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Research article

Solutions for gauged nonlinear Schrödinger equations on $\mathbb{R}^2$ involving sign-changing potentials

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Abstract: This study focused on establishing the existence and multiplicity of solutions for gauged nonlinear Schrödinger equations set on the plane with sign-changing potentials. Our findings contribute to the extension of recent advancements in this area of research. Initially, we examined scenarios where the potential function V is lower-bounded and the function space has a compact embedding into Lebesgue spaces. Subsequently, we addressed more complex cases characterized by a sign-changing potential V and a function space that fails to compactly embed into Lebesgue spaces. The proofs of our results are based on the Trudinger-Moser inequality, the application of variational methods, and the utilization of Morse theory.

Keywords: Schrödinger equation; variational method; sign-changing potentival; Morse theory; Chern-Simons gauge term
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1. Introduction

Jackiw and Pi [1,2] introduced a nonrelativistic model in which the nonlinear Schrödinger dynamics are coupled with the Chern-Simons gauge terms as follows:

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi = -\lambda|\phi|^{p-2}\phi, \\ \partial_0 A_1 - \partial_1 A_0 = -Im(\bar{\phi}D_2\phi), \\ \partial_0 A_2 - \partial_2 A_0 = Im(\bar{\phi}D_1\phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \tag{1.1}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_\mu : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative for $\mu = 0, 1, 2$, and λ is a positive constant representing the strength of interaction potential. This system is very useful in studying the high-temperature superconductor, Aharovnov-Bohm scattering, and fractional quantum Hall effect. For more information on system (1.1), we refer the reader to [3–5]. System (1.1) is invariant under the following transformation

$$\phi \to \phi e^{i\chi}, \;\; A_\mu \to A_\mu - \partial_\mu \chi, \tag{1.2}$$

where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary C^∞ function; this system was studied in [6]. The existence of stationary states to system (1.1) with general $p > 2$ has been studied in [7] by using the ansatz

$$\begin{aligned} &\phi(t, x) = u(|x|)e^{i\omega t}, \;\; A_0(x, t) = A_0(|x|), \\ &A_1(x, t) = \frac{x_2}{|x|} H(|x|), \;\; A_2(x, t) = -\frac{x_1}{|x|} h(|x|). \end{aligned} \tag{1.3}$$

Then the ansatz (1.3) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Inserting (1.3) into (1.1), the authors in [7] found that u satisfies the following nonlocal elliptic equation

$$-\Delta u + \omega u + \left(\xi + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right) u + \frac{h^2(|x|)}{|x|^2} u = \lambda |u|^{p-2} u \;\; in\, \mathbb{R}^2, \tag{1.4}$$

where $h(s) = \frac{1}{2} \int_0^s \tau u^2(\tau) d\tau$, ξ is a constant, $\omega > 0$.

As mentioned in [7], taking $\chi = ct$ in the gauge invariance (1.1), we derive another stationary solution for any given stationary solution; the functions $A_1(x)$, $A_2(x)$, $u(x)$ are preserved, and

$$\omega \to \omega + c, \;\; A_0(x) \to A_0(x) - c,$$

which means that the constant $\omega + \xi$ is a gauge invariant of the stationary solutions of the problem. Thus, we can choose $\xi = 0$ in what follows, i.e.,

$$\lim_{|x| \to \infty} A_0(x) = 0,$$

which was indeed assumed in [6]. Under this case, (1.1) turns into

$$-\Delta u + \omega u + \left(\int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2}\right) u = \lambda |u|^{p-2} u \;\; \text{in} \;\; \mathbb{R}^2. \tag{1.5}$$

In [8], by combing the constraint minimization method and quantitative deformation lemma, the authors proved that problem (1.5) possesses at least one energy sign-changing solution. In [9], the authors treated the problem (1.5) via a perturbation approach and the method of invariant sets of descending flow in $H^1_{rad}(\mathbb{R}^2)$ for $p \in (4, 6)$. They overcame the difficulty of the boundedness of PS-sequences and proved the existence and multiplicity of sign-changing solutions. More results on nonlinear Chern-Simons-Schrödinger equations can be found in [10–15] and references therein.

In this paper, when $\lambda = 1$, we will replace $|u|^{p-2}u$ and $\omega(x)$ of problem (1.5) with a more general nonlinearity $f(x, u)$ and sign-changing potential $V(x)$, respectively, as follows:

$$\begin{cases} -\Delta u + V(x)u + \kappa\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(x) ds\right) = f(x, u) \;\; \text{in} \;\; \mathbb{R}^2, \\ u(x) = u(|x|) \in H^1(\mathbb{R}^2), \end{cases} \tag{1.6}$$

where $V \in C(\mathbb{R}^2, \mathbb{R})$ and $f \in (\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$. The hypothesis on V is the following.

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf V(x) > -\infty$. Furthermore, there exists a constant $a_0 > 0$ such that

$$\lim_{|y|\to\infty} \text{meas}\{x \in \mathbb{R}^N : |x-y| \le a_0, V(x) \le K\} = 0, \quad \forall K > 0.$$

Remark 1.1. *The hypothesis of* (V1) *was first introduced by Bartsch and Wang [16], where* $\inf V(x) > 0$ *was required. By virtue of* (V1)*, we know that the potential* $V(x)$ *is allowed to be sign-changing. Furthermore, lots of papers give the following hypothesis.*
($\tilde{V}$*):* $V \in C(\mathbb{R}^N, \mathbb{R})$ *satisfies* $\inf V(x) \ge V_0 > 0$ *and* $\lim_{|x|\to\infty} V(x) = +\infty$ *(see [17]).*

Under this condition, their working space can compactly embed into Lebesgue spaces. Then, we can see that the condition (V1) is much weaker than ($\tilde{V}$).

From hypothesis (V1) we see that $V \in C(\mathbb{R}^N)$ is bounded from below. Then, we can take a constant $W_0 > 0$ such that $\hat{V}(x) = V(x) + W_0 \ge 1$ for $x \in \mathbb{R}^N$ and set $l(x,u) = f(x,u) + W_0 u$. Thus, (1.6) is equivalent to the following equation

$$\begin{cases} -\Delta u + \hat{V}(x)u + \kappa\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(x) ds\right) = l(x,u) \ \ in\ \mathbb{R}^2, \\ u(x) = u(|x|) \in H^1(\mathbb{R}^2). \end{cases} \tag{1.7}$$

In order to give our main results, we make the following hypotheses on the function $l(x,u)$ and its primitive function L, and introduce our working space.

(l1) $|l(x,t)| \le C_1|t| + C_2(e^{4\pi t^2} - 1)$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$;
(l2) $\lim_{|t|\to\infty} \frac{L(x,t)}{t^6} = +\infty$ and $\lim_{|t|\to 0} \frac{l(x,t)}{t} = 0$ for all $x \in \mathbb{R}^2$;
(l3) $tl(x,t) - 6L(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$;
(l4) There are constants $p > 6$ and $C_p > 0$ such that $l(x,t) \ge C_p t^{p-1}$ for all $(x,t) \in \mathbb{R}^2 \times [0,+\infty)$, where $C_p > 3^{\frac{p-2}{2}} \left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_p^p$.

Let $H = H^1_{rad}(\mathbb{R}^2)$ be the standard Sobolev space. Given the linear subspace

$$E = \left\{ u \in H : \int_{\mathbb{R}^2} \hat{V}(x)u^2 < \infty \right\},$$

we endow with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \hat{V}(x)uv)$$

and the corresponding norm $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$. Then, $(E, \|\cdot\|)$ is a Hilbert space that will be denoted by E for simplicity.

If (V1) holds, from a well-known compact embedding theorem established by Bartsch-Wang [16], we have that the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in [2,+\infty)$. It follows from the spectral theory of self-adjoint compact operators that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \ \ u \in E, \tag{1.8}$$

has a sequence of eigenvalues

$$-\infty < \lambda_1 \le \lambda_2 \le \cdots < \lambda_k < \cdots, \qquad \lambda_k \to +\infty.$$

Every λ_k has been repeated in the sequence according to its finite multiplicity. Denote by φ_k the eigenfunction of λ_k with $|\varphi_k|_2 = 1$, where $|\cdot|_r$ is the L^r-norm. The energy functional of problem (1.7) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{1}{2}\kappa\int_{\mathbb{R}^2}\frac{u^2}{|x|^2}h^2(|x|)\mathrm{d}x - \int_{\mathbb{R}^2}L(x,u)\mathrm{d}x,$$

for simplicity, in what follows, denote by

$$B(u) := \frac{1}{2}\int_{\mathbb{R}^2}\frac{u^2}{|x|^2}h^2(|x|)dx, \tag{1.9}$$

then $B \in C^1(E,\mathbb{R})$ and

$$\langle B'(u), v\rangle = \int_{\mathbb{R}^2}\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)ds\right)u(x)v(x)dx.$$

For any $u, v \in E$, we have

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^2}(\nabla u\nabla v + V(x)uv) + \kappa\langle B'(u), v\rangle - \int_{\mathbb{R}^2}l(x,u)v. \tag{1.10}$$

Consequently, the critical points of I are weak solutions of problem (1.6).

If $\kappa = 0$, problem (1.6) does not depend on the Chern-Simons term any more; then, it becomes the following Schrödinger equation:

$$-\Delta u + V(x)u = f(x,u) \ \ in\ \mathbb{R}^2. \tag{1.11}$$

Problem (1.11) was extensively discussed by lots of authors since 1970, see [18–26] and references therein. Comparing with the above equation, problem (1.6) is nonlocal, which means that it is not a pointwise identity with the appearance of the Chern-Simons term

$$\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)ds + \frac{h^2(|x|)}{|x|^2}\right)u.$$

Based on such a character, people call it a nonlocal problem and it is quite different from the usual semilinear Schrödinger equation. The nonlocal term brings some mathematical difficulties and makes this problem rough and particularly interesting. One of the main difficulties is to prove the boundedness of PS-sequences if one tries to employ directly the mountain pass theorem to derive critical points of $I(u)$ in E. Furthermore, in order to find critical points of functionals with an indefinite quadratic part, the commonly used method is the linking theorem. More precisely, let

$$\Omega_1 = \{u \in X^+ : \|u\| = \rho\},\ \ \Omega_2 = \{u \in X^{-1} \oplus \mathbb{R}^+\varphi : \|u\| \leq R\},$$

where $\varphi \in X^+ \setminus \{0\}$. If I satisfies the PS-condition and for some $0 < \rho < R$,

$$\alpha = \inf_{\Omega_1} I > \max_{\partial\Omega_2} I, \tag{1.12}$$

then, from the linking Theorem [23, Theorem 5.3], it gives rise to a nontrivial critical of I. In order to check (1.12), one usually needs to prove that $I \leq 0$ on X^-. However, since the integral $\int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)ds$

in our energy functional is positive for $u \neq 0$, it seems impossible to derive $I\,|_{X^-} \leq 0$ even if we suppose $F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Thus, unlike many other indefinite problems (see e.g., [29, 30, 37]), the usual linking theorem is not suitable for our case. Fortunately, we notice that the functional I has a local linking at the origin. Thus, we can combine the local linking theorem [28, 33] with infinite dimensional Morse theory [35] to prove our main results. Moreover, to the best of our knowledge, there have been few results on the Chern-Simons-Schrödinger system with critical exponential growth until now, that is, it behaves like $exp(\alpha|u|^2)$ as $|u| \to \infty$. More precisely, there is $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(x,s)|}{e^{\alpha s^2}} = 0, \ \ \forall\alpha > \alpha_0, \ \ and \ \ \lim_{|s|\to\infty} \frac{|f(x,s)|}{e^{\alpha s^2}} = +\infty, \ \ \forall\alpha < \alpha_0.$$

Then, in order to discuss this class of problems, the Trudinger-Moser inequalities play an important role in overcoming the difficulty of the critical case.

Our main results are the following:

Theorem 1.1. *If (V1), (l1)–(l4) hold, and 0 is not an eigenvalue of (1.8), then problem (1.6) has a nontrivial solution.*

Theorem 1.2. *If (V1), (l1)–(l4) hold, $f(x,\cdot)$ is odd for all $x \in \mathbb{R}^2$ and 0 is not an eigenvalue of (1.8), then problem (1.6) has a sequence of solutions $\{u_n\}$ such that $I(u_n) \to +\infty$.*

Next, we give an other common hypothesis on the potential V.

(V2) $V \in C(\mathbb{R}^N, \mathbb{R})$ is a bounded function such that the quadratic form $\mathscr{A} : E \to \mathbb{R}$,

$$\mathscr{A}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) \tag{1.13}$$

is non-degenerate and the negative space of $\mathscr{A}$ is finite-dimensional.

It is easy to see that, under the hypothesis (V2), the working space E cannot be compactly embedded into Lebesgue space $L^q(\mathbb{R}^2)$ for $[2, +\infty)$. In order to better discuss the problem (1.6), without loss of generality, we set $f(x,u) = |u|^{p-2}u$, $\kappa = 1$, and $p > 6$, then problem (1.6) turns into

$$\begin{cases} -\Delta u + V(x)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(x)ds\right) = |u|^{p-2}u \ \ in\ \mathbb{R}^2, \\ u(x) = u(|x|) \in H^1(\mathbb{R}^2). \end{cases} \tag{1.14}$$

Although we losse the compactness of embedding, under the condition $f(x,u) = |u|^{p-2}u$, we still have the following result.

Theorem 1.3. *Suppose that 0 is not an eigenvalue of (1.8), $p > 6$, and (V2) holds, the problem (1.14) possesses a nontrivial solution $u \in E$.*

Remark 1.2. *In the literature [36], by combining the constraint minimization method with the quantitative deformation lemma, the authors obtained at least one least energy sign-changing solution for Eq (1.5) under some assumptions. However, our Theorems 1.1–1.3 extend beyond these constraints. We consider ω not merely as a constant but as a variable function that changes sign, and we replace $|u|^{p-1}u$ with $f(x,u)$, which means that there are lots of functions that satisfy our hypotheses. Furthermore, Theorem 1.2 demonstrates the existence of infinitely many solutions. This broader scope of our study indicates a wider applicability and a more comprehensive understanding of the equations under consideration.*

This paper is structured as follows: Section 2 commences with a comprehensive exposition of the foundational concepts and preliminary notions pertinent to our investigation. Subsequently, Section 3 delineates the principal theorems, which are rigorously established through the adept application of Morse theory and variational techniques.

2. Preliminaries

First, we give some notations. $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ denotes its topological dual. C and $C_i (i = 1, 2, ...)$ denote estimated constants (the concrete values may be different from one to another one). '$\to$' means the stronger convergence in X and '$\rightharpoonup$' stands for the weak convergence in X. $|u|_p$ denotes the norm of $L^p(\mathbb{R}^2)$.

Now, we define the negative space of $\mathscr{A}$, defined in (1.8),

$$E^- = span\{\varphi_1, ..., \varphi_k\}.$$

Have E^+ be the orthogonal complement of E^+, thus $E = E^+ \oplus E^-$ and there is a constant $\delta > 0$ such that

$$\pm\mathscr{A}(u) \geq \delta\|u\|^2 \qquad for\ u \in E^\pm. \tag{2.1}$$

In the following, we give some properties, which are very important in proving our main results.

Lemma 2.1. *(see [34]) Set $\alpha > 0$ and $k > 1$. Then, for each $\beta > k$, there exists a positive constant $C = C(\beta)$ such that for all $t \in \mathbb{R}$,*

$$(e^{\alpha t^2} - 1)^k \leq C(e^{\alpha\beta t^2} - 1).$$

Moreover, if $u \in H^1(\mathbb{R}^2)$, then $(e^{\alpha t^2} - 1)^k \in L^1(\mathbb{R}^2)$.

Lemma 2.2. *(see [34]) Assume $u \in H^1(\mathbb{R}^2)$, $\alpha > 0$, $q > 0$ and $\|u\| \leq M$ with $\alpha M^2 < 4\pi$, then there is $C = C(\alpha, M, q) > 0$ such that*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)|u|^q \leq C\|v\|^q.$$

Lemma 2.3. *(see [7]) If $u_n \rightharpoonup u$ in $H^1_{rad}(\mathbb{R}^2)$ as $n \to +\infty$, then*

(i) $\lim_{n\to+\infty} B(u_n) = B(u)$;
(ii) $\lim_{n\to+\infty}\langle B'(u_n), u_n\rangle = \langle B'(u), u\rangle$;
(iii) $\lim_{n\to+\infty}\langle B'(u_n), v\rangle = \langle B'(u), v\rangle$.

Furthermore, for any $u \in H^1_{rad}(\mathbb{R}^2)$,

(iv) $B(u) = \frac{1}{2}\int_{\mathbb{R}^2} u^2 \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds dx$;
(v) $\langle B'(u), u\rangle = 6B(u)$.

Lemma 2.4. *For any $u \in H^1_{rad}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, $0 \leq B(u) \leq C\|u\|^6$.*

Proof. For any $p > 2$ and $x \in \mathbb{R}^2$, one has

$$h(|x|) = \int_{|y|\leq|x|} \frac{1}{4\pi} u^2(y) dy \leq c_p|x|^{\frac{2(p-2)}{p}}|u|_p^2.$$

Hence, if $|x| \le 1$, then for any $p \in (2,4)$ and $p' \in (4,+\infty)$,

$$\begin{aligned}\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)ds &= \int_{|x|}^{1} \frac{h(s)}{s} u^2(s)ds + \int_{1}^{\infty} \frac{h(s)}{s} u^2(s)ds \\ &\le C\|u\|_4^2 \int_{|x|}^{1} u^2(s)ds + C\|u\|_p^2 \int_1^{\infty} s^{\frac{p-4}{p}} u^2(s)ds \\ &\le C\|u\|_4^2 \left(\int_{|x|}^{1} s^{\frac{-2}{p'-2}} ds\right)^{\frac{p'-2}{p'}} \left(\int_{|x|}^{1} |u(s)|^{p'} s ds\right)^{\frac{2}{p'}} \\ &\quad + C\|u\|_p^2 \left(\int_1^{\infty} s^{\frac{p-8}{p}} ds\right)^{\frac{1}{2}} \left(\int_1^{\infty} |u(s)|^4 s ds\right)^{\frac{1}{2}} \\ &\le \|u\|_4^2 (C_{p'}\|u\|_{p'}^2 + C_p\|u\|_p^2).\end{aligned}$$

If $|x| > 1$, the above inequality is also true.

Consequently,

$$\begin{aligned}B(u) &= \frac{1}{2} \int_{\mathbb{R}^2} u^2 \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)dsdx \\ &\le \frac{1}{2}\|u\|_4^2 (C_{p'}\|u\|_{p'}^2 + C_p\|u\|_p^2) \int_{\mathbb{R}^2} u^2 dx \\ &\le C\|u\|^6.\end{aligned}$$

□

Set X be a Banach space $J : X \to \mathbb{R}$ be a C^1-functional, u is an isolated critical point of J and $J(u) = c$. Then

$$C_i(J,u) := H_i(J_c, J_c \setminus \{0\}), \quad i \in \mathbb{N} = \{0,1,2,...\},$$

is called the i-th critical group of J at u, where $J_c := J^{-1}(-\infty, c]$ and H_* denotes the singular homology with coefficients in $\mathbb{Z}$.

If J satisfies the (PS)-condition and the critical values of J are bounded from below by Θ, then, from Bartsch and Li [38], we give the i-th critical group of J at infinity by

$$C_i(J,\infty) := H_i(X, J_\Theta), \quad i \in \mathbb{N},$$

since we know that the homology on the right-hand side does not depend on the choice of Θ.

Proposition 2.1. *(see [38]) If $J \in C^1(X,\mathbb{R})$ satisfies the PS-condition, and $C_k(J,0) \ne C_k(J,\infty)$ for some $k \in \mathbb{N}$, then J has a nonzero critical point.*

Proposition 2.2. *(see [32]) Assume that $J \in C^1(X,\mathbb{R})$ has a local linking at 0 with respect to the decomposition $X = Y \oplus Z$, i.e., for some $\epsilon > 0$,*

$$\begin{aligned}J(u) \le 0 \ \ for\ u \in Y \cap B_\epsilon, \\ J(u) > 0 \ \ for\ u \in (Z \setminus \{0\}) \cap B_\epsilon,\end{aligned}$$

where $B_\epsilon = \{u \in X : \|u\| \le \epsilon\}$. If $k = dimY < \infty$, then $C_k(J,0) \ne 0$.

The following Lemma shows that I has a local linking at 0.

Lemma 2.5. *If (V1), (l1), and (l2) hold, 0 is not an eigenvalue of (1.8), then I has a local linking at* 0 *with respect to the decomposition* $E = E^- \oplus E^+$.

Proof. From (l1)-(l2), for all $\epsilon > 0$, $q > 2$, there exists $C_\epsilon > 0$ such that

$$|L(x,u)| \leq \epsilon u^2 + C_\epsilon(exp(4\pi u^2) - 1)|u|^q \ \ for\ all\ x \in \mathbb{R}^2. \tag{2.2}$$

Thus, by Lemma 2.4 and Lemma 2.2, we have that as $\|u\| \to 0$,

$$B(u) = o(\|u\|^2), \quad \int_{\mathbb{R}^2} L(x,u) = o(\|u\|^2).$$

Then, when $\|u\| \to 0$,

$$I(u) = \mathscr{A}(u) + \kappa B(u) - \int_{\mathbb{R}^2} L(x,u) = \mathscr{A}(u) + o(\|u\|^2).$$

From this equality and (2.1), one can derive the conclusion of this lemma. □

3. Proof of the main results

Remember that I satisfies the $(PS)_c$ condition, if any sequence $\{u_n\} \subset E$ along with $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence. If I satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$, then, I satisfies the (PS) condition.

Lemma 3.1. *Assume that (l1) and (l4) hold. Then there is* $\lambda_0^* > 0$ *such that for any* $0 < \lambda < \lambda_0^*$, $c < \frac{1}{3}$.

Proof. Fix a positive function $u_p \in E$,

$$S_p = \inf_{u \in E\backslash\{0\}} \frac{(\int_{\mathbb{R}^2}(|\nabla u_p|^2 + |u_p|^2))^{\frac{1}{2}}}{(\int_{\mathbb{R}^2}|u_p|^p)^{\frac{1}{p}}}.$$

It is easy to obtain that

$$\begin{aligned}\max_{t\geq 0} I_0(tu_p) &\leq \max_{t\geq 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^2}(|\nabla u_p|^2 + |u_p|^2) - \frac{C_p}{p}t^p\int_{\mathbb{R}^2}|u_p|^p\right\}\\ &= \frac{p-2}{2p}\frac{S_p^{\frac{2p}{p-2}}}{C_p^{\frac{2}{p-2}}},\end{aligned}$$

where $I_0(u) = \mathscr{A}(u) - \int_{\mathbb{R}^2} L(x,u)$. Thus, from (l4), there exists $\kappa_0^* > 0$ such that for any $0 < \kappa < \kappa_0^*$, one has

$$\max_{t\geq 0} I(tu_p) \leq \frac{p-2}{p}\frac{S_p^{\frac{2p}{p-2}}}{C_p^{\frac{2}{p-2}}} < \frac{1}{3}.$$

□

Lemma 3.2. *Assume that (V1) and (l3) hold. If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of I, i.e.,* $I(u_n) \to c$, $I'(u_n) \to 0$ *as* $n \to +\infty$, *then* $\{u_n\}$ *is bounded and* $\|u_n\| < 1$.

Proof. From (l3), for n large enough, one has

$$\begin{aligned}6c+\epsilon\|u_n\| &\geq 6I(u_n)-I'(u_n)u_n\\ &= 2\|u_n\|^2+\int_{\mathbb{R}^2}(u_n l(x,u_n)-6L(x,u_n))\\ &\geq 2\|u_n\|^2,\end{aligned}$$

where $\epsilon_n \to 0$. Then, this deduces the boundedness of $\{u_n\}$. According to Lemma 3.1, we infer that $\|u_n\| \leq 1$. □

Lemma 3.3. *Assume that (V1) and (l1)–(l3) hold. Then, any bounded PS-sequence of I has a strongly convergent subsequence in E.*

Proof. Let $\{u_n\} \subset E$ be any bounded PS-sequence of I. Passing to a subsequence if necessary, one has

$$I(u_n)\to c,\quad I'(u_n)\to 0 \ \ and \ \ \sup_n \|u_n\| < +\infty. \tag{3.1}$$

Noting that the embedding

$$E \hookrightarrow L^q(\mathbb{R}^2),\quad 2\leq q<+\infty$$

is compact, up to a subsequence if necessary, there exists $u_0 \in E$ such that

$$\begin{aligned}&u_n \rightharpoonup u_0 \quad in \quad E,\\ &u_n \to u_0 \quad in \quad L^q(\mathbb{R}^2)(2\leq q<+\infty),\\ &u_n(x)\to u_0(x) \quad a.e. \quad in \ \ \mathbb{R}^2.\end{aligned} \tag{3.2}$$

Set $u_n = u_0 + w_n$, then $w_n \rightharpoonup 0$ in E and $w_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in [2,+\infty)$. It follows from Brézis-Lieb lemma [32] that we have

$$\|u_n\|_E^2 = \|u_0\|_E^2 + \|w_n\|_E^2 + o_n(1). \tag{3.3}$$

In the following we prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} l(x,u_n)u_0 \to \int_{\mathbb{R}^2} l(x,u_0)u_0. \tag{3.4}$$

Indeed, since $C_0^\infty(\mathbb{R}^2)$ is dense in E, for any $\epsilon>0$, there is $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\|u_0-\psi\|<\epsilon$. Note that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} l(x,u_n)u_0-\int_{\mathbb{R}^2} l(x,u_0)u_0\right| &\leq \left|\int_{\mathbb{R}^2} l(x,u_n)(u_0-\psi)\right|+\left|\int_{\mathbb{R}^2} l(x,u_0)(u_0-\psi)\right|\\ &\quad+\|\psi\|_\infty\int_{supp\psi}|l(x,u_n)-l(x,u_0)|.\end{aligned} \tag{3.5}$$

For the first integral, using $|I'(u_n)(u_0-\psi)| \leq \epsilon_n\|u_0-\psi\|$ with $\epsilon_n \to 0$ as $n\to\infty$ and Lemma 2.3, we derive

$$\begin{aligned}\left|\int_{\mathbb{R}^2} l(x,u_n)(u_0-\psi)\right| &\leq \epsilon_n\|u_0-\psi\|+\kappa|\langle B'(u_n),u_0-\psi\rangle|\\ &\quad+\left|\int_{\mathbb{R}^2}|\nabla u_n\nabla(u_0-\psi)\right|\\ &\leq \epsilon_n\|u_0-\psi\|+\|u_n\|\|u_0-\psi\|+\kappa|\langle B'(u_n),u_0-\psi\rangle|\\ &\leq C\|u_0-\psi\|\leq C\epsilon\end{aligned}$$

for n large enough. Similarly, by $I'(u_0)(u_0-\psi)=0$, we derive that

$$\left|\int_{\mathbb{R}^2} l(x,u_0)(u_0-\psi)\right| \le C\epsilon.$$

Since $\lim_{n\to\infty}\int_{\mathbb{R}^2} l(x,u_n)\psi = \int_{\mathbb{R}^2} l(x,u_0)\psi \ \ \forall\psi\in C_0^\infty(\mathbb{R}^2)$, we obtain

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^2} l(x,u_n)u_0 - \int_{\mathbb{R}^2} l(x,u_0)u_0\right| \le C\epsilon.$$

Since ϵ is arbitrary, the above inequalities deduce that (3.4) is true. From (3.3) and Lemma 2.3, one has

$$\begin{aligned}
I'(u_n)u_n &= \|u_n\|^2 + \kappa\langle B'(u_n),u_0\rangle - \int_{\mathbb{R}^2} l(x,u_n)u_n\\
&= \|u_0\|^2 + \|w_n\|^2 + \kappa\langle B'(u_0),u_0\rangle - \int_{\mathbb{R}^2} l(x,u_0)u_0 - \int_{\mathbb{R}^2} l(x,u_n)w_n + o_n(1)\\
&= I'(u_0)u_0 + \|w_n\|^2 - \int_{\mathbb{R}^2} l(x,u_n)w_n + o_n(1),
\end{aligned}$$

which means that

$$\|w_n\|^2 = \int_{\mathbb{R}^2} l(x,u_n)w_n + o_n(1).$$

It follows from Lemma 2.1 and Hölder inequality that

$$\begin{aligned}
\int_{\mathbb{R}^2} l(x,u_n)w_n &\le C\int_{\mathbb{R}^2}|u_nw_n| + C_4\int_{\mathbb{R}^2}(e^{4\pi u_n^2(x)}-1)|w_n|\\
&\le C_3|u_n|_2|w_n|_2 + C_4\left(\int_{\mathbb{R}^2}(e^{4\pi u_n^2(x)}-1)^s\right)^{\frac{1}{s}}|w_n|_{s'}\\
&\le C_3|u_n|_2|w_n|_2 + C_4\left(\int_{\mathbb{R}^2}(e^{4\pi\tau u_n^2(x)}-1)\right)^{\frac{1}{s}}|w_n|_{s'}\\
&= C_3|u_n|_2|w_n|_2 + C_4\left(\int_{\mathbb{R}^2}(e^{4\pi\tau\|u_n\|^2\frac{u_n^2(x)}{\|u_n\|^2}}-1)\right)^{\frac{1}{s}}|w_n|_{s'},
\end{aligned}$$

where $s>1$, $\frac{1}{s}+\frac{1}{s'}=1$. From $\limsup_{n\to\infty}\|u_n\|^2=\varsigma\le 3c<1$, we obtain $\|u_n\|<1$ for n enough large. Now, we choose $\tau>1$ and s close to 1 such that $4\pi\tau\|u_n\|^2<4\pi$. It follows from (3.2) that

$$\int_{\mathbb{R}^2} l(x,u_n)w_n \to 0$$

as $n\to\infty$. Consequently, we have $\lim_{n\to\infty}\|w_n\|=0$, which means that $u_n\to u_0$ in E. This completes the proof. □

Lemma 3.4. *If (V1) and (l1)–(l3) hold, and 0 is not an eigenvalue of (1.8), then there exists* $A>0$ *such that if* $I(u)\le -A$*, then* $\frac{d}{dt}|_{t=1} I(tu)<0$.

Proof. Suppose this lemma is false. Then, we would have $\{u_n\}\subset E$ such that $I(u_n)\le -n$, but

$$\langle I'(u_n),u_n\rangle = \frac{d}{dt}\Big|_{t=1} I(tu_n)\ge 0.$$

Consequently, we have $\|u_n\| \to \infty$ and

$$\begin{aligned}\|u_n^+\|^2 - \|u_n^-\|^2 &\le (\|u_n^+\|^2 - \|u_n^-\|^2) + \int_{\mathbb{R}^2} (l(x,u_n)u_n - 6L(x,u_n)) \\ &\le 6I(u_n) - \langle I'(u_n), u_n\rangle \le -6n.\end{aligned} \tag{3.6}$$

Set $v_n = \frac{u_n}{\|u_n\|}$ and $v_n^\pm$ be the orthogonal projection of v_n on $E^\pm$. Then, passing to a subsequence, $v_n \to v^-$ for some $v^- \in E$ as $dimE^- < \infty$. If $v_n^- \neq 0$, then $v_n \to v$ in E for some $v \in E \setminus \{0\}$. By (l2) and (l3) one has

$$\frac{l(x,t)t}{t^6} \ge \frac{6L(x,t)}{t^6} \to +\infty$$

as $t \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. Then meas$(\{v \neq 0\}) > 0$. Hence,

$$\frac{1}{\|u_n\|_E^6} \int_{\mathbb{R}^2} l(x,u_n)u_n \ge \int_{\mathbb{R}^2} \frac{6L(x,u_n)}{u_n^6} v_n^6(x) \to +\infty. \tag{3.7}$$

By $B(u_n) \le C\|u_n\|^6$, we derive a contradiction. Note that

$$\begin{aligned} 0 &\le \frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^6} \\ &= \frac{1}{\|u_n\|^6}(\|u_n^+\|^2 - \|u_n^-\|^2) + \langle B'(u_n), u_n\rangle - \int_{\mathbb{R}^2} l(x,u_n)u_n \\ &\le o_n(1) + c - \frac{1}{\|u_n\|^6}\int_{\mathbb{R}^2} l(x,u_n)u_n \to -\infty,\end{aligned}$$

from which it follows that $v^- = 0$. But $\|u_n^+\|^2 + \|u_n^-\|^2 = 1$, one derives $\|v_n^+\| \to 1$. Now, for large n, one has

$$\|u_n^+\| = \|u_n\|\|v_n^+\| \ge \|u_n\|\|v_n^-\| = \|u_n^-\|,$$

which is a contradiction to (3.6). □

Remark 3.1. *We need to emphasize that the proof of this lemma does not depend on the compactness of the embedding $E \hookrightarrow L^2(\mathbb{R}^2)$. Thus, this result remains valid if we replace (V1) with (V2).*

Lemma 3.5. $C_i(I, \infty) = 0$ *for all* $i = 0, 1, 2, ...$

Proof. Let $B = \{v \in E : \|v\| \le 1\}$, $S = \partial B$ be the unit sphere in E, and $A > 0$ be the number given in Lemma 3.4. Without loss of generality, we may suppose that

$$-A < \inf_{\|u\| \le 2} I(u).$$

By (l2), it follows that for any $v \in S$,

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|^2 + \frac{t^6}{2}B(v) - \int_{\mathbb{R}^2} L(x,tv) \\ &= t^6\Big(\frac{\|v\|^2}{2t^4} + \frac{1}{2}B(v) - \int_{\mathbb{R}^2}\frac{L(x,tv)}{t^6}\Big) \to -\infty\end{aligned}$$

as $t \to +\infty$. Thus, there exists $t_v > 0$ such that $I(t_v v) = -A$. Let $u = t_v v$. From a simple computation, one has

$$\frac{d}{dt}\Big|_{t=t_v} I(tv) = \frac{1}{t_v}\frac{d}{ds}\Big|_{s=1} I(su) < 0.$$

By the implicit function, there exists a map T such that $T : v \mapsto t_v$ is a continuous function on S. Applying the function T, as in [25,26], one can construct a strong deformation retract $\eta : E \setminus B \to I_{-A}$,

$$\eta(u) = \begin{cases} u, & \text{if } I(u) \le -A, \\ T(\frac{u}{\|u\|})\frac{u}{\|u\|}, & \text{if } I(u) > -A, \end{cases}$$

and obtain

$$C_i(I,\infty) = H_i(E, I_{-A}) \cong H_i(E, E \setminus B) = 0 \;\; for\ all\ i \in \mathbb{N}.$$

□

Proof of Theorem 1.1. From Lemma 3.3 and Lemma 2.5, we have proved that I_λ satisfies the (PS)-condition and has a local linking at 0 with respect to the decomposition $E = E^+ \oplus E^-$. Since $E^- = k$, Proposition 2.2 yields $C_k(I, 0) \neq 0$. By Lemma 3.5, we derive that

$$C_k(I, 0) \neq C_k(I, \infty).$$

Consequently, it follows from Proposition 2.1 that I has a nonzero critical point u, which is a nontrivial solution of problem (1.6). □

In order to prove Theorem 1.2, we give the following symmetric mountain pass theorem due to Ambrosetti-Rabinowitz [31]

Proposition 3.1. *([27]) Let X be an infinite dimensional Banach space. $I(0) = 0$, $I \in C^1(X,\mathbb{R})$ satisfies the (PS)-condition and is even. If $X = Y \oplus Z$ with $dimY < \infty$, and I satisfies*

(i) *there are constants $\rho, \alpha > 0$ such that $I\ |_{\partial B_\rho \cap Z} \geq \alpha$,*
(ii) *for any finite dimensional subspace $W \subset X$, there exists an $R = R(W)$ such that $I \le 0$ on $W \setminus B_{R(W)}$,*

then I has a sequence of critical values $c_j \to +\infty$.

Lemma 3.6. *For $u \in E_i$, let $E_i = \overline{span}\{\varphi_i, \varphi_{i+1}, ...\}$ and $\beta_i = \sup_{u\in E_i, \|u\|=1} |u|_2$. Then $\beta_i \to 0$ as $i \to \infty$.*

Proof. For $u \in E_i$ with $\|u\| = 1$, one has

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \geq \lambda_i \int_{\mathbb{R}^2} u^2,$$

or equivalently, since $\hat{V}(x) = V(x) + W_0$,

$$\begin{aligned} 1 = \|u\|^2 &= \int_{\mathbb{R}^2} (|\nabla u|^2 + \hat{V}(x)u^2) \\ &\geq (\lambda_i + W_0)\int_{\mathbb{R}^2} u^2 = (\lambda_i + W_0)|u|_2^2. \end{aligned}$$

Consequently,

$$|\beta_i| \le \frac{1}{\sqrt{\lambda_i + W_0}} \to 0\ as\ \lambda_i \to +\infty.$$

Proof of Theorem 1.2. Under the hypotheses of Theorem 1.2, the functional I satisfies the PS-condition and is even. We only need to verify the assumptions (i) and (ii) of Lemma 3.1.

Verification of (i). It follows from (l1) that there exist $C_5, C_6 > 0$ and $q > 6$ such that

$$|L(x,t)| \leq C_5|t|^2 + C_6(e^{4\pi t^2} - 1)|t|^q \tag{3.8}$$

for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. For $i \in \mathbb{N}$, let E_i and β_i as in Lemma 3.6. Then, one has $\beta_i \to 0$ as $i \to \infty$. Take $k \in \mathbb{N}$ such that

$$\mu = \frac{1}{2} - C_5\beta_k^2 > 0,$$

and set

$$Y = span\{\varphi_1, ..., \varphi_{k-1}\}, \quad Z = \overline{span}\{\varphi_k, \varphi_{k+1}, ...\}.$$

So $E = Y \oplus Z$, by (3.8) and Lemma 2.2, we derive

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \kappa B(u) - \int_{\mathbb{R}^2} L(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} L(x,u) \\ &\geq \left(\frac{1}{2} - C_5\beta_k^2\right)\|u\|^2 - C_6\int_{\mathbb{R}^2}(e^{4\pi u^2} - 1)|u|^q \\ &\geq \left(\frac{1}{2} - C_5\beta_k^2\right)\|u\|^2 - C\|u\|^q \\ &= \mu\|u\|^2 + o(\|u\|^2) \end{aligned}$$

as $\|u\| \to 0$, which is easy to see that (i) is satisfied.

Verification of (ii). We only need to check that I is anti-coercive on any finite dimensional subspace $\hat{E}$. If, otherwise, there are $\{u_n\} \subset \hat{E}$ and $A > 0$ such that $\|u_n\| \to \infty$, but $I(u_n) \geq -A$. Set $v_n = \frac{u_n}{\|u_n\|}$. Passing to a subsequence if necessary, then $v_n \to v$ for some $v \in \hat{E} \setminus \{0\}$ as $\dim\hat{E} < \infty$. Similar to (3.7), one has

$$\frac{1}{\|u_n\|^6}\int_{\mathbb{R}^2} L(x,u)\mathrm{d}x \to +\infty.$$

According to Lemma 2.3, we derive

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \kappa B(u) - \int_{\mathbb{R}^2} L(x,u)\mathrm{d}x \\ &\leq \|u_n\|^6\left(\frac{1}{2\|u_n\|^4} + \kappa C - \frac{\int_{\mathbb{R}^2} L(x,u)}{\|u_n\|^6}\right) \to -\infty, \end{aligned}$$

contrary to $I(u_n) \geq -A$. Then, the proof of Theorem 1.2 is completed. □

In the following, we now assume that V satisfies (V2), then the embedding $E \hookrightarrow L^2(\mathbb{R}^2)$ is not compact anymore. Thus, we need to recover the PS-condition.

Lemma 3.7. *Let $\{u_n\}$ be a PS-sequence of I, i.e., $\sup_n |I(u_n)| < \infty$, $I'(u_n) \to 0$. Then $\{u_n\}$ is bounded in E.*

Proof. Proceeding by contradiction, we may assume that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. Then

$$v_n = v_n^+ + v_n^- \rightharpoonup v = v^+ + v^- \in E,\ v_n^\pm, v^\pm \in E^\pm.$$

If $v = 0$, then $v_n^- \to v^- = 0$ as $\dim E^- < \infty$. Noting that

$$\|v_n^+\|^2 + \|v_n^-\|^2 = 1$$

for n large enough, we obtain

$$\|v_n^+\|^2 - \|v_n^-\| \geq \frac{1}{1+\delta} \tag{3.9}$$

for any $\delta > 0$. From (13), we infer that

$$\begin{aligned} 1 + \sup_n |I(u_n)| + \|u_n\| &\geq I(u_n) - \frac{1}{6}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{3}\|u_n\|^2(\|v_n^+\|^2 - \|v_n^-\|^2) + \left(\frac{1}{6} - \frac{1}{p}\right)\int_{\mathbb{R}^2} |u_n|^p \\ &\geq \frac{1}{3(1+\delta)}\|u_n\|^2, \end{aligned}$$

contradicting to $\|u_n\| \to \infty$.

Next, we assume $v \neq 0$. Then, the set $\beth = \{v(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \beth$, one has $|u_n(x)| \to \infty$ and

$$\frac{|u_n|^p}{\|u_n\|^6} = \frac{|u_n|^p v_n^6(x)}{u_n^6(x)} \to +\infty.$$

It follows from Fatou's Lemma that

$$\int_{\mathbb{R}^2} \frac{|u_n|^p}{\|u_n\|^6} \geq \int_{\beth} \frac{|u_n|^p}{\|u_n\|^6} \to +\infty. \tag{3.10}$$

On the other hand, for large enough n,

$$\begin{aligned} &\int_{\beth} \frac{|u_n|^p}{u_n^6} v_n^6 = \frac{1}{\|u_n\|^6}\int_{\beth} |u_n|^p \leq \frac{1}{\|u_n\|^6}\int_{\mathbb{R}^2} |u_n|^p \\ &= \frac{1}{\|u_n\|^6}\left(\frac{1}{2}\|u\|^2 + \kappa B(u) - \int_{\mathbb{R}^2} L(x,u)\mathrm{d}x\right) \\ &\leq 1 + C, \end{aligned}$$

which is a contradiction to (3.10). Then, we derive that the sequence $\{u_n\}$ is bounded. □

Lemma 3.8. *If (V2) holds, then I satisfies the PS-condition.*

Proof. Let $\{u_n\}$ be a PS-sequence. It follows from Lemma 3.8 that $\{u_n\}$ is bounded in E. Passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in E. Then

$$\int_{\mathbb{R}^2} (\nabla u_n \cdot \nabla u + V(x)u_n u) \to \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) = \|u^+\|^2 - \|u^-\|^2.$$

Consequently, we have

$$
\begin{aligned}
o_n(1) &= \langle I'(u_n), u_n - u\rangle \\
&= \int_{\mathbb{R}^2} [\nabla u_n \cdot \nabla(u_n - u) + V(x)u_n(u_n - u)] + \langle B'(u_n), u_n - u\rangle \\
&\quad - \int_{\mathbb{R}^2} |u_n|^{p-2}u_n(u_n - u) \\
&= \|u_n^+\|^2 - \|u_n^-\|^2 - (\|u^+\|^2 - \|u^-\|^2) \\
&\quad + \langle B'(u_n), u_n - u\rangle - \int_{\mathbb{R}^2} |u_n|^{p-2}u_n(u_n - u).
\end{aligned}
$$

Since $\dim X^- < \infty$, we have $u_n^- \to u^-$, i.e., $\|u_n^-\| \to \|u^-\|$. Collecting all infinitesimal terms, one has

$$
\|u_n^+\|^2 - \|u^+\|^2 = o(1) + \int_{\mathbb{R}^2} |u_n|^{p-2}u_n(u_n - u) - \langle B'(u_n), u_n - u\rangle.
$$

Since $\int_{\mathbb{R}^2} |u_n|^{p-2}u_n(u_n - u) \to 0$ and $\langle B'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$, we obtain that $\|u_n^+\| \to \|u^+\|$ as $n \to \infty$, from which we infer that $u_n \to u$ in E. □

Proof of Theorem 1.3. From Remark 3.1, we know that Lemma 3.4 remains true if (V1) is replaced by (V2). Hence, under the hypotheses of Theorem 1.3, there is $A > 0$ such that $I(u) \le -A$, then

$$
\frac{d}{dt}\Big|_{t=1} I(tu_n) \ge 0. \tag{3.11}
$$

Similar to the proof of Lemma 3.5, we can obtain that $C_i(I_\lambda, \infty) = 0$ for all $i \in \mathbb{N}$. On the other hand, by an analysis similar to that in the proof of Lemma 2.5, we can prove that I also has a local linking at 0 with respect to the decomposition $E = E^- \oplus E^+$; therefore, for $k = dimE^-$, we derive $C_k(I, 0) \ne 0$, which means that

$$
C_k(I, 0) \ne C_k(I, \infty).
$$

It follows from Lemma 3.4 and Proposition 2.1 that I has a nonzero critical point, which completes the proof. □

4. Conclusions

In this study, we establish the existence and multiplicity of solutions for gauged nonlinear Schrödinger equations with sign-changing potentials on the plane. Our approach, which combines the Trudinger-Moser inequality, variational methods, and Morse theory, has proven effective in handling the complexities introduced by nonlocal terms and critical exponential growth. The theorems presented in this paper not only extend the existing knowledge in this research area but also provide new insights into the behavior of solutions under different conditions.

Looking forward, there are several promising directions for future research. Firstly, exploring the stability and dynamics of the solutions found in this study could yield valuable insights into the physical implications of these equations. Second, extending the analysis to higher dimensions or to different types of potentials could broaden the applicability of our results. Additionally, investigating the interplay between the nonlocal terms and the nonlinearities in the equation could lead to the development of more sophisticated analytical tools. Lastly, considering the impact of external fields or boundary conditions on the solutions could enrich the theoretical framework and possibly lead to new applications in physics and engineering.

Overall, this research opens up new avenues for studying nonlinear Schrödinger equations and contributes to a deeper understanding of the underlying mathematical structures and their physical relevance. □

Author contributions

Both authors contributed equally to the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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