s_j}\right)\int_{a_j}^{\sigma(\xi_j)} \left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta} \Delta\eta_j \\ =\ & \left(\frac{\mu - 1}{\mu - s_j}\right)\left[V_j^{\sigma}(\xi_j)\right]^{\frac{\mu-s_j}{\mu-1}},\ j = 1, 2, ..., m, \end{aligned} \tag{3.13}$$

and then we have from (3.13) that

$$\begin{aligned} &\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} \left[V_1^{\sigma}(\eta_1)\right]^{\frac{1-s_1}{\mu-1}} ... \left[V_m^{\sigma}(\eta_m)\right]^{\frac{1-s_m}{\mu-1}} \\ &\times \left[v_1(\eta_1)\right]^{\frac{-1}{\mu-1}} ... \left[v_m(\eta_m)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_1\right) - a_1\right)^{\frac{1}{\mu-1}} ... \left(\sigma\left(\eta_m\right) - a_m\right)^{\frac{1}{\mu-1}} \Delta\eta \\ =\ & \left(\int_{a_1}^{\sigma(\xi_1)} \left[V_1^{\sigma}(\eta_1)\right]^{-\frac{(s_1-1)}{\mu-1}} \left[v_1(\eta_1)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_1\right) - a_1\right)^{\frac{1}{\mu-1}} \Delta\eta_1\right) \\ &\times .... \times \left(\int_{a_m}^{\sigma(\xi_m)} \left[V_m^{\sigma}(\eta_m)\right]^{-\frac{(s_m-1)}{\mu-1}} \left[v_m(\eta_m)\right]^{\frac{-1}{\mu-1}} \left(\sigma\left(\eta_m\right) - a_m\right)^{\frac{1}{\mu-1}} \Delta\eta_m\right) \\ \leq\ & \left(\frac{\mu - 1}{\mu - s_1}\right) ... \left(\frac{\mu - 1}{\mu - s_m}\right)\left[V_1^{\sigma}(\xi_1)\right]^{\frac{\mu-s_1}{\mu-1}} ... \left[V_m^{\sigma}(\xi_m)\right]^{\frac{\mu-s_m}{\mu-1}}. \end{aligned}$$

Substituting into (3.8), we see that

$$\begin{aligned} &\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left[\psi\left(A_k h\left(\xi\right)\right)\right]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1) - a_1) ... (\sigma(\xi_m) - a_m)} \Delta\xi \\ \leq\ & \left(\frac{\mu - 1}{\mu - s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}} ... \left(\frac{\mu - 1}{\mu - s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}} \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left[V_1^{\sigma}(\xi_1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}} ... \left[V_m^{\sigma}(\xi_m)\right]^{\frac{\varrho(\mu-s_m)}{\mu}} \end{aligned}$$

$$\times\left(\int_{a_1}^{\sigma(\xi_1)}\ldots\int_{a_m}^{\sigma(\xi_m)} k^{\mu}(\xi,\eta)\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\ldots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\Delta\eta\right)^{\frac{\varrho}{\mu}}$$
$$\times\frac{u(\xi)}{K^{\varrho}(\xi)\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi. \tag{3.14}$$

Applying (2.4) on the term

$$\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\left(\int_{a_1}^{\sigma(\xi_1)}\ldots\int_{a_m}^{\sigma(\xi_m)} k^{\mu}(\xi,\eta)\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\ldots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\Delta\eta\right)^{\frac{\varrho}{\mu}}$$
$$\times\left[V_1^{\sigma}(\xi_1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}}\ldots\left[V_m^{\sigma}(\xi_m)\right]^{\frac{\varrho(\mu-s_m)}{\mu}}\frac{u(\xi)}{K^{\varrho}(\xi)\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi,$$

with $\varrho/\mu > 1$, we observe that

$$\left[\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\left(\int_{a_1}^{\sigma(\xi_1)}\ldots\int_{a_m}^{\sigma(\xi_m)} k^{\mu}(\xi,\eta)\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\ldots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\Delta\eta\right)^{\frac{\varrho}{\mu}}\right.$$
$$\left.\times\left[V_1^{\sigma}(\xi_1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}}\ldots\left[V_m^{\sigma}(\xi_m)\right]^{\frac{\varrho(\mu-s_m)}{\mu}}\frac{u(\xi)}{K^{\varrho}(\xi)\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi\right]^{\frac{\mu}{\varrho}}$$
$$\leq \int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\ldots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}$$
$$\times\left[\int_{\eta_1}^{b_1}\ldots\int_{\eta_m}^{b_m} k^{\varrho}(\xi,\eta)\left[V_1^{\sigma}(\xi_1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}}\ldots\left[V_m^{\sigma}(\xi_m)\right]^{\frac{\varrho(\mu-s_m)}{\mu}}\right.$$
$$\left.\times\frac{u(\xi)}{K^{\varrho}(\xi)\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi\right]^{\frac{\mu}{\varrho}}\Delta\eta. \tag{3.15}$$

Substituting (3.15) into (3.14), we obtain

$$\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\left[\psi\left(A_k h(\xi)\right)\right]^{\varrho}\frac{u(\xi)}{\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi$$
$$\leq\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\ldots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}\left[\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\ldots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\right.$$
$$\times\left(\int_{\eta_1}^{b_1}\ldots\int_{\eta_m}^{b_m} k^{\varrho}(\xi,\eta)\left[V_1^{\sigma}(\xi_1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}}\ldots\left[V_m^{\sigma}(\xi_m)\right]^{\frac{\varrho(\mu-s_m)}{\mu}}\right.$$
$$\left.\left.\times\frac{u(\xi)}{K^{\varrho}(\xi)\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi\right)^{\frac{\mu}{\varrho}}\Delta\eta\right]^{\frac{\varrho}{\mu}}. \tag{3.16}$$

Using the assumptions (3.1) and (3.2), the inequality (3.16) becomes

$$\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\left[\psi\left(A_k h(\xi)\right)\right]^{\varrho}\frac{u(\xi)}{\left(\sigma(\xi_1)-a_1\right)\ldots\left(\sigma(\xi_m)-a_m\right)}\Delta\xi$$
$$\leq A^{\varrho}(s_1,\ldots,s_m)\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\ldots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}\left[\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\Psi(\eta)\Delta\eta\right]^{\frac{\varrho}{\mu}}$$

$$
\begin{aligned}
&= \left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}} \ldots \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}} A^{\varrho}(s_1,...,s_m) \\
&\quad \times \left[\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} \psi^{\mu}(h(\eta)) \frac{v_1(\eta_1)...v_m(\eta_m)}{(\sigma(\eta_1)-a_1)\ldots(\sigma(\eta_m)-a_m)} \Delta\eta\right]^{\frac{\varrho}{\mu}} \\
&= \left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}} \ldots \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}} A^{\varrho}(s_1,...,s_m) \\
&\quad \times \left[\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} \psi^{\mu}(h(\eta)) \frac{v(\eta_1,..,\eta_m)}{(\sigma(\eta_1)-a_1)\ldots(\sigma(\eta_m)-a_m)} \Delta\eta\right]^{\frac{\varrho}{\mu}}, \qquad (3.17)
\end{aligned}
$$

and then

$$
\begin{aligned}
&\left(\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} \left[\psi\left(A_k h\left(\xi\right)\right)\right]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1)-a_1)\ldots(\sigma(\xi_m)-a_m)} \Delta\xi\right)^{\frac{1}{\varrho}} \\
&\leq \left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}} \ldots \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}} A(s_1,...,s_m) \\
&\quad \times \left[\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} \psi^{\mu}(h(\eta)) \frac{v(\eta_1,..,\eta_m)}{(\sigma(\eta_1)-a_1)\ldots(\sigma(\eta_m)-a_m)} \Delta\eta\right]^{\frac{1}{\mu}},
\end{aligned}
$$

which is (3.3). □

Remark 3.2. *If $\mathbb{T}=\mathbb{R}$, then (3.3) gives the inequality (1.3) proved by Oguntuase, Persson, and Essel [4].*

Corollary 3.3. *In Theorem 3.1, let $\mathbb{T}=\mathbb{Z}$, $a_j, b_j \in \mathbb{Z}$, $1<\mu\leq\varrho<\infty$, $s_j\in(1,\mu)$, $j=1,2,...,m$, and ψ be a nonnegative and convex sequence on (a,d), $-\infty\leq a<d\leq\infty$. Define $u(\xi_1,...,\xi_m)$ and $v(\xi_1,...,\xi_m)$ as nonnegative weighted sequences such that*

$$
v(\xi_1,...,\xi_m) = v_1(\xi_1)v_2(\xi_2)...v_m(\xi_m).
$$

Then

$$
\begin{aligned}
&\left(\sum_{\xi_1=a_1}^{b_1-1} \ldots \sum_{\xi_m=a_m}^{b_m-1} \left[\psi\left(A_k h\left(\xi\right)\right)\right]^{\varrho} \frac{u(\xi)}{(\xi_1+1-a_1)\ldots(\xi_m+1-a_m)}\right)^{\frac{1}{\varrho}} \\
&\leq \left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}} \ldots \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}} A(s_1,...,s_m) \\
&\quad \times \left(\sum_{\eta_1=a_1}^{b_1-1} \ldots \sum_{\eta_m=a_m}^{b_m-1} \psi^{\mu}(h(\eta)) \frac{v(\eta)}{(\eta_1+1-a_1)\ldots(\eta_m+1-a_m)}\right)^{\frac{1}{\mu}},
\end{aligned}
$$

provided that

$$
A(s_1,...,s_m) = \sup_{a_j<\eta_j<b_j} \left(\sum_{\xi_1=\eta_1}^{b_1-1} \ldots \sum_{\xi_m=\eta_m}^{b_m-1} \left(\frac{k(\xi,\eta)}{K(\xi)}\right)^{\varrho} \left[V_1(\xi_1+1)\right]^{\frac{\varrho(\mu-s_1)}{\mu}} \ldots \left[V_m(\xi_m+1)\right]^{\frac{\varrho(\mu-s_m)}{\mu}}\right.
$$

$$\times \frac{u(\xi)}{(\xi_1+1-a_1)\dots(\xi_m+1-a_m)}\Bigg)^{\frac{1}{\varrho}} [V_1(\eta_1+1)]^{\frac{s_1-1}{\mu}} \dots [V_m(\eta_m+1)]^{\frac{s_m-1}{\mu}} < \infty,$$

where

$$A_k h(\xi_1,...,\xi_m) = \frac{1}{K(\xi_1,...,\xi_m)} \sum_{\eta_1=a_1}^{\xi_1} \dots \sum_{\eta_m=a_m}^{\xi_m} k(\xi_1,...,\xi_m,\eta_1,...,\eta_m) h(\eta_1,...,\eta_m),$$

$$K(\xi_1,...,\xi_m) = \sum_{\tau_1=a_1}^{\xi_1} \dots \sum_{\tau_m=a_m}^{\xi_m} k(\xi_1,...,\xi_m,\tau_1,...,\tau_m),$$

and

$$V_j(\eta_j) = \sum_{\tau_j=a_j}^{\eta_j-1} \left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}} \left(\tau_j+1-a_j\right)^{\frac{1}{\mu-1}}, \quad j=1,2,....,m.$$

Example 3.4. *If we put $m = 1$, $k(\xi,\eta) = 1$, $\psi(x) = x$, $f(\xi) = \frac{u(\xi)}{(\sigma(\xi)-a)^{\varrho+1}}$, and $g(\eta) = \frac{v(\eta)}{(\sigma(\eta)-a)^{\mu+1}}$, in Theorem 3.1, then we get the inequality (1.5) proved by Saker et al. [6].*

Theorem 3.5. *Let $a_j, b_j \in \mathbb{T}$, $1 < \mu \le \varrho < \infty$, $s_j \in (1,\mu)$, $j = 1,2,...,m$, and ϕ be a nonnegative function on (a,d), $-\infty \le a < d \le \infty$ such that*

$$A\psi(\xi) \le \phi(\xi) \le B\psi(\xi), \tag{3.18}$$

holds for constants $0 < A \le B < \infty$, and ψ is a nonnegative and convex function. We define $u(\xi_1,...,\xi_m)$ and $v(\xi_1,...,\xi_m)$ as nonnegative weighted functions such that

$$v(\xi_1,...,\xi_m) = v_1(\xi_1)v_2(\xi_2)...v_m(\xi_m). \tag{3.19}$$

If (3.1) holds, then

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\phi(A_k h(\xi))]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)} \Delta\xi\right)^{\frac{1}{\varrho}} \\ \le\ & \frac{B}{A}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}} \dots \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}} A(s_1,...,s_m) \\ &\times\left(\int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \phi^{\mu}(h(\eta)) \frac{v(\eta_1,...,\eta_m)}{(\sigma(\eta_1)-a_1)\dots(\sigma(\eta_m)-a_m)} \Delta\eta\right)^{\frac{1}{\mu}}. \end{aligned} \tag{3.20}$$

Proof. From (3.18) and by applying (2.3), we see that

$$\begin{aligned} &\int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\phi(A_k h(\xi))]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)} \Delta\xi \\ \le\ & B^{\varrho} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\psi(A_k h(\xi))]^{\varrho} \frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)} \Delta\xi \\ =\ & B^{\varrho} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \left[\psi\left(\frac{1}{K(\xi)} \int_{a_1}^{\sigma(\xi_1)} \dots \int_{a_m}^{\sigma(\xi_m)} k(\xi,\eta)h(\eta)\Delta\eta\right)\right]^{\varrho} \end{aligned}$$

$$\times\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi$$
$$\leq B^{\varrho}\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\left(\frac{1}{K(\xi)}\int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)\psi(h(\eta))\Delta\eta\right)^{\varrho}$$
$$\times\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi$$
$$= B^{\varrho}\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\frac{u(\xi)}{K^{\varrho}(\xi)}J^{\varrho}(\xi)\frac{\Delta\xi}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}, \tag{3.21}$$

where

$$J(\xi)=\int_{a_1}^{\sigma(\xi_1)}\dots\int_{am}^{\sigma(\xi_m)}k(\xi,\eta)\psi(h(\eta))\Delta\eta. \tag{3.22}$$

Denote $\psi^{\mu}(h(\eta))\frac{v_1(\eta_1)..v_m(\eta_m)}{(\sigma(\eta_1)-a_1)..(\sigma(\eta_m)-a_m)}=\Psi(\eta)$ and substitute it into (3.22) to obtain that

$$\begin{aligned}J(\xi) &= \int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)\left[\Psi(\eta)\right]^{\frac{1}{\mu}}\left[v_1(\eta_1)\right]^{\frac{-1}{\mu}}\dots\left[v_m(\eta_m)\right]^{\frac{-1}{\mu}}\\
&\quad\times(\sigma(\eta_1)-a_1)^{\frac{1}{\mu}}\dots(\sigma(\eta_m)-a_m)^{\frac{1}{\mu}}\Delta\eta\\
&= \int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}k(\xi,\eta)\left[\Psi(\eta)\right]^{\frac{1}{\mu}}\left[V_1^{\sigma}(\eta_1)\right]^{\frac{s_1-1}{\mu}}\dots\left[V_m^{\sigma}(\eta_m)\right]^{\frac{s_m-1}{\mu}}\\
&\quad\times\left[V_1^{\sigma}(\eta_1)\right]^{\frac{1-s_1}{\mu}}\dots\left[V_m^{\sigma}(\eta_m)\right]^{\frac{1-s_m}{\mu}}\left[v_1(\eta_1)\right]^{\frac{-1}{\mu}}\dots\left[v_m(\eta_m)\right]^{\frac{-1}{\mu}}\\
&\quad\times(\sigma(\eta_1)-a_1)^{\frac{1}{\mu}}\dots(\sigma(\eta_m)-a_m)^{\frac{1}{\mu}}\Delta\eta,\end{aligned} \tag{3.23}$$

where $V_j(\eta_j)=\int_{a_j}^{\eta_j}\left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\tau_j\right)-a_j\right)^{\frac{1}{\mu-1}}\Delta\tau_j,\ j=1,2,....,m$. Applying (2.2) with $\mu>1$ and $\mu/(\mu-1)$ on (3.23), we see that

$$\begin{aligned}J(\xi) &\leq \left(\int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}k^{\mu}(\xi,\eta)\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\dots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\Delta\eta\right)^{\frac{1}{\mu}}\\
&\quad\times\left[\int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}\left[V_1^{\sigma}(\eta_1)\right]^{\frac{1-s_1}{\mu-1}}\dots\left[V_m^{\sigma}(\eta_m)\right]^{\frac{1-s_m}{\mu-1}}\right.\\
&\quad\left.\times\left[v_1(\eta_1)\right]^{\frac{-1}{\mu-1}}\dots\left[v_m(\eta_m)\right]^{\frac{-1}{\mu-1}}(\sigma(\eta_1)-a_1)^{\frac{1}{\mu-1}}\dots(\sigma(\eta_m)-a_m)^{\frac{1}{\mu-1}}\Delta\eta\right]^{\frac{\mu-1}{\mu}}.\end{aligned} \tag{3.24}$$

Substituting (3.24) into (3.21), we have

$$\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\left[\phi\left(A_k h\left(\xi\right)\right)\right]^{\varrho}\frac{u(\xi)}{(\sigma(\varkappa_1)-a_1)\dots(\sigma(\varkappa_m)-a_m)}\Delta\xi$$
$$\leq B^{\varrho}\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)K^{\varrho}(\xi)}$$
$$\times\left(\int_{a_1}^{\sigma(\xi_1)}\dots\int_{a_m}^{\sigma(\xi_m)}k^{\mu}(\xi,\eta)\Psi(\eta)\left[V_1^{\sigma}(\eta_1)\right]^{s_1-1}\dots\left[V_m^{\sigma}(\eta_m)\right]^{s_m-1}\Delta\eta\right)^{\frac{\varrho}{\mu}}$$

$$\times\left[\int_{a_1}^{\sigma(\xi_1)}...\int_{a_m}^{\sigma(\xi_m)}[V_1^{\sigma}(\eta_1)]^{\frac{1-s_1}{\mu-1}}...[V_m^{\sigma}(\eta_m)]^{\frac{1-s_m}{\mu-1}}\right.$$

$$\left.\times[v_1(\eta_1)]^{\frac{-1}{\mu-1}}...[v_m(\eta_m)]^{\frac{-1}{\mu-1}}(\sigma(\eta_1)-a_1)^{\frac{1}{\mu-1}}...(\sigma(\eta_m)-a_m)^{\frac{1}{\mu-1}}\Delta\eta\right]^{\frac{\varrho(\mu-1)}{\mu}}\Delta\xi. \tag{3.25}$$

Since

$$V_j(\eta_j)=\int_{a_j}^{\eta_j}\left[v_j(\tau_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\tau_j\right)-a_j\right)^{\frac{1}{\mu-1}}\Delta\tau_j,\ \ j=1,2,...,m,$$

then

$$V_j^{\Delta}(\eta_j)=\left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\eta_j\right)-a_j\right)^{\frac{1}{\mu-1}}>0. \tag{3.26}$$

Therefore, the function V_j is increasing. Applying the chain rule formula (2.1) on $\left[V_j(\eta_j)\right]^{1-(s_j-1/(\mu-1))}$, we obtain

$$\left[\left[V_j(\eta_j)\right]^{1-\frac{(s_j-1)}{\mu-1}}\right]^{\Delta}=\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta}=\left(\frac{\mu-s_j}{\mu-1}\right)\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}}V_j^{\Delta}(\eta_j), \tag{3.27}$$

where $\xi_j\in[\eta_j,\sigma(\eta_j)]$, $j=1,2,...,m$. Thus, by substituting (3.26) into (3.27), we see that

$$\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta}=\left(\frac{\mu-s_j}{\mu-1}\right)\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}}\left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\eta_j\right)-a_j\right)^{\frac{1}{\mu-1}}. \tag{3.28}$$

Since $\xi_j\leq\sigma(\eta_j)$ and V_j is increasing, we have

$$V_j\left(\xi_j\right)\leq V_j^{\sigma}(\eta_j).$$

Using the relation $1<s_j<\mu$, $j=1,2,..,m$, we see that

$$\left[V_j(\xi_j)\right]^{-\frac{(s_j-1)}{\mu-1}}\geq\left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}}. \tag{3.29}$$

Substituting (3.29) into (3.28), we have

$$\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta}\geq\left(\frac{\mu-s_j}{\mu-1}\right)\left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}}\left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\eta_j\right)-a_j\right)^{\frac{1}{\mu-1}},$$

and then

$$\int_{a_j}^{\sigma(\xi_j)}\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta}\Delta\eta_j\geq\left(\frac{\mu-s_j}{\mu-1}\right)\int_{a_j}^{\sigma(\xi_j)}\left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}}\left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\eta_j\right)-a_j\right)^{\frac{1}{\mu-1}}\Delta\eta_j.$$

Thus, we have (note $V_j(a_j)=0$) that

$$\int_{a_j}^{\sigma(\xi_j)}\left[V_j^{\sigma}(\eta_j)\right]^{-\frac{(s_j-1)}{\mu-1}}\left[v_j(\eta_j)\right]^{\frac{-1}{\mu-1}}\left(\sigma\left(\eta_j\right)-a_j\right)^{\frac{1}{\mu-1}}\Delta\eta_j$$

$$\leq\left(\frac{\mu-1}{\mu-s_j}\right)\int_{a_j}^{\sigma(\xi_j)}\left[\left[V_j(\eta_j)\right]^{\frac{\mu-s_j}{\mu-1}}\right]^{\Delta}\Delta\eta_j$$

$$= \left(\frac{\mu-1}{\mu-s_j}\right)\left[V_j^\sigma(\xi_j)\right]^{\frac{\mu-s_j}{\mu-1}}, \; j=1,2,...,m, \tag{3.30}$$

and then we have from (3.30) that

$$\begin{aligned}
& \int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} [V_1^\sigma(\eta_1)]^{\frac{1-s_1}{\mu-1}} ... [V_m^\sigma(\eta_m)]^{\frac{1-s_m}{\mu-1}} \\
& \times [v_1(\eta_1)]^{\frac{-1}{\mu-1}} ... [v_m(\eta_m)]^{\frac{-1}{\mu-1}} (\sigma(\eta_1)-a_1)^{\frac{1}{\mu-1}} ... (\sigma(\eta_m)-a_m)^{\frac{1}{\mu-1}} \Delta\eta \\
= & \left(\int_{a_1}^{\sigma(\xi_1)} [V_1^\sigma(\eta_1)]^{-\frac{(s_1-1)}{\mu-1}} [v_1(\eta_1)]^{\frac{-1}{\mu-1}} (\sigma(\eta_1)-a_1)^{\frac{1}{\mu-1}} \Delta\eta_1\right) \\
& \times ... \times \left(\int_{a_m}^{\sigma(\xi_m)} [V_m^\sigma(\eta_m)]^{-\frac{(s_m-1)}{\mu-1}} [v_m(\eta_m)]^{\frac{-1}{\mu-1}} (\sigma(\eta_m)-a_m)^{\frac{1}{\mu-1}} \Delta\eta_m\right) \\
\leq & \left(\frac{\mu-1}{\mu-s_1}\right) ... \left(\frac{\mu-1}{\mu-s_m}\right) [V_1^\sigma(\xi_1)]^{\frac{\mu-s_1}{\mu-1}} ... [V_m^\sigma(\xi_m)]^{\frac{\mu-s_m}{\mu-1}}.
\end{aligned}$$

Substituting into (3.25), we see

$$\begin{aligned}
& \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} [\phi(A_k h(\xi))]^\varrho \frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)} \Delta\xi \\
\leq & B^\varrho \left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}} ... \left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}} \\
& \times \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left(\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k^\mu(\xi,\eta)\Psi(\eta) [V_1^\sigma(\eta_1)]^{s_1-1} ... [V_m^\sigma(\eta_m)]^{s_m-1} \Delta\eta\right)^{\frac{\varrho}{\mu}} \\
& \times [V_1^\sigma(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}} ... [V_m^\sigma(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}} \frac{u(\xi)}{K^\varrho(\xi)(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)} \Delta\xi.
\end{aligned} \tag{3.31}$$

Applying (2.4) on the term

$$\begin{aligned}
& \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left(\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k^\mu(\xi,\eta)\Psi(\eta) [V_1^\sigma(\eta_1)]^{s_1-1} ... [V_m^\sigma(\eta_m)]^{s_m-1} \Delta\eta\right)^{\frac{\varrho}{\mu}} \\
& \times [V_1^\sigma(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}} ... [V_m^\sigma(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}} \frac{u(\xi)}{K^\varrho(\xi)(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)} \Delta\xi,
\end{aligned}$$

with $\varrho/\mu > 1$, we observe

$$\begin{aligned}
& \left[\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \left(\int_{a_1}^{\sigma(\xi_1)} ... \int_{a_m}^{\sigma(\xi_m)} k^\mu(\xi,\eta)\Psi(\eta) [V_1^\sigma(\eta_1)]^{s_1-1} ... [V_m^\sigma(\eta_m)]^{s_m-1} \Delta\eta\right)^{\frac{\varrho}{\mu}}\right. \\
& \left.\times [V_1^\sigma(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}} ... [V_m^\sigma(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}} \frac{u(\xi)}{K^\varrho(\xi)(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)} \Delta\xi\right]^{\frac{\mu}{\varrho}} \\
\leq & \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \Psi(\eta) [V_1^\sigma(\eta_1)]^{s_1-1} ... [V_m^\sigma(\eta_m)]^{s_m-1} \\
& \times \left[\int_{\eta_1}^{b_1} ... \int_{\eta_m}^{b_m} k^\varrho(\xi,\eta) [V_1^\sigma(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}} ... [V_m^\sigma(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}}\right.
\end{aligned}$$

$$\times \frac{u(\xi)}{K^{\varrho}(\xi)\,(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi\Bigg]^{\frac{\mu}{\varrho}}\Delta\eta. \tag{3.32}$$

Substituting (3.32) into (3.31), we obtain

$$\begin{aligned}
&\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}[\phi\,(A_k h\,(\xi))]^{\varrho}\,\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi\\
&\le B^{\varrho}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}\Bigg[\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\Psi(\eta)\,[V_1^{\sigma}(\eta_1)]^{s_1-1}\dots[V_m^{\sigma}(\eta_m)]^{s_m-1}\\
&\times\Bigg(\int_{\eta_1}^{b_1}\dots\int_{\eta_m}^{b_m}k^{\varrho}(\xi,\eta)\,[V_1^{\sigma}(\xi_1)]^{\frac{\varrho(\mu-s_1)}{\mu}}\dots[V_m^{\sigma}(\xi_m)]^{\frac{\varrho(\mu-s_m)}{\mu}}\\
&\times\frac{u(\xi)}{K^{\varrho}(\xi)\,(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi\Bigg)^{\frac{\mu}{\varrho}}\Delta\eta\Bigg]^{\frac{\varrho}{\mu}}.
\end{aligned} \tag{3.33}$$

Using the assumptions (3.1), (3.18), and (3.19), the inequality (3.33) becomes

$$\begin{aligned}
&\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}[\phi\,(A_k h\,(\xi))]^{\varrho}\,\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi\\
\le\;& B^{\varrho}A^{\varrho}(s_1,\dots,s_m)\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}\left[\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\Psi(\eta)\Delta\eta\right]^{\frac{\varrho}{\mu}}\\
=\;& B^{\varrho}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}A^{\varrho}(s_1,\dots,s_m)\\
&\times\left[\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\psi^{\mu}(h(\eta))\frac{v_1(\eta_1)\dots v_m(\eta_m)}{(\sigma(\eta_1)-a_1)\dots(\sigma(\eta_m)-a_m)}\Delta\eta\right]^{\frac{\varrho}{\mu}}\\
\le\;& \frac{B^{\varrho}}{A^{\varrho}}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}A^{\varrho}(s_1,\dots,s_m)\\
&\times\left[\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\phi^{\mu}(h(\eta))\frac{v_1(\eta_1)\dots v_m(\eta_m)}{(\sigma(\eta_1)-a_1)\dots(\sigma(\eta_m)-a_m)}\Delta\eta\right]^{\frac{\varrho}{\mu}}\\
=\;& \frac{B^{\varrho}}{A^{\varrho}}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\varrho(\mu-1)}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\varrho(\mu-1)}{\mu}}A^{\varrho}(s_1,\dots,s_m)\\
&\times\left[\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\phi^{\mu}(h(\eta))\frac{v(\eta_1,..,\eta_m)}{(\sigma(\eta_1)-a_1)\dots(\sigma(\eta_m)-a_m)}\Delta\eta\right]^{\frac{\varrho}{\mu}},
\end{aligned} \tag{3.34}$$

and then

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}[\phi\,(A_k h\,(\xi))]^{\varrho}\,\frac{u(\xi)}{(\sigma(\xi_1)-a_1)\dots(\sigma(\xi_m)-a_m)}\Delta\xi\right)^{\frac{1}{\varrho}}\\
\le\;& \frac{B}{A}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}}\dots\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}}A(s_1,\dots,s_m)
\end{aligned}$$

$$\times\left[\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\phi^{\mu}(h(\eta))\frac{v(\eta_1,..,\eta_m)}{(\sigma(\eta_1)-a_1)...(\sigma(\eta_m)-a_m)}\Delta\eta\right]^{\frac{1}{\mu}},$$

which is (3.20). □

Remark 3.6. *If $\mathbb{T}=\mathbb{R}$, then (3.20) gives the inequality (1.4) proved by Oguntuase and Durojaye [5].*

Remark 3.7. *If $A=B=1$ in Theorem 3.5, then we get Theorem 3.1.*

Remark 3.8. *It is obvious that we can use another technique to prove the inequality (3.20) in Theorem 3.5 by using Theorem 3.1 with (3.1) and (3.18) as follows:*

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\left[\phi\left(A_k h\left(\xi\right)\right)\right]^{\varrho}\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\right)^{\frac{1}{\varrho}}\\
\leq\ & B\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\left[\psi\left(A_k h\left(\xi\right)\right)\right]^{\varrho}\frac{u(\xi)}{(\sigma(\xi_1)-a_1)...(\sigma(\xi_m)-a_m)}\Delta\xi\right)^{\frac{1}{\varrho}}\\
\leq\ & B\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}}...\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}}A(s_1,...,s_m)\\
&\times\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\psi^{\mu}(h(\eta))\frac{v(\eta_1,...,\eta_m)}{(\sigma(\eta_1)-a_1)...(\sigma(\eta_m)-a_m)}\Delta\eta\right)^{\frac{1}{\mu}}\\
\leq\ & \frac{B}{A}\left(\frac{\mu-1}{\mu-s_1}\right)^{\frac{\mu-1}{\mu}}...\left(\frac{\mu-1}{\mu-s_m}\right)^{\frac{\mu-1}{\mu}}A(s_1,...,s_m)\\
&\times\left(\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\phi^{\mu}(h(\eta))\frac{v(\eta_1,...,\eta_m)}{(\sigma(\eta_1)-a_1)...(\sigma(\eta_m)-a_m)}\Delta\eta\right)^{\frac{1}{\mu}}.
\end{aligned}$$

4. Conclusions

In this work, new multidimensional Hardy-type inequalities with general kernels have been developed in the context of time scales, a mathematical theory that unifies continuous and discrete analysis. These inequalities were proven using the n-dimensional time scale versions of Holder's inequality, Jensen's inequality, and Minkowski's inequality. Special cases were derived for $\mathbb{T}=\mathbb{N}$, which are essentially novel contributions to the field. These results extend the applicability of Hardy-type inequalities, providing new insights and tools that bridge discrete and continuous mathematical analysis.

Author contributions

Ghada AlNemer: Writing-review editing and Funding; M. Zakarya: Writing-review editing and Funding; H. M. Rezk: Investigation, Software, Supervision, Writing-original draft; A. I. Saied: Investigation, Software, Supervision, Writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflict of interest in this paper.

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