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Research article

On the structure of irreducible Yetter-Drinfeld modules over D

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Abstract: A class of algebras $D(m, d, \xi)$ introduced by [22] were not pointed and generated by the coradical of $D(m, d, \xi)$. Let D be the quotient of $D(m, d, \xi)$ module the principle ideal $(g^m - 1)$. First, we describe all simple left modules of D. Then, according to Radford's method, we construct the Yetter-Drinfeld module over D by the tensor product of a simple module of D and D itself. Hence, we find some simple left Yetter-Drinfeld modules over D, and the relevant braidings are of a triangular type.

Keywords: Yetter-Drinfeld module; Hopf algebra
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1. Introduction

Let $\Bbbk$ be an algebraically closed field of a zero characteristic. Kaplansky first posed conjecture about the classification of finite dimensional Hopf algebras over $\Bbbk$ up to an isomorphism. A general technique is the lifting method [4] proposed by Andruskiewitsch and Schneider, which works well to classify Hopf algebras with the Chevalley Property. More recently, Andruskiewitsch and Cuadra [1] proposed a generalized lifting method to classify Hopf algebras without the Chevalley Property.

Let us briefly recall the generalized lifting method. Let H be a Hopf algebra. Assume the antipode of H is injective; then, the standard filtration $\{H_{[n]}\}_{n\geq 0}$ is a Hopf algebra filtration, where $H_{[0]}$ is the subalgebra generated by H_0. In particular, the associated graded coalgebra $grH = \bigoplus_{n\geq 0} H_{[n]}/H_{[n-1]}$ with $H_{[-1]} = 0$ is a Hopf algebra. In addition, consider a projection $\pi : grH \to H_{[0]}$. By a theorem of Radford [16], there exists a unique connected graded braided Hopf algebra $R = \bigoplus_{n\geq 0} R(n)$ in the monoidal category ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$ such that $grH \cong R\sharp H_{[0]}$. If the coradical H_0 is a Hopf subalgebra, then the generalized lifting method coincides with the lifting method. Let A be an arbitrary Hopf algebra. We say that H is a Hopf algebra over A if $H_{[0]} \simeq A$.

The lifting method has been applied to classify some finite-dimensional pointed and copointed Hopf algebras [2, 3, 5–7], etc.; additionally, it has been used to classify finite dimensional Hopf algebras whose coradicals are neither group algebras nor the duals of group algebras, for instance, [23–25]. The generalized lifting method works well for Hopf algebras without the Chevalley Property. Nevertheless, there are a few classification results such as those in [8, 9].

A class of Hopf algebras $D(m, d, \xi)$ were first introduced in [22] as affine prime regular Hopf algebras of GK-dimension one. Moreover, $D(m, d, \xi)$ are not pointed and the coradical of $D(m, d, \xi)$ is not a Hopf subalgebra; however, the Hopf subalgebra generated by the coradical is the Hopf algebra $D(m, d, \xi)$ [21].

Let $I = (g^m - 1)$, $D := D(m, d, \xi)/I$. Then, D are finite dimensional Hopf algebras. Our work is devoted to classifying finite dimensional Hopf algebras over D based on the generalized lifting method. In general, each step of the method constitutes a difficult problem to solve. This paper seeks to describe simple Yetter-Drinfeld modules over D.

Yetter-Drinfeld modules over a bialgebra were introduced by Yetter [19] in 1990. For any finite dimensional Hopf algebra H over a field k, Majid [15] identified the YetterDrinfeld modules with the modules over the Drinfeld double $D(H^{cop})$ by giving the category equivalences ${}^H_H\mathcal{YD} \approx_{H^{cop}} \mathcal{YD}^{H^{cop}} \approx_{D(H^{cop})} \mathcal{M}$. Many mathematicians have contributed to the construction of Yetter-Drinfeld modules, such as [12, 18]. We take Radford's method. First, we obtain simple modules of D by the category equivalences $\mathcal{M}^{D^*} \cong_D \mathcal{M}$. Then, according to Radford's method, we construct the Yetter-Drinfeld module over D by the tensor product of a simple module V of D and D itself. Moreover, by the structure of D, $V \otimes D = (V \otimes B) \oplus (V \otimes C)$ as Yetter-Drinfeld modules(see Section 4). For $V \otimes B$, we find the simple left Yetter-Drinfeld modules, see the Propositions 4.3 and 4.5. For $V \otimes C$, we have not found simple left Yetter-Drinfeld modules because it is difficult to describe $u_q u_l u_{p-q}$.

The paper is organized as follows. In Section 2, we recall some basics and notations of Yetter-Drinfeld modules, the structure of $D(m, d, \xi)$, and multiplicative matrices. In Section 3, we describe all simple modules of D. In Section 4, we determine simple Yetter-Drinfeld modules over D by Radford's method. In Section 5, we conclude that the braidings of simple Yetter-Drinfeld modules mentioned in Section 4 are of a triangular type.

2. Preliminaries

Conventions. Let n be a positive integer. Throughout the paper, we denote condition $0 \le i \le n-1$ simply by "$i \in n$".

If H is a Hopf algebra over $\Bbbk$, then Δ, ε, S denote the comultiplication, the counit, and the antipode, respectively. We use Sweedler's notation for the comultiplication and coaction (*i.e.*, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$). For the references of the Hopf algebra theory, one can consult [10, 13, 18, 19], etc.

If V is a $\Bbbk$-vector space, $v \in V$, $f \in V^*$, we use either $f(v)$, $\langle f, v\rangle$, or $\langle v, f\rangle$ to denote the evaluation. Throughout the paper, $\lambda^{\frac{1}{m}}$ denotes an mth root of $\lambda \in \Bbbk^*$ for a positive integer m.

2.1. Yetter-Drinfeld modules

Let H be a Hopf algebra with a bijective antipode. A left-left Yetter-Drinfeld module V over H is a left H-module $(V, \cdot)$ and a left H-comodule (V, δ) with $\delta(v) = v_{(-1)} \otimes v_{(0)} \in H \otimes V$ for all $v \in V$, satisfying the following:

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad \forall\ v \in V,\ h \in H.$$

We denote ${}^H_H\mathcal{YD}$ by the category of left-left Yetter-Drinfeld modules over H. It is a braided monoidal category: for $V,\ W \in {}^H_H\mathcal{YD}$, the braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \quad \forall\ v \in V,\ w \in W. \tag{2.1}$$

In particular, $(V, c_{V,V})$ is a braided vector space, that is, $c := c_{V,V}$ is a linear isomorphism that satisfies the following braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

A braided vector space (V, c) is of a diagonal type if there exist a basis $x_1, \dots, x_\theta$ of V and scalars $q_{ij} \in \Bbbk^\times$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad 1 \le i, j \le \theta.$$

In such case, $q = (q_{ij})_{i,j\in\mathbb{I}_{1,\theta}}$ is called a braiding matrix of (V, c).

Definition 2.1. [20, Definition 5.1] Let (M, c) be a finite dimensional vector space with a totally ordered basis X. (M, c) will be called right triangular (with respect to the basis X) if for all $x, y, z \in X$ with $z>x$, there exist $\beta_{x,y} \in \Bbbk\backslash\{0\}$ and $\omega_{x,y} \in M$ such that

$$c(x \otimes y) = \beta_{x,y} y \otimes x + \sum_{z>x} \omega_{x,y} \otimes z \quad for\ all\ x, y \in X.$$

2.2. The Hopf algebra $D(m, d, \xi)$

Let m, d be positive integers such that $(1+m)d$ is even and ξ is a primitive $2m$th root of unity. Define

$$\omega := md, \quad \gamma := \xi^2.$$

As an algebra, $D(m, d, \xi)$ is generated by $x^{\pm 1}, g, y$ and $u_0, u_1, \cdots, u_{m-1}$ with the following relations:

$$\begin{aligned}
&xx^{-1} = x^{-1}x = 1,\ gx = xg,\ yx = xy,\\
&yg = \gamma gy,\ y^m = 1 - x^\omega = 1 - g^m,\\
&u_i x = x^{-1} u_i,\ yu_i = \phi_i u_{i+1} = \xi x^d u_i y,\ u_i g = \gamma^i x^{-2d} g u_i,\\
&u_i u_j = \begin{cases}
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \phi_i\phi_{i+1}\cdots\phi_{m-2-j} y^{i+j} g\\
if\ i + j \le m - 2,\\
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} y^{i+j} g\\
if\ i + j = m - 1,\\
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j} y^{i+j-m} g\\
if\ i + j \ge m,
\end{cases}
\end{aligned}$$

where $\phi_i := 1 - \gamma^{-i-1} x^d$ and $i, j \in m$.

Then, $D(m, d, \xi)$ becomes a Hopf algebra with comultiplication, a counit, and the antipode given by the following:

$$\Delta(x) = x \otimes x,\ \Delta(g) = g \otimes g,\ \Delta(y) = y \otimes g + 1 \otimes y,$$

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j},$$
$$\varepsilon(x) = \varepsilon(g) = \varepsilon(u_0) = 1,\ \varepsilon(y) = \varepsilon(u_l) = 0,$$
$$S(x) = x^{-1},\ S(g) = g^{-1},\ S(y) = -yg^{-1} = -\gamma^{-1}g^{-1}y,$$
$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-i-1} u_i,$$

for $i \in m$ and $1 \leq l \leq m-1$.

Moreover, $D(m, d, \xi)$ has a linear basis $\{x^i g^j y^l, x^i g^j u_l | i \in \omega, j \in \mathbb{Z}, l \in m\}$ ([21, Lemma 3.3] and [22, Eq 4.7]). The authors in [11] defined the following elements in $D(m, d, \xi)^*$:

$$\zeta_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0} \lambda^{\frac{i}{\omega}} \lambda^{\frac{j}{m}} \\ x^i g^j u_l \mapsto 0 \end{cases}, \quad \chi_{\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}}} : \begin{cases} x^i g^j y^l \mapsto 0 \\ x^i g^j u_l \mapsto \delta_{l,0} \lambda^{\frac{i}{\omega}} \lambda^{\frac{j}{m}} \end{cases},$$
$$E_1 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,1} \\ x^i g^j u_l \mapsto \frac{\xi}{1-\gamma^{-1}} \delta_{l,1} \end{cases}, \quad E_2 : \begin{cases} x^i g^j y^l \mapsto \delta_{l,0}(\frac{i}{\omega} + \frac{j}{m}) \\ x^i g^j u_l \mapsto \delta_{l,0}(\frac{i}{\omega} + \frac{j}{m}) \end{cases},$$

for any $i \in \omega, j \in \mathbb{Z}, l \in m$ and $\lambda^{\frac{1}{\omega}}, \lambda^{\frac{1}{m}} \in \mathbb{k}^*$.

Additionally, [11] denoted the following:

$$E_1^{[k]} := \frac{1}{k!_\gamma} E_1^k : \begin{cases} x^i g^j y^l \mapsto \delta_{l,k} \\ x^i g^j u_l \mapsto \frac{1}{k!_\gamma} \frac{\xi^{k^2}}{(1-\gamma^{-1})^k} \delta_{l,k} = \frac{\xi^k}{(1-\gamma^{-1})(1-\gamma^{-2})\cdots(1-\gamma^{-k})} \delta_{l,k} \end{cases}.$$

2.3. *Multiplicative matrices*

The notation of the multiplicative matrices over coalgebras was once introduced in [14]. Let us recall some notations and definitions.

Notation 2.2. Let V and W be vector spaces.

1) For any matrix $\mathcal{A} := (v_{ij})_{m \times n}$ over V and matrix $\mathcal{B} := (w_{ij})_{n \times l}$ over W, denote the following matrix
$$\mathcal{A} \tilde{\otimes} \mathcal{B} := \left(\sum_{k=1}^{n} v_{ik} \otimes w_{kl} \right)_{m \times l};$$
2) For any linear map $f : V \to W$ and a matrix $\mathcal{A} := (v_{ij})_{m \times n}$ over V, denote the following matrix
$$f(\mathcal{A}) := (f(v_{ij}))_{m \times n}.$$

Then, the multiplicative matrices can be simply defined as follows.

Definition 2.3. Let (H, Δ, ε) be a coalgebra over $\mathbb{k}$.

1) A square matrix $\mathcal{G}$ over H is said to be multiplicative if $\Delta(\mathcal{G}) = \mathcal{G} \tilde{\otimes} \mathcal{G}$ and $\varepsilon(\mathcal{G}) = I$ (the identity matrix over $\mathbb{k}$) both hold;
2) A multiplicative matrix C is said to be basic if its entries are linearly independent.

Clearly, all the entries of a basic multiplicative matrix C span a simple subcoalgebra C of H. Conversely, when the base field $\mathbb{k}$ is algebraically closed, any simple coalgebra C has a basic multiplicative matrix C whose entries span C.

2.4. Combinatorial notions

The well-known quantum binomial coefficients for a parameter $q \in \Bbbk^*$ are defined as

$$\begin{pmatrix} l \\ k \end{pmatrix}_q := \frac{l!_q}{k!_q(l-k)!_q}$$

for integers $l \geq k \geq 0$, where $l!_q := 1_q 2_q \cdots l_q$ and $l_q := 1 + q + \cdots + q^{l-1}$.

Lemma 2.4. [10, Proposition IV.2.7] *Fix an invertible element q of the field k. For any scalar a, we have the following:*

$$(a - z)(a - qz)\cdots(a - q^{n-1}z) = \sum_{l=0}^{n} (-1)^l \binom{n}{l}_q q^{\frac{l(l-1)}{2}} a^{n-l} z^l.$$

3. The simple modules of D

Let $I = (g^m - 1)$, $D := D(m, d, \xi)/I$. In this section, we will determine all simple D-modules.

It is clear that D has a linear basis

$$\{x^i g^j y^l, x^i g^j u_l \mid i \in \omega, j, l \in m\}.$$

Let η be an primitive ωth root of 1. [11, Proposition 5.6] shows that

$$\{\zeta_{\eta,1}^i \zeta_{1,\gamma}^j E_1^l, \chi_{\eta,1}^i \chi_{1,\gamma}^j E_1^l \mid i \in \omega, j, l \in m\}$$

is a linear basis of the $2\omega m^2$-dimensional space D^*, where $\zeta_{\eta,1}$, $\zeta_{1,\gamma}$, $\chi_{\eta,1}$, $\chi_{1,\gamma}$ are described in Section 2.2.

Whence $\lambda^{\frac{1}{\omega}} = \eta$, $\lambda^{\frac{1}{m}} = \gamma$. Thus $\lambda = 1$ in this case. Therefore, in this paper, we let $\lambda = 1$.

Because D is finite dimensional, we have $\mathcal{M}^{D^*} \cong_D \mathcal{M}$. In order to determine all simple left D modules, we first get all simple right D^* comodules. We know that a right D^* comodule isomorphic to a minimal right coideal of D^*, which contains a simple subcoalgebra of D^*. Next, we describe all simple subcoalgebras of D^* by providing all basic multiplicative matrices in Section 2.4.

For convience, for each $j \in m$, we denote the following:

$$\begin{aligned}
&\varphi_j := 1 - \gamma^j \lambda^{\frac{1}{m}}, \;\; \varphi'_j := 1 - \gamma^j, \;\; \varphi'_0 := \frac{1}{m}, \\
&\theta_j := \frac{\varphi_j}{\varphi'_j} \;\Rightarrow\; \theta_0 \theta_1 \cdots \theta_{m-1} = 1 - \lambda, \\
&\Lambda := \lambda^{\frac{(1-m)d/2}{\omega}}.
\end{aligned}$$

Then, by [11, Lemma 5.4],

$$\begin{aligned}
&\Delta(\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}) \\
=&\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \otimes \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} + (1-\lambda)\sum_{k=1}^{m-1} \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} E_1^{[k]} \otimes \zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}} \zeta_{1,\gamma}^k E_1^{[m-k]}
\end{aligned}$$

$$+\Lambda(1-\lambda)(\theta_0^{-1}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\otimes\chi_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}$$
$$+\sum_{k=1}^{m-1}\theta_{m-k}^{-1}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1^{[k]}\otimes\chi_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}\xi^k\chi_{1,\gamma}^kE_1^{[m-k]}),$$
$$\Delta(\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}})$$
$$=\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\otimes\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}-\theta_0\sum_{k=1}^{m-1}\theta_1\cdots\theta_{k-1}\zeta_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1^{[k]}\otimes\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\xi^k\chi_{1,\gamma}^kE_1^{[m-k]}$$
$$+\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}\otimes\zeta_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}$$
$$-\theta_0\sum_{k=1}^{m-1}\lambda^{\frac{-(m-k)}{m}}\theta_1\cdots\theta_{m-k-1}\chi_{\lambda^{\frac{1}{\omega}},\lambda^{\frac{1}{m}}}E_1^{[k]}\otimes\zeta_{\lambda^{\frac{-1}{\omega}},\lambda^{\frac{-1}{m}}}\zeta_{1,\gamma}^kE_1^{[m-k]}.$$

Next, we consider the basic multiplicative matrices over D^*.

Proposition 3.1. Since $\lambda = 1$, let $\lambda^{\frac{1}{m}} = \gamma^{m-k}$ for some $0 \le k \le m-1$.

1) When $1 \le k \le m-1$, there are m multiplicative matrices of size 2:

$$\begin{pmatrix} \zeta_{\lambda^{\frac{1}{\omega}},\gamma^{m-k+j}} & \Lambda\theta_{1+j}\cdots\theta_{k-1}\xi^j\chi_{\lambda^{\frac{1}{\omega}},\gamma^{m-k+j}}E_1^{[m-k]} \\ \theta_{k+1}\cdots\theta_j\xi^{m-k+j}\chi_{\lambda^{\frac{-1}{\omega}},\gamma^j}E_1^{[k]} & \zeta_{\lambda^{\frac{-1}{\omega}},\gamma^j} \end{pmatrix}$$

and
$$\begin{pmatrix} \zeta_{\lambda^{\frac{1}{\omega}},\gamma^{j'}} & -\Lambda\theta_{k+1+j'}\cdots\theta_{m+k-1}\xi^{k+j'}\chi_{\lambda^{\frac{1}{\omega}},\gamma^{j'}}E_1^{[m-k]} \\ \theta_{k+1}\cdots\theta_{k+j'}\xi^{j'}\chi_{\lambda^{\frac{-1}{\omega}},\gamma^{k+j'}}E_1^{[k]} & \zeta_{\lambda^{\frac{-1}{\omega}},\gamma^{k+j'}} \end{pmatrix}$$
for $0 \le j \le k-1$ and $0 \le j' \le m-k-1$.

2) When $k = 0$, there are m multiplicative matrices of size 2:

$$\begin{pmatrix} \zeta_{\lambda^{\frac{1}{\omega}},\gamma^j} & \xi^j\chi_{\lambda^{\frac{1}{\omega}},\gamma^j} \\ \xi^j\chi_{\lambda^{\frac{-1}{\omega}},\gamma^j} & \zeta_{\lambda^{\frac{-1}{\omega}},\gamma^j} \end{pmatrix}$$

for $0 \le j \le m-1$.

- If $\lambda^{\frac{2}{\omega}} = 1$, then the above 2×2 multiplicative matrices are not basic with group-like entries $\zeta_{\lambda^{\frac{1}{\omega}},\gamma^j} \pm \xi^j\chi_{\lambda^{\frac{1}{\omega}},\gamma^j}$.

3) If $k = 0$, $\lambda^{\frac{2}{\omega}} \neq 1$ or $k \neq 0$, then the above 2×2 multiplicative matrices are basic.

Proof. The claim follows [11, Lemmas 5.3 and 5.4] and the definition of the multiplicative matrices. □

Remark 3.2. [11, Lemma 5.4] showed that $(\zeta_{1,\gamma} + \chi_{1,\gamma})^k$ ($k \in 2m$) were group-like elements. The case (2) in Proposition 3.1 contains these group-like elements. In fact, the case (3) in Proposition 3.1 contains all basic multiplicative matrices over D^* by the coalgebra structure of D^*.

Since all the entries of a basic multiplicative matrix span a simple subcoalgebra of D^*, then every line of basic multiplicative matrices are simple right comodules of D^*. By $\mathcal{M}^{D^*} \cong_D \mathcal{M}$, we conclude all simple left D-modules.

Proposition 3.3. Let $\lambda^{\frac{1}{m}} = 1$, $\lambda^{\frac{2}{\omega}} = 1$, $0 \le j \le m-1$. Then, there are $4m$ one-dimensional modules $\Bbbk\{v\}$ of D, and the actions of D are as follows:

$$x^i g^n y^l \cdot v = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v, \quad x^i g^n u_l \cdot v = \delta_{l,0}\xi^j\lambda^{\frac{i}{\omega}}\gamma^{jn}v,$$

or

$$x^i g^n y^l \cdot v = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v, \quad x^i g^n u_l \cdot v = -\delta_{l,0}\xi^j\lambda^{\frac{i}{\omega}}\gamma^{jn}v,$$

where $i \in \omega, n, l \in m$.

Proof. By Proposition 3.1, $\zeta_{\lambda^{\frac{1}{\omega}},\gamma^j} \pm \xi^j\chi_{\lambda^{\frac{1}{\omega}},\gamma^j}$ are group-like elements of D^*. Let $v = \zeta_{\lambda^{\frac{1}{\omega}},\gamma^j} \pm \xi^j\chi_{\lambda^{\frac{1}{\omega}},\gamma^j}$, $V = \Bbbk\{v\}$. Then, V is a right D^*-comodule with a structure map Δ. Thus (V, ψ_Δ) is a left D module by [19, Proposition 2.1.1]:

$$x^i g^n y^l \cdot v = \left\langle v, x^i g^n y^l \right\rangle v = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v,$$
$$x^i g^n u_l \cdot v = \left\langle v, x^i g^n u_l \right\rangle v = \pm\delta_{l,0}\xi^j\lambda^{\frac{i}{\omega}}\gamma^{jn}v.$$

Then, we obtain the claim. □

Proposition 3.4. Let $0 \le j \le m-1$, $1 \le k \le m-1$, or $k = 0$, $\lambda^{\frac{2}{\omega}} \ne 1$. Then, there exist two-dimensional simple D-modules $\Bbbk\{v_1, v_2\}$, and the actions of D are as follows:

$$x^i g^n y^l \cdot v_1 = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v_1, \quad x^i g^n y^l \cdot v_2 = \delta_{l,0}\lambda^{\frac{-i}{\omega}}\gamma^{(j+k)n}v_2,$$
$$x^i g^n u_l \cdot v_1 = -\delta_{l,k}\Lambda\theta_{k+j+1}\cdots\theta_{m+k-1}\theta_{k+1}\cdots\theta_{k+j}\frac{\xi^{2k+2j}\lambda^{\frac{-i}{\omega}}\gamma^{(j+k)n}}{(1-\gamma^{-1})\cdots(1-\gamma^{-k})}v_2,$$
$$x^i g^n u_l \cdot v_2 = -\delta_{l,m-k}\frac{\xi^{-k}\lambda^{\frac{i}{\omega}}\gamma^{jn}}{(1-\gamma^{-1})\cdots(1-\gamma^{-(m-k)})}v_1,$$

where $i \in \omega, n, l \in m$.

Proof. By Proposition 3.1(3), when $k = 0$, $\lambda^{\frac{2}{\omega}} \ne 1$ or $k \ne 0$, we have 2×2 basic multiplicative matrices. It is easy to know that every row of the basic multiplicative matrix spans the isomorphic minimal right codeal of D^*; then, it is a simple right D^*-comodule. First, let $1 \le k \le m-1$, $v_1 = \zeta_{\lambda^{\frac{1}{\omega}},\gamma^j}$, and $v_2 = \chi_{\lambda^{\frac{1}{\omega}},\gamma^j}E_1^{[m-k]}$, where $0 \le j \le m-k-1$. Then, $V = \Bbbk\{v_1, v_2\}$ is a right D^*-comodule with the structure map Δ :

$$\Delta(v_1) = v_1 \otimes v_1 - \Lambda\theta_{k+j+1}\cdots\theta_{m+k-1}\xi^{j+k}v_2 \otimes \theta_{k+1}\cdots\theta_{k+j}\xi^j\chi_{\lambda^{\frac{-1}{\omega}},\gamma^{j+k}}E_1^{[k]},$$
$$\Delta(v_2) = v_1 \otimes v_2 + v_2 \otimes \zeta_{\lambda^{\frac{-1}{\omega}},\gamma^{j+k}}.$$

Thus, we can obtain the action of D on V by [19, Proposition 2.1.1]:

$$x^i g^n y^l \cdot v_1 = \left\langle v_1, x^i g^n y^l \right\rangle v_1 = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v_1,$$
$$x^i g^n y^l \cdot v_2 = \left\langle \zeta_{\lambda^{\frac{-1}{\omega}},\gamma^{j+k}}, x^i g^n y^l \right\rangle v_2 = \delta_{l,0}\lambda^{\frac{-i}{\omega}}\gamma^{(j+k)n}v_2,$$
$$x^i g^n u_l \cdot v_1 = -\Lambda\theta_{k+j+1}\cdots\theta_{m+k-1}\theta_{k+1}\cdots\theta_{k+j}\xi^{k+2j}\left\langle \chi_{\lambda^{\frac{-1}{\omega}},\gamma^{j+k}}E_1^{[k]}, x^i g^n u_l \right\rangle v_2$$

$$= -\delta_{l,k}\Lambda\theta_{k+j+1}\cdots\theta_{m+k-1}\theta_{k+1}\cdots\theta_{k+j}\frac{\xi^{2k+2j}\lambda^{\frac{-i}{\omega}}\gamma^{(j+k)n}}{(1-\gamma^{-1})\cdots(1-\gamma^{-k})}v_2,$$

$$x^ig^nu_l\cdot v_2 = \left\langle v_2, x^ig^nu_l\right\rangle v_1 = -\delta_{l,m-k}\frac{\xi^{-k}\lambda^{\frac{i}{\omega}}\gamma^{jn}}{(1-\gamma^{-1})\cdots(1-\gamma^{-(m-k)})}v_1.$$

Similarly, for the first 2×2 basic multiplicative matrix in Proposition 3.1(1), we obtain the module structure for $m-k\leq j\leq m-1$ by choosing the appropriate parameters.

When $k=0$, $\lambda^{\frac{2}{\omega}}\neq 1$, the module structures are as follows:

$$x^ig^ny^l\cdot v_1 = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v_1,\quad x^ig^ny^l\cdot v_2 = \delta_{l,0}\lambda^{\frac{-i}{\omega}}\gamma^{jn}v_2,$$
$$x^ig^nu_l\cdot v_1 = \delta_{l,0}\xi^{2j}\lambda^{\frac{-i}{\omega}}\gamma^{jn}v_2,\quad x^ig^nu_l\cdot v_2 = \delta_{l,0}\lambda^{\frac{i}{\omega}}\gamma^{jn}v_1.$$

Then, we obtain the claim. □

4. Yetter-Drinfeld modules over D

Similarly, according to Radford's method [17, Proposition 2], any simple left Yetter-Drinfeld module over D can be constructed by the submodule of the tensor product of a left simple module V of D and D itself, where the module and comodule structures are given by the following:

$$\begin{aligned} &h\cdot(v\otimes g) = (h_{(2)}\cdot v)\otimes h_{(1)}gS(h_{(3)}),\\ &\rho(v\otimes g) = h_{(1)}\otimes(v\otimes h_{(2)}),\quad \forall\, h,g\in D, v\in V. \end{aligned}\tag{4.1}$$

Let $B=\Bbbk\{x^ig^jy^l\mid i\in\omega, j,l\in m\}$, and $C=\Bbbk\{x^ig^ju_l\mid i\in\omega, j,l\in m\}$. Then, $D=B\oplus C$ as coalgebras. By the Hopf algebra structure of D, we conclude that

$$B^2+C^2\subseteq B,\ BC+CB\subseteq C,\ S(B)\subseteq B,\ S(C)\subseteq C.$$

Then, as the Yetter-Drinfeld modules over D, $V\otimes D=(V\otimes B)\oplus(V\otimes C)$.

First, we consider the Yetter-Drinfeld modules $V\otimes B$. Take t to be an arbitrary integer, and define $\bar{t}$ to be the unique element in $\{0,1,\ldots,m-1\}$ that satisfies $\bar{t}\equiv t(mod\ m)$.

Proposition 4.1. Let $V=\Bbbk\{v\}$ be the one-dimensional module of D that appeared in Proposition 3.3. Then, $V\otimes B$ is a Yetter-Drinfeld module over D with the module and comodule structures given by the following:

$$x\cdot(v\otimes x^ig^ny^l) = \lambda^{\frac{1}{\omega}}(v\otimes x^ig^ny^l),\quad g\cdot(v\otimes x^ig^ny^l) = \gamma^{j-l}(v\otimes x^ig^ny^l),$$
$$y\cdot(v\otimes x^ig^ny^l) = \gamma^{-(l+1)}(\gamma^{j+n}-1)(v\otimes x^ig^{n-1}y^{l+1}),$$
$$u_p\cdot(v\otimes x^ig^ny^l) = \begin{cases} 0, & l\geq m-p,\\ 0, & 1\leq p\leq m-1,\ 0\leq\overline{n+j}\leq p-1,\\ \pm(-1)^{\overline{n+j}-p}\binom{m-1-p}{\overline{n+j}-p}_{\gamma^{-1}}\xi^{j-l-2lp-(\overline{n+j}-p)(\overline{n+j}+1-p)}(v\otimes x^{-i+(j-n-l)d}g^{n-p}y^{l+p}), & otherwise.\end{cases}$$
$$\rho(v\otimes x^ig^ny^l) = \sum_{k=0}^{l}\binom{l}{k}_{\gamma}x^ig^ny^k\otimes(v\otimes x^ig^{n+k}y^{l-k}),$$

for $0\leq p\leq m-1$, and some $i\in\omega, n,l\in m$.

Proof. We only verfy the action of u_p, since others can be easily obtained from (4.1). After a direct computation, we have that

$$\begin{aligned}
\Delta^2(u_p) &= \sum_{q=0}^{m-1} \gamma^{q(p-q)}(\sum_{k=0}^{m-1} \gamma^{k(q-k)} u_k \otimes x^{-kd} g^k u_{q-k}) \otimes x^{-qd} g^q u_{p-q},\\
S(x^{-qd} g^q u_{p-q}) &= (-1)^{p-q} \xi^{q-p} \gamma^{-\frac{(p-q)(p-q+1)}{2}} \gamma^{q(q-p)} x^{pd+\frac{3}{2}(1-m)d} g^{-p-1} u_{p-q},\\
u_q u_{p-q} &= \begin{cases} (-1)^{q-p} \xi^{q-p} \gamma^{\frac{(p-q)(p-q+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \phi_q \phi_{q+1} \cdots \phi_{m-2-(p-q)} y^p g, & (p \le m-2),\\ (-1)^{q-p} \xi^{q-p} \gamma^{\frac{(p-q)(p-q+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} y^p g & (p = m-1). \end{cases}
\end{aligned}$$

Then, by (4.1), we know that

$$\begin{aligned}
u_p \cdot (v \otimes x^i g^n y^l) &= \sum_{q=0}^{m-1} \gamma^{q(p-q)} x^{-qd} g^q u_0 \cdot v \otimes u_q x^i g^n y^l S(x^{-qd} g^q u_{p-q})\\
&= \pm\, \xi^j \sum_{q=0}^{m-1} (-1)^{p-q} \xi^{q-p-l} \gamma^{l(-p-1)+q(n+j-p-1)} \gamma^{-\frac{(p-q)(p-q+1)}{2}}\\
&\quad (v \otimes x^{-[i+(2n+l-p-\frac{1}{2}+\frac{m}{2})d]} g^{n-p-1} y^l u_q u_{p-q})\\
&= \begin{cases} \pm\frac{1}{m} \xi^{j-l-2lp} v \otimes (\sum_{q=0}^{m-1} \gamma^{q(n+j-p)} \phi_q \phi_{q+1} \cdots \phi_{m-2-(p-q)}) x^{-[i+(2n+l-p)d]} g^{n-p} y^{l+p}, & (p \le m-2)\\ \pm\frac{1}{m} \xi^{j-l-2lp} v \otimes (\sum_{q=0}^{m-1} \gamma^{q(n+j-p)}) x^{-[i+(2n+l-p)d]} g^{n-p} y^{l+p}. & (p = m-1) \end{cases}
\end{aligned}$$

By Lemma 2.4, we have that

$$\begin{aligned}
&\sum_{q=0}^{m-1} \gamma^{q(n+j-p)} \phi_q \phi_{q+1} \cdots \phi_{m-2-(p-q)}\\
&= \sum_{q=0}^{m-1} \gamma^{q(n+j-p)} \sum_{l=0}^{m-1-p} (-1)^l \binom{m-1-p}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}} \gamma^{-lq} x^{ld}\\
&= \sum_{l=0}^{m-1-p} (-1)^l \binom{m-1-p}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}} x^{ld} \sum_{q=0}^{m-1} \gamma^{q(n+j-p-l)}\\
&= \begin{cases} 0, & 1 \le p \le m-1,\ 0 \le \overline{n+j} \le p-1,\\ m(-1)^{\overline{n+j}-p} \binom{m-1-p}{\overline{n+j}-p}_{\gamma^{-1}} \gamma^{-\frac{(\overline{n+j}-p)(\overline{n+j}-p+1)}{2}} x^{(n+j-p)d}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Where the last equation holds because $\sum_{j=0}^{m-1} \gamma^{kj} = 0$ for $1 \le k \le m-1$. Therefore, we obtain the action of u_p. □

Using the same method, we have the following proposition.

Proposition 4.2. Let $W = \Bbbk\{v_1, v_2\}$ be the two-dimensional simple module of D that appeared in Proposition 3.4. Then, $W \otimes B$ is a Yetter-Drinfeld module over D with the module and the comodule structures given by the following:

$$x \cdot (v_1 \otimes x^i g^n y^l) = \lambda^{\frac{1}{\omega}} (v_1 \otimes x^i g^n y^l), \quad g \cdot (v_1 \otimes x^i g^n y^l) = \gamma^{j-l} (v_1 \otimes x^i g^n y^l),$$

$$
\begin{aligned}
&y \cdot (v_1 \otimes x^i g^n y^l) = \gamma^{-(l+1)}(\gamma^{j+n} - 1)(v_1 \otimes x^i g^{n-1} y^{l+1}),\\
&x \cdot (v_2 \otimes x^i g^n y^l) = \lambda^{-\frac{1}{\omega}}(v_2 \otimes x^i g^n y^l), \quad g \cdot (v_2 \otimes x^i g^n y^l) = \gamma^{j+k-l}(v_2 \otimes x^i g^n y^l),\\
&y \cdot (v_2 \otimes x^i g^n y^l) = \gamma^{-(l+1)}(\gamma^{j+n+k} - 1)(v_2 \otimes x^i g^{n-1} y^{l+1}),\\
&u_p \cdot (v_1 \otimes x^i g^n y^l) = \begin{cases} 0, & l \geq m - \overline{p-k},\\ 0, & 1 \leq \overline{p-k} \leq m-1,\ 0 \leq \overline{n+j} \leq \overline{p-k} - 1,\\ -(-1)^{\overline{n+j}-\overline{p-k}} \Lambda \theta_{k+j+1} \cdots \theta_{m+k-1} \theta_{k+1} \cdots \theta_{k+j} & \\ \binom{m-1-\overline{p-k}}{\overline{n+j}-\overline{p-k}}_{\gamma^{-1}} \frac{\xi^{2j+2k-l-2lp-(\overline{n+j}-\overline{p-k})(\overline{n+j}-\overline{p-k}+1)}}{(1-\gamma^{-1})\cdots(1-\gamma^{-k})} & \\ (v_2 \otimes x^{-i+(j-n-l+k)d} g^{n-p} y^{l+\overline{p-k}}), & otherwise. \end{cases}\\
&u_p \cdot (v_2 \otimes x^i g^n y^l) = \begin{cases} 0, & l \geq m - \overline{p+k-m},\\ 0, & 1 \leq \overline{p+k-m} \leq m-1,\ \overline{n+j} \geq m-k, \overline{n+j} \leq p-1,\\ -(-1)^{\overline{n+j}-p} \binom{m-1-\overline{p+k-m}}{\overline{n+j}-p}_{\gamma^{-1}} \frac{\xi^{-k-l-2lp-(\overline{n+j}-p)(\overline{n+j}+1-p)}}{(1-\gamma^{-1})\cdots(1-\gamma^{-(m-k)})} & \\ (v_1 \otimes x^{-i+(j-n-l)d} g^{n-p} y^{l+\overline{p+k-m}}), & otherwise. \end{cases}\\
&\rho(v \otimes x^i g^n y^l) = \sum_{q=0}^{l} \binom{l}{q}_{\gamma} x^i g^n y^q \otimes (v \otimes x^i g^{n+q} y^{l-q}), \qquad v \in \{v_1, v_2\}.
\end{aligned}
$$

for $0 \leq p \leq m-1$, and some $i \in \omega, n, l \in m$.

Proposition 4.3. The simple Yetter-Drinfeld submodule $V_{i,n,l}$ of $V \otimes B$ in Proposition 4.1 is

$$\Bbbk\{v \otimes x^i g^{n+t} y^{l-t}, v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t} | 0 \leq t \leq l, n+j = 0\},$$

for some $i \in \omega, n, l, j \in m$.

Proof. For a fix j in Proposition 4.1, we can always find $n \in m$ such that $n + j = 0$. Let V' be a Yetter-Drinfeld submodule of $V \otimes B$, and $v \otimes x^i g^n y^l \in V'$for some $i \in \omega, l \in m$ and $n + j = 0$. Since

$$u_0 \cdot (v \otimes x^i g^n y^l) \in \Bbbk\{v \otimes x^{-i-(2n+l)d} g^n y^l\},$$

then $v \otimes x^{-i-(2n+l)d} g^n y^l \in V'$. For the comodule structure of $v \otimes x^i g^n y^l$ and $v \otimes x^{-i-(2n+l)d} g^n y^l$, we conclude that

$$v \otimes x^i g^{n+t} y^{l-t} \in V', \ \ v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t} \in V', \ \ 0 \leq t \leq l.$$

By Proposition 4.1, the action of x, g on $v \otimes x^i g^{n+t} y^{l-t}$ and $v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t}$ are closed and

$$
\begin{aligned}
&y \cdot (v \otimes x^i g^{n+t} y^{l-t}) \in \Bbbk\{v \otimes x^i g^{n+t-1} y^{l-(t-1)}\}, && 1 \leq t \leq l,\\
&y \cdot (v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t}) \in \Bbbk\{v \otimes x^{-i-(2n+l)d} g^{n+t-1} y^{l-(t-1)}\}, && 1 \leq t \leq l,\\
&y \cdot (v \otimes x^i g^n y^l) = 0, \ \ y \cdot (v \otimes x^{-i-(2n+l)d} g^n y^l) = 0,\\
&for\ the\ following\ formula,\ 0 \leq t \leq l,\\
&u_p \cdot (v \otimes x^i g^{n+t} y^{l-t}) \begin{cases} = 0, & t < p \leq m-1,\\ \in \Bbbk\{v \otimes x^{-i-(2n+l)d} g^{n+t-p} y^{l-(t-p)}\}, & 0 \leq p \leq t. \end{cases}\\
&u_p \cdot (v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t}) \begin{cases} = 0, & t < p \leq m-1,\\ \in \Bbbk\{v \otimes x^i g^{n+t-p} y^{l-(t-p)}\}, & 0 \leq p \leq t. \end{cases}
\end{aligned}
$$

Then, $V' = \Bbbk\{v \otimes x^i g^{n+t} y^{l-t}, v \otimes x^{-i-(2n+l)d} g^{n+t} y^{l-t} | 0 \leq t \leq l, n + j = 0\}$. Next, we prove that V' is simple.

We know that the simple subcomodules of V' are $\Bbbk\{v \otimes x^i g^{n+l}\}$ and $\Bbbk\{v \otimes x^{-i-(2n+l)d} g^{n+l}\}$. Let M be a simple Yetter-Drinfeld submodule of V'. By [17, Proposition 2], there exists a simple subcomodule $N \subseteq M$ such that $M = D \cdot N$. However, N is also a simple subcomodule of V'; then, either $N = \Bbbk\{v \otimes x^i g^{n+l}\}$ or $N = \Bbbk\{v \otimes x^{-i-(2n+l)d} g^{n+l}\}$. It follows that $M = V'$. Hence, V' is simple.

□

Remark 4.4. Let $V_{i,n,l}$ and $V_{i',n',l'}$ be simple Yetter-Drinfeld submodules of $V \otimes B$ in Proposition 4.3. Then, $V_{i,n,l} \simeq V_{i',n',l'}$ in case of $\lambda^{\frac{1}{\omega}}$ is fixed, and both $n = n', l = l'$, and both have either $i \equiv -i - (2n + l)d \mod \omega$, $i' \equiv -i' - (2n + l)d \mod \omega$ or $i \not\equiv -i - (2n + l)d \mod \omega$, $i' \not\equiv -i' - (2n + l)d \mod \omega$.

Proposition 4.5. The simple Yetter-Drinfeld submodule $W_{i,n,l,k}$ of $W \otimes B$ in Proposition 4.2 is

$$\Bbbk\{v_1 \otimes x^i g^{n+t} y^{l-t}, v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k+t} y^{l-t} | 0 \leq t \leq l, n + j = 0\},$$

for some $i \in \omega, n, l, j, k \in m$.

Proof. For a fixed j in Proposition 4.2, we can always find $n \in m$ such that $n + j = 0$. Let W' be a Yetter-Drinfeld submodule of $W \otimes B$, and $v_1 \otimes x^i g^n y^l \in W'$ for some $i \in \omega, l \in m$ and $n + j = 0$. For $0 \leq k \leq m - 1$,

$$u_k \cdot (v_1 \otimes x^i g^n y^l) \in \Bbbk\{v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k} y^l\},$$

then $v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k} y^l \in W'$. By the comodule structure of $v_1 \otimes x^i g^n y^l$ and $v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k} y^l$, we conclude that

$$v_1 \otimes x^i g^{n+t} y^{l-t} \in W', \quad v_2 \otimes x^{-i-(2n+l-k)d} g^{n+t-k} y^{l-t} \in W', \quad 0 \leq t \leq l.$$

By Proposition 4.2, the action of x, g on

$$v_1 \otimes x^i g^{n+t} y^{l-t} \; and \; v_2 \otimes x^{-i-(2n+l-k)d} g^{n+t-k} y^{l-t}$$

are closed, and

$$\begin{aligned}
&y \cdot (v_1 \otimes x^i g^{n+t} y^{l-t}) \in \Bbbk\{v_1 \otimes x^i g^{n+t-1} y^{l-(t-1)}\}, && 1 \leq t \leq l,\\
&y \cdot (v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k+t} y^{l-t}) \in \Bbbk\{v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k+t-1} y^{l-(t-1)}\},\\
& && 1 \leq t \leq l,\\
&y \cdot (v_1 \otimes x^i g^n y^l) = 0, \;\; y \cdot (v_2 \otimes x^{-i+-(2n+l-k)d} g^{n-k} y^l) = 0,\\
&for\ the\ following\ formula,\ 0 \leq t \leq l,\\
&u_p \cdot (v_1 \otimes x^i g^{n+t} y^{l-t}) \begin{cases} = 0, & \overline{p-k} > t,\\ \in \Bbbk\{v_2 \otimes x^{-i-(2n+l-k)d} g^{n+t-p} y^{l-t+\overline{p-k}}\}, & 0 \leq \overline{p-k} \leq t.\end{cases}\\
&u_p \cdot (v_2 \otimes x^{-i-(2n+l-k)d} g^{n+t-k} y^{l-t}) \begin{cases} = 0, & \overline{p+k-m} > t,\\ \in \Bbbk\{v_1 \otimes x^i g^{n-k+t-p} y^{l-t+\overline{p+k-m}}\}, & 0 \leq \overline{p+k-m} \leq t.\end{cases}
\end{aligned}$$

Then, as a Yetter-Drinfeld submodule of $W \otimes B$,

$$W' = \Bbbk\{v_1 \otimes x^i g^{n+t} y^{l-t}, v_2 \otimes x^{-i-(2n+l-k)d} g^{n-k+t} y^{l-t} | 0 \leq t \leq l, n + j = 0\}.$$

The proof of W' is similar to Proposition 4.3. □

Remark 4.6. Let $W_{i,n,l,k}$ and $W_{i',n',l',k'}$ be simple Yetter-Drinfeld submodules of $W \otimes B$ in Proposition 4.5. Then, $W_{i,n,l,k} \simeq W_{i',n',l',k'}$ in the case where $\lambda^{\frac{1}{\omega}}$ is fixed, $n = n', l = l', k = k'$, and both have either $i \equiv -i - (2n + l - k)d \mod \omega$, $i' \equiv -i' - (2n + l - k)d \mod \omega$ or $i \not\equiv -i - (2n + l - k)d \mod \omega$, $i' \not\equiv -i' - (2n + l - k)d \mod \omega$.

Let $V = \Bbbk\{v\}$ be the one-dimensional module of D that appeared in Proposition 3.3. Then, $V \otimes C$ is a Yetter-Drinfeld module over D with the module and the comodule structures given by the following:

$$\begin{aligned}
x \cdot (v \otimes x^i g^n u_l) =& \lambda^{\frac{1}{\omega}} (v \otimes x^{i+2} g^n u_l), \quad g \cdot (v \otimes x^i g^n u_l) = \gamma^{j-l} (v \otimes x^{i+2d} g^n u_l),\\
y \cdot (v \otimes x^i g^n u_l) =& v \otimes [-\gamma^{j+n-2(l+1)} x^{i+3d} + (\gamma^{j+n-l-1} + \xi^{-1}\gamma^{-2(l+1)}) x^{i+2d}\\
& - \xi^{-1}\gamma^{-l-1} x^{i+d}] g^{n-1} u_{l+1},\\
\rho(v \otimes x^i g^n u_l) =& \sum_{k=0}^{l} \gamma^{k(l-k)} x^i g^n u_k \otimes (v \otimes x^{i-kd} g^{n+k} u_{l-k}),
\end{aligned}$$

where $i \in \omega, n, l \in m, 0 \leq p \leq m - 1$.

Problem. The action of x, y can easily be obtained from (4.1). After a direct computation, we have the following:

$$\begin{aligned}
u_p \cdot (v \otimes x^i g^n u_l) =& \sum_{q=0}^{m-1} \gamma^{q(p-q)} x^{-qd} g^q u_0 \cdot v \otimes u_q x^i g^n u_l S(x^{-qd} g^q u_{p-q})\\
&= \pm (-1)^p \sqrt{\Lambda} \xi^{j-p} \gamma^{-l(p+1)} \sum_{q=0}^{m-1} (-1)^q \xi^q \gamma^{q(n+j-p-1) - \frac{(p-q)(p-q+1)}{2}}\\
&(v \otimes x^{-i+[p-2n+\frac{3}{2}(1-m)]d} g^{n-p-1} y^l u_q u_l u_{p-q})
\end{aligned}$$

and

$$u_q u_l u_{p-q} = \begin{cases} (-1)^{-l} \xi^{-l} \gamma^{\frac{l(l+1)}{2}+q+l} \frac{1}{m} \phi_q \phi_{q+1} \cdots \phi_{m-2-l} \phi_{p-q} \phi_{p-q+1} \cdots \phi_{p+l-1} x^{-\frac{1+m}{2}d} g u_{p+l} & (q + l \leq m - 2)\\ (-1)^{-l} \xi^{-l} \gamma^{\frac{l(l+1)}{2}+q+l} \frac{1}{m} \phi_{p-q} \phi_{p-q+1} \cdots \phi_{p+l-1} x^{-\frac{1+m}{2}d} g u_{p+l} & (q + l = m - 1)\\ (-1)^{-l} \xi^{-l} \gamma^{\frac{l(l+1)}{2}+q+l-m} \frac{1}{m} \phi_q \cdots \phi_{m-1} \phi_0 \cdots \phi_{m-2-l} \phi_{p-q} \phi_{p-q+1} \cdots \phi_{p+l-1} x^{-\frac{1+m}{2}d} g u_{p+l-m} & (q + l \geq m) \end{cases}$$

Then, the difficulty in handling the action of u_p is how to describe $u_q u_l u_{p-q}$.

5. Braidings of simple Yetter-Drinfeld modules over D

In this section, we provide the braiding of $V_{i,n,l}$ and $W_{i,n,l,k}$. By formula (2.1), we easily have the following propositions.

Proposition 5.1. Let $V_{i,n,l}$ be simple Yetter-Drinfeld submodules of $V \otimes B$ in Proposition 4.3, and

$$v_t = v \otimes x^i g^{n+t-1} y^{l-(t-1)}, \;\; v'_t = v \otimes x^{-i-(2n+l)d} g^{n+t-1} y^{l-(t-1)},$$

where $1 \leq t \leq l + 1$. Then, the braiding of $V_{i,n,l}$ is given by the following:

$$c(v_t \otimes v_s) = \lambda^{\frac{i}{\omega}} \gamma^{(-n-l+s-1)(n+t-1)} \sum_{k=0}^{min\{s-1, l-(t-1)\}} \binom{l-(t-1)}{k}_\gamma$$

$$\gamma^{-k(n+t+l)+(s-1)+(s-2)+\cdots(s-k)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-k}-1)v_{s-k}\otimes v_{k+t},$$
$$c(v_t\otimes v'_s)=\lambda^{\frac{i}{\omega}}\gamma^{(-n-l+s-1)(n+t-1)}\sum_{k=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{k}_\gamma$$
$$\gamma^{-k(n+t+l)+(s-1)+(s-2)+\cdots(s-k)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-k}-1)v'_{s-k}\otimes v_{k+t},$$
$$c(v'_t\otimes v_s)=\lambda^{-\frac{i}{\omega}}\gamma^{(-n-l+s-1)(n+t-1)}\sum_{k=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{k}_\gamma$$
$$\gamma^{-k(n+t+l)+(s-1)+(s-2)+\cdots(s-k)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-k}-1)v_{s-k}\otimes v'_{k+t},$$
$$c(v'_t\otimes v'_s)=\lambda^{-\frac{i}{\omega}}\gamma^{(-n-l+s-1)(n+t-1)}\sum_{k=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{k}_\gamma$$
$$\gamma^{-k(n+t+l)+(s-1)+(s-2)+\cdots(s-k)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-k}-1)v'_{s-k}\otimes v'_{k+t},$$

where $1\le t,s\le l+1$.

Proposition 5.2. Let $W_{i,n,l,k}$ be simple Yetter-Drinfeld submodules of $W\otimes B$ in Proposition 4.5, and

$$\omega_t=v_1\otimes x^ig^{n+t-1}y^{l-(t-1)},\ \ \omega'_t=v_2\otimes x^{-i-(2n+l-k)d}g^{n-k+t-1}y^{l-(t-1)},$$

where $1\le t\le l+1$. Then, the braiding of $W_{i,n,l,k}$ is given by the following:

$$c(\omega_t\otimes\omega_s)=\lambda^{\frac{i}{\omega}}\gamma^{(-n-l+s-1)(n+t-1)}\sum_{p=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{p}_\gamma$$
$$\gamma^{-p(n+t+l)+(s-1)+(s-2)+\cdots(s-p)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-p}-1)\omega_{s-p}\otimes\omega_{p+t},$$
$$c(\omega_t\otimes\omega'_s)=\lambda^{-\frac{i}{\omega}}\gamma^{(-n-l+k+s-1)(n+t-1)}\sum_{p=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{p}_\gamma$$
$$\gamma^{-p(n+t+l)+(s-1)+(s-2)+\cdots(s-p)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-p}-1)\omega'_{s-p}\otimes\omega_{p+t},$$
$$c(\omega'_t\otimes\omega_s)=\lambda^{\frac{-i+(k-2n-l)d}{\omega}}\gamma^{(-n-l+s-1)(n-k+t-1)}\sum_{p=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{p}_\gamma$$
$$\gamma^{-p(n-k+t+l)+(s-1)+(s-2)+\cdots(s-p)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-p}-1)\omega_{s-p}\otimes\omega'_{p+t},$$
$$c(\omega'_t\otimes\omega'_s)=\lambda^{\frac{i-(k-2n-l)d}{\omega}}\gamma^{(-n-l+k+s-1)(n-k+t-1)}\sum_{p=0}^{min\{s-1,l-(t-1)\}}\binom{l-(t-1)}{p}_\gamma$$

$$\gamma^{-p(n-k+t+l)+(s-1)+(s-2)+\cdots(s-p)}$$
$$(\gamma^{s-1}-1)(\gamma^{s-2}-1)\cdots(\gamma^{s-p}-1)\omega'_{s-p}\otimes\omega'_{p+t},$$

where $1 \le t, s \le l+1$.

Remark 5.3. The braiding c in the above two propositions is right triangular by Definition 2.1. Additionally, by [20, Theorem 5.7], there is a pointed Hopf algebra H with an abelian coradical having V as a Yetter-Drinfeld module such that the induced braiding is c and $G(H)$ acts diagonally on V.

6. Conclusions

In this paper, we describe all simple left modules of D. Then, according to Radford's method, we construct the Yetter-Drinfeld module over D by the tensor product of a simple module of D and D itself. Hence, we find some simple left Yetter-Drinfeld modules over D, and the relevant braidings are of a triangular type.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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