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Research article

New extension of quasi-M-hypnormal operators

Ohud Bulayhan Almutairi[1,*] **and Sid Ahmed Ould Ahmed Mahmoud**[2]

[1] Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin, 31991, Saudi Arabia
[2] Mathematics Department, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia; Email: Sidahmed@ju.edu.sa

* **Correspondence:** Email: ohudbalmutairi@gmail.com.

Abstract: This study introduces a new class of operators called polynomilally quasi-M-hyponormal, which combining M-hyponormal, quasi-M-hyponormal, and k-quasi-M-hyponormal operators. We will demonstrate several properties of this class that correspond to those of k-quasi-M-hyponormal operators.

Keywords: M-hyponormal; quasi-M-hyponormal
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1. Introduction

Let $\mathcal{B}[\mathcal{K}]$ be the the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{K}$ with inner product $\langle .\,|.\,\rangle$ and

$$\mathcal{B}[\mathcal{K}]^+ := \{\mathcal{U} \in \mathcal{B}[\mathcal{K}] \;\; /\langle \mathcal{U}\omega \mid \omega\rangle \geq 0 \;\; \forall\, \omega \in \mathcal{K}\;\}.$$

The elements of $\mathcal{B}(\mathcal{K})$ are called positive operators. For every $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$, $\ker(\mathcal{U})$, $\mathcal{R}(\mathcal{U})$, and $\overline{\mathcal{R}(\mathcal{U})}$ represent, respectively, the null space, the range, and the closure of the range of $\mathcal{U}$. Its adjoint operator is denoted by $\mathcal{U}^*$. In addition, if $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}[\mathcal{K}]$, then $\mathcal{U}_1 \geq \mathcal{U}_2$ means that $\mathcal{U}_1 - \mathcal{U}_2 \in \mathcal{B}[\mathcal{K}]^+$.

For $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$, let $\sigma_p(\mathcal{U})$, $\sigma(\mathcal{U})$, $\sigma_s(\mathcal{U})$, and $\sigma_a(\mathcal{U})$ denote the point spectrum, the spectrum, the surjective spectrum, and, approximate point spectrum of $\mathcal{U}$. If $\mu \in \sigma_p(\mathcal{U})$ and $\overline{\mu} \in \sigma_p(\mathcal{U}^*)$, then μ is in the joint point spectrum, $\sigma_{jp}(\mathcal{U})$. If $\mu \in \sigma_a(\mathcal{U})$ and $\overline{\mu} \in \sigma_a(\mathcal{U}^*)$, then we say that μ is in the joint approximate point spectrum, $\sigma_{ja}(\mathcal{U})$. The following classes of operators have been studied by many authors. An operator $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$ is said to be
(1) normal if $\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^*$ [1–3],
(2) hyponormal operator if $\mathcal{U}^*\mathcal{U} \geq \mathcal{U}\mathcal{U}^*$ [4, 5],

(3) M-hyponormal operator [6] if there exists a real positive number M such that

$$M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho) \geq (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*, \quad \forall\, \varrho \in \mathbb{C}, \tag{1.1}$$

or, equivalently, if

$$M^2|\mathcal{U}-\varrho|^2 \geq |(\mathcal{U}-\varrho)^*|^2 \quad \forall\, \varrho \in \mathbb{C},$$

(4) quasi-M-hyponormal [7, 8] if there exits a real positive number M such that

$$\mathcal{U}^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)\Big)\mathcal{U} \geq \mathcal{U}^*\Big((\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)\mathcal{U}, \quad \forall\, \varrho \in \mathbb{C}, \tag{1.2}$$

(5) k-quasi-M-hyponormal operator if there exists a real positive number M such that

$$\mathcal{U}^{*k}\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)\Big)\mathcal{U}^k \geq \mathcal{U}^{*k}\Big((\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)\mathcal{U}^k, \quad \forall\, \varrho \in \mathbb{C}, \tag{1.3}$$

where k is a natural number [9].

In the papers [10, 11], the authors have introduced the class of polynomially normal as follows: An operator $\mathcal{U}$ is said to be polynomially normal if there exists a nontrivial polynomial $P = \sum_{0\leq k\leq n} b_k z^k \in C[z]$ with

$$P(\mathcal{U})\mathcal{U}^* - \mathcal{U}^*P(\mathcal{U}) = \sum_{0\leq k\leq n} b_k\Big(\mathcal{U}^k\mathcal{U}^* - \mathcal{U}^*\mathcal{U}^k\Big) = 0.$$

In the following, we introduce a new class of operators called the class of polynomially quasi-M-hyponormal operators denoted by $[\mathcal{PQK}]_M$.

Definition 1.1. An operator $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$ is said to be a polynomially quasi-M-hyponormal operator if there exists a nonconstant polynomial $P \in \mathbb{C}[z]$ and a positive constant M such that,

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)\Big)P(\mathcal{U}) \geq P(\mathcal{U})^*\Big((\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \tag{1.4}$$

for all $\varrho \in \mathbb{C}$ or

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho) - (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0.$$

Remark 1.1. (1) Every M-hyponormal operator is in $[\mathcal{PQK}]_M$.
(2) Every quasi-M-hyponormal operator is in $[\mathcal{PQK}]_M$ with $P(z) = z$.
(3) Every k-quasi-M-hyponormal operator is in $[\mathcal{PQK}]_M$. with $P(z) = z^k$.

2. Main results

In this section, we will show several properties of the class $[\mathcal{PQK}]_M$.

Theorem 2.1. *Let $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$. Then $\mathcal{U} \in [\mathcal{PQK}]_M$ if and only if there exists a real number M that is positive, such that*

$$M^2\|(\mathcal{U}-\varrho)P(\mathcal{U})\omega\| \geq \|(\mathcal{U}-\varrho)^*P(\mathcal{U})\omega\| \quad \forall\, \omega \in \mathcal{K}.$$

Proof. Assume that $\mathcal{U} \in [\mathcal{PQK}]_M$, then there exists $P \in \mathbb{C}[z]$ and $M > 0$ for which

$$\begin{aligned}
\|(\mathcal{U}-\varrho)^* P(\mathcal{U})\omega\|^2 &= \langle (\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \rangle \\
&= \langle P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \rangle \\
&= \langle \omega \mid P(\mathcal{U})^*((\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*)P(\mathcal{U})\omega \rangle \\
&\le M^2 \left\langle \omega \mid P(\mathcal{U})^*(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U})\omega \right\rangle \\
&= M^2 \left\langle P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U})\omega \right\rangle \\
&= M^2 \langle (\mathcal{U}-\varrho)P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)P(\mathcal{U})\omega \rangle \\
&= M^2\|(\mathcal{U}-\varrho)P(\mathcal{U})\omega\|^2.
\end{aligned}$$

Conversely, assume that $\mathcal{U}$ satisfies

$$\|(\mathcal{U}-\varrho)^* P(\mathcal{U})\omega\|^2 \le M^2\|(\mathcal{U}-\varrho)P(\mathcal{U})\omega\|^2$$

for each $\omega \in \mathcal{K}$, so one can obtain that

$$\begin{aligned}
\langle (\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \rangle &= \langle P(\mathcal{U})\omega \mid (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^* P(\mathcal{U})\omega \rangle \\
&= \langle \omega \mid P(\mathcal{U})^*((\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*)P(\mathcal{U})\omega \rangle \\
&\le M^2\langle \omega \mid P(\mathcal{U})^*(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U})\omega \rangle.
\end{aligned}$$

So one can obtain that

$$\langle \omega \mid M^2P(\mathcal{U})^*(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U})\omega - P(\mathcal{U})^*(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*P(\mathcal{U})\omega \rangle \ge 0.$$

Therefore

$$M^2P(\mathcal{U})^*(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U}) - P(\mathcal{U})^*(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*P(\mathcal{U}) \ge 0.$$

Hence, one can obtain

$$M^2P(\mathcal{U})^*((\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)P(\mathcal{U}) \ge P(\mathcal{U})^*(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*P(\mathcal{U}).$$

Therefore, $\mathcal{U}$ is polynomially quasi-M-hyponormal operator. □

Corollary 2.1. *Let $\mathcal{U} \in [\mathcal{PQK}]_M$ such that $\overline{\mathcal{R}(P(\mathcal{U}))} = \mathcal{K}$, then $\mathcal{U}$ is an M-hyponormal operator.*

Proof. Supposing $\overline{\mathcal{R}(P(\mathcal{U}))} = \mathcal{K}$, let $\omega \in \mathcal{K}$. Then there is a sequence $\omega_n \in \mathcal{K}$ such that $P(\mathcal{U})\omega_n \longrightarrow \omega$ as $n \longrightarrow \infty$. Now, from the hypothesis of this corollary and Theorem 2.1, we have

$$M^2\|(\mathcal{U}-\varrho)P(\mathcal{U})\omega\| \ge \|(\mathcal{U}-\varrho)^*P(\mathcal{U})\omega\|, \ \forall\ \omega \in \mathcal{K}.$$

This implies

$$M^2\|(\mathcal{U}-\varrho)P(\mathcal{U})\omega_n\| \ge \|(\mathcal{U}-\varrho)^*P(\mathcal{U})\omega_n\|.$$

By taking the limit $n \to \infty$ we obtain

$$M^2\|(\mathcal{U}-\varrho)\omega\| \ge \|(\mathcal{U}-\varrho)^*\omega\|, \qquad \omega \in \mathcal{K}.$$

Therefore, $\mathcal{U}$ is M-hyponormal operator. □

A characterization of some members of $[\mathcal{PQK}]_M$ will be given in the following theorem.

Theorem 2.2. *Let $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$ such that $\overline{\mathcal{R}(P(\mathcal{U}))} \neq \mathcal{K}$ for some $P \in \mathbb{C}[z]$, then the following are equivalent.*

(1) $\mathcal{U} \in [\mathcal{PQK}]_M$.

(2) $\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix}$ *on* $\mathcal{K} = \overline{\mathcal{R}(P(\mathcal{U}))} \oplus \ker(P(\mathcal{U})^*)$, *where* $\mathcal{U}_1 = \mathcal{U}|_{\overline{\mathcal{R}(P(\mathcal{U}))}}$ *satisfies*

$$M^2(\mathcal{U}_1 - \varrho)^*(\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2\mathcal{U}_2^* \geq 0, \quad \forall\, \varrho \in \mathbb{C},$$

and $P(\mathcal{U}_3) = 0$. *Furthermore,* $\sigma(\mathcal{U}) = \sigma(\mathcal{U}_1) \cup \sigma(\mathcal{U}_3)$.

Proof. (1) $\Rightarrow$ (2). By taking into account the matrix representation of $\mathcal{U}$ with respect to the decomposition $\mathcal{K} = \overline{\mathcal{R}(P(\mathcal{U}))} \oplus \ker(P(\mathcal{U})^*)$: $\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix}$. Let $P|_{\overline{\mathcal{R}(P(\mathcal{U})}}$ be the projection onto $\overline{\mathcal{R}(P(\mathcal{U})}$. Then $\begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{U}P|_{\overline{\mathcal{R}(P(\mathcal{U}))}} = P|_{\overline{\mathcal{R}(P(\mathcal{U}))}}\mathcal{U}P|_{\overline{\mathcal{R}(P(\mathcal{U}))}}$. Since $\mathcal{U} \in [\mathcal{PQK}]_M$, from Definition 2.1, we have

$$P|_{\overline{\mathcal{R}(P(\mathcal{U})}}\Big(M^2(\mathcal{U} - \varrho)^*(\mathcal{U} - \varrho) - (\mathcal{U} - \varrho)(\mathcal{U} - \varrho)^*\Big)P|_{\overline{\mathcal{R}(P(\mathcal{U}))}} \geq 0.$$

That is

$$M^2(\mathcal{U}_1 - \varrho)^*(\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2\mathcal{U}_2^* \geq 0,$$

for all $\varrho \in \mathbb{C}$.

On the other hand, let $\omega = \omega_1 + \omega_2 \in \mathcal{K} = \overline{\mathcal{R}(P(\mathcal{U}))} \oplus \ker(P(\mathcal{U})^*)$. A simple computation shows that

$$\begin{aligned} \langle P(\mathcal{U}_3)\omega_2 \mid \omega_2\rangle &= \langle P(\mathcal{U})(I - P|_{\overline{\mathcal{R}(P(\mathcal{U})}})\omega \mid (I - P|_{\overline{\mathcal{R}(P(\mathcal{U})}})\omega\rangle \\ &= \langle (I - P|_{\overline{\mathcal{R}(P(\mathcal{U}))}})\omega \mid P(\mathcal{U})^*(I - P|_{\overline{\mathcal{R}(P(\mathcal{U}))}})\omega\rangle = 0. \end{aligned}$$

So, $P(\mathcal{U}_3) = 0$. The proof of the identity $\sigma(\mathcal{U}) = \sigma(\mathcal{U}_1) \cup \sigma(\mathcal{U}_3)$ is deduced by an argument similar to the one given in [12, Corollaries 7 and 8].

(2) $\Rightarrow$ (1) Suppose that $\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix}$ onto $\mathcal{K} = \overline{\mathcal{R}(P(\mathcal{U}))} \oplus \ker(P(\mathcal{U})^*)$, with

$$M^2(\mathcal{U}_1 - \varrho)^*(\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2\mathcal{U}_2^* \geq 0,$$

for all $\varrho \in \mathbb{C}$ and $P(\mathcal{U}_3) = 0$.

Since $\mathcal{U}^m = \begin{pmatrix} \mathcal{U}_1^m & \sum_{j=0}^{m-1} \mathcal{U}_1^j\mathcal{U}_2\mathcal{U}_3^{k-1-j} \\ 0 & \mathcal{U}_3^m \end{pmatrix}$, $P(\mathcal{U}) = \begin{pmatrix} P(\mathcal{U}_1) & X \\ 0 & 0 \end{pmatrix}$,

$$(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)=\begin{pmatrix}(\mathcal{U}_1-\varrho)^*(\mathcal{U}_1-\varrho) & (\mathcal{U}_1-\varrho)^*\mathcal{U}_2\\ \mathcal{U}_2^*(\mathcal{U}_1-\varrho) & \mathcal{U}_2^*\mathcal{U}_2+(\mathcal{U}_3-\varrho)^*(\mathcal{U}_3-\varrho)\end{pmatrix},$$

and

$$(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*=\begin{pmatrix}(\mathcal{U}_1-\varrho)(\mathcal{U}_1-\varrho)^*+\mathcal{U}_2\mathcal{U}_2^* & \mathcal{U}_2(\mathcal{U}_1-\varrho)^*\\ (\mathcal{U}_1-\varrho)\mathcal{U}_2^* & (\mathcal{U}_3-\varrho)(\mathcal{U}_3-\varrho)^*\end{pmatrix}.$$

Further

$$\begin{aligned}P(\mathcal{U})P(\mathcal{U})^* &= \begin{pmatrix}p(\mathcal{U}_1)P(\mathcal{U}_1)^*+XX^* & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix},\end{aligned}$$

where $D=P(\mathcal{U}_1)P(\mathcal{U}_1)^*+XX^*=D^*$.
Hence, for all $\varrho\in\mathbb{C}$, we have

$$\begin{aligned}&P(\mathcal{U})P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*((\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})P(\mathcal{U})^*\\ =&\begin{pmatrix}D\Big(M^2(\mathcal{U}_1-\varrho)^*(\mathcal{U}_1-\varrho)-(\mathcal{U}_1-\varrho)(\mathcal{U}_1-\varrho)^*-\mathcal{U}_2\mathcal{U}_2^*\Big)D & 0\\ 0 & 0\end{pmatrix}\\ \geq&\ 0.\end{aligned}$$

It follows that

$$P(\mathcal{U})P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})P(\mathcal{U})^*\geq 0.$$

This means that

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})\geq 0,$$

on $\mathcal{K}=\overline{\mathcal{R}(P(\mathcal{U})^*)}\oplus\ker(P(\mathcal{U}))$. Consequently, $\mathcal{U}\in[\mathcal{PQK}]_M$.

□

In the following theorem we prove that part of $[\mathcal{PQK}]_M$ on a closed subspace is again $[\mathcal{PQK}]_M$.

Theorem 2.3. *Let $\mathcal{U}\in[\mathcal{PQK}]_M$. If $\mathcal{M}\subset\mathcal{K}$ is a closed invariant subspace for $\mathcal{U}$, then the restriction $\mathcal{U}_{|_\mathcal{M}}$ is in $[\mathcal{PQK}]_M$.*

Proof. With respect to the decomposition $\mathcal{K}=\mathcal{M}\oplus\mathcal{M}^\perp$, $\mathcal{U}$ can be written

$$\mathcal{U}=\begin{pmatrix}\mathcal{U}_1 & \mathcal{U}_2\\ 0 & \mathcal{U}_3\end{pmatrix}.$$

Hence, for all integer k, $k \geq 1$, we get

$$\mathcal{U}^k = \begin{pmatrix} \mathcal{U}_1^k & \sum_{p=0}^{k-1} \mathcal{U}_1^{k-1-p} \mathcal{U}_2 \mathcal{U}_3^p \\ 0 & \mathcal{U}_3^k \end{pmatrix}, \quad P(\mathcal{U}) = \begin{pmatrix} P(\mathcal{U}_1) & X \\ 0 & P(\mathcal{U}_3) \end{pmatrix},$$

for some $X \in \mathcal{B}[\mathcal{K}]$ and

$$(\mathcal{U} - \varrho)^*(\mathcal{U} - \varrho) = \begin{pmatrix} (\mathcal{U}_1 - \varrho)^*(\mathcal{U}_1 - \varrho) & (\mathcal{U}_1 - \varrho)^* \mathcal{U}_2 \\ \mathcal{U}_2^*(\mathcal{U}_1 - \varrho) & \mathcal{U}_2^* \mathcal{U}_2 + (\mathcal{U}_3 - \varrho)^*(\mathcal{U}_3 - \varrho) \end{pmatrix},$$

and

$$(\mathcal{U} - \varrho)(\mathcal{U} - \varrho)^* = \begin{pmatrix} (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* + \mathcal{U}_2 \mathcal{U}_2^* & \mathcal{U}_2(\mathcal{U}_1 - \varrho)^* \\ (\mathcal{U}_1 - \varrho)\mathcal{U}_2^* & (\mathcal{U}_3 - \varrho)(\mathcal{U}_3 - \varrho)^* \end{pmatrix}.$$

Since $\mathcal{U} \in [\mathcal{PQK}]_M$, there exists $P \in \mathbb{C}[z]$ and $M \geq 0$ such that for all $\varrho \in \mathbb{C}$

$$P(\mathcal{U})^* \Big(M^2 (\mathcal{U} - \varrho)^* (\mathcal{U} - \varrho) - (\mathcal{U} - \varrho)(\mathcal{U} - \varrho)^* \Big) P(\mathcal{U}) \geq 0.$$

Hence, we obtain

$$P(\mathcal{U})^* \Big(M^2 (\mathcal{U} - \varrho)^* (\mathcal{U} - \varrho) - (\mathcal{U} - \varrho)(\mathcal{U} - \varrho)^* \Big) P(\mathcal{U}) = \begin{pmatrix} \Phi & \Psi \\ \Psi^* & Z \end{pmatrix},$$

where

$$\Phi = P(\mathcal{U}_1)^* \Big(M^2 (\mathcal{U}_1 - \varrho)^* (\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2 \mathcal{U}_2^* \Big) P(\mathcal{U}_1)$$

$$\Psi = P(\mathcal{U}_1)^* \Big(M^2 (\mathcal{U}_1 - \varrho)^* (\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2 \mathcal{U}_2^* \Big) X +$$

$$+ P(\mathcal{U}_1)^* \Big(M^2 (\mathcal{U}_1 - \varrho)^* \mathcal{U}_2 - \mathcal{U}_2 (\mathcal{U}_1 - \varrho)^* \Big) P(\mathcal{U}_3)$$

and some operator $Z \in \mathcal{B}[\mathcal{K}]$. By [13, Theorem 6], $\begin{pmatrix} \Phi & \Psi \\ \Psi^* & Z \end{pmatrix} \geq 0$ if and only if $\Phi, Z \geq 0$ and $\Psi = \Phi^{\frac{1}{2}} W Z^{\frac{1}{2}}$ for some contraction W. Thus,

$$\Phi = P(\mathcal{U}_1)^* \Big(M^2 (\mathcal{U}_1 - \varrho)^* (\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* - \mathcal{U}_2 \mathcal{U}_2^* \Big) P(\mathcal{U}_1) \geq 0.$$

According to $\mathcal{U}_2 \mathcal{U}_2^* \geq 0$, it follows that

$$\Phi = P(\mathcal{U}_1)^* \Big(M^2 (\mathcal{U}_1 - \varrho)^* (\mathcal{U}_1 - \varrho) - (\mathcal{U}_1 - \varrho)(\mathcal{U}_1 - \varrho)^* \Big) P(\mathcal{U}_1) \geq 0.$$

Consequently, the restriction $\mathcal{U}_1 = \mathcal{U}|_{\mathcal{M}} \in [\mathcal{PQK}]_M$.

□

Theorem 2.4. *Let $P \in \mathbb{C}[z]$ be a polynomial and $\mathcal{U} = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$. If $\mathcal{U}_1 \in [\mathcal{PQK}]_M$, $P(\mathcal{U}_3) = 0$ and $\sigma_s(\mathcal{U}_1) \cap \sigma_a(\mathcal{U}_3) = \emptyset$, then $\mathcal{U}$ is similar to a direct sum of a member of $[\mathcal{PQK}]_M$ and an algebraic operator.*

Proof. According to the condition $\sigma_s(\mathcal{U}_1) \cap \sigma_a(\mathcal{U}_3) = \emptyset$, it follows from [14, Theorem 3.5.1, (c)] that there exists $B \in \mathcal{B}(\mathcal{K})$ such that $\mathcal{U}_1 B - B\mathcal{U}_3 = \mathcal{U}_2$. In view of the equality,

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 \\ 0 & \mathcal{U}_3 \end{pmatrix} = \begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_3 \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}.$$

It is clear that $\mathcal{U}$ is similar to $\Psi = \begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_3 \end{pmatrix}$. From the assumption that $\mathcal{U}_1 \in [\mathcal{PQK}]_M$ and $P(\mathcal{U}_3) = 0$, we get by easy calculation that

$$\begin{aligned}
& P(\Psi)^*\Big(M^2(\Psi-\varrho)^*(\Psi-\varrho) - (\Psi-\varrho)(\Psi-\varrho)^*\Big)P(\Psi) \\
= & \begin{pmatrix} P(\mathcal{U}_1)^* & 0 \\ 0 & 0 \end{pmatrix} \\
& \left\{\begin{pmatrix} (M^2(\mathcal{U}_1-\varrho)^*(\mathcal{U}_1-\varrho) - (\mathcal{U}_1-\varrho)(\mathcal{U}_1-\varrho)^* & 0 \\ 0 & (M^2(\mathcal{U}_3-\varrho)^*(\mathcal{U}_3-\varrho) - (\mathcal{U}_3-\varrho)(\mathcal{U}_3-\varrho)^* \end{pmatrix}\right\} \\
& \begin{pmatrix} P(\mathcal{U}_1) & 0 \\ 0 & 0 \end{pmatrix} \\
= & \begin{pmatrix} P(\mathcal{U}_1)^*((M^2(\mathcal{U}_1-\varrho)^*(\mathcal{U}_1-\varrho) - (\mathcal{U}_1-\varrho)(\mathcal{U}_1-\varrho)^*)P(\mathcal{U}_1) & 0 \\ 0 & 0 \end{pmatrix} \\
\geq & \ 0.
\end{aligned}$$

Following this, $\mathcal{U}$ is similar to a member of $[\mathcal{PQK}]_M$ and an algebraic operator.

□

Theorem 2.5. *Let $N \in \mathcal{B}[\mathcal{K}]$ be an invertible operator and $\mathcal{U} \in \mathcal{B}[\mathcal{K}]$ be an operator such that $\mathcal{U}$ commutes with N^*N. Then $\mathcal{U} \in [\mathcal{PQK}]_M$ if and only if $N\mathcal{U}N^{-1} \in [\mathcal{PQK}]_M$.*

Proof. Assume that $\mathcal{U} \in [\mathcal{PQK}]_M$. Then there exists $P \in \mathbb{C}[z]$ and $M > 0$ such that

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho) - (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0.$$

From this, we have that

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho) - (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^* \geq 0.$$

A computation shows that

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho) - (\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*(NN^*)$$

$$= NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})(N^*N)N^*$$
$$= N(N^*N)P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*$$
$$= (NN^*)NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*.$$

This shows that the operator NN^* commutes with the operator

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*.$$

Hence, the operator $(NN^*)^{-1}$ also commutes with the operator

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*.$$

Using the fact that the operators $(NN^*)^{-1}$ and

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*$$

are positive, and since they commute with each other. We get that their product is also a positive operator

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^*(NN^*)^{-1}\geq 0.$$

This implies that

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})(N^{-1}\geq 0.$$

From the fact that $\mathcal{U}N^*N = N^*N\mathcal{U}$, it follows that

$$(N\mathcal{U}N^{-1})^{*k} = (N\mathcal{U}N^{-1})^*(N\mathcal{U}N^{-1})^*\cdots(N\mathcal{U}N^{-1})^* = (N^*)^{-1}\mathcal{U}^{*k}N^*,$$
$$(N\mathcal{U}N^{-1})^k = N\mathcal{U}^kN^{-1}.$$

Hence,

$$P(N\mathcal{U}N^{-1})^* = (N^*)^{-1}P(\mathcal{U})^*N^* \text{ and } P(N\mathcal{U}N^{-1}) = NP(\mathcal{U})N^{-1}.$$

On the other hand,

$$(N\mathcal{U}N^{-1}-\varrho)^*(N\mathcal{U}N^{-1}-\varrho) = (N^*)^{-1}(\mathcal{U}-\varrho)^*N^*N(\mathcal{U}-\varrho)N^{-1} = N(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)N^{-1},$$
$$(N\mathcal{U}N^{-1}-\varrho)(N\mathcal{U}N^{-1}-\varrho)^* = N(\mathcal{U}-\varrho)N^{-1}(N^*)^{-1}(\mathcal{U}-\varrho)^*N^* = N(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*N^{-1}.$$

Now we show that $N\mathcal{U}N^{-1}\in[\mathcal{PQK}]_M$. Indeed

$$P(N\mathcal{U}N^{-1})^*\Big(M^2((N\mathcal{U}N^{-1}-\varrho)^*(N\mathcal{U}N^{-1}-\varrho))-(N\mathcal{U}N^{-1}-\varrho)(N\mathcal{U}N^{-1}-\varrho)^*\Big)P(N\mathcal{U}N^{-1})$$
$$= (N^*)^{-1}P(\mathcal{U})^*N^*\Big(M^2N(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho))N^{-1}-N(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*N^{-1}\Big)NP(\mathcal{U})N^{-1}$$

$$
\begin{aligned}
&= (N^*)^{-1}P(\mathcal{U})^*N^*N\Big(M^2(\mathcal{U}-\varrho))^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)N^{-1}NP(\mathcal{U})N^{-1}\\
&= (N^*)^{-1}N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho))^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^{-1}\\
&= NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^{-1}\\
&\geq 0.
\end{aligned}
$$

Based on these calculations, we conclude that $N\mathcal{U}N^{-1} \in [\mathcal{PQK}]_M$.

Conversely, assume that $N\mathcal{U}N^{-1} \in [\mathcal{PQK}]_M$. Then

$$P(N\mathcal{U}N^{-1})^*\Big(M^2(N\mathcal{U}N^{-1}-\varrho)^*(N\mathcal{U}N^{-1}-\varrho)-(N\mathcal{U}N^{-1}-\varrho)(N\mathcal{U}N^{-1}-\varrho))^*\Big)P(N\mathcal{U}N^{-1}) \geq 0.$$

Similar to before, we get

$$NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^{-1} \geq 0.$$

Hence,

$$N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho))^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})N^{-1}N \geq 0$$

or equivalently

$$N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0.$$

By using that, N^*N commutes with operator $\mathcal{U}$, and hence commutes with operators

$$N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho))^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}).$$

It follows that $(N^*N)^{-1}$ commute with

$$N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}).$$

By observing that $(N^*N)^{-1}$ and

$$N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})$$

are positive, and since they commutes with each other, we have

$$(N^*N)^{-1}N^*NP(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0.$$

Therefore,

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0.$$

Whit does it mean that $\mathcal{U} \in [\mathcal{PQK}]_M$. □

Theorem 2.6. *Let $\mathcal{U} \in [\mathcal{PQK}]_M$ for $P \in \mathbb{C}[z]$. Then*

$$\ker(\mathcal{U}-\mu) \subseteq \ker(\mathcal{U}-\mu)^* = \ker(\mathcal{U}^*-\overline{\mu}),$$

for all $\mu \in \mathbb{C}$ such that $P(\mu) \neq 0$.

Proof. Let $\omega \in \ker(\mathcal{U}-\mu)$. Since $\mathcal{U} \in [\mathcal{PQK}]_M$ for $P \in \mathbb{C}[z]$, it follows in view of Theorem 2.1,

$$M\|(\mathcal{U}-\mu)P(\mathcal{U})\omega\| \geq \|(\mathcal{U}-\mu)^*P(\mathcal{U})\omega\|.$$

Since $\mathcal{U}\omega = \mu\omega$, we get $P(\mathcal{U})\omega = P(\mu)\omega$, and therefore

$$M\|(\mathcal{U}-\mu)P(\mu)\omega\| \geq \|(\mathcal{U}-\mu)^*P(\mu)\omega\|.$$

According to $(\mathcal{U}-\mu)\omega = 0$ we obtain $\|(\mathcal{U}-\mu)^*P(\mu)\omega\| = 0$ or
$|P(\mu)|\|(\mathcal{U}-\mu)^*\omega\| = 0$. Since $P(\mu) \neq 0$ we get $(\mathcal{U}-\mu)^*\omega = 0$. Therefore, the proof is complete. □

Remark 2.1. When $P(z) = z$, Theorem 2.6 coincides with [8, Proposition 1.9].

Corollary 2.2. *Let $\mathcal{U} \in [\mathcal{PQK}]_M$ for some $P \in C[z]$. If $\alpha, \beta \in \sigma_p(\mathcal{U}) - \{0\}$ with $\alpha \neq \beta$ and $P(\beta) \neq 0$. Then*

$$\ker(\mathcal{U}-\alpha) \perp \ker(\mathcal{U}-\beta).$$

Proof. Let $\omega_1 \in \ker(\mathcal{U}-\alpha)$ and $\omega_2 \in \ker(\mathcal{U}-\beta)$, then $\mathcal{U}\omega_1 = \alpha\omega_1$ and $\mathcal{U}\omega_2 = \beta\omega_2$. Therefore

$$\begin{aligned}
\alpha\langle\omega_1 \mid \omega_2\rangle &= \langle\alpha\omega_1 \mid \omega_2\rangle \\
&= \langle\mathcal{U}\omega_1 \mid \omega_2\rangle \\
&= \langle\omega_1 \mid \mathcal{U}^*\omega_2\rangle \\
&= \left\langle\psi_1 \mid \overline{\beta}\omega_2\right\rangle \\
&= \beta\langle\omega_1 \mid \omega_2\rangle.
\end{aligned}$$

We deduce that $(\alpha-\beta)\langle\omega_1 \mid \omega_2\rangle = 0$ and so that $\langle\omega_1 \mid \omega_2\rangle = 0$ $(\alpha \neq \beta)$. Thus, $\ker(\mathcal{U}-\alpha) \perp \ker(\mathcal{U}-\beta)$. □

Remark 2.2. When $P(z) = z$, Corollary 2.2 coincides with [8, Corollary 1.10].

Theorem 2.7. *[15] Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ satisfying the following properties for every $T, S \in \mathcal{B}(\mathcal{H})$ and $\varrho, \mu \in \mathbb{C}$.*
(1) $\psi(T^*) = \psi(T)^*$, $\psi(I_{\mathcal{H}}) = I_{\mathcal{K}}, \psi(\varrho T + \mu S) = \varrho\psi(T) + \mu\psi(S)$,
(2) $\psi(TS) = \psi(T)\psi(S)$, $\|\psi(T)\| = \|T\|, \psi(T) \geq \psi(S)$, for $T \geq S$,
(3) $\psi(T) \geq 0$ if $T \geq 0$,
(4) $\sigma_a(T) = \sigma_a(\psi(T)) = \sigma_p(\psi(T))$,
(5) $\sigma_{ja}(T) = \sigma_{jp}(\psi(T))$.

Theorem 2.8. *Let $\mathcal{U} \in [\mathcal{PUK}]_M$ for some $P \in C[z]$ such that $P(\mu) \neq 0$ for all $\mu \in \sigma_a(\mathcal{U})$. Then $\sigma_a(\mathcal{U}) = \sigma_{ja}(\mathcal{U})$.*

Proof. Since $\mathcal{U} \in [\mathcal{PQK}]_M$, then there exists $P \in \mathbb{C}[z]$ and constant $M > 0$ such that

$$P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U}) \geq 0 \quad \forall\, \varrho \in \mathbb{C}. \tag{2.1}$$

In view of Theorem 2.7, we have

$$\begin{aligned} & P(\psi(\mathcal{U}))^*\Big(M^2(\psi(\mathcal{U})-\varrho)^*(\psi(\mathcal{U})-\varrho)-(\psi(\mathcal{U})-\varrho)(\psi(\mathcal{U})-\varrho)^*\Big)P(\psi(\mathcal{U})) \\ = \;& \psi(P(\mathcal{U})^*)\Big(M^2(\psi(\mathcal{U}-\varrho)^*(\psi(\mathcal{U}-\varrho)-(\psi(\mathcal{U}-\varrho)(\psi(\mathcal{U}-\varrho)^*\Big)\psi(P((\mathcal{U})) \\ = \;& \psi\Big(P(\mathcal{U})^*\Big(M^2(\mathcal{U}-\varrho)^*(\mathcal{U}-\varrho)-(\mathcal{U}-\varrho)(\mathcal{U}-\varrho)^*\Big)P(\mathcal{U})\Big) \\ \geq \;& 0 \text{ (by Theorem 2.7 } \textit{and } (2.1)). \end{aligned}$$

Hence $\psi(\mathcal{U}) \in [\mathcal{PQK}]_M$.

From Theorem 2.7, we have $\sigma_a(\mathcal{U}) = \sigma_p(\psi(\mathcal{U}))$. Since $\psi(\mathcal{U}) \in [\mathcal{PQK}]_M$, we have $\ker(\psi(\mathcal{U})-\mu) \subset \ker(\psi(\mathcal{U})-\mu)^*$ (from Theorem 2.6). Hence $\sigma_p(\psi(\mathcal{U})) = \sigma_{jp}(\psi(\mathcal{U}))$. According to Theorem 2.7, we have $\sigma_{jp}(\psi(\mathcal{U})) = \sigma_{ja}(\mathcal{U})$. Hence, $\sigma_a(\mathcal{U}) = \sigma_{ja}(\mathcal{U})$. □

In the following theorem, we will prove, under suitable conditions, the stability of the class $[\mathcal{PQK}]_M$ under the sum of operators.

Theorem 2.9. *Let $\mathcal{U}_k \in [\mathcal{PQK}]_M$ for $k = 1, 2$. If $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy the following conditions for some $P \in \mathbb{C}[z]$:*

$$\begin{cases} (\mathcal{U}_1-\varrho)P(\mathcal{U}_2) = (\mathcal{U}_2-\varrho)P(\mathcal{U}_1) = 0, \\ P(\mathcal{U}_2)^*(\mathcal{U}_1-\varrho) = P(\mathcal{U}_1)^*(\mathcal{U}_2-\varrho) = 0, \\ (\mathcal{U}_2-\varrho)^*(\mathcal{U}_1-\varrho) = 0, \\ \mathcal{U}_1\mathcal{U}_2 = \mathcal{U}_2\mathcal{U}_1 = 0. \end{cases}$$

Then $\mathcal{U}_1 + \mathcal{U}_2 \in [\mathcal{PQK}]_M$.

Proof. Set $P(z) = \sum\limits_{0\leq k\leq n} a_k z^k$. Since $\mathcal{U}_1\mathcal{U}_2 = \mathcal{U}_2\mathcal{U}_1 = 0$, we obtain

$$\begin{aligned} P(\mathcal{U}_1+\mathcal{U}_2) &= \sum_{0\leq k\leq n} a_k(\mathcal{U}_1+\mathcal{U}_2)^k \\ &= \sum_{0\leq k\leq n} a_k\Big(\mathcal{U}_1^k + \binom{k}{1}\mathcal{U}_1^{k-1}\mathcal{U}_2 + \cdots + \binom{k}{k-1}\mathcal{U}_1\mathcal{U}_2^{k-1} + \mathcal{U}_2^k\Big) \\ &= \sum_{0\leq k\leq n} a_k(\mathcal{U}_1^k + \mathcal{U}_2^k) \\ &= \sum_{0\leq k\leq n} a_k\mathcal{U}_1^k + \sum_{0\leq k\leq n} a_k\mathcal{U}_2^k \\ &= P(\mathcal{U}_1) + P(\mathcal{U}_2). \end{aligned}$$

From the hypothesis that $\mathcal{U}_1$ and $\mathcal{U}_2$ are in $[\mathcal{PQK}]_M$, then both of them satisfy (2.1), and by our hypothesis

$$(\mathcal{U}_1-\varrho)P(\mathcal{U}_2)=(\mathcal{U}_2-\varrho)P(\mathcal{U}_1)=0,$$
$$P(\mathcal{U}_2)^*(\mathcal{U}_1-\varrho)=P(\mathcal{U}_1)^*(\mathcal{U}_2-\varrho)=0,$$

and

$$(\mathcal{U}_2-\varrho)^*(\mathcal{U}_1-\varrho)=0.$$

To show that $\mathcal{U}_1+\mathcal{U}_2\in[\mathcal{PQK}]_M$, we have

$$\begin{aligned}
&P(\mathcal{U}_1+\mathcal{U}_2)^*\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big(\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]P(\mathcal{U}_1+\mathcal{U}_2)\\
=\;&\Big(P(\mathcal{U}_1)^*+P(\mathcal{U}_2)^*\Big)\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]\Big(P(\mathcal{U}_1)+P(\mathcal{U}_2)\Big)\\
=\;&P(\mathcal{U}_1)^*\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]P(\mathcal{U}_1)\\
&+P(\mathcal{U}_1)^*\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]P(\mathcal{U}_2)\\
&+P(\mathcal{U}_2)^*\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]P(\mathcal{U}_1)\\
&+P(\mathcal{U}_2)^*\Big[M^2\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\\
&-\Big((\mathcal{U}_1-\varrho)+(\mathcal{U}_2-\varrho)\Big)\Big((\mathcal{U}_1-\varrho)^*+(\mathcal{U}_2-\varrho)^*\Big)\Big]P(\mathcal{U}_2)\\
=\;&P(\mathcal{U}_1)^*\Big(M^2(\mathcal{U}_1-\varrho)^*(\mathcal{U}_1-\varrho)-(\mathcal{U}_1-\varrho)(\mathcal{U}_1-\varrho)^*\Big)P(\mathcal{U}_1)+\\
&+P(\mathcal{U}_2)^*\Big(M^2(\mathcal{U}_2-\varrho)^*(\mathcal{U}_2-\varrho)-(\mathcal{U}_2-\varrho)(\mathcal{U}_2-\varrho)^*\Big)P(\mathcal{U}_2)\\
\geq\;&0.
\end{aligned}$$

Therefore, $\mathcal{U}_1+\mathcal{U}_2\in[\mathcal{PQK}]_M$. □

3. Conclusions

In this paper, we have presented a study of new class of operators which considered as an extension of previous work in this field. This study will contribute to further studies in the field of operator theory.

Author contributions

O. B. Almutairi and S. A. O. A. Mahmoud: Conceptualization, Validation, Formal analysis, Supervision, Writing-review and Editing. All authors contributed equally to the writing of this article. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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