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Research article

Applications of fuzzy differential subordination theory on analytic p-valent functions connected with $\mathfrak{q}$-calculus operator

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Abstract: In recent years, the concept of fuzzy set has been incorporated into the field of geometric function theory, leading to the evolution of the classical concept of differential subordination into that of fuzzy differential subordination. In this study, certain generalized classes of p-valent analytic functions are defined in the context of fuzzy subordination. It is highlighted that for particular functions used in the definitions of those classes, the classes of fuzzy p-valent convex and starlike functions are obtained, respectively. The new classes are introduced by using a $\mathfrak{q}$-calculus operator defined in this investigation using the concept of convolution. Some inclusion results are discussed concerning the newly introduced classes based on the means given by the fuzzy differential subordination theory. Furthermore, connections are shown between the important results of this investigation and earlier ones. The second part of the investigation concerns a new generalized $\mathfrak{q}$-calculus operator, defined here and having the $(p, \mathfrak{q})$-Bernardi operator as particular case, applied to the functions belonging to the new classes introduced in this study. Connections between the classes are established through this operator.

Keywords: fuzzy differential subordination; p-valent functions; convolution; $\mathfrak{q}$-analogue multiplier-Ruscheweyh operator; $\mathfrak{q}$-catas operator; $\mathfrak{q}$-Bernardi operator
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Lotfi A. Zadeh published a paper in 1965 [1] that developed the theory of fuzzy sets. Fuzzy sets theory was included in 2011 [2], in the study of complex-valued functions related to subordination.

The connection of this theory with the field of complex analysis was motivated by the many successful attempts of researchers to connect fuzzy sets with established fields of mathematical study. The differential subordination concept was first presented by the writers of [3, 4]. Fuzzy differential subordination was first proposed in 2012 [5]. A publication released in 2017 [6] provides a good overview of the evolution of the fuzzy set concept and its connections to many scientific and technical domains. It also includes references to the research done up until that moment in the context of fuzzy differential subordination theory. The research revealed in 2012 [7] showed how to adapt the well-established theory of differential subordination to the specific details that characterizes fuzzy differential subordination and provided techniques for analyzing the dominants as well as for providing the best dominants of fuzzy differential subordinations. Also, some researchers applied fuzzy differential subordination to different function classes; see [8, 9]. After that, the specific Briot–Bouquet fuzzy differential subordinations were taken into consideration for the studies [10].

Let $\mathcal{A}_p$ (p a positive integer) denote the class of functions of the form:

$$\mathfrak{f}(\eta) = \eta^p + \sum_{n=p+1}^{\infty} a_n \eta^n, \quad (p \in \mathbb{N} = \{1, 2, 3, ..\}, \ \eta \in \mathbf{U}), \tag{1.1}$$

which are analytic and multivalent (or p-valent) in the open unit disk $\mathbf{U}$ given by

$$\mathbf{U} = \{\eta : |\eta| < 1\}.$$

For $p = 1$, the class $\mathcal{A}_p = \mathcal{A}$ represents the class of normalized analytic and univalent functions in U.

Jackson [11,12] was the first to employ the $\mathfrak{q}$-difference operator in the context of geometric function theory. Carmichael [13], Mason [14], Trijitzinsky [15], and Ismail et al. [16] presented for the first time some features connected to the $\mathfrak{q}$-difference operator. Moreover, many writers have studied different $\mathfrak{q}$-calculus applications for generalized subclasses of analytic functions; see [17–26].

The Jackson's $\mathfrak{q}$-difference operator $\mathfrak{d}_{\mathfrak{q}} : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$\mathfrak{d}_{\mathfrak{q},\mathfrak{p}}\mathfrak{f}(\eta) := \begin{cases} \frac{\mathfrak{f}(\eta)-\mathfrak{f}(\mathfrak{q}\eta)}{(1-\mathfrak{q})\eta} & (\eta \neq 0; \ 0 < \mathfrak{q} < 1), \\ \\ \mathfrak{f}'(0) & (\eta = 0), \end{cases} \tag{1.2}$$

provided that $\mathfrak{f}'(0)$ exists. From (1.1) and (1.2), we deduce that

$$\mathfrak{d}_{\mathfrak{q},\mathfrak{p}}\mathfrak{f}(\eta) := [p]_{\mathfrak{q}} \eta^{p-1} + \sum_{\kappa=p+1}^{\infty} [\kappa]_{\mathfrak{q}} a_\kappa \eta^{\kappa-1}, \tag{1.3}$$

where

$$\begin{aligned} [\kappa]_{\mathfrak{q}} &= \frac{1-\mathfrak{q}^\kappa}{1-\mathfrak{q}} = 1 + \sum_{n=1}^{\kappa-1} \mathfrak{q}^n, \qquad [0]_{\mathfrak{q}} = 0, \\ [\kappa]_{\mathfrak{q}}! &= \begin{cases} [\kappa]_{\mathfrak{q}} [\kappa-1]_{\mathfrak{q}} \,......... \, [2]_{\mathfrak{q}} [1]_{\mathfrak{q}} & \kappa = 1, 2, 3, ... \\ 1 & \kappa = 0. \end{cases} \end{aligned} \tag{1.4}$$

We observe that

$$\lim_{\mathfrak{q}\to g^-} \mathfrak{d}_{\mathfrak{q},\mathfrak{p}}\mathfrak{f}(\eta) := \lim_{\mathfrak{q}\to g^-} \frac{\mathfrak{f}(\eta) - \mathfrak{f}(\mathfrak{q}\eta)}{(1-\mathfrak{q})\eta} = \mathfrak{f}^{'}(\eta).$$

The $\mathfrak{q}$-difference operator is subject to the following basic laws:

$$\eth_{\mathfrak{q}}\left(c\mathfrak{f}(\eta) \pm d\hbar(\eta)\right) = c\eth_{\mathfrak{q}}\mathfrak{f}(\eta) \pm d\eth_{\mathfrak{q}}\hbar(\eta) \tag{1.5}$$

$$\eth_{\mathfrak{q}}\left(\mathfrak{f}(\eta)\hbar(\eta)\right) = \mathfrak{f}(\mathfrak{q}\eta)\eth_{\mathfrak{q}}\left(\hbar(\eta)\right) + \hbar(\eta)\eth_{\mathfrak{q}}\left(\mathfrak{f}(\eta)\right) \tag{1.6}$$

$$\eth_{\mathfrak{q}}\left(\frac{\mathfrak{f}(\eta)}{\hbar(\eta)}\right) = \frac{\eth_{\mathfrak{q}}\left(\mathfrak{f}(\eta)\right)\hbar(\eta) - \mathfrak{f}(\eta)\eth_{\mathfrak{q}}\left(\hbar(\eta)\right)}{\hbar(\mathfrak{q}\eta)\hbar(\eta)}, \quad \hbar(\mathfrak{q}\eta)\hbar(\eta) \neq 0 \tag{1.7}$$

$$\eth_{\mathfrak{q}}\left(\log\mathfrak{f}(\eta)\right) = \frac{\ln\mathfrak{q}}{\mathfrak{q}-1}\frac{\eth_{\mathfrak{q}}\left(\mathfrak{f}(\eta)\right)}{\mathfrak{f}(\eta)}, \tag{1.8}$$

where $\mathfrak{f}, \hbar \in \mathcal{A}$, and c and d are real or complex constants.

Jackson in [12] introduced the $\mathfrak{q}$-integral of $\mathfrak{f}$ as:

$$\int_0^\eta \mathfrak{f}(t)\eth_{\mathfrak{q}}t = \eta(1-\mathfrak{q})\sum_{\kappa=0}^{\infty}\mathfrak{q}^\kappa\mathfrak{f}(\eta\mathfrak{q}^\kappa),$$

and

$$\lim_{q\to 1-}\int_0^\eta \mathfrak{f}(t)\eth_{\mathfrak{q}}t = \int_0^\eta \mathfrak{f}(t)\eth t,$$

where $\int_0^\eta \mathfrak{f}(t)\eth t$, is the ordinary integral.

The study of linear operators is an important topic for research in the field of geometric function theory. Several prominent scholars have recently expressed interest in the introduction and analysis of such linear operators with regard to $\mathfrak{q}$-analogues. The Ruscheweyh derivative operator's $\mathfrak{q}$-analogue was examined by the writers of [27], who also examined some of its properties. It was Noor et al. [28] who originally introduced the $\mathfrak{q}$-Bernardi integral operator.

In [29], Aouf and Madian investigate the $\mathfrak{q}$-p-valent Cătas operator $\mathrm{I}^s_{\mathfrak{q},\mathfrak{p}}(\lambda,\ell) : \mathcal{A}_p \to \mathcal{A}_p$ ($s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1, p \in \mathbb{N}$) as follows:

$$\begin{aligned} \mathcal{I}^s_{\mathfrak{q},\mathfrak{p}}(\lambda,\ell)\mathfrak{f}(\eta) &= \eta^p + \sum_{\kappa=p+1}^{\infty}\left(\frac{[p+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [p+\ell]_{\mathfrak{q}})}{[p+\ell]_{\mathfrak{q}}}\right)^s a_\kappa\eta^\kappa \\ (s &\in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1, p \in \mathbb{N}). \end{aligned}$$

Also, Arif et al. [30] introduced the extended $\mathfrak{q}$-derivative operator $\mathfrak{R}^{\mu+p-1}_{\mathfrak{q}} : \mathcal{A}_p \to \mathcal{A}_p$ for p-valent analytic functions is defined as follows:

$$\begin{aligned} \mathfrak{R}^{\mu+p-1}_{\mathfrak{q}}\mathfrak{f}(\eta) &= Q_p(\mathfrak{q},\mu+p;\eta) * \mathfrak{f}(\eta) \quad (\mu > -p), \\ &= \eta^p + \sum_{\kappa=p+1}^{\infty}\frac{[\mu+p,\mathfrak{q}]_{\kappa-p}}{[\kappa-p,\mathfrak{q}]!}a_\kappa\eta^\kappa. \end{aligned}$$

Setting

$$\mathfrak{G}^{s,p}_{\mathfrak{q},\lambda,\ell}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty}\left(\frac{[p+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [p+\ell]_{\mathfrak{q}})}{[p+\ell]_{\mathfrak{q}}}\right)^s\eta^\kappa.$$

Now, we define a new function $\mathfrak{G}^{s,\mu}_{\mathfrak{q},\mathfrak{p},\lambda,\ell}(\eta)$ in terms of the Hadamard product (or convolution) by:

$$\mathfrak{G}^{s,p}_{\mathfrak{q},\lambda,\ell}(\eta) * \mathfrak{G}^{s,\mu}_{\mathfrak{q},\mathfrak{p},\lambda,\ell}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty} \frac{[\mu+p,\mathfrak{q}]_{\kappa-p}}{[\kappa-p,\mathfrak{q}]!}\eta^\kappa \quad (p \in \mathbb{N}).$$

Then, motivated essentially by the $\mathfrak{q}$-analogue of the Ruscheweyh operator and the $\mathfrak{q}$-analogue Cătas operator, we now introduce the operator $\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell) : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta) = \mathfrak{G}^{s,\mu}_{\mathfrak{q},\mathfrak{p},\lambda,\ell}(\eta) * \mathfrak{f}(\eta), \tag{1.9}$$

where $s \in \mathbb{N}_0, \ell, \lambda \geq 0, \mu > -p, 0 < \mathfrak{q} < 1, p \in \mathbb{N}$. For $\mathfrak{f} \in \mathcal{A}_p$ and (1.9), it is clear that

$$\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty} \left(\frac{[p+\ell]_{\mathfrak{q}}}{[p+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [p+\ell]_{\mathfrak{q}})} \right)^s \frac{[\mu+p]_{\kappa-p,\mathfrak{q}}}{[\kappa-p]_{\mathfrak{q}}!} a_\kappa \eta^\kappa. \tag{1.10}$$

We use (1.10) to deduce the following:

$$\eta\, \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right) = \frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^\ell} \mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta) - \left(\frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^\ell} - 1 \right) \mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta),$$
$$(\lambda > 0), \tag{1.11}$$

$$\mathfrak{q}^\mu \eta\, \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right) = [\mu+p]_{\mathfrak{q}}\, \mathcal{I}^{s,p}_{\mathfrak{q},\mu+1}(\lambda,\ell)\mathfrak{f}(\eta) - [\mu]_{\mathfrak{q}}\, \mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta). \tag{1.12}$$

We note that :

(*i*) If $s = 0$ and $\mathfrak{q} \to 1^-$ the operator defined in (1.10) reduces to the differential operator investigated by Goel and Sohi [31], and further, by making $p = 1$, we get the familiar Ruscheweyh operator [32] (see also [33]). Also, for more details on the $\mathfrak{q}$-analogue of different differential operators, see the works [34, 35];

(*ii*) If we set $\mathfrak{q} \to 1^-$, $p = 1$, we obtain $\mathcal{I}^s_{\lambda,\ell,\mu}\mathfrak{f}(\eta)$ that was defined by Aouf and El-Ashwah [36];

(*iii*) If we set $\mu = 0$, and $\mathfrak{q} \to 1^-$, we obtain $J^s_{\mathfrak{p}}(\lambda,\ell)\mathfrak{f}(\eta)$ that was introduced by El-Ashwah and Aouf [37];

(*iv*) If $\mu = 0$, $\ell = \lambda = 1$, $p = 1$, and $\mathfrak{q} \to 1^-$, we obtain $\mathcal{I}^s\mathfrak{f}(\eta)$ that was investigated by Jung et al. [38];

(*v*) If $\mu = 0$, $\lambda = 1, \ell = 0$, $p = 1$, and $\mathfrak{q} \to 1^-$, we obtain $\mathcal{I}^s\mathfrak{f}(\eta)$ that was defined by Sălăgean [39];

(*vi*) If we set $\mu = 0$, $\lambda = 1$, and $p = 1$, we obtain $\mathcal{I}^\ell_{\mathfrak{q},s}\mathfrak{f}(\eta)$ that was presented by Shah and Noor [40];

(*vii*) If we set $\mu = 0$, $\lambda = 1$, $p = 1$, and $\mathfrak{q} \to 1^-$, we obtain $J^s_{\mathfrak{q},\ell}$, the Srivastava–Attiya operator; see [41, 42];

(*vii*) $\mathcal{I}^{1,1}_{\mathfrak{q},0}(1,0) = \int_0^\eta \frac{\mathfrak{f}(t)}{t}\mathfrak{d}_{\mathfrak{q}}t$ ($\mathfrak{q}$-Alexander operator [40]);

(*viii*) $\mathcal{I}^{1,1}_{\mathfrak{q},0}(1,\ell) = \frac{[1+\varrho]_{\mathfrak{q}}}{\eta^\varrho} \int_0^\eta t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t$ ($\mathfrak{q}$-Bernardi operator [28]);

(*ix*) $\mathcal{I}^{1,1}_{\mathfrak{q},0}(1,1) = \frac{[2]_{\mathfrak{q}}}{\eta} \int_0^\eta \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t$ ($\mathfrak{q}$-Libera operator [28]).

We also observe that:

(*i*) $I^{s,p}_{\mathfrak{q},\mu}(1,0)\mathfrak{f}(\eta) = I^{s,p}_{\mathfrak{q},\mu}\mathfrak{f}(\eta)$

$$\mathfrak{f}(\eta) \in \mathbf{A} : \mathcal{I}^{s,p}_{\mathfrak{q},\mu}\mathfrak{f}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty} \left(\frac{[p]_{\mathfrak{q}}}{[\kappa]_{\mathfrak{q}}} \right)^s \frac{[\mu+p]_{\kappa-p,\mathfrak{q}}}{[\kappa-p]_{\mathfrak{q}}!} a_\kappa \eta^\kappa,$$
$$(s \in \mathbb{N}_0, \mu \geq 0, 0 < \mathfrak{q} < 1, p \in \mathbb{N}, \eta \in \mathbf{U}).$$

(*ii*) $I^{s,p}_{\mathfrak{q},\mu}(1,\ell)\mathfrak{f}(\eta) = I^{s,p,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(\eta)$

$$\mathfrak{f}(\eta) \in \mathbf{A} : \mathcal{I}^{s,p,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty} \left(\frac{[p+\ell]_{\mathfrak{q}}}{[\kappa+\ell]_{\mathfrak{q}}}\right)^s \frac{[\mu+p]_{\kappa-p,\mathfrak{q}}}{[\kappa-p]_{\mathfrak{q}}!} a_\kappa \eta^\kappa,$$
$$(s \in \mathbb{N}_0, \ell > 0, \mu \geq 0, 0 < \mathfrak{q} < 1, p \in \mathbb{N}, \eta \in \mathbf{U}).$$

(*iii*) $\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,0)\mathfrak{f}(\eta) = \mathcal{I}^{s,p,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(\eta)$

$$\mathfrak{f}(\eta) \in \mathbf{A} : \mathcal{I}^{s,p,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(\eta) = \eta^p + \sum_{\kappa=p+1}^{\infty} \left(\frac{[p]_{\mathfrak{q}}}{[p]_{\mathfrak{q}} + \lambda([\kappa]_{\mathfrak{q}} - [p]_{\mathfrak{q}})}\right)^s \frac{[\mu+p]_{\kappa-p,\mathfrak{q}}}{[\kappa-p]_{\mathfrak{q}}!} a_\kappa \eta^\kappa,$$
$$(s \in \mathbb{N}_0, \lambda > 0, \mu \geq 0, 0 < \mathfrak{q} < 1, p \in \mathbb{N}, \eta \in \mathbf{U}).$$

2. Preliminaries

We provide an overview of a number of fundamental ideas that are important to our research.

Definition 2.1. [43] A mapping $\mathfrak{F}$ is said to be a fuzzy subset on $\mathfrak{Y} \neq \phi$, if it maps from $\mathfrak{Y}$ to $[0, 1]$.

Alternatively, it is defined as:

Definition 2.2. [43] A pair $(\mathbf{U}, \mathfrak{F}_{\mathbf{U}})$ is said to be a fuzzy subset on $\mathfrak{Y}$, where $\mathfrak{F}_{\mathbf{U}} : \mathfrak{Y} \to [0, 1]$ is the membership function of the fuzzy set $(\mathbf{U}, \mathfrak{F}_{\mathbf{U}})$ and $U = \{x \in \mathfrak{Y} : 0 < \mathfrak{F}_{\mathbf{U}}(x) \leq 1\} = \sup(\mathbf{U}, \mathfrak{F}_{\mathbf{U}})$ is the support of fuzzy set $(\mathbf{U}, \mathfrak{F}_{\mathbf{U}})$.

Definition 2.3. [43] Let $(\mathbf{U}_1, \mathfrak{F}_{\mathbf{U}_1})$ and $(\mathbf{U}_2, \mathfrak{F}_{\mathbf{U}_2})$ be two subsets of $\mathfrak{Y}$. Then, $(\mathbf{U}_1, \mathfrak{F}_{\mathbf{U}_1}) \subseteq (\mathbf{U}_2, \mathfrak{F}_{\mathbf{U}_2})$ if and only if $\mathfrak{F}_{\mathbf{U}_1}(t) \leq \mathfrak{F}_{\mathbf{U}_2}(t)$, $t \in \mathfrak{Y}$, whereas $(\mathbf{U}_1, \mathfrak{F}_{\mathbf{U}_1})$ and $(\mathbf{U}_2, \mathfrak{F}_{\mathbf{U}_2})$ of $\mathfrak{Y}$ are equal if and only if $\mathbf{U}_1 = \mathbf{U}_2$.

The subordination method for two analytic functions $\mathfrak{f}$ and $\mathfrak{h}$ was established by Miller and Mocanu [44]. Specifically, if $\mathfrak{f}(\eta) = \mathfrak{h}(\kappa(\eta))$, where $\kappa(\eta)$ is a Schwartz function in $\mathbf{U}$, then, $\mathfrak{f}$ is subordinate to $\mathfrak{h}$, symbolized by $\mathfrak{f} \prec \mathfrak{h}$.

According to Oros [5], the subordination technique of analytic functions can be generalized to fuzzy notions as follows:

Definition 2.4. If $\mathfrak{f}(\eta_0) = \mathfrak{h}(\eta_0)$ and $\mathfrak{F}(\mathfrak{f}(\eta)) \leq \mathfrak{F}(\mathfrak{h}(\eta))$, $(\eta \in \mathbf{U} \subset \mathbb{C})$, where $\eta_0 \in U$ be a fixed point, then $\mathfrak{f}$ is fuzzy subordinate to $\mathfrak{h}$ and is denoted by $\mathfrak{f} \prec_{\mathfrak{F}} \mathfrak{h}$.

Definition 2.5. [5] Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, and let $\mathfrak{h}$ be univalent in $\mathbb{U}$. If ω is analytic in $\mathbb{U}$ and satisfies the (second-order) fuzzy differential subordination:

$$\mathfrak{F}_{\psi(\mathbb{C}^3\times\mathbb{U})}(\psi(\omega(\eta), \eta\omega'(\eta), \eta^2\omega''(\eta); \eta)) \leq \mathfrak{F}_{\mathfrak{h}(\mathbb{U})}(\mathfrak{h}(\eta)), \tag{2.1}$$

i.e.,

$$\psi(\omega(\eta), \eta\omega'(\eta), \eta^2\omega''(\eta); \eta)) \leq_{\mathfrak{F}} (\mathfrak{h}(\zeta)), \quad \eta \in \mathbf{U},$$

then, ω is called a fuzzy solution of the fuzzy differential subordination. The univalent function ω is called a fuzzy dominant if $\omega(\eta) \prec_{\mathfrak{F}} \chi(\eta)$, for all ω satisfying (2.1). A fuzzy dominant $\widetilde{\chi}$ that satisfies $\widetilde{\chi}(\eta) \prec_{\mathfrak{F}} \chi(\eta)$ for all fuzzy dominant χ of (2.1) is said to be the fuzzy best dominant of (2.1).

Using the concept of fuzzy subordination, certain special classes are next defined.

The class of analytic functions $\mathfrak{h}(\eta)$ that are univalent convex functions in $\mathbf{U}$ with $\mathfrak{h}(0) = 1$ and $\mathrm{Re}(\mathfrak{h}(\eta)) > 0$ is denoted by Ω. We define the following for $\mathfrak{h}(\eta) \in \Omega$, $\mathfrak{F} : \mathbb{C} \to [0,1]$, $s \in \mathbb{N}_0$, $\ell,\ \lambda \geq 0$, $\mu > -p$, $0 < \mathfrak{q} < 1$, and $p \in \mathbb{N}$:

Definition 2.6. When $\mathfrak{f} \in \mathcal{A}_p$, we say that $\mathfrak{f} \in \mathfrak{F}\mathcal{M}_\gamma^p(\mathfrak{h})$ if and only if

$$\frac{(1-\gamma)}{[p]_\mathfrak{q}} \frac{\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)}{\mathfrak{f}(\eta)} + \frac{\gamma}{[p]_\mathfrak{q}} \frac{\mathfrak{d}_\mathfrak{q}\left(\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)\right)}{\mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}(\eta).$$

Furthermore,

$$\mathfrak{F}\mathcal{M}_0^p(\mathfrak{h}) = \mathfrak{F}\mathcal{ST}_p(\mathfrak{h}) = \left\{\mathfrak{f} \in \mathcal{A}_p : \frac{\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)}{[p]_\mathfrak{q} \mathfrak{f}(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}(\eta)\right\},$$

and

$$\mathfrak{F}\mathcal{M}_1^p(\mathfrak{h}) = \mathfrak{F}\mathcal{CV}_p(\mathfrak{h}) = \left\{\mathfrak{f} \in \mathcal{A}_p : \frac{\mathfrak{d}_\mathfrak{q}\left(\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)\right)}{[p]_\mathfrak{q} \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}(\eta)\right\}.$$

It is noted that

$$\mathfrak{f} \in \mathfrak{F}\mathcal{CV}_p(\mathfrak{h}) \Leftrightarrow \frac{\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)}{[p]_\mathfrak{q}} \in \mathfrak{F}\mathcal{ST}_p(\mathfrak{h}). \tag{2.2}$$

Particularly, for $\mathfrak{h}(\eta) = \frac{1+\eta}{1-\eta}$, the classes $\mathfrak{F}\mathcal{CV}_p(\mathfrak{h})$ and $\mathfrak{F}\mathcal{ST}_p(\mathfrak{h})$ reduce to the classes $\mathfrak{F}\mathcal{CV}_p$, and $\mathfrak{F}\mathcal{ST}_p$, of the fuzzy p-valent convex and fuzzy p-valent starlike functions, respectively.

With the operator $\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda,\ell)$, specified by (1.10), certain new classes of fuzzy p-valent functions are defined as follows:

Definition 2.7. Let $\mathfrak{f} \in \mathcal{A}_p, \ell, \lambda \geq 0, \mu > -p, 0 < \mathfrak{q} < 1, p \in \mathbb{N}$ and s be a real. Then,

$$\mathfrak{F}\mathcal{M}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \left\{\mathfrak{f} \in \mathcal{A}_p : \mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda,\ell)\mathfrak{f}(\eta) \in \mathfrak{F}\mathcal{M}_\gamma^p(\mathfrak{h})\right\},$$

$$\mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \left\{\mathfrak{f} \in \mathcal{A}_p : \mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda,\ell)\mathfrak{f}(\eta) \in \mathfrak{F}\mathcal{ST}_p(\mathfrak{h})\right\},$$

and

$$\mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \left\{\mathfrak{f} \in \mathcal{A}_p : \mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda,\ell)\mathfrak{f}(\eta) \in \mathfrak{F}\mathcal{CV}_p(\mathfrak{h})\right\}.$$

It is clear that

$$\mathfrak{f} \in \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) \Leftrightarrow \frac{\eta \mathfrak{d}_\mathfrak{q} \mathfrak{f}(\eta)}{[p]_\mathfrak{q}} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}). \tag{2.3}$$

Particularly, if $s = 0$, $\mu = 1$, then $\mathfrak{F}\mathcal{M}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \mathfrak{F}\mathcal{M}_\gamma^p(\mathfrak{h})$, $\mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \mathfrak{F}\mathcal{ST}_p(\mathfrak{h})$, and $\mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\gamma,\lambda,\ell;\mathfrak{h}) = \mathfrak{F}\mathcal{CV}_p(\mathfrak{h})$. Moreover, if $p = 1$, then the classes $\mathfrak{F}\mathcal{M}_\gamma^p(\mathfrak{h})$, $\mathfrak{F}\mathcal{ST}_p(\mathfrak{h})$, and $\mathfrak{F}\mathcal{CV}_p(\mathfrak{h})$ reduce to the classes $\mathfrak{F}\mathcal{M}_\gamma(\mathfrak{h})$, $\mathfrak{F}\mathcal{ST}(\mathfrak{h})$, and $\mathfrak{F}\mathcal{CV}(\mathfrak{h})$ studied in [45].

In the first part of this investigation, the goal is to establish certain inclusion relations between the classes seen in Definitions 2.6 and 2.7 using the properties of fuzzy differential subordination and then to obtain connections between the newly introduced subclasses by applying a new generalized $\mathfrak{q}$-calculus operator, which will be defined in the second part of this study. This research follows the line established by recent publications like [46–48].

3. Main results

The proofs of the main results require the following lemma:

Lemma 3.1. *[47] Let r_1, $r_2 \in \mathbb{C}$, $r_1 \neq 0$, and a convex function $\mathfrak{h}$ satisfies*

$$Re\,(r_1\mathfrak{h}(t) + r_2) > 0,\, t \in \mathbf{U}.$$

If $\mathfrak{g}$ is analytic in $\mathbf{U}$ with $\mathfrak{g}(0) = \mathfrak{h}(0)$, and $\Omega\left(\mathfrak{g}(t), t\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}(t); t\right) = \mathfrak{g}(t) + \frac{t\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}(t)}{r_1\mathfrak{g}(t)+r_2}$ is analytic in $\mathbf{U}$ with $\Omega\left(\mathfrak{h}(0), 0; 0\right) = \mathfrak{h}(0)$, then,

$$\mathfrak{F}_{\Omega(\mathbb{C}^2\times\mathbf{U})}\left[\mathfrak{g}(t) + \frac{t\mathfrak{d}_{\mathfrak{q}}\mathfrak{g}(t)}{r_1\mathfrak{g}(t) + r_2}\right] \leq \mathfrak{F}_{\mathfrak{h}(\mathbf{U})}\left(\mathfrak{h}(t)\right)$$

implies

$$\mathfrak{F}_{\mathfrak{g}(\mathbf{U})}\left(\mathfrak{g}(t)\right) \leq \mathfrak{F}_{\mathfrak{h}(\mathbf{U})}\left(\mathfrak{h}(t)\right),\ t \in \mathbf{U}.$$

3.1. Inclusion properties

In this section, we are going to discuss some inclusion properties for the classes defined above.

Theorem 3.1. *Let $\mathfrak{h} \in \Omega$, $0 \leq \gamma \leq 1, \ell, \lambda \geq 0, \mu > -p, 0 < \mathfrak{q} < 1, p \in \mathbb{N}$, and s be a real. Then,*

$$\mathfrak{F}\mathcal{M}_{\mathfrak{q},\mu}^{s,p}\left(\gamma, \lambda, \ell; \mathfrak{h}\right) \subset \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}\left(\lambda, \ell; \mathfrak{h}\right).$$

Proof. Let $\mathfrak{f} \in \mathfrak{F}\mathcal{M}_{\mathfrak{q},\mu}^{s,p}\left(\gamma, \lambda, \ell; \mathfrak{h}\right)$, and let

$$\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\,\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} = \chi(\eta), \tag{3.1}$$

with $\chi(\eta)$ being analytic in $\mathbf{U}$ and $\chi(0) = 1$.

We take logarithmic differentiation of (2.1) to get

$$\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)\right)}{\eta\mathfrak{d}_{\mathfrak{q}}\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} - \frac{\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} = \frac{\mathfrak{d}_{\mathfrak{q}}(\chi(\eta))}{\chi(\eta)}.$$

Equivalently,

$$\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)\right)}{[p]_{\mathfrak{q}}\,\mathfrak{d}_{\mathfrak{q}}\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} = \chi(\eta) + \frac{1}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}(\chi(\eta))}{\chi(\eta)}. \tag{3.2}$$

Since $\mathfrak{f} \in \mathfrak{F}\mathcal{M}_{\mathfrak{q},\mu}^{s,p}\left(\gamma, \lambda, \ell; \mathfrak{h}\right)$, from (2.1) and (3.2), we get

$$\frac{(1-\gamma)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} + \frac{\gamma}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)} = \chi(\eta) + \frac{\gamma}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}(\chi(\eta))}{\chi(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}\left(\eta\right). \tag{3.3}$$

We obtain $\chi(\eta) \prec_{\mathfrak{F}} \mathfrak{h}\left(\eta\right)$ by applying (3.3) and Lemma 3.1. Hence, $\mathfrak{f} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}\left(\lambda, \ell; \mathfrak{h}\right)$. □

Corollary 3.1. *$\mathfrak{F}M^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}^{s}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$, if $p=1$. Furthermore, if $s=0,\ \mu=1$, we obtain $\mathfrak{F}\mathcal{M}_{\gamma}(\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}(\mathfrak{h})$, and if $\gamma=1$, then $\mathfrak{F}\mathcal{CV}(\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}(\mathfrak{h})$. Additionally, for $\mathfrak{h}(\eta)=\frac{1+\eta}{1-\eta}$, we obtain $\mathfrak{F}\mathcal{CV} \subset \mathfrak{F}\mathcal{ST}$.*

Theorem 3.2. *Let $\mathfrak{h}\in\Omega$, $\gamma>1, \ell,\ \lambda\geq 0,\ \mu>-p,\ 0<\mathfrak{q}<1,\ p\in\mathbb{N}$, and s be a real. Then,*

$$\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h}) \subset \mathfrak{F}\mathcal{CV}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h}).$$

Proof. Let $\mathfrak{f}\in\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})$. Then, by definition, we write

$$\frac{(1-\gamma)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}+\frac{\gamma}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}=p_1(\eta)\prec_{\mathfrak{F}}\mathfrak{h}(\eta).$$

Now,

$$\begin{aligned}\frac{\gamma}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} &= \frac{(1-\gamma)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}+\frac{\gamma}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}\\ +\frac{(\gamma-1)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} &= \frac{(\gamma-1)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}+p_1(\eta).\end{aligned}$$

This implies

$$\begin{aligned}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{[p]_{\mathfrak{q}}\,\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} &= \frac{1}{\gamma}p_1(\eta)+\left(1-\frac{1}{\gamma}\right)\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\,\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}\\ &= \frac{1}{\gamma}p_1(\eta)+\left(1-\frac{1}{\gamma}\right)p_2(\eta).\end{aligned}$$

Since $p_1, p_2\prec_{\mathfrak{F}}\mathfrak{h}(\eta)$, $\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{[p]_q\eta\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}\prec_{\mathfrak{F}}\mathfrak{h}(\eta)$. This is the expected outcome. □

In particular, if $p=1$, we get $\mathfrak{F}\mathcal{M}^{s}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})\subset\mathfrak{F}\mathcal{CV}^{s}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})$. Additionally, when $s=0,\ \mu=1$, we have $\mathfrak{F}\mathcal{M}_{\gamma}(\mathfrak{h})\subset\mathfrak{F}\mathcal{CV}(\mathfrak{h})$, and considering $\mathfrak{h}(\eta)=\frac{1+\eta}{1-\eta}$, we obtain $\mathfrak{F}\mathcal{M}^{p}_{\gamma}\subset\mathfrak{F}\mathcal{CV}$.

Theorem 3.3. *Let $\mathfrak{h}\in\Omega$, $0\leq\gamma_1<\gamma_2<1,\ \ell,\ \lambda\geq 0,\ \mu>-p,\ 0<\mathfrak{q}<1,\ p\in\mathbb{N}$, and s be a real number. Then,*

$$\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma_2,\lambda,\ell;\mathfrak{h})\subset\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma_1,\lambda,\ell;\mathfrak{h}).$$

Proof. For $\gamma_1=0$, it is obviously true, based on the preceding theorem.

Let $\mathfrak{f}\in\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma_2,\lambda,\ell;\mathfrak{h})$. Then, by definition, we have

$$\frac{(1-\gamma_2)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}+\frac{\gamma_2}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\eta\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}=\mathfrak{g}_1(\eta)\prec_{\mathfrak{F}}\mathfrak{h}(\eta). \tag{3.4}$$

Now, we can easily write

$$\frac{(1-\gamma_1)}{[p]_{\mathfrak{q}}}\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}+\frac{\gamma_1}{[p]_{\mathfrak{q}}}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}=\frac{\gamma_1}{\gamma_2}\mathfrak{g}_1(\eta)+\left(1-\frac{\gamma_1}{\gamma_2}\right)\mathfrak{g}_2(\eta), \tag{3.5}$$

where we have used (3.4), and $\frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} = \mathfrak{g}_2(\eta) \prec_{\mathfrak{F}} \mathfrak{h}(\eta)$. Since $\mathfrak{g}_1, \mathfrak{g}_2 \prec_{\mathfrak{F}} \mathfrak{h}(\eta)$, (3.5) implies

$$\frac{(1-\gamma_1)}{[p]_{\mathfrak{q}}} \frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} + \frac{\gamma_1}{[p]_{\mathfrak{q}}} \frac{\mathfrak{d}_{\mathfrak{q}}\left(\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)\right)}{\mathfrak{d}_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}(\eta).$$

This proves the theorem. □

Remark 3.1. If $\gamma_2 = 1$, and $\mathfrak{f} \in \mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(1,\lambda,\ell;\mathfrak{h}) = \mathfrak{F}\mathcal{CV}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$, then the previously proved result shows that

$$\mathfrak{f} \in \mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma_1,\lambda,\ell;\mathfrak{h}), \text{ for } 0 \le \gamma_1 < 1.$$

Consequently, by using Theorem 3.1, we get $\mathfrak{F}\mathcal{CV}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(1,\lambda,\ell;\mathfrak{h})$.

Now, certain inclusion results are discussed for the subclasses given by Definition 2.1.

Theorem 3.4. *Let* $\mathfrak{h} \in \Omega$, ℓ, $\lambda \ge 0$, $\mu > -p$, $0 < \mathfrak{q} < 1$, $p \in \mathbb{N}$, *and* s *be a real with* $[\ell+p]_{\mathfrak{q}} > \lambda \mathfrak{q}^{\ell}$. *Then,*

$$\mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu+1}(\lambda,\ell;\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h}) \subset \mathfrak{F}\mathcal{ST}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h}).$$

Proof. Let $\mathfrak{f} \in \mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$. Then,

$$\frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} \prec_{\mathfrak{F}} \mathfrak{h}(\eta).$$

Now, we set

$$\frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} = \mathfrak{P}(\eta), \tag{3.6}$$

with analytic $\mathfrak{P}(\eta)$ in $\mathbf{U}$ and $\mathfrak{P}(0) = 1$.
From (1.11) and (3.6), we get

$$\frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} = \frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} \frac{\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} - \frac{1}{[p]_{\mathfrak{q}}}\left(\frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} - 1\right),$$

equivalently,

$$\frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} \frac{\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s+1,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} = \mathfrak{P}(\eta) + \xi_{\mathfrak{q},p}.$$

where $\xi_{\mathfrak{q},p} = \frac{1}{[p]_{\mathfrak{q}}}\left(\frac{[\ell+p]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} - 1\right)$.

On $\mathfrak{q}$-logarithmic differentiation yields,

$$\frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)} = \mathfrak{P}(\eta) + \frac{\eta \mathfrak{d}_{\mathfrak{q}}(\mathfrak{P}(\eta))}{[p]_{\mathfrak{q}}(\mathfrak{P}(\eta) + \xi_{\mathfrak{q},p})}. \tag{3.7}$$

Since $\mathfrak{f} \in \mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$, (3.7) implies

$$\mathfrak{P}(\eta) + \frac{\eta \mathfrak{d}_{\mathfrak{q}}(\mathfrak{P}(\eta))}{[p]_{\mathfrak{q}}(\mathfrak{P}(\eta) + \xi_{\mathfrak{q},p})} \prec_{\mathfrak{F}} \mathfrak{h}(\eta). \tag{3.8}$$

We conclude that $\mathfrak{P}(\eta) \prec_{\mathfrak{F}} \mathfrak{h}(\eta)$ by applying (3.8) and Lemma 3.1. Hence, $\mathfrak{f} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell; \mathfrak{h})$. To prove the first part, let $\mathfrak{f} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu+1}^{s,p}(\lambda, \ell; \mathfrak{h})$, and set

$$\chi(\eta) = \frac{\eta \mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}} \mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)},$$

where χ is analytic in $\mathbf{U}$ with $\chi(0) = 1$. Then, it follows $\chi \prec_{\mathfrak{F}} \mathfrak{h}(\eta)$ that by applying the same arguments as those described before with (1.12). Theorem 3.4's proof is now complete. □

Theorem 3.5. *Let* $\mathfrak{h} \in \Omega$, $\ell, \lambda \geq 0, \mu > -p, 0 < \mathfrak{q} < 1, p \in \mathbb{N}$, *and* s *be a real. Then,*

$$\mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu+1}^{s,p}(\lambda, \ell; \mathfrak{h}) \subset \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell; \mathfrak{h}) \subset \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell; \mathfrak{h}).$$

Proof. Let $\mathfrak{f} \in \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell; \mathfrak{h})$. Applying (2.3), we show that

$$\begin{aligned}
\mathfrak{f} \in \ & \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell; \mathfrak{h}) \Leftrightarrow \frac{\eta(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})}{[p]_{\mathfrak{q}}} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell; \mathfrak{h}) \\
\Leftrightarrow \ & \frac{\eta}{[p]_{\mathfrak{q}}}\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s,p}(\lambda, \ell)\mathfrak{f}(\eta)\right) \in \mathfrak{F}\mathcal{ST}_p(\mathfrak{h}) \\
\Leftrightarrow \ & \frac{\eta(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})}{[p]_{\mathfrak{q}}} \in \mathfrak{F}\mathcal{ST}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell; \mathfrak{h}) \\
\Leftrightarrow \ & \frac{\eta}{[p]_{\mathfrak{q}}}\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell)\mathfrak{f}(\eta)\right) \in \mathfrak{F}\mathcal{ST}_p(\mathfrak{h}) \\
\Leftrightarrow \ & \mathcal{I}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell)\left(\frac{\eta(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})}{[p]_{\mathfrak{q}}}\right) \in \mathfrak{F}\mathcal{ST}_p(\mathfrak{h}) \\
\Leftrightarrow \ & \mathcal{I}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell)\mathfrak{f}(\eta) \in \mathfrak{F}\mathcal{CV}_p(\mathfrak{h}) \\
\Leftrightarrow \ & \mathfrak{f} \in \mathfrak{F}\mathcal{CV}_{\mathfrak{q},\mu}^{s+1,p}(\lambda, \ell; \mathfrak{h}).
\end{aligned}$$

We can demonstrate the first part using arguments similar to those described above. Theorem 3.5's proof is now complete. □

3.2. Properties involving integral operator.

For $\mathfrak{f}(\eta) \in \mathcal{A}_p$, the generalized $(p, \mathfrak{q})$-Bernardi integral operator for p-valent functions $\mathfrak{B}_{n,\mathfrak{q}}^p\mathfrak{f}(\eta) : \mathcal{A}_p \to \mathcal{A}_p$ is defined by

$$\mathfrak{B}_{n,\mathfrak{q}}^p\mathfrak{f}(\eta) = \begin{cases} \mathfrak{B}_{1,\mathfrak{q}}^p(\mathfrak{B}_{n-1,\mathfrak{q}}^p\mathfrak{f}(\eta)), & (n \in \mathbb{N}), \\ \mathfrak{f}(\eta), & (n = 0), \end{cases}$$

where $\mathfrak{B}_{1,\mathfrak{q}}^p\mathfrak{f}(\eta)$ is given by

$$\begin{aligned}
\mathfrak{B}_{1,\mathfrak{q}}^p\mathfrak{f}(\eta) &= \frac{[p+\varrho]_{\mathfrak{q}}}{\eta^{\varrho}}\int_0^{\eta} t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \\
&= \eta^p + \sum_{\kappa=p+1}^{\infty}\left(\frac{[p+\varrho]_{\mathfrak{q}}}{[\kappa+\varrho]_{\mathfrak{q}}}\right)a_{\kappa}z^{\kappa}, \quad (\varrho > -p, \eta \in \mathbf{U}).
\end{aligned} \tag{3.9}$$

From $\mathfrak{B}^{p}_{1,\mathfrak{q}}\mathfrak{f}(\eta)$, we deduce that

$$\begin{aligned}\mathfrak{B}^{p}_{2,\mathfrak{q}}\mathfrak{f}(\eta) &= \mathfrak{B}^{p}_{1,\mathfrak{q}}(\mathfrak{B}^{p}_{1,\mathfrak{q}}\mathfrak{f}(\eta))\\ &= \eta^{p}+\sum_{\kappa=p+1}^{\infty}\left(\frac{[p+\varrho]_{\mathfrak{q}}}{[\kappa+\varrho]_{\mathfrak{q}}}\right)^{2}a_{\kappa}z^{\kappa}, \qquad (\varrho>-p),\end{aligned}$$

and

$$\mathfrak{B}^{p}_{n,\mathfrak{q}}\mathfrak{f}(\eta)=\eta^{p}+\sum_{\kappa=p+1}^{\infty}\left(\frac{[p+\varrho]_{\mathfrak{q}}}{[\kappa+\varrho]_{\mathfrak{q}}}\right)^{n}a_{\kappa}z^{\kappa}, \qquad (n\in\mathbb{N},\ \varrho>-p),$$

which are defined in [49].

If $n=1$, we obtain the $\mathfrak{q}$-Bernardi integral operator for a p-valent function [50].

Theorem 3.6. *Let* $\mathfrak{f}\in\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})$*, and define*

$$\mathfrak{B}^{p}_{\mathfrak{q},\varrho}(\eta)=\frac{[p+\varrho]_{\mathfrak{q}}}{\eta^{\varrho}}\int_{0}^{\eta}t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \quad (\varrho>0). \tag{3.10}$$

Then, $\mathfrak{B}^{p}_{\mathfrak{q},\varrho}\in\mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$.

Proof. Let $\mathfrak{f}\in\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})$, and $\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta)=\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\mathfrak{B}^{p}_{\mathfrak{q},\varrho}(\eta)\right)$. We assume

$$\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta)\right)}{[p]_{\mathfrak{q}}\,\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta)}=\mathfrak{R}(\eta), \tag{3.11}$$

where $\mathfrak{R}(\eta)$ is analytic in $\mathbf{U}$ with $\mathfrak{R}(0)=1$.

From (3.10), we obtain

$$\frac{\mathfrak{d}_{\mathfrak{q}}(\eta^{\varrho}\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta))}{[p+\varrho]_{\mathfrak{q}}}=\eta^{\varrho-1}\mathfrak{f}(\eta).$$

This implies

$$\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta)\right)=\left([p]_{\mathfrak{q}}+\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}\right)\mathfrak{f}(\eta)-\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}\mathfrak{B}^{s,p}_{\mathfrak{q},\mu,\varrho}(\lambda,\ell)(\eta). \tag{3.12}$$

We use (3.11), (3.12), and (1.10), to obtain

$$\mathfrak{R}(\eta)=\left([p]_{\mathfrak{q}}+\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}\right)\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\,\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\mathfrak{B}^{p}_{\mathfrak{q},\varrho}(\eta)\right)}-\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}\,[p]_{\mathfrak{q}}}.$$

We use logarithmic differentiation to obtain

$$\frac{\eta\mathfrak{d}_{\mathfrak{q}}\left(\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)\right)}{[p]_{\mathfrak{q}}\,\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\eta)}=\mathfrak{R}(\eta)+\frac{\eta\mathfrak{d}_{\mathfrak{q}}\mathfrak{R}(\eta)}{[p]_{\mathfrak{q}}\,\mathfrak{R}(\eta)+\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}}. \tag{3.13}$$

Since $\mathfrak{f}\in\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}(\gamma,\lambda,\ell;\mathfrak{h})\subset\mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda,\ell;\mathfrak{h})$, (3.13) implies

$$\mathfrak{R}(\eta)+\frac{\eta\mathfrak{d}_{\mathfrak{q}}\mathfrak{R}(\eta)}{[p]_{\mathfrak{q}}\,\mathfrak{R}(\eta)+\frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}}\prec_{\mathfrak{F}}\mathfrak{h}(\eta).$$

The intended outcome follows from applying Lemma 3.1. □

When $p = 1$, the following corollary can be stated:

Corollary 3.2. *Let* $\mathfrak{f} \in \mathfrak{F}\mathcal{M}^{s}_{\mathfrak{q},\mu}(\gamma, \lambda, \ell; \mathfrak{h})$*, and define*

$$\mathfrak{B}_{\mathfrak{q},\varrho}\mathfrak{f}(\eta) = \frac{[1+\varrho]_{\mathfrak{q}}}{\eta^{\varrho}} \int_0^{\eta} t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \ \ (\varrho > 0).$$

Then, $\mathfrak{B}_{\mathfrak{q},\varrho}\mathfrak{f}(\eta) \in \mathfrak{F}\mathcal{ST}^{s}_{\mathfrak{q},\mu}(\lambda, \ell; \mathfrak{h})$.

4. Conclusions

The means of the fuzzy differential subordiantion theory are employed in order to introduce and initiate investigations on certain subclasses of multivalent functions. The $\mathfrak{q}$-p-analogue multiplier-Ruscheweyh operator $\mathcal{I}^{s,p}_{\mathfrak{q},\mu}(\lambda, \ell)$ is developed using the notion of a $\mathfrak{q}$-difference operator and the concept of convolution. The $\mathfrak{q}$-analogue of the Ruscheweyh operator and the $\mathfrak{q}$-p-analogue of the Cătas operator are further used to introduce a new operator applied for defining particular subclasses. In the second section, we obtained some inclusion properties between the classes $\mathfrak{F}\mathcal{M}^{s,p}_{\mathfrak{q},\mu}\gamma, \lambda, \ell; \mathfrak{h})$, $\mathfrak{F}\mathcal{ST}^{s,p}_{\mathfrak{q},\mu}(\lambda, \ell; \mathfrak{h})$, and $\mathfrak{F}\mathcal{CV}^{s,p}_{\mathfrak{q},\mu}(\lambda, \ell; \mathfrak{h})$. The investigations concern the $(p, \mathfrak{q})$-Bernardi integral operator for the p-valent function preservation property and certain inclusion outcomes for the newly defined classes. Another new generalized $\mathfrak{q}$-calculus operator is defined in this investigation that helps establish connections between the classes introduced and investigated in this study. For instance, many researchers used fuzzy theory in different branches of mathematics [51–54].

This work is intended to motivate future studies that would contribute to this direction of study by developing other generalized subclasses of $\mathfrak{q}$-close-to-convex and quasi-convex multivalent functions as well as by presenting other generalized $\mathfrak{q}$-calculus operators.

Author contributions

Ekram E. Ali1, Georgia Irina Oros, Rabha M. El-Ashwah and Abeer M. Albalahi: conceptualization, methodology, validation, formal analysis, formal analysis, writing-review and editing; Ekram E. Ali1, Rabha M. El-Ashwah and Abeer M. Albalahi: writing-original draft preparation; Ekram E. Ali1 and Georgia Irina Oros: supervision; Ekram E. Ali1: project administration; Georgia Irina Oros: funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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