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Research article

On Kirchhoff type problems with singular nonlinearity in closed manifolds

Nanbo Chen[1,2], **Honghong Liang**[1] **and Xiaochun Liu**[3,*]

[1] School of Mathematics and Computing Science, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin, 541002, China

[2] Center for Applied Mathematics of Guangxi (GUET), Guilin, 541002, China

[3] School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China

* **Correspondence:** Email: xcliu@whu.edu.cn.

Abstract: This paper dealt with a class of Kirchhoff type equations involving singular nonlinearity in a closed Riemannian manifold (M, g) of dimension $n \geq 3$. Existence and uniqueness of a positive weak solution were obtained under certain assumptions with the help of the variation methods and some analysis techniques.

Keywords: Kirchhoff equations; singularity; critical exponent; closed manifolds; variational methods
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1. Introduction

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ and $h \in L^\infty(M)$. Let $\mathcal{L}_g$ be the stationary Schrödinger operator given by

$$\mathcal{L}_g = \Delta_g + h,$$

where $\Delta_g = -\operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator with respect to g and ∇_g is the gradient operator. We consider the following Kirchhoff type equations involving singular nonlinearity:

$$\left(a + b\int_M \left(|\nabla_g u|_g^2 + hu^2\right) dv_g\right)\mathcal{L}_g u = f(x)u^{-\gamma} - \lambda u^p \tag{$\mathcal{K}_g$}$$

in M, where $a, b, \lambda \geq 0$, $0 < \gamma \leq 1$, $0 < p \leq 2^* - 1$, $f(x)$ is a positive function in M, and dv_g is the canonical volume element in (M, g). Here, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of Sobolev spaces $H^1(M)$ into Lebesgue spaces.

The Kirchhoff equation was proposed by Kirchhoff [1] in 1883, which is an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. Almost one century later, Jacques Louis Lions [2] put forward an abstract framework for these kinds of problems and, after that, the Kirchhoff type problems began to receive significant attention. The problems of Kirchhoff-type are often referred to as being nonlocal because of the appearance of the integration term $\int_\Omega |\nabla u|^2 dx$, which implies that the problem is no longer a pointwise equation. Numerous intriguing studies on such problems can be found in the literature. We refer the reader to the works by Arosio-Panizzi [3], Alves-Corrêa-Figueiredo [4], Fang-Liu [5], Fiscella [6], He [7], Sun-Tan [8] and Naimen [9, 10], and Faraci-Silva [11], and we quote only few of them.

In the Euclidean setting, Liu and Sun [12] investigated the existence of solutions for the following problem with singular and superlinear terms:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x)u^{-\gamma} + \lambda w(x)\frac{u^p}{|x|^s}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $0 < \gamma < 1$, $0 \le s < 1$, $3 < p < 5 - 2s$. They obtained two positive solutions with the help of the Nehari manifold.

Moreover, Lei et al. [13] considered the Kirchhoff equations with the nonlinearity containing both singularity and critical exponents:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u^{-\gamma} + u^5, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $\lambda > 0$, and $\gamma \in (0, 1)$. By the variational and perturbation methods, they also obtained two positive solutions.

Furthermore, Duan et al. [14] studied the p-Kirchhoff type problem with singularity:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^p dx\right)\Delta_p u = f(x)u^{-\gamma} - \lambda u^q, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \ge 3$. Here, $a, b \ge 0$ with $a + b > 0$, $0 < \gamma < 1$, $\lambda \ge 0$, $0 < q \le p^* - 1$, and f is a positive function. Under appropriate conditions, it is shown that problem (1.1) has a unique positive solution by the variational method and some analysis techniques.

It should be noted that the aforementioned results hold true when $0 < \gamma < 1$. When $\gamma = 1$, Wang and Yan [15] considered a class of Kirchhoff type equations with singularity and nonlinearity:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x)u^{-1} - \mu u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \ge 3$, a, b, μ are real numbers, $1 < p < 2^* - 1$, and $f(x) \in L^2(\Omega)$. Using the approximation method, they proved that problem (1.2) has a unique positive solution.

In the realm of Riemannian manifolds, nonlinear analysis has experienced significant development in recent decades. Some recent research works can be found in [16–19] and the references therein. For Kirchhoff equations and stationary Kirchhoff systems, we refer the reader to the works by Hebey [20–

22], Hebey-Thizy [23,24], and the recent paper of Bai et al. [25]. They discussed existence of solutions, compactness, and stability properties of the critical Kirchhoff equations in closed manifolds. It is worth noting that there are limited results available for Kirchhoff equations with singularity. Motivated by the above papers, we investigate the existence and uniqueness of the solution to problem ($\mathcal{K}_g$). To the best of our knowledge, no previous studies have explored the existence of solutions for problem ($\mathcal{K}_g$) in Riemannian manifolds. Our work somehow extends the main results in [15,26] from Euclidean case to any closed Riemannian manifold.

Our main results can be stated as follows. We first consider the case when $0 < \gamma < 1$.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Assume that $a, b \geq 0$ with $a + b > 0$, $\lambda \geq 0$, $0 < \gamma < 1$, $0 < p \leq 2^* - 1$, and $f \in L^{\frac{2^*}{2^*+\gamma-1}}(M)$ satisfying $f > 0$. Let $h \in L^\infty(M)$ be such that $\mathcal{L}_g$ is positive. Then, problem ($\mathcal{K}_g$) possesses a unique positive weak solution in $H^1(M)$. Moreover, this solution is a global minimum solution.*

It should be noted that Theorem 1.1 encompasses the critical case. Additionally, we give the case when $\gamma = 1$ below.

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Assume that $a > 0$, $b \geq 0$, $\lambda \geq 0$, $\gamma = 1$, $1 < p < 2^* - 1$, and $f \in L^2(M)$ is positive. Let $h \in L^\infty(M)$ be such that $\mathcal{L}_g$ is positive. Then, problem ($\mathcal{K}_g$) has a unique positive weak solution in $H^1(M)$.*

Remark 1.3. In particular, when $a = 1, b = 0$, problem ($\mathcal{K}_g$) reduces to the following semilinear singular problem:

$$\Delta_g u + hu = f(x)u^{-\gamma} - \lambda u^p \quad \text{in} \quad M.$$

We mention that Theorems 1.1 and 1.2 are also ture. Moreover, when $\lambda = 0$, the counterpart results for the singular boundary value problem in $\mathbb{R}^n$ can be found in [27,28].

Remark 1.4. The energy functional associated with problem ($\mathcal{K}_g$) fails to be Fréchet differentiable because of the presence of the singular term. Therefore, the direct application of critical point theory to establish the existence of solutions is not viable. To overcome the difficulties caused by the nonlocal term and the singularity, we will follow some ideas similar to those developed in [26,28,29].

The paper is organized as follows. In Section 2, we give some definitions related to the Sobolev space and properties of energy functionals. In Section 3, we establish a series of lemmas and then give the proof of Theorem 1.1. Finally, in Section 4, we present several lemmas, followed by the proof of Theorem 1.2.

2. Preliminaries

In this section, we provide several main definitions and properties of functionals that will be useful for our subsequent analysis. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with a metric g. Given $1 \leq p < \infty$, we denote by $L^p(M)$ the usual Lebesgue space of p-th power integrable functions with the standard L^p-norm $\|u\|_{L^p}^p = \int_M |u|^p dv_g$. The Sobolev space $H^1(M)$ is defined as the completion of $C^\infty(M)$ with respect to the Sobolev norm given by

$$\|u\|_{H^1} = \left(\int_M |\nabla_g u|_g^2 dv_g + \int_M u^2 dv_g \right)^{\frac{1}{2}}, \tag{2.1}$$

where ∇_g is the gradient operator and dv_g is the canonical volume element in (M, g). Precisely, in local coordinates $\{x^i\}$, we have $dv_g = \sqrt{|g|}dx^1 \ldots dx^n$, $\nabla u = g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}$, and

$$\Delta_g u = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x^j}\right),$$

where (g_{ij}) is the metric matrix, (g^{ij}) is the inverse matrix of (g_{ij}), and $|g| = \det(g_{ij})$ is the determinant of g. Here, the Einstein's summation convention is adopted. With the norm (2.1), $H^1(M)$ becomes a Hilbert space with the inner product

$$\langle u, v\rangle = \int_M \left(\langle\nabla_g u, \nabla_g v\rangle_g + huv\right) dv_g,$$

where $\langle\nabla_g u, \nabla_g v\rangle_g$ is the pointwise scalar product of $\nabla_g u$ and $\nabla_g v$ with respect to g. We assume that $\mathcal{L}_g$ is positive, where by positive we mean that its minimum eigenvalue is positive. In other words, we assume that for $u \in H^1(M)$,

$$\lambda_1 = \inf_{\int_M u^2 dv_g = 1} \int_M \left(|\nabla_g u|_g^2 + hu^2\right) dv_g > 0. \tag{2.2}$$

Clearly, this happens if $h(x) \in C^0(M)$ with $h > 0$. Consequently, we get a natural equivalent norm $\|\cdot\|$ on H^1 given by

$$\|u\| = \left(\int_M \left(|\nabla_g u|_g^2 + hu^2\right) dv_g\right)^{\frac{1}{2}} \quad \text{for all } u \in H^1(M). \tag{2.3}$$

We denote by the first eigenfunction φ_1 with $\Delta_g\varphi_1 + h\varphi_1 = \lambda\varphi_1$ in M, $\|\varphi_1\| = 1$. By the maximum principle and elliptic regularity, we know that $\varphi_1 > 0$ in M and $\varphi_1 \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$ (see, for instance, [30] and references therein).

By the Rellich-Kondrachov theorem, since $p < 2^*$, $H_1(M)$ embeds compactly into $L^p(M)$. For $p = 2^*$, let $S = S(M, g, h)$ be the sharp Sobolev constant of (M, g) associated to $\|\cdot\|$, that is, the largest positive constant S such that the Sobolev inequality

$$S\|u\|_{L^{2^*}}^2 \leq \|u\|^2 \tag{2.4}$$

holds true for all $u \in H^1(M)$.

The energy functional corresponding to problem $(\mathcal{K}_g)$ is defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{1+p}\int_M |u|^{1+p} dv_g - \frac{1}{1-\gamma}\int_M f(x)|u|^{1-\gamma} dv_g,$$

for $u \in H^1(M)$ and $0 < \gamma < 1$. Note that, the functional I is only a continuous functional on $H^1(M)$ because of the presence of the singular term. In general, we say that a function u is a positive weak solution of problem $(\mathcal{K}_g)$ if $u \in H^1(M)$ such that $u > 0$ a.e. in M and

$$(a + b\|u\|^2)\int_M (\langle\nabla_g u, \nabla_g\varphi\rangle_g + hu\varphi) dv_g + \lambda\int_M u^p\varphi dv_g - \int_M f(x)u^{-\gamma}\varphi dv_g = 0 \tag{2.5}$$

for all $\varphi \in H^1(M)$.

3. Existence of solutions for $0 < \gamma < 1$

In this section, we consider the existence and uniqueness of positive weak solutions to equation $(\mathcal{K}_g)$ for $0 < \gamma < 1$. We first give some useful lemmas, which will be used in the proof of Theorem 1.1.

Lemma 3.1. *The functional I is coercive and bounded from below on $H^1(M)$.*

Proof. By Hölder inequality and (2.4), we have

$$\begin{aligned}\int_M f(x)|u|^{1-\gamma}dv_g &\leqslant \left(\int_M |f|^{\frac{2^*}{2^*+\gamma-1}}dv_g\right)^{\frac{2^*+\gamma-1}{2^*}}\left(\int_M |u|^{(1-\gamma)\cdot\frac{2^*}{1-\gamma}}dv_g\right)^{\frac{1-\gamma}{2^*}}\\ &= \|f\|_{L^{\frac{2^*}{2^*+\gamma-1}}}\cdot\|u\|_{L^{2^*}}^{1-\gamma}\\ &\leqslant \|f\|_{L^{\frac{2^*}{2^*+\gamma-1}}}\cdot S^{\frac{\gamma-1}{2}}\cdot\|u\|^{1-\gamma}.\end{aligned}$$

Notice that $\lambda \geq 0$, hence

$$\begin{aligned}I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{1+p}\int_M |u|^{1+p}dv_g - \frac{1}{1-\gamma}\int_M f(x)|u|^{1-\gamma}dv_g\\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{1-\gamma}\int_M f(x)|u|^{1-\gamma}dv_g\\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{1-\gamma}\|f\|_{L^{\frac{2^*}{2^*+\gamma-1}}}S^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma},\end{aligned} \tag{3.1}$$

which implies that I is coercive and bounded from below on $H^1(M)$. □

Let m be given by

$$m = \inf\{I(u)|u \in H^1(M)\}.$$

According to Lemma 3.1, m is well-defined. Moreover, since $0 < \gamma < 1$ and $f(x) > 0$ in M, we can easily get $m < 0$.

Lemma 3.2. *Given the assumptions of Theorem 1.1, the functional I attains the global minimizer in $H^1(M)$, i.e., there exists $u_* \in H^1(M)$ such that $I(u_*) = m$.*

Proof. From the definition of the number m, there exists a minimizing sequence $\{u_n\} \subset H^1(M)$ such that

$$\lim_{n\to+\infty} I(u_n) = m < 0.$$

By standard properties of Sobolev spaces on manifolds, if $u \in H^1(M)$, then $|u| \in H^1(M)$, and $|\nabla_g|u||_g = |\nabla_g u|_g$ a.e. Up to changing u_n into $|u_n|$, we may assume that $u_n \geq 0$ in M. By Lemma 3.1, I is coercive, so that $\{u_n\}$ is bounded in $H^1(M)$. Being bounded, we get that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } H^1(M),\\ u_n \to u_* & \text{strongly in } L^s(M), \text{where } 1 \leq s < 2^*,\\ u_n \to u_* & \text{a.e. in } M,\end{cases} \tag{3.2}$$

as $n \to +\infty$ for some $u_* \in H^1(M)$. Next, we are going to prove that $u_n \to u_*$ as $n \to +\infty$ in $H^1(M)$.

By Vitali's theorem (see [26]), we have

$$\int_M f(x)|u_n|^{1-\gamma}dv_g = \int_M f(x)|u_*|^{1-\gamma}dv_g + o(1). \tag{3.3}$$

Let $\omega_n = u_n - u_*$. From the weak convergence of $\{u_n\}$ in $H^1(M)$ and Brézis-Lieb's lemma [31], it follows that

$$\|u_n\|^2 = \|\omega_n\|^2 + \|u_*\|^2 + o(1), \tag{3.4}$$

$$\|u_n\|^4 = \|\omega_n\|^4 + \|u_*\|^4 + 2\|\omega_n\|^2\|u_*\|^2 + o(1), \tag{3.5}$$

$$\int_M |u_n|^{1+p}dv_g = \int_M |\omega_n|^{1+p}dv_g + \int_M |u_*|^{1+p}dv_g + o(1), \tag{3.6}$$

where $o(1)$ is an infinitesimal as $n \to \infty$. Hence, in the case that $0 < p \le 2^* - 1$, from (3.4)–(3.6), we deduce that

$$\begin{aligned} m &= \lim_{n\to+\infty}\left(\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{\lambda}{1+p}\int_M |u_n|^{1+p}dv_g - \frac{1}{1-\gamma}\int_M f(x)|u_n|^{1-\gamma}dv_g\right) \\ &= I(u_*) + \lim_{n\to+\infty}\left(\frac{a}{2}\|\omega_n\|^2 + \frac{b}{4}\|\omega_n\|^4 + \frac{b}{2}\|\omega_n\|^2\|u_*\|^2 + \frac{\lambda}{1+p}\int_M |\omega_n|^{1+p}dv_g\right) \\ &\ge I(u_*), \end{aligned}$$

which implies that $I(u_*) = m$ and $\lim_{n\to+\infty}\|\omega_n\| = 0$. This completes the proof of Lemma 3.2. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 We divide the proof into three steps.

Step 1. We show that $u_* > 0$ a.e in M. It follows from Lemma 3.2 that $I(u_*) = m < 0$, and then $u_* \not\equiv 0$ in M. Let $\phi \in H^1(M)$ and $\phi \ge 0$. Since u_* is a global minimizer in $H^1(M)$, for $t > 0$ we have

$$\begin{aligned} 0 \le& \frac{1}{t}(I(u_* + t\phi) - I(u_*)) \\ =& \frac{a}{2t}\left(\|u_* + t\phi\|^2 - \|u_*\|^2\right) + \frac{b}{4t}\left(\|u_* + t\phi\|^4 - \|u_*\|^4\right) \\ &+ \frac{\lambda}{(1+p)t}\int_M \left((u_* + t\phi)^{1+p} - u_*^{1+p}\right)dv_g - \frac{1}{(1-\gamma)t}\int_M f(x)\left((u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}\right)dv_g. \end{aligned} \tag{3.7}$$

Obviously, one gets

$$\frac{1}{1+p}\int_M \frac{(u_* + t\phi)^{1+p} - u_*^{1+p}}{t}dv_g = \int_M (u_* + \theta t\phi)^p \phi dv_g$$

and

$$\frac{1}{(1-\gamma)}\int_M \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t}f(x)dv_g = \int_M f(x)(u_* + \zeta t\phi)^{-\gamma}\phi dv_g,$$

where $0 < \theta, \zeta < 1$. Moreover,

$$(u_* + \theta t\phi)^p\phi \to u_*^p\phi \ \text{ and } \ (u_* + \zeta t\phi)^{-\gamma}\phi \to u_*^{-\gamma}\phi \ \text{ for a.e } x \in M$$

as $t \to 0^+$. We note that

$$f(x)(u_* + \zeta t\phi)^{-\gamma}\phi \geq 0 \text{ for all } x \in M,$$

and, thus, by Fatou's Lemma, we conclude that

$$\begin{aligned}
&\limsup_{t\to 0^+} \frac{1}{1-\gamma} \int_M \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} f(x) dv_g \\
&\geq \liminf_{t\to 0^+} \frac{1}{1-\gamma} \int_M \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} f(x) dv_g \\
&= \liminf_{t\to 0^+} \int_M f(x)(u_* + \zeta t\phi)^{-\gamma}\phi dv_g \\
&\geq \int_M f(x) u_*^{-\gamma}\phi dv_g.
\end{aligned} \tag{3.8}$$

Moreover, by Lebesgue's dominated convergence theorem, we get

$$\lim_{t\to 0^+} \frac{\lambda}{1+p} \int_M \frac{(u_* + t\phi)^{1+p} - u_*^{1+p}}{t} dv_g = \int_M u_*^p \phi dv_g. \tag{3.9}$$

Taking the lower limit in (3.7), we obtain

$$(a + b\|u_*\|^2) \int_M \left(\langle \nabla_g u_*, \nabla_g \phi\rangle_g + h u_* \phi\right) dv_g + \lambda \int_M u_*^p \phi dv_g - \int_M f(x) u_*^{-\gamma}\phi dv_g \geq 0 \tag{3.10}$$

for all $\phi \in H^1(M)$ with $\phi \geq 0$. Let φ_1 be the first eigenfunction of the operator $\mathcal{L}_g$ with $\varphi_1 > 0$ and $\|\varphi_1\| = 1$. Choosing, in particular, $\phi = \varphi_1$ in (3.10), we have

$$\int_M f(x) u_*^{-\gamma}\varphi_1 dv_g \leq (a + b\|u_*\|^2) \int_M (\langle \nabla_g, u_* \nabla_g \varphi_1\rangle_g + h u_* \varphi_1) dv_g + \lambda \int_M \varphi_1 u_*^p dv_g < \infty,$$

which implies that $u_* > 0$ a.e. in M.

Step 2. We prove that u_* is a solution of problem $(\mathcal{K}_g)$. To this end, we need to check that u_* satisfies (2.5). We claim that the inequality (3.10) holds true for all $\phi \in H^1(M)$. Define $\varphi : [-\delta, \delta] \to \mathbb{R}$ by $\varphi(t) = I((1+t)u_*)$, then φ attains its minimum at $t = 0$. Thus, we get

$$\varphi'(t)|_{t=0} = a\|u_*\|^2 + b\|u_*\|^4 + \lambda \int_M u_*^{1+p} dv_g - \int_M f(x) u_*^{1-\gamma} dv_g = 0. \tag{3.11}$$

Let $\phi \in H^1(M)$ and $\varepsilon > 0$. We define

$$\psi = (u_* + \varepsilon\phi)^+ \in H^1(M),$$

where $(u_* + \varepsilon\phi)^+ = \max\{0, u_* + \varepsilon\phi\}$. Using (3.11) and inserting ψ into (3.10), we deduced that

$$\begin{aligned}
0 &\leq \left(a + b\|u_*\|^2\right) \int_M \left(\langle \nabla_g u_*, \nabla_g \psi\rangle_g + h u_* \psi\right) dv_g + \lambda \int_M u_*^p \psi dv_g - \int_M f(x) u_*^{-\gamma}\psi dv_g \\
&= \int_{\{u_* + \varepsilon\phi > 0\}} \Big[\left(a + b\|u_*\|^2\right)\left(\langle \nabla_g u_*, \nabla_g (u_* + \varepsilon\phi)\rangle_g + h u_* (u_* + \varepsilon\phi)\right) + \lambda u_*^p (u_* + \varepsilon\phi) - f(x) u_*^{-\gamma}(u_* + \varepsilon\phi)\Big] dv_g
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \Big[\Big(a + b\|u_*\|^2\Big)\Big(\langle\nabla_g u_*, \nabla_g(u_* + \varepsilon\phi)\rangle_g + hu_*(u_* + \varepsilon\phi)\Big) + \lambda u_*^p(u_* + \varepsilon\phi) - f(x)u_*^{-\gamma}(u_* + \varepsilon\phi)\Big]\, dv_g \\
&\quad - \int\limits_{\{u_*+\varepsilon\phi\le 0\}} \Big[\Big(a + b\|u_*\|^2\Big)\Big(\langle\nabla_g u_*, \nabla_g(u_* + \varepsilon\phi)\rangle_g + hu_*(u_* + \varepsilon\phi)\Big) + \lambda u_*^p(u_* + \varepsilon\phi) - f(x)u_*^{-\gamma}(u_* + \varepsilon\phi)\Big]\, dv_g \\
&\le \varepsilon \int_M \Big[\Big(a + b\|u_*\|^2\Big)\Big(\langle\nabla_g u_*, \nabla_g\phi\rangle_g + hu_*\phi\Big) + \lambda u_*^p\phi - f(x)u_*^{-\gamma}\phi\Big]\, dv_g \\
&\quad - \varepsilon \int\limits_{\{u_*+\varepsilon\phi\le 0\}} \Big[\Big(a + b\|u_*\|^2\Big)\Big(\langle\nabla_g u_*, \nabla_g\phi\rangle_g + hu_*\phi\Big) + \lambda u_*^p\phi\Big]\, dv_g + \varepsilon^2\Big(a + b\|u_*\|^2\Big)\|h\|_\infty \int\limits_M \phi^2 dv_g.
\end{aligned} \tag{3.12}
$$

Since the measure of the domain of integration $\{u_* + \varepsilon\phi \le 0\}$ tends to zero as $\varepsilon \to 0^+$, it follows that

$$
\lim_{\varepsilon\to 0} \int\limits_{\{u_*+\varepsilon\phi\le 0\}} (\langle\nabla_g u_*, \nabla_g\phi\rangle_g + hu_*\phi)dv_g = 0
$$

and

$$
\lim_{\varepsilon\to 0} \int\limits_{\{u_*+\varepsilon\phi\le 0\}} u_*^p\phi dv_g = 0.
$$

Hence, dividing (3.12) by ε and letting $\varepsilon \to 0^+$, one has

$$
(a + b\|u_*\|^2)\int_M (\langle\nabla_g u_*, \nabla_g\phi\rangle_g + hu_*\phi)dv_g + \lambda\int_M u_*^p\phi dv_g - \int_M f(x)u_*^{-\gamma}\phi dv_g \ge 0.
$$

By the arbitrariness of ϕ, the above inequality also holds equally well for $-\phi$. Thus, u_* is a solution of problem $(\mathcal{K}_g)$. Furthermore, by Lemma 3.2, one has

$$
I(u_*) = \inf_{u\in H^1(M)} I(u),
$$

which means that u_* is a positive global minimizer solution.

Step 3. We prove the uniqueness of solutions of problem $(\mathcal{K}_g)$. Suppose $v_* \in H^1(M)$ is also a solution of problem $(\mathcal{K}_g)$. Then, u_* and v_* satisfy (2.5). Taking $\varphi = u_* - v_*$ in (2.5), we get

$$
\begin{aligned}
(a + b\|u_*\|^2)&\int_M (\langle\nabla_g u_*, \nabla_g(u_* - v_*)\rangle_g + hu_*(u_* - v_*))dv_g + \lambda\int_M u_*^p(u_* - v_*)dv_g \\
&- \int_M f(x)u_*^{-\gamma}(u_* - v_*)dv_g = 0,
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
(a + b\|v_*\|^2)&\int_M (\langle\nabla_g v_*, \nabla_g(u_* - v_*)\rangle_g + hv_*(u_* - v_*))dv_g + \lambda\int_M v_*^p(u_* - v_*)dv_g \\
&- \int_M f(x)v_*^{-\gamma}(u_* - v_*)dv_g = 0.
\end{aligned} \tag{3.14}
$$

Denote

$$J(u_*, v_*) = \|u_*\|^4 + \|v_*\|^4 - (\|u_*\|^2 + \|v_*\|^2) \int_M (\langle \nabla_g u_*, \nabla_g v_* \rangle_g + h u_* v_*) dv_g.$$

Subtracting (3.13) from (3.14), we obtain

$$a\|u_* - v_*\|^2 + bJ(u_*, v_*) + \lambda \int_M (u_*^p - v_*^p)(u_* - v_*) dv_g - \int_M f(x)(u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) dv_g = 0. \quad (3.15)$$

Using the Cauchy-Schwarz inequality, we get

$$\int_M (\langle \nabla_g u_*, \nabla_g v_* \rangle_g + h u_* v_*) dv_g \leq \|u_*\|\|v_*\| \leq \frac{1}{2}(\|u_*\|^2 + \|v_*\|^2). \quad (3.16)$$

This implies that

$$\begin{aligned} J(u_*, v_*) =& \|u_*\|^4 + \|v_*\|^4 - \frac{1}{2}\left(\|u_*\|^2 + \|v_*\|^2\right)^2 \\ =& \frac{1}{2}\left(\|u_*\|^2 - \|v_*\|^2\right)^2 \geq 0. \end{aligned} \quad (3.17)$$

On the other hand, for $0 < \gamma < 1$ and $p > 0$, we have

$$(m^{-\gamma} - n^{-\gamma})(m - n) \leq 0 \ \text{ and } \ (m^p - n^p)(m - n) \geq 0 \text{ for all } m, n > 0,$$

which thus implies

$$\int_M f(x)(u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) dv_g \leq 0 \ \text{ and } \ \int_M (u_*^p - v_*^p)(u_* - v_*) dv_g \geq 0. \quad (3.18)$$

Hence, if $a > 0$, it follows from (3.15) that $a\|u_* - v_*\|^2 \leq 0$ and then $\|u_* - v_*\|^2 = 0$. If $a = 0, b > 0$, inequalities (3.15) and (3.17) imply that $J(u_*, v_*) = 0$ and $\|u_*\|^2 = \|v_*\|^2$. Consequently,

$$\begin{aligned} J(u_*, v_*) &= \|u_*\|^2 \left(2\|u_*\|^2 - 2\int_M (\langle \nabla_g u_*, \nabla_g v_* \rangle_g + h u_* v_*) dv_g\right) \\ &= \|u_*\|^2 \left(\int_M (|\nabla_g u_*|_g^2 + h u_*^2) dv_g - 2\int_M (\langle \nabla_g u_*, \nabla_g v_* \rangle_g + h u_* v_*) dv_g + \int_M (|\nabla_g v_*|_g^2 + h v_*^2) dv_g\right) \\ &= \|u_*\|^2 \int_M (|\nabla_g (u_* - v_*)|_g^2 + h(u_* - v_*)^2) dv_g \\ &= \|u_*\|^2 \|u_* - v_*\|^2 = 0, \end{aligned}$$

which implies $\|u_* - v_*\|^2 = 0$. Thus, $u_* = v_*$ a.e. in M. This completes the proof of Theorem 1.1.

4. Existence of solutions for $\gamma = 1$

In this section, we establish the existence and uniquness of a positive weak solution to the problem $(\mathcal{K}_g)$ for $\gamma = 1$ in $H^1(M)$.

We begin with some auxiliary lemmas that will be used in the proof of Theorem 1.2.

Lemma 4.1. *Let $q \in L^{\frac{n}{2}}(M)$ satisfy $q(x) \geq 0$ a.e. in M. Then, for every $g \in L^{\frac{2n}{n+2}}(M)$, the problem*

$$\mathcal{L}_g u + q(x)u = g(x) \quad in \;\; M, \tag{4.1}$$

has a unique solution in $H^1(M)$.

Proof. For $u \in H^1(M)$, define $J : H^1(M) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_M qu^2 dv_g - \int_M gudv_g,$$

which is differentiable. By Hölder inequality and (2.4), we find

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \int_M gudv_g \geq \frac{1}{2}\|u\|^2 - \|g\|_{L^{\frac{2n}{n+2}}}\|u\|_{L^{\frac{2n}{n-2}}} \\ &\geq \frac{1}{2}\|u\|^2 - S^{-\frac{1}{2}}\|g\|_{L^{\frac{2n}{n+2}}}\|u\|. \end{aligned}$$

This implies that $J(u)$ is coercive and bounded from below in $H^1(M)$. Then, J achieves its minimum at some $u_0 \in H^1(M)$, which is its critical point. Thus, u_0 is a solution of (4.1). Since, for $u \neq v$,

$$\begin{aligned} \langle J'(u) - J'(v), u - v\rangle &= \int_M (|\nabla_g(u - v)|_g^2 + h(u - v)^2)dv_g + \int_M q(u - v)^2 dv_g \\ &= \|u - v\|^2 + \int_M q(u - v)^2 dv_g > 0, \end{aligned}$$

J is strictly convex. Therefore, the problem (4.1) has a unique solution. □

Remark 4.2. Clearly, the sign condition on q in Lemma 4.1 is not necessary to obtain the desired properties. Indeed, the same conclusion holds provided q is "not too negative". For instance, $q \in L^{\frac{n}{2}}(M)$ satisfies $\|q\|_{L^{\frac{n}{2}}} < S$.

We make use of a well-known approximating scheme for this problem. To this end, let $n \in \mathbb{N}$ and $f_n(x) = \max\{\frac{1}{n}, \min\{f(x), n\}\}$. We consider the following approximating equation

$$(a + b\|u_n\|^2)\mathcal{L}_g u_n = \frac{f_n(x)}{(u_n + \frac{1}{n})^\gamma} - \lambda u_n^p \quad \text{in} \;\; M. \tag{4.2}$$

Lemma 4.3. *Problem (4.2) has a nonnegative solution u_n in $H^1(M)$ with $\gamma > 0$.*

Proof. Given $n \in N$, let v be a function in $H^1(M)$. By Lemma 4.1, define $\omega = Q(v)$ to be the unique solution of

$$(a + b\|v\|^2)\mathcal{L}_g\omega = \frac{f_n(x)}{(|v| + \frac{1}{n})^\gamma} - \lambda|v|^{p-1}\omega \quad \text{in} \;\; M. \tag{4.3}$$

Taking ω as a test function, we have

$$\|\omega\|^2 \leq \frac{1}{a}\int_M \frac{f_n(x)\omega}{(|v| + \frac{1}{n})^\gamma}dv_g \leq \frac{n^{\gamma+1}}{a}\int_M |\omega|dv_g.$$

Using Hölder inequality and (2.4), we infer

$$\|\omega\|^2 \leq \frac{n^{\gamma+1}}{a} \int_M |\omega| dv_g \leq \frac{n^{\gamma+1}}{a} \left(\int_M |\omega|^{2^*} dv_g \right)^{\frac{1}{2^*}} \left(\int_M 1 dv_g \right)^{\frac{n+2}{2n}} \leq Cn^{\gamma+1} \|\omega\|,$$

where C is a constant independent on v. Then, one has

$$\|\omega\| \leq Cn^{\gamma+1}.$$

Hence, the ball of radius $Cn^{\gamma+1}$ in $H^1(M)$ is invariant for Q.

We now prove the continuity and compactness of Q from $H^1(M)$ to $H^1(M)$. Indeed, if $v_k \to v$ in $H^1(M)$, recalling $\omega_k = Q(v_k)$ satisfies (4.3), and one has

$$(a + b\|v_k\|^2) \int_M (\langle \nabla_g \omega_k, \nabla_g \varphi \rangle_g + h\omega_k \varphi) dv_g = \int_M \frac{f_n(x)}{(|v_k| + \frac{1}{n})^\gamma} \varphi dv_g - \lambda \int_M |v_k|^{p-1} \omega_k \varphi dv_g, \tag{4.4}$$

for each $\varphi \in H^1(M)$. Moreover, since ω_k is bounded in $H^1(M)$, there exist a subsequence (still denoted by $\{\omega_k\}$) and a function $\omega \in H^1(M)$ such that $\omega_k \rightharpoonup \omega$ in $H^1(M)$ and $\omega_k \to \omega$ in $L^s(M)(1 \leq s < 2^*)$. Letting $k \to +\infty$ in (4.4), we see

$$(a + b\|v\|^2) \int_M (\langle \nabla_g \omega, \nabla_g \varphi \rangle_g + h\omega \varphi) dv_g = \int_M \frac{f_n(x)\varphi}{(|v| + \frac{1}{n})^\gamma} dv_g - \lambda \int_M |v|^{p-1} \omega \varphi dv_g. \tag{4.5}$$

It shows that $\omega = Q(v)$. Furthermore, taking $\varphi = \omega_k$ in (4.4) and letting $k \to +\infty$, we have

$$(a + b\|v\|^2) \lim_{k \to +\infty} \|\omega_k\|^2 = \int_M \frac{f_n(x)\omega}{(|v| + \frac{1}{n})^\gamma} dv_g - \lambda \int_M |v|^{p-1} \omega^2 dv_g. \tag{4.6}$$

On the other hand, taking $\varphi = \omega$ in (4.5), one gets

$$(a + b\|v\|^2)\|\omega\|^2 = \int_M \frac{f_n(x)\omega}{(|v| + \frac{1}{n})^\gamma} dv_g - \lambda \int_M |v|^{p-1} \omega^2 dv_g. \tag{4.7}$$

Using (4.6) and (4.7), we deduce that

$$\lim_{k \to +\infty} \|\omega_k\|^2 = \|\omega\|^2.$$

Hence $\omega_k \to \omega$ strongly in $H^1(M)$, and then Q is continuous. In order to obtain the compactness of Q, we apply the above argument again, with $\|v\|^2$ replaced by $\lim\limits_{k \to +\infty} \|v_k\|^2$ in (4.5)–(4.7). By the Schauder fixed point theorem, we infer that Q has a fixed point $u_n \in H^1(M)$, which solves

$$(a + b\|u_n\|^2)\mathcal{L}_g u_n = \frac{f_n(x)}{(|u_n| + \frac{1}{n})^\gamma} - \lambda |u_n|^{p-1} u_n \quad \text{in } M. \tag{4.8}$$

Choosing $u_n^- = max\{-u_n, 0\}$ as a test function in (4.8), we have

$$\begin{aligned} 0 &\geq (a + b\|u_n\|^2) \int_M (\langle \nabla_g u_n, \nabla_g u_n^- \rangle_g + h u_n u_n^-) dv_g \\ &= \int_M \frac{f_n(x) u_n^-}{(|u_n| + \frac{1}{n})^\gamma} dv_g - \lambda \int_M |u_n|^{p-1} u_n u_n^- dv_g \geq 0. \end{aligned}$$

Therefore, u_n is a nonnegative solution of (4.2). □

Remark 4.4. When $\gamma = 1$, problem (4.2) becomes:

$$(a + b\|u_n\|^2)\mathcal{L}_g u_n = \frac{f_n(x)}{|u_n| + \frac{1}{n}} - \lambda u_n^p \quad \text{in } M. \tag{4.9}$$

Obviously, Lemma 4.3 is also correct for problem (4.9), which is the approximated problem of $(\mathcal{K}_g)$.

Lemma 4.5. *Let u_n be the solution of* (4.9)*. Then, u_n is bounded in $H^1(M)$. Moreover, there exists a constant $c_\lambda > 0$ such that*

$$u_n > c_\lambda, \quad a.e. \quad x \in M. \tag{4.10}$$

Proof. (i) Taking u_n as a test function in (4.9) and recalling $0 \le f_n \le f + 1$, we have

$$\|u_n\|^2 \le \frac{1}{a}\int_M \frac{f_n u_n}{u_n + \frac{1}{n}} dv_g \le \frac{1}{a}\int_M f_n dv_g \le \frac{1}{a}\int_M (f+1) dv_g := c_f.$$

Therefore, u_n is bounded in $H^1(M)$.

(ii) By (i), we know $\|u_n\|^2 \le c_f$, and then

$$\mathcal{L}_g u_n \ge \frac{f_n}{a + bc_f}\frac{1}{u_n + \frac{1}{n}} - \frac{\lambda}{a}u_n^p. \tag{4.11}$$

Consider the following equation:

$$\mathcal{L}_g \omega_n = \frac{f_n}{a + bc_f}\frac{1}{\omega_n + \frac{1}{n}} - \frac{\lambda}{a}\omega_n^p \qquad \text{in } M. \tag{4.12}$$

Combining (4.11) and (4.12), we infer

$$\mathcal{L}_g(\omega_n - u_n) \le \frac{f_n}{a + bc_f}\frac{u_n - \omega_n}{(\omega_n + \frac{1}{n})(u_n + \frac{1}{n})} - \frac{\lambda}{a}(\omega_n^p - u_n^p).$$

Choosing $(\omega_n - u_n)^+$ as a test function, noticing that

$$(\omega_n^p - u_n^p)(\omega_n - u_n)^+ \ge 0,$$

and recalling that $f_n > 0$, we have

$$\|(\omega_n - u_n)^+\|^2 = \int_M \left(|\nabla_g(\omega_n - u_n)^+|_g^2 + h((\omega_n - u_n)^+)^2\right) dv_g \le 0.$$

Hence, $(\omega_n - u_n)^+ = 0$ a.e. in M, which implies $\omega_n \le u_n$. Let φ_1 be an eigenfunction associated to the first eigenvalue λ_1 of $\mathcal{L}_g$. Define $k : [0, \infty) \to \mathbb{R}$ by

$$k(\varepsilon) = \frac{f_n}{a + bc_f}\frac{1}{\varepsilon\varphi_1 + \frac{1}{n}} - \frac{\lambda}{a}(\varepsilon\varphi_1)^p - \lambda_1\varepsilon\varphi_1.$$

Obviously, $k(\varepsilon)$ is decreasing on $[0, +\infty)$ and satisfies $k(0) > 0$. Moreover, by the continuity of the function k, we can choose $\varepsilon_\lambda > 0$ small enough such that

$$k(\varepsilon_\lambda) = \frac{f_n}{a + bc_f}\frac{1}{\varepsilon_\lambda\varphi_1 + \frac{1}{n}} - \frac{\lambda}{a}(\varepsilon_\lambda\varphi_1)^p - \lambda_1\varepsilon_\lambda\varphi_1 \ge 0,$$

which implies

$$\mathcal{L}_g(\varepsilon_\lambda\varphi_1) \leq \frac{f_n}{a+bc_f}\frac{1}{\varepsilon_\lambda\varphi_1+\frac{1}{n}} - \frac{\lambda}{a}(\varepsilon_\lambda\varphi_1)^p.$$

Thus, we obtain that $\varepsilon_\lambda\varphi_1$ is a sub-solution of (4.12). By the comparison principle, we infer $\omega_n \geq \varepsilon_\lambda\varphi_1$. Since $\varphi_1 > 0$ in M and $\varphi_1 \in C^{1,\alpha}(M)$, $0 < \alpha < 1$, there exists a positive constant c such that $\varphi_1 > c$. Thus, we conclude that

$$u_n \geq \omega_n \geq \varepsilon_\lambda\varphi_1 > \varepsilon_\lambda c := c_\lambda, \quad \text{a.e.} \quad x \in M.$$

□

Now, we are in a position to present the proof of Theorem 1.2.

Proof of Theorem 1.2 (i) We first show the existence of a positive weak solution of problem ($\mathcal{K}_g$). By Lemma 4.5, $\{u_n\}$ is bounded in $H^1(M)$, and we can choose a subsequence (still called $\{u_n\}$) and $u \in H^1(M)$ such that

$$\lim_{n\to+\infty}\int_M(\langle\nabla_g u_n, \nabla_g\varphi\rangle_g + hu_n\varphi)dv_g = \int_M(\langle\nabla_g u, \nabla_g\varphi\rangle_g + hu\varphi)dv_g \tag{4.13}$$

for every φ in $H^1(M)$. Furthermore, since u_n satisfies (4.10), we have

$$0 \leq \left|\frac{f_n\varphi}{u_n+\frac{1}{n}}\right| \leq \frac{(f+1)|\varphi|}{c_\omega}.$$

Thus, by Lebesgue convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_M \frac{f_n\varphi}{u_n+\frac{1}{n}}dv_g = \int_M \frac{f\varphi}{u}dv_g. \tag{4.14}$$

On the other hand, u_n is the solution of (4.9), namely,

$$(a+b\|u_n\|^2)\int_M(\langle\nabla_g u_n, \nabla_g\varphi\rangle_g + hu_n\varphi)dv_g = \int_M \frac{f_n(x)\varphi}{(u_n+\frac{1}{n})}dv_g - \lambda\int_M u_n^p\varphi dv_g \tag{4.15}$$

for every φ in $H^1(M)$. Then, by (4.13)–(4.15), one has

$$(a+b\lim_{n\to+\infty}\|u_n\|^2)\int_M(\langle\nabla_g u, \nabla_g\varphi\rangle_g + hu\varphi)dv_g = \int_M \frac{f\varphi}{u}dv_g - \lambda\int_M u^p\varphi dv_g. \tag{4.16}$$

Choosing $\varphi = u_n$ in (4.15) and letting $n \to +\infty$, we get

$$(a+b\lim_{n\to+\infty}\|u_n\|^2)\lim_{n\to+\infty}\|u_n\|^2 = \int_M f dv_g - \lambda\int_M u^{p+1}dv_g. \tag{4.17}$$

Replacing φ by u in (4.16), we infer

$$(a+b\lim_{n\to+\infty}\|u_n\|^2)\|u\|^2 = \int_M f dv_g - \lambda\int_M u^{p+1}dv_g. \tag{4.18}$$

Combining (4.17) with (4.18), we deduce that $\lim_{n\to+\infty} \|u_n\|^2 = \|u\|^2$. Thus, substituting this into (4.16) leads to

$$(a + b\|u\|^2)\int_M (\langle \nabla_g u, \nabla_g \varphi\rangle_g + hu\varphi)dv_g = \int_M \frac{f\varphi}{u}dv_g - \lambda \int_M u^p \varphi dv_g,$$

which shows that u is a solution of $(\mathcal{K}_g)$. Furthermore, recalling Lemma 4.5, the solution is positive.

(ii) We prove the uniqueness of solutions of $(\mathcal{K}_g)$. Suppose that v is another solution of $(\mathcal{K}_g)$. Denote

$$J(u,v) = \|u\|^4 - \|u\|^2 \int_M (\langle \nabla_g u, \nabla_g v\rangle_g + huv)dv_g - \|v\|^2 \int_M (\langle \nabla_g u, \nabla_g v\rangle_g + huv)dv_g + \|v\|^4.$$

By (3.17), we have

$$J(u,v) \geq 0.$$

Since

$$(a + b\|u\|^2)\int_M (\langle \nabla_g u, \nabla_g \varphi\rangle_g + hu\varphi)dv_g = \int_M \frac{f}{u}\varphi dv_g - \lambda \int_M u^p \varphi dv_g \tag{4.19}$$

and

$$(a + b\|v\|^2)\int_M (\langle \nabla_g v, \nabla_g \varphi\rangle_g + hv\varphi)dv_g = \int_M \frac{f}{v}\varphi dv_g - \lambda \int_M v^p \varphi dv_g, \tag{4.20}$$

we subtract (4.19) from (4.20) and obtain

$$a\|u-v\|^2 + bJ(u,v) + \lambda \int_M (u^p - v^p)(u-v)dv_g - \int_M f\left(\frac{1}{u} - \frac{1}{v}\right)(u-v)dv_g = 0. \tag{4.21}$$

Moreover, it is easy to get

$$\int_M (u^p - v^p)(u-v)dv_g \geq 0, \quad \int_M f\left(\frac{1}{u} - \frac{1}{v}\right)(u-v)dv_g \leq 0.$$

Therefore, it follows from (4.21) that $\|u - v\| = 0$, which implies $u = v$. This ends the proof.

5. Conclusions

This paper investigates Kirchhoff-type equations with singular nonlinear terms on closed Riemannian manifolds. Currently, results for Kirchhoff-type equations are mostly established in Euclidean spaces. This paper establishes the existence and uniqueness of solutions to nonlinear Kirchhoff equations with strong and weak singularities on closed Riemannian manifolds. This is achieved through the application of minimization techniques and approximation methods. The results obtained in this study are novel.

Author contributions

All authors contributed equally to the writing of this article. Additionally, all authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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