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Research article

Fuzzy differential subordination and superordination results for the Mittag-Leffler type Pascal distribution

Madan Mohan Soren[1] **and Luminiţa-Ioana Cotîrlă**[2,*]

[1] Department of Mathematics, Berhampur University, Bhanja Bihar 760007, Odisha, India
[2] Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca 400114, Romania

* **Correspondence:** Email: luminita.cotirla@math.utcluj.ro.

Abstract: In this paper, we derive several fuzzy differential subordination and fuzzy differential superordination results for analytic functions $\mathcal{M}_{\xi,\beta}^{s,\gamma}$, which involve the extended Mittag-Leffler function and the Pascal distribution series. We also investigate and introduce a class $\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$ of analytic and univalent functions in the open unit disc $\mathcal{D}$ by employing the newly defined operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$. We determine a specific relationship of inclusion for this class. Further, we establish prerequisites for a function role in serving as both the fuzzy dominant and fuzzy subordinant of the fuzzy differential subordination and superordination, respectively. Some novel results that are sandwich-type can be found here.

Keywords: analytic function; fuzzy differential subordinations and fuzzy differential superordinations; Mittag-Leffler functions; Mittag-Leffler type Pascal distribution; sandwich-type results
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1. Introduction

Let $\mathcal{K}(\mathcal{D})$ denote the family of functions that are analytic in the open unit disk

$$\mathcal{D} := \{w : w \in \mathbb{C} \quad \text{and} \quad |w| < 1\}.$$

Note that $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, define the class of functions as follows:

$$\mathcal{K}[a,n] := \{f : f \in \mathcal{K}(\mathcal{D}) \quad \text{and} \quad f(w) = a + a_n w^n + \cdots\}.$$

Suppose the class $\mathcal{A}$ of functions $f \in \mathcal{K}(\mathcal{D})$, which are analytic in $\mathcal{D}$ and normalized by

$$f(w) = w + \sum_{k=2}^{\infty} a_k w^k, \quad (w \in \mathcal{D}). \tag{1.1}$$

Given two functions $f, g \in \mathcal{K}(\mathcal{D})$. If the functions f is subordinate to g, or g is superordinate to f, written by $f(w) \prec g(w)$, then there exists a function $u \in \mathcal{D}$ with $u(0) = 0$ and $|u(w)| < 1$ such that $f(w) = g(u(w))$. Furthermore, if the function g is univalent in $\mathcal{D}$, then

$$f(w) \prec g(w) \Longleftrightarrow f(0) = g(0)$$

and $f(\mathcal{D}) \subset g(\mathcal{D})$ (see, for details [1,2]).

For the functions $f, g \in \mathcal{A}$, the Hadamard product (or convolution) is defined by

$$(f \star g)(w) := w + \sum_{k=2}^{\infty} a_k b_k w^k =: (g \star f)(w), \quad (w \in \mathcal{D}).$$

Mittag-Leffler [3,4] introduced and studied the functions

$$E_{\xi}(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\xi k + 1)} \quad \text{and} \quad E_{\xi,\beta}(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\xi k + \beta)}, \quad (w, \xi, \beta \in \mathbb{C};\ \Re(\xi) > 0).$$

The Mittag-Leffler function of the two-parameter version contains several elementary functions as well as their special cases, like the hyperbolic, trigonometric, and exponential functions. Many authors studied the generalized Mittag-Leffler function, Attiya [5] some applications in the unit disk, Frasin et al. [6] some properties of a linear operator, Srivastava et al. [7] fractional integral operators involving a certain generalized multi-index Mittag-Leffler function and its properties studied by Agarwal [8] and Wiman [9] and the references therein.

Moreover, Prabhakar [10] introduced the function $E^{\gamma}_{\xi,\beta}(w)$ in the form

$$E^{\gamma}_{\xi,\beta}(w) := \sum_{k=0}^{\infty} \frac{(\gamma)_k\, w^k}{\Gamma(\xi k + \beta) k!}, \quad (w, \xi, \beta, \gamma \in \mathbb{C};\ \Re(\xi) > 0),$$

where $(\lambda)_n$ is the Pochhammer symbol

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda + 1)...(\lambda + n - 1). & \end{cases}$$

For $\gamma = 1$, it becomes the Mittag-Leffler function. In order to study, we define the function $\mathcal{E}^{\gamma}_{\xi,\beta}(w)$ by

$$\begin{aligned} \mathcal{E}^{\gamma}_{\xi,\beta}(w) &:= \frac{\Gamma(\xi + \beta)}{(\gamma)_1} \left(E^{\gamma}_{\xi,\beta}(w) - \frac{1}{\Gamma(\beta)} \right) \\ &= w + \sum_{k=2}^{\infty} \frac{\Gamma(\gamma + k)\Gamma(\xi + \beta)\, w^k}{k!\, \Gamma(\gamma + 1)\Gamma(\xi k + \beta)}, \end{aligned}$$

where $w, \xi, \beta, \gamma \in \mathbb{C}$ and $\Re(\xi) > 0$. Throughout this paper, unless otherwise specified, we only take the case when the values of parameters ξ, β, and γ are real-valued and for $w \in \mathcal{D}$.

Moreover, a random variable y defines a Pascal distribution if it takes the non-negative integer values $k = 0, 1, 2, \dots$ accordingly with the formula

$$P(y = k) = \binom{k+s-1}{s-1} r^k (1-r)^s, \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where $0 < r < 1$ and the integers $s > 0$ are parameters. We denote $\mathcal{P}_r^s(w)$ the power series in which coefficients are Pascal distribution probabilities

$$\mathcal{P}_r^s(w) := w + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} r^{k-1} (1-r)^s w^k, \quad (k \in \mathbb{N}_0,\ s \geq 1,\ 0 \leq r \leq 1).$$

The power series $\mathcal{P}_r^s(w)$ whose radius of convergence is at least $\frac{1}{r} \geq 1$ by the ratio test; therefore, the series $\mathcal{P}_r^s(w) \in \mathcal{A}$.

We define the Mittag-Leffler-type Pascal distribution series by the Hadamard product

$$\mathcal{M}_{\xi,\beta}^{s,\gamma}(w) = w + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} \frac{\Gamma(\gamma+k)\Gamma(\xi+\beta)}{k!\,\Gamma(\gamma+1)\Gamma(\xi k+\beta)}\, r^{k-1}(1-r)^s w^k.$$

Now, we introduce a new operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$ defined, for $f(w) \in \mathcal{A}$ by

$$\begin{aligned}\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w) &:= \mathcal{M}_{\xi,\beta}^{s,\gamma}(w) \star f(w) \\ &:= w + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} \frac{\Gamma(\gamma+k)\Gamma(\xi+\beta)}{k!\,\Gamma(\gamma+1)\Gamma(\xi k+\beta)}\, r^{k-1}(1-r)^s a_k w^k,\end{aligned}$$

where $\xi, \beta, \gamma \in \mathbb{C};\ \Re(\xi) > 0;\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \geq 1;\ 0 \leq r \leq 1$.

This investigation was motivated by several recent works [11–13] that exploited different categories of probability distributions of the class of analytic functions like convex functions and starlike functions, which are defined and normalized in $\mathcal{D}$.

The main objective of the present article is to investigate several fruitful results of fuzzy differential subordinations and fuzzy differential superordinations and their applications in geometric function theory and that results are closely associated with the Mittag-Leffler-type Pascal distribution series.

In 1965, Zadeh [14] introduced the notation of fuzzy set theory. Now it has grown exponentially and has applications in many areas, such as scientific and technological fields [15]. There has been extensive research done on fractional calculus of fuzzy functions with fuzzy data, both theoretically and experimentally. Oros and Oros [16] brought to light different usages of this idea in geometric function theory with the notion of fuzzy differential subordination in 2012 [17, 18]. In 2017, Atshan and Hussain [19] introduced the concept of fuzzy differential superordination. In this direction, many researchers have studied different properties of analytic functions using the concepts of fuzzy differential subordinations and superordinations: Wanas operator [20, 21], Sălăgean and Ruscheweyh operators [22–24], generalized Noor-Salagean operator [25]. For more details on this subject, we refer the reader to see fuzzy differential subordinations obtained for strong Janowski functions [26], spiral-like functions [27], λ-pseudo starlike and λ-pseudo convex functions [28].

Since Srivastavas general context was introduced in 1989 [29], many applications of quantum calculus in geometric function theory have emerged in recent years. Some aspects of the application of quantum calculus to geometric functions theory are highlighted in geometric function theory and quantum calculus in [30], while other developments are highlighted in the work of Srivastava [31] in 2020, as well as a large number of q-operators derived by including known differential and integral operators. For example, q-analog operators involving analytic functions are studied regarding fuzzy theory: q-hypergeometric function and fractional calculus [32], q-analogue of multiplier transformation [33], q-difference operator [34]. Fuzzy differential subordinations were obtained using fractional integrals applied to the Mittag-Leffler function [35], the fractional derivative [36], the fractional integral of confluent hypergeometric function [37, 38], the Riemann-Liouville fractional integral of the Ruscheweyh and Salagean operators [39], the Atangana-Baleanu fractional integral [40], and the fractional integral of the Gaussian hypergeometric function [41].

This paper is divided into sections as follows: In Section 2, we reminded ourselves some useful definitions and preliminaries that provide the foundation of our paper. New fuzzy differential subordinations are proved, and the fuzzy best dominants are resolved in Section 3. Dual results regarding fuzzy differential superordinations are established, and the fuzzy best subordinates are given in Section 4. In Section 5, we present the sandwich-type results based on our work. Lastly, in Section 6, we completed our study after giving the conclusion of the work.

2. Definitions and preliminaries

To prove our results, we shall need the following definitions and lemmas:

Definition 2.1. ([42]) Suppose $\mathtt{X}$ is a non-empty set. A pair $(\mathcal{J}, F_{\mathcal{J}})$, where

$$F_{\mathcal{J}} : \mathtt{X} \to [0, 1]$$

and

$$\mathcal{J} = \{y \in \mathtt{X} : 0 < F_{\mathcal{J}}(y) \leq 1\} = supp(\mathcal{J}, F_{\mathcal{J}})$$

be known as a fuzzy subset of $\mathtt{X}$. The membership function of the fuzzy set $(\mathcal{J}, F_{\mathcal{J}})$ is named after the set function $F_{\mathcal{J}}$.

We introduce and apply the concept of membership functions of moduli of complex-valued functions on the set $\mathbb{C}$ given by

$$z = u + iv, \quad (u, v \in \mathbb{R})$$

and

$$|z| = \sqrt{u^2 + v^2} \geq 0, \quad (z \in \mathbb{C}).$$

Definition 2.2. ([24]) Assume that E: $\mathbb{C} \to \mathbb{R}_+$ is a function such that

$$E_{\mathbb{C}}(\mathbb{C}) = |E(w)|, \quad (w \in \mathcal{D}).$$

The fuzzy subset of the set $\mathbb{C}$ of complex numbers is denoted by

$$E_{\mathbb{C}}(\mathbb{C}) = \{w : w \in \mathbb{C} \text{ and } 0 < |E(w)| \leq 1\} = supp(\mathbb{C}, E_{\mathbb{C}}).$$

We call the following subset:

$$F_{\mathbb{C}}(\mathbb{C}) = \{w : w \in \mathbb{C} \text{ and } 0 < |E(w)| \leq 1\} = \mathcal{D}_F(0,1)$$

the fuzzy unit disk. Note that $(\mathbb{C}, F_{\mathbb{C}})$ is the same as its fuzzy unit disk, $\mathcal{D}_F(0,1)$.

Definition 2.3. ([16]) Each of the following statements holds true.

(1) If $(\mathcal{J}, F_{\mathcal{J}}) = (\mathcal{V}, F_{\mathcal{V}})$, then $\mathcal{J} = \mathcal{V}$, where

$$\mathcal{J} = supp(\mathcal{J}, F_{\mathcal{J}}) \text{ and } \mathcal{V} = supp(\mathcal{V}, F_{\mathcal{V}}).$$

(2) If $(\mathcal{J}, F_{\mathcal{J}}) \subseteq (\mathcal{V}, F_{\mathcal{V}})$, then $\mathcal{J} \subseteq \mathcal{V}$, where

$$\mathcal{J} = supp(\mathcal{J}, F_{\mathcal{J}}) \text{ and } \mathcal{V} = supp(\mathcal{V}, F_{\mathcal{V}}).$$

Remark 2.4. [24] Set $\Pi \subset \mathbb{C}$, and f and g are analytic functions in $\mathcal{K}(\Pi)$. We are usually signified by

$$f(\Pi) = supp(f(\Pi), F_{f(\Pi)}) = \{f(w) : 0 < |F_{f(\Pi)}(f(w))| \leq 1, w \in \Pi\} \tag{2.1}$$

and

$$g(\Pi) = supp(g(\Pi), F_{g(\Pi)}) = \{g(w) : 0 < |F_{g(\Pi)}(g(w))| \leq 1, w \in \Pi\}. \tag{2.2}$$

Then, for $w \in \Pi$, we have the following properties:

(i) For any $\delta \in C$, $F_{(\delta f)(\Pi)}(\delta f)(w) = F_{f(\Pi)} f(w)$.

(ii) $F_{(f+g)(\Pi)}(f+g)(w) = \frac{F_{f(\Pi)} f(w) + F_{g(\Pi)} g(w)}{2}$.

(iii) If $0 < |F_{f(\Pi)} f(w)| \leq 1$ and $0 < |F_{g(\Pi)} g(w)| \leq 1$, then $0 < |F_{(f+g)(\Pi)}(f+g)(w)| \leq 1$.

Definition 2.5. ([16]) Assume that $w_0 \in \Pi$ is a constant value, and $f, g \in \mathcal{K}(\Pi)$. We claim that f is a fuzzy subordinate to g, written as $f \prec_F g$ or $f(w) \prec_F g(w)$, if the following requirements are satisfied:

(i) $f(w_0) = g(w_0)$;

(ii) $f(\Pi) \subseteq g(\Pi)$ and $|F_{f(\Pi)}(f(w))| \leq |F_{g(\Pi)}(g(w))|, \quad (w \in \Pi)$,

where (i) and (ii) are given in (2.1) and (2.2), respectively.

Definition 2.6. ([17]) Let χ: $\mathbb{C}^3 \times \mathcal{D} \to \mathbb{C}$ and h be univalent in $\mathcal{D}$. If p is analytic in $\mathcal{D}$ and the (second-order) fuzzy differential subordination is satisfied

$$|F_{\chi(\mathbb{C}^3 \times \mathcal{D})}(\chi(p(w), wp'(w), w^2 p''(w); w))| \leq |F_{h(\mathcal{D})}(h(w))|,$$

that is,

$$\chi(p(w), wp'(w), w^2 p''(w); w) \prec_F h(w), \quad (w \in \mathcal{D}), \tag{2.3}$$

then $p(w)$ is named a fuzzy solution of the fuzzy differential subordination. The univalent function $q(w)$ is named a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or simply a fuzzy dominant, if

$$|F_{p(\mathcal{D})}(p(w))| \leq |F_{q(\mathcal{D})}(q(w))|,$$

i.e., $p(w) \prec_F q(w)$, $(w \in \mathcal{D})$ for all functions p that satisfy (2.3). A fuzzy dominant $\tilde{q}$ satisfying the following condition:

$$|F_{\tilde{q}(\mathcal{U})}(\tilde{q}(z))| \le |F_{q(\mathcal{D})}(q(w))|,$$

i.e., $\tilde{q}(w) \prec_F q(w)$, $(w \in \mathcal{D})$ for all fuzzy dominants q of (2.3), is called the fuzzy best dominant of (2.3).

Definition 2.7. ([19]) Let $\chi \colon \mathbb{C}^3 \times \mathcal{D} \to \mathbb{C}$ and h be analytic in $\mathcal{D}$. If p and $\chi(p(w), wp'(w), w^2p''(w); w)$ are univalent in $\mathcal{D}$ and satisfy the (second-order) fuzzy differential superordination

$$|F_{h(\mathcal{D})}(h(w))| \le |F_{\chi(\mathbb{C}^3\times\mathcal{D})}(\chi(p(w), wp'(w), w^2p''(w); w))|,$$

that is,

$$h(w) \prec_F \chi(p(w), wp'(w), w^2p''(w); w) \quad (w \in \mathcal{D}), \tag{2.4}$$

then $p(w)$ is named a fuzzy solution of the fuzzy differential superordination. An analytic function $q(w)$ is named a fuzzy subordinant of the fuzzy solutions of the fuzzy differential superordination, or simply a fuzzy subordinant, if

$$|F_{q(\mathcal{D})}(q(w))| \le |F_{p(\mathcal{D})}(p(w))|,$$

i.e., $q(w) \prec_F p(w)$, $(w \in \mathcal{D})$ for all functions p that satisfy (2.4). A univalent fuzzy subordination $\tilde{q}$ that satisfy

$$|F_{q(\mathcal{D})}(q(w))| \le |F_{\tilde{q}(\mathcal{U})}(\tilde{q}(w))|,$$

i.e., $q(w) \prec_F \tilde{q}(w)$, $(w \in \mathcal{D})$ for all fuzzy subordinate q of (2.4) is called the fuzzy best subordinate of (2.4).

Denote by Q the set of all functions $q(w)$ that are analytic and injective as a function of w on $\overline{\mathcal{D}} \setminus E(q(w))$, where

$$E(q(w)) = \left\{\zeta \in \partial\mathcal{D} : \lim_{z\to\zeta} q(w) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{D} \setminus E(q)$. The subclass of Q for which $q(0) = a$ is denoted by $\mathsf{Q}(a)$.

We need the following lemmas to prove the main results:

Lemma 2.8. *([2]) Let $\psi \in \mathcal{A}$ and assume that*

$$\varrho(w) = \frac{1}{w}\int_0^w \psi(\tau)d\tau, \quad (w \in \mathcal{D}).$$

If

$$\Re\left(1 + \frac{w\psi''(w)}{\psi'(w)}\right) > -\frac{1}{2}, \quad (w \in \mathcal{D}),$$

then ϱ is a convex function.

Lemma 2.9. *([18]) Let $g \in \mathcal{D}$ be a convex function, and the function*

$$h(w) = g(w) + n\alpha wg'(w), \quad (w \in \mathcal{D});\ \alpha > 0,$$

and let n be a positive integer. If the function

$$p(w) = g(0) + p_n w^n + p_{n+1} w^{n+1} + \cdots$$

is holomorphic in $\mathcal{D}$, and the fuzzy differential subordination

$$|F_{p(\mathcal{D})}(p(w) + \alpha w p'(w))| \leq |F_{h(\mathcal{D})} h(w)|,$$

then

$$|F_{p(\mathcal{D})}(p(w))| \leq |F_{g(\mathcal{D})} g(w)|, \ \textit{i.e.}, p(w) \prec_F g(w), \quad (w \in \mathcal{D}).$$

The result is sharp.

Lemma 2.10. *([18]) Let h be a univalent convex function in $\mathcal{D}$ with $h(0) = a$ and $\beta \in \mathbb{C}$, with $\Re(\beta) \geq 0$. If $p(w) \in \mathcal{K}[a, n]$ with $p(0) = a$ and χ: $\mathbb{C}^2 \times \mathcal{D} \to \mathbb{C}$,*

$$\chi(p(w), wp'(w); w) = p(w) + \frac{1}{\beta} wp'(w)$$

is univalent in $\mathcal{D}$, and the fuzzy differential subordination

$$\left| F_{\chi(\mathbb{C}^2 \times \mathcal{D})}[p(w) + \frac{1}{\beta} wp'(w)] \right| \leq |F_{h(\mathcal{D})}(h(w))| \Rightarrow p(w) + \frac{1}{\beta} wp'(w) \prec_F h(w), \quad (w \in \mathcal{D}),$$

then

$$|F_{p(\mathcal{D})} p(w)| \leq |F_{q(\mathcal{D})} q(w)| \leq |F_{h(\mathcal{D})} h(w)| \Rightarrow p(w) \prec_F q(w), \quad (w \in \mathcal{D}),$$

where

$$q(w) = \frac{\beta}{n w^{\frac{\beta}{n}}} \int_0^\infty h(t) t^{\frac{\beta}{n} - 1} dt$$

is convex, and the fuzzy best dominant.

Lemma 2.11. *([19]) Let h be a univalent convex function in $\mathcal{D}$ with $h(0) = a$ and $\beta \in \mathbb{C}$, with $\Re(\beta) \geq 0$. If $p(w) \in \mathtt{Q} \cap \mathcal{K}[a, n]$ and χ: $\mathbb{C}^2 \times \mathcal{D} \to \mathbb{C}$,*

$$\chi(p(w), wp'(w); w) = p(w) + \frac{1}{\beta} wp'(w)$$

is univalent in $\mathcal{D}$, and satisfy the fuzzy differential superordination

$$\left| F_{h(\mathcal{D})}(h(w)) \right| \leq \left| F_{\chi(\mathbb{C}^2 \times \mathcal{D})}[p(w) + \frac{1}{\beta} wp'(w)] \right| \Rightarrow h(w) \prec_F p(w) + \frac{1}{\beta} wp'(w), \quad (w \in \mathcal{D}),$$

then

$$|F_{h(\mathcal{D})} h(w)| \leq |F_{q(\mathcal{D})} q(w)| \leq |F_{p(\mathcal{D})} p(w)| \Rightarrow q(w) \prec_F p(w), \quad (w \in \mathcal{D}),$$

where

$$q(w) = \frac{\beta}{n w^{\frac{\beta}{n}}} \int_0^\infty h(t) t^{\frac{\beta}{n} - 1} dt$$

is convex and the fuzzy best subordinant.

In this paper, we have used the generalized Mittag-Leffler functions with the general Pascal-type probability distribution, which is symmetric in $\mathcal{D}$. We derive several fuzzy differential subordinations and fuzzy differential superordinations from the results of analytic functions involving the operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$. Afterward, some sandwich-type results are also presented.

3. Fuzzy differential subordination

In this section, we first defined the new class of normalized analytic functions in the open unit disk by the operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$.

Definition 3.1. The function $f(w) \in \mathcal{A}$, given by (1.1), is said to be in the class $\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$ with $0 \leq \rho < 1$ if

$$\left| F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)' \right| > \rho, \quad (w \in \mathcal{D}). \tag{3.1}$$

We establish the convexity of the class $\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$.

Theorem 3.2. *The class $\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$ is a convex set.*

Proof. Suppose that

$$f_1, f_2 \in \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$$

and

$$f(w) = b_1 f_1(w) + b_2 f_2(w),$$

where b_1, b_2 are positive numbers with

$$b_1 + b_2 = 1.$$

We have to prove that $f \in \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$, $(w \in \mathcal{D})$. Now

$$f'(w) = b_1 f_1'(w) + b_2 f_2'(w)$$

and

$$\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)' = b_1 \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1(w)\right)' + b_2 \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2(w)\right)'.$$

Since

$$f_1,\ f_2 \in \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho),$$

we have

$$\rho < \left| F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1(w)\right)' \right| \leq 1$$

and

$$\rho < \left| F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2(w)\right)' \right| \leq 1,$$

which implies

$$\rho = (b_1 + b_2)\rho < \left| \frac{F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1(w)\right)' + F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2(w)\right)'}{2} \right| \leq 1. \tag{3.2}$$

Again, by applying fuzzy theory, we get

$$\begin{aligned} F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)' &= F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}(b_1 f_1 + b_2 f_2)\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}(b_1 f_1 + b_2 f_2)(w)\right)' \\ &= F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}(b_1 f_1 + b_2 f_2)\right)'(\mathcal{D})} \left(b_1 \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_1(w)\right)' + b_2 \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f_2(w)\right)'\right) \end{aligned} \tag{3.3}$$

$$= \frac{F_{\left(b_1\mathcal{M}^{s,\gamma}_{\xi,\beta}f_1\right)'(\mathcal{D})}\left(b_1\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_1(w)\right)'\right)+F_{\left(b_2\mathcal{M}^{s,\gamma}_{\xi,\beta}f_2\right)'(\mathcal{D})}\left(b_2\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_2(w)\right)'\right)}{2}$$

$$= \frac{F_{\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_1\right)'(\mathcal{D})}\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_1(w)\right)'+F_{\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_2\right)'(\mathcal{D})}\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f_2(w)\right)'}{2}.$$

From Definition 3.1, along with Eqs (3.2) and (3.3), we obtain the required result. □

We next investigate the fuzzy differential subordination results involving the convex functions and the operator $\mathcal{M}^{s,\gamma}_{\xi,\beta}$.

Theorem 3.3. *Suppose that $h \in \mathcal{K}(\mathcal{D})$ is a convex function with $h(0) = 1$ such that*

$$\Re\left(1+\frac{wh''(w)}{h'(w)}\right) > -\frac{1}{2}, \quad (w \in \mathcal{D}). \tag{3.4}$$

If $f \in \mathcal{A}$ satisfies the fuzzy differential subordination

$$\left|F_{\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f\right)'(\mathcal{D})}\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)\right)'\right| \leq \left|F_{h(\mathcal{D})}h(w)\right| \Longrightarrow \left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)\right)' \prec_F h(w), \quad (w \in \mathcal{D}), \tag{3.5}$$

then

$$\left|F_{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(\mathcal{D})}\frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w} \prec_F g(w), \quad (w \in \mathcal{D}),$$

where

$$g(w) = \frac{1}{w}\int_0^w h(t)dt$$

is a convex function, and the fuzzy is the best dominant.

Proof. Let the function

$$p(w) = \frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w} = 1+\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}\frac{\Gamma(\gamma+k)\Gamma(\xi+\beta)}{k!\,\Gamma(\gamma+1)\Gamma(\xi k+\beta)}\,r^{k-1}(1-r)^s a_k z^{k-1}. \tag{3.6}$$

It is clear that $p(w) \in \mathcal{K}[1, 1]$. We observe that

$$p(w) + wp'(w) = \left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)\right)', \quad (w \in \mathcal{D}). \tag{3.7}$$

Further, suppose $h \in \mathcal{K}(\mathcal{D})$ with $h(0) = 1$ such that

$$\Re\left(1+\frac{wh''(w)}{h'(w)}\right) > -\frac{1}{2}, \quad (w \in \mathcal{D}).$$

In view of the Lemma 2.8, we obtain

$$g(w) = \frac{1}{w}\int_0^w h(t)dt, \quad (w \in \mathcal{D}).$$

Hence, $g(w)$ is the convex univalent function in $\mathcal{D}$, and we can easily compute that

$$g(w) + wg'(w) = h(w).$$

The fuzzy differential subordination (3.5) can be written as

$$\left|F_{p(\mathcal{D})}(p(w) + wp'(w))\right| \leq \left|F_{g(\mathcal{D})}g(w) + wg'(w)\right|, \quad (w \in \mathcal{D}).$$

By using Lemma 2.9, we have

$$\left|F_{p(\mathcal{D})}p(w)\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right| \leq \left|F_{g(\mathcal{D})}g(w)\right|,$$

implies that

$$\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w} \prec_F g(w), \quad (w \in \mathcal{D}),$$

where

$$g(w) = \frac{1}{w}\int_0^w h(t)dt$$

is the fuzzy best dominant. The proof of Theorem 3.3 is completed. □

Taking

$$h(w) = \frac{1 + (2\alpha - 1)w}{1 + w}, \quad (w \in \mathcal{D})$$

in Theorem 3.3, we can get the following result:

Corollary 3.4. *Let*

$$h(w) = \frac{1 + (2\alpha - 1)w}{1 + w}, \quad h(0) = 1$$

be the normalized convex function, and $0 \leq \alpha < 1$. *If* $f \in \mathcal{A}$ *satisfies the fuzzy differential subordination:*

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)'\right| \leq \left|F_{h(\mathcal{D})}h(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)' \prec_F h(w), \tag{3.8}$$

then

$$\left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w} \prec_F g(w), \quad (w \in \mathcal{D}),$$

where

$$g(w) = 2\alpha - 1 + \frac{2(1-\alpha)}{w}\log(1 + w)$$

is convex, and the fuzzy is best dominant.

Proof. Following from Theorem 3.3, the relation (3.8) can be expressed as

$$\left|F_{p(\mathcal{D})}(p(w)+wp'(w))\right| \leq \left|F_{h(\mathcal{D})}h(w)\right|, \quad (w \in \mathcal{D}).$$

Application of Lemma 2.9, we obtain

$$\left|F_{p(\mathcal{D})}p(w)\right| \leq \left|F_{g(\mathcal{D})}g(w)\right|,$$

that means

$$\left|F_{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(\mathcal{D})}\frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}\right| \leq \left|F_{g(\mathcal{D})}g(w)\right|, \quad (w \in \mathcal{D}),$$

where

$$\begin{aligned} g(w) &= \frac{1}{w}\int_0^w h(t)dt \\ &= \frac{1}{w}\int_0^w \frac{1+(2\alpha-1)t}{t+1}dt \\ &= 2\alpha - 1 + \frac{2(1-\alpha)}{w}\log(1+w) \end{aligned}$$

is the convex function and the fuzzy best dominant. □

Example 3.5. *Let*

$$h(w) = \frac{1-w}{w+1}$$

be a convex function in $\mathcal{D}$ with $h(0) = 1$. Suppose that

$$f(w) = w + w^2, \quad w \in \mathcal{D}.$$

For $\xi = 0,\ \beta = \gamma = r = 1,\ k = 2$ and $s = \frac{1}{2}$, we have

$$\mathcal{M}^{1,1}_{0,1}f(w) = w + \frac{1}{4}w^2.$$

Then

$$(\mathcal{M}^{1,1}_{0,1}f(w))' = 1 + \frac{1}{2}w$$

and

$$\frac{\mathcal{M}^{1,1}_{0,1}f(w)}{w} = 1 + \frac{1}{4}w.$$

Because

$$g(w) = \frac{1}{w}\int_0^w \frac{1-t}{t+1}dt = -1 + \frac{2ln(w+1)}{w}.$$

From Theorem 3.3, we obtain

$$1 + \frac{1}{2}w \prec_F \frac{1-w}{w+1},$$

then

$$1 + \frac{1}{4}w \prec_F -1 + \frac{2\ln(w+1)}{w}, \quad (w \in \mathcal{D}).$$

By using Lemma 2.9, we obtain the following theorem:

Theorem 3.6. *Suppose g is a convex function with*

$$g(0) = 1 \ \text{and} \ h(w) = g(w) + wg'(w), \quad (w \in \mathcal{D}).$$

If $f \in \mathcal{A}$ satisfies the fuzzy differential subordination:

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)'\right| \leq \left|F_{h(\mathcal{D})}h(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)' \prec_F h(w), \quad (w \in \mathcal{D}),$$

then

$$\left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w} \prec_F g(w), \quad (w \in \mathcal{D}).$$

This result is seen to be sharp when the inequality is satisfied for a suitably specified function.

Proof. The proof follows from Theorem 3.3 by taking

$$h(w) = g(w) + wg'(w), \quad (w \in \mathcal{D})$$

and relation (3.7). Hence, by applying Lemma 2.9 with $\alpha = 1$, we deduce the required result. This result is sharp. Thus, the proof of Theorem 3.6 is complete. □

We now have the following theorem:

Theorem 3.7. *Suppose g is a normalized convex function and*

$$h(w) = g(w) + \frac{1}{b+2}wg'(w), \quad (w \in \mathcal{D};\ b > -2).$$

Let

$$G(w) = T^b f(w) = \frac{b+2}{w^{b+1}}\int_0^w t^b f(t)dt, \quad (w \in \mathcal{D}). \tag{3.9}$$

If $f \in \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$ and satisfies the fuzzy differential subordination:

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)'\right| \leq \left|F_{h(\mathcal{D})}h(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)' \prec_F h(w), \tag{3.10}$$

then

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)'\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)' \prec_F g(w), \quad (w \in \mathcal{D}),$$

where $g(w)$ is the fuzzy best dominant.

Proof. From (3.9), we write

$$w^{b+1}G(w) = (b+2)\int_0^w t^b f(t)dt, \quad (w \in \mathcal{D}). \tag{3.11}$$

Differentiating (3.11) with respect to w, we have

$$(b+1)G(w) + wG'(w) = (b+2)f(w). \tag{3.12}$$

Thus, by applying the operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$ on both sides of (3.12) and differentiating, after simplifications, we get

$$\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)' + \frac{1}{b+2}w\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)'' = \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)', \quad (w \in \mathcal{D}). \tag{3.13}$$

The fuzzy differential subordination (3.10) can be written as follows:

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)' + \frac{1}{b+2}w\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)''\right)\right| \leq \left|F_{h(\mathcal{D})}\left(g(w) + \frac{1}{b+2}wg'(w)\right)\right|. \tag{3.14}$$

Let

$$p(w) = \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)'. \tag{3.15}$$

Then p is an analytic function with $p(0) = 1$. By substituting (3.15) into (3.14), yields

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(p(w) + \frac{1}{b+2}wp'(w)\right)\right| \leq \left|F_{h(\mathcal{D})}\left(g(w) + \frac{1}{b+2}wg'(w)\right)\right|.$$

Now, applying Lemma 2.10, we have

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)'\right| \leq \left|F_{g(\mathcal{D})}g(w)\right|.$$

Therefore,

$$\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)' \prec_F g(w),$$

where g is the fuzzy best dominant. Thus, the proof of Theorem 3.7 is complete. □

The following result is an immediate consequence of Theorem 3.7.

Theorem 3.8. *Suppose $h \in \mathcal{K}(\mathcal{D})$ is a convex function with $h(0) = 1$. Let the operator T^b be given by* (3.9). *If $f \in \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$ and satisfies the fuzzy differential subordination*

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)'\right| \leq \left|F_{h(\mathcal{D})}h(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)\right)' \prec_F h(w),$$

then

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)'\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \Longrightarrow \left(\mathcal{M}_{\xi,\beta}^{s,\gamma}G(w)\right)' \prec_F g(w), \quad (w \in \mathcal{D}),$$

where

$$g(w) = \frac{b+2}{w^{b+2}} \int_0^w t^{b+1}h(t)dt$$

is the convex function, and the fuzzy best dominant.

Example 3.9. *Let*

$$h(w) = \frac{1-w}{w+1}$$

be a convex function in $\mathcal{D}$ *with* $h(0) = 1$. *Suppose that*

$$f(w) = w + w^2, \quad w \in \mathcal{D}.$$

For $\xi = 0,\ \beta = \gamma = r = 1,\ k = 2$, *and* $s = \frac{1}{2}$, *we have*

$$\mathcal{M}_{0,1}^{1,1} f(w) = w + \frac{1}{4}w^2.$$

Then

$$(\mathcal{M}_{0,1}^{1,1} f(w))' = 1 + \frac{1}{2}w.$$

Now for $b = 4$, *we get*

$$G(w) = T^4 f(w) = \frac{6}{w^5}\int_0^w t^4(t+t^2)dt = w + \frac{6}{7}w^2.$$

Hence,

$$\mathcal{M}_{0,1}^{1,1} G(w) = w + \frac{3}{14}w^2$$

and

$$(\mathcal{M}_{0,1}^{1,1} G(w))' = 1 + \frac{3}{7}w.$$

We deduce that

$$g(w) = \frac{6}{w^6}\int_0^w \frac{1-t}{t+1}t^5 dt = -\frac{12\ln(1+w)}{w^6} - \frac{48}{w^5} - \frac{66}{w^4} - \frac{16}{w^3} - \frac{3}{w^2} + \frac{12}{5w} - 1.$$

Using Theorem 3.7, we obtain

$$1 + \frac{1}{2}w \prec_F \frac{1-w}{w+1}$$

implies

$$1 + \frac{3}{7}w \prec_F -\frac{12\ln(1+w)}{w^6} - \frac{48}{w^5} - \frac{66}{w^4} - \frac{16}{w^3} - \frac{3}{w^2} + \frac{12}{5w} - 1, \quad (w \in \mathcal{D}).$$

Our next result will demonstrate some significant inclusion relation for the class $\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)$.

Theorem 3.10. *Suppose that the function*

$$h(w) = \frac{1+(2\eta-1)w}{1+w}, \quad \eta \in [0,1)$$

and $b > 0$. *Let the operator* T^b *be given by* (3.9). *Then*

$$T^b\left[\mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho)\right] \subset \mathcal{MB}_{\xi,\beta}^{F,s,\gamma}(\rho^\star), \tag{3.16}$$

where

$$\rho^\star := 2\eta - 1 + 2(b+2)(1-\eta)\int_0^1 \frac{t^{b+2}}{t+1}dt.$$

Proof. Since $h \in \mathcal{D}$ is the convex function and uses the same line as in the proof of Theorem 3.7, we obtain from the hypothesis of Theorem 3.10 that

$$\left|F_{p(\mathcal{D})}\left(p(w) + \frac{1}{b+2} w p'(w)\right)\right| \leq \left|F_{h(\mathcal{D})} h(w)\right|,$$

where $p(w)$ is defined by (3.15). By applying Lemma 2.10, we have

$$\left|F_{p(\mathcal{D})} p(w)\right| \leq \left|F_{g(\mathcal{D})} g(w)\right| \leq \left|F_{h(\mathcal{D})} h(w)\right|,$$

implies that

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G(w)\right)'\right| \leq \left|F_{g(\mathcal{U})} g(w)\right| \leq \left|F_{h(\mathcal{U})} h(w)\right|,$$

where $g(w)$ is given by

$$\begin{aligned} g(w) &= \frac{b+2}{w^{b+2}} \int_0^w t^{b+1} \frac{1 + (2\eta - 1)t}{1+t} dt \\ &= (2\eta - 1) + \frac{2(b+2)(1-\eta)}{w^{b+2}} \int_0^w \frac{t^{b+1}}{1+t} dt, \end{aligned}$$

belongs to the convex function class C in $\mathcal{D}$, and $g(\mathcal{D})$ is symmetric with respect to the real axis. Thus, we get

$$\left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G\right)'(\mathcal{D})}\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G(w)\right)'\right| \geq \min_{|w|=1} \left\{\left|F_{g(\mathcal{D})} g(w)\right|\right\} = \left|F_{g(\mathcal{D})} g(1)\right|$$

and

$$\rho^\star := g(1) = 2\eta - 1 + (b+2)(2 - 2\eta) \int_0^1 \frac{t^{b+2}}{t+1} dt.$$

This completes the proof of Theorem 3.10. □

4. Fuzzy differential superordination

In this section, we state and prove the following fuzzy differential superordination results involving the convex function and the operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$.

Theorem 4.1. *Considering h as a convex function with $h(0) = 1$. Suppose that $(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w))'$ is a univalent function in $\mathcal{D}$ and*

$$\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)}{w} \in \mathcal{Q} \cap \mathcal{K}[1,1].$$

If $f \in \mathcal{A}$ satisfies the fuzzy differential superordination:

$$\left|F_{h(\mathcal{D})} h(w)\right| \leq \left|F_{(\mathcal{M}_{\xi,\beta}^{s,\gamma} f)'(\mathcal{D})}(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w))'\right| \Longrightarrow h(w) \prec_F (\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w))', \quad (w \in \mathcal{D}), \tag{4.1}$$

then

$$\left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(\mathcal{D})}\frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}\right| \Longrightarrow g(w) \prec_F \frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}, \quad (w \in \mathcal{D}),$$

where the convex function

$$g(w) = \frac{1}{w}\int_0^w h(t)dt$$

is the fuzzy best subordinant.

Proof. Let the function $p(w)$ be defined by (3.6). Using the relation (3.7) in (4.1), we have

$$\left|F_{g(\mathcal{D})}g(w) + wg'(w)\right| \leq \left|F_{p(\mathcal{D})}(p(w) + wp'(w))\right|, \quad (w \in \mathcal{D}).$$

Since $h \in \mathcal{K}(\mathcal{D})$ with $h(0) = 1$ such that

$$\Re\left(1 + \frac{wh''(w)}{h'(w)}\right) > -\frac{1}{2}, \quad (w \in \mathcal{D}).$$

From Lemma 2.8, we obtain

$$g(w) = \frac{1}{w}\int_0^w h(t)dt, \quad (w \in \mathcal{D}).$$

Hence, $g(w)$ is the convex univalent function in $\mathcal{D}$ and we can easily compute that

$$g(w) + wg'(w) = h(w).$$

By using Lemma 2.11, we have

$$\left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{p(\mathcal{D})}p(w)\right| \Rightarrow \left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(\mathcal{D})}\frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}\right|$$

implies that

$$g(w) \prec_F \frac{\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)}{w}, \quad (w \in \mathcal{D}),$$

where

$$g(w) = \frac{1}{w}\int_0^w h(t)dt$$

is the fuzzy best subordinant. This completes the proof of Theorem 4.1. □

Taking

$$h(w) = \frac{1 + (2\eta - 1)w}{1 + w}, \quad (w \in \mathcal{D})$$

in Theorem 4.1, we get the following result:

Corollary 4.2. *Let the function*

$$h(w)=\frac{1+(2\eta-1)w}{1+w},\quad \eta\in[0,1),\ w\in\mathcal{D}.$$

Suppose that $(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(z))'$ *is a univalent function in* $\mathcal{D}$ *and*

$$\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\in Q\cap\mathcal{K}[1,1].$$

If $f\in\mathcal{A}$ *satisfies the fuzzy differential superordination:*

$$\left|F_{h(\mathcal{D})}h(w)\right|\le\left|F_{(\mathcal{M}_{\xi,\beta}^{s,\gamma}f)'(\mathcal{D})}(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w))'\right|\Longrightarrow h(w)\prec_F(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w))',\quad(w\in\mathcal{D}),\tag{4.2}$$

then

$$\left|F_{g(\mathcal{D})}g(w)\right|\le\left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right|\Longrightarrow g(w)\prec_F\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w},\quad(w\in\mathcal{D}),$$

where the convex function

$$g(w)=2\alpha-1+2(1-\alpha)\frac{\ln(1+w)}{w}$$

is the fuzzy best subordinant.

Proof. Following from Theorem 4.1, the fuzzy differential subordination (4.2) in the form

$$\left|F_{h(\mathcal{D})}h(w)\right|\le\left|F_{p(\mathcal{D})}(p(w)+wp'(w))\right|,\quad(w\in\mathcal{D}).$$

Application of Lemma 2.11, we have

$$\left|F_{g(\mathcal{D})}g(w)\right|\le\left|F_{p(\mathcal{D})}p(w)\right|,$$

that means

$$\left|F_{g(\mathcal{D})}g(w)\right|\le\left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right|,\quad(w\in\mathcal{D}),$$

where

$$\begin{aligned}g(w)&=\frac{1}{w}\int_0^w h(t)dt\\&=\frac{1}{w}\int_0^w\frac{1+(2\alpha-1)t}{t+1}dt\\&=2\alpha-1+\frac{2(1-\alpha)}{w}\log(1+w)\end{aligned}$$

is the convex function and the fuzzy best subordinant. □

Example 4.3. *Let*

$$h(w)=\frac{1-w}{w+1}$$

be a convex function in $\mathcal{D}$ with $h(0)=1$. Suppose that

$$f(w)=w+w^2,\ w\in\mathcal{D}.$$

For $\xi=0,\ \beta=\gamma=r=1,\ k=2$, and $s=\frac{1}{2}$, we have

$$\mathcal{M}_{0,1}^{1,1}f(w)=w+\frac{1}{4}w^2.$$

Then,

$$(\mathcal{M}_{0,1}^{1,1}f(w))'=1+\frac{1}{2}w$$

is univalent in $\mathcal{D}$ and

$$\frac{\mathcal{M}_{0,1}^{1,1}f(w)}{w}=1+\frac{1}{4}w\in Q\cap\mathcal{K}[1,1].$$

We deduce that

$$g(w)=\frac{1}{w}\int_0^w\frac{1-t}{t+1}dt=-1+\frac{2ln(w+1)}{w}.$$

From Theorem 4.1, we obtain

$$\frac{1-w}{w+1}\prec_F 1+\frac{1}{2}w,$$

then

$$-1+\frac{2\ln(w+1)}{w}\prec_F 1+\frac{1}{4}w,\quad (w\in\mathcal{D}).$$

We next establish a series of fuzzy differential superordination results involving the convex functions and the operator $\mathcal{M}_{\xi,\beta}^{s,\gamma}$.

Theorem 4.4. *Let g be a convex function in $\mathcal{D}$, and the function*

$$h(w)=g(w)+wg'(w).$$

Suppose that $(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w))'$ is a univalent function in $\mathcal{D}$ and

$$\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\in Q\cap\mathcal{K}[1,1].$$

If $f\in\mathcal{A}$ satisfies the fuzzy differential superordination:

$$\left|F_{h(\mathcal{D})}h(w)\right|\leq\left|F_{(\mathcal{M}_{\xi,\beta}^{s,\gamma}f)'(\mathcal{D})}(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w))'\right|\Longrightarrow h(w)\prec_F(\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w))',\quad (w\in\mathcal{D}),$$

then

$$\left|F_{g(\mathcal{D})}g(w)\right|\leq\left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(\mathcal{D})}\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w}\right|\Longrightarrow g(w)\prec_F\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma}f(w)}{w},\quad (w\in\mathcal{D}).$$

This result is sharp.

Proof. Using arguments similar to those in the proof of Theorem 4.1, we have

$$\left|F_{g(\mathcal{D})}g(w) + wg'(w)\right| \leq \left|F_{p(\mathcal{D})}(p(w) + wp'(w))\right|, \quad (w \in \mathcal{D}),$$

then by applying Lemma 2.11, we obtain the required result. This result is seen to be sharp when the inequality satisfies a suitably specific function. Thus, the proof of the Theorem 4.4 is completed. □

We now have the following theorem:

Theorem 4.5. *Suppose g is a convex function and*

$$h(w) = g(w) + \frac{wg'(w)}{b+2},$$

with $\Re(b) > -2,\ w \in \mathcal{D}$. *Suppose* $G(w)$ *is defined in* (3.9). *Let* $(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w))'$ *be a univalent function in* $\mathcal{D}$, *and*

$$(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))' \in Q \cap \mathcal{K}[1,1].$$

If $f \in \mathcal{A}$ *satisfies the fuzzy differential superordination:*

$$\left|F_{h(\mathcal{D})}h(w)\right| \leq \left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}f)'(\mathcal{D})}(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w))'\right| \Longrightarrow h(w) \prec_F (\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w))', \quad (w \in \mathcal{D}), \tag{4.3}$$

then

$$\left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}G)'(\mathcal{D})}(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))'\right| \Longrightarrow g(w) \prec_F (\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))', \quad (w \in \mathcal{D})$$

where the convex function $g(w) = \frac{b+2}{w^{b+2}} \int_0^w t^{b+1}h(t)dt$ *is the fuzzy best subordinant.*

Proof. The proof of this theorem is similar to that of Theorem 3.7. By using the relations (3.13) and (3.15), the fuzzy differential superordination (4.3) can be written as follows:

$$\left|F_{h(\mathcal{D})}\left(g(w) + \frac{1}{b+2}wg'(w)\right)\right| \leq \left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}f)'(\mathcal{D})}\left(p(w) + \frac{1}{b+2}wp'(w)\right)\right|.$$

Now applying Lemma 2.11 with

$$\Re(\beta) = b + 2 > 0,$$

we have

$$\left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}G)'(\mathcal{D})}(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))'\right|.$$

Therefore,

$$g(w) \prec_F \mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))',$$

where g is the fuzzy best dominant. Thus, the proof of Theorem 4.5 is complete. □

We now have the following theorem:

Theorem 4.6. *Let*

$$h(w)=\frac{1+(2\eta-1)w}{1+w},\quad \eta\in[0,1).$$

Suppose $G(z)$ is defined in (3.9), $(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(z))'$ *is a univalent function in $\mathcal{D}$, and*

$$(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(z))'\in\mathcal{Q}\cap\mathcal{K}[1,1].$$

If $f\in\mathcal{A}$ satisfies the fuzzy differential superordination:

$$\left|F_{h(\mathcal{D})}h(w)\right|\le\left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}f)'(\mathcal{D})}(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w))'\right|\Longrightarrow h(w)\prec_F(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w))',\quad (w\in\mathcal{D}),\tag{4.4}$$

then

$$\left|F_{g(\mathcal{D})}g(w)\right|\le\left|F_{(\mathcal{M}^{s,\gamma}_{\xi,\beta}G)'(\mathcal{D})}(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))'\right|\Longrightarrow g(w)\prec_F(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w))',\quad (w\in\mathcal{D}),$$

where the convex function

$$g(w)=2\eta-1+\frac{(b+2)(2-2\eta)}{w^{b+2}}\int_0^w\frac{t^{b+1}}{1+t}dt$$

is the fuzzy best subordinant.

Proof. Since

$$G(w)=T^bf(w)=\frac{b+2}{w^{b+1}}\int_0^w t^bf(t)dt.$$

We can be written as

$$w^{b+1}G(w)=(b+2)\int_0^w t^bf(t)dt,\quad (w\in\mathcal{D}).\tag{4.5}$$

Differentiating (4.5) with respect to w, we have

$$(b+1)G(w)+wG'(w)=(b+2)f(w).\tag{4.6}$$

Thus, by applying operator $\mathcal{M}^{s,\gamma}_{\xi,\beta}$ on both sides of (4.6) and differentiating, after simplifications, we get

$$\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)'+\frac{1}{b+2}w\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)''=\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f(w)\right)',\quad (w\in\mathcal{D}).$$

The fuzzy differential suerordination (4.4) can be written as follows:

$$\left|F_{h(\mathcal{D})}h(w)\right|\le\left|F_{\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}f\right)'(\mathcal{D})}\left(\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)'+\frac{1}{b+2}w\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)''\right)\right|.\tag{4.7}$$

Set

$$p(w)=\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)'.\tag{4.8}$$

By substituting (4.8) into (4.7), yields

$$\left|F_{h(\mathcal{D})}h(w)\right|\le\left|F_{p(\mathcal{D})}\left(p(w)+\frac{1}{b+2}wp'(w)\right)\right|.$$

Thus, by applying Lemma 2.10, we have

$$\left|F_{h(\mathcal{D})}h(w)\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{p(\mathcal{D})}p(w)\right|$$

implies that

$$\left|F_{h(\mathcal{D})}h(w)\right| \leq \left|F_{g(\mathcal{D})}g(w)\right| \leq \left|F_{\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G\right)'(\mathcal{D})}\left(\mathcal{M}^{s,\gamma}_{\xi,\beta}G(w)\right)'\right|,$$

where $g(w)$ is given by

$$\begin{aligned} g(w) &= \frac{b+2}{w^{b+2}}\int_0^w t^{b+1}\frac{1+(2\eta-1)t}{1+t}dt \\ &= (2\eta-1) + \frac{2(b+2)(1-\eta)}{w^{b+2}}\int_0^w \frac{t^{b+1}}{1+t}dt, \end{aligned}$$

belongs to convex function class C in $\mathcal{D}$, and $g(\mathcal{D})$ is symmetric with respect to the real axis. This completes the proof of the theorem. □

Example 4.7. *Let*

$$h(w) = \frac{1-w}{w+1}$$

be a convex function in $\mathcal{D}$ with $h(0) = 1$. Suppose that

$$f(w) = w + w^2, \quad w \in \mathcal{D}.$$

For $\xi = 0,\ \beta = \gamma = r = 1,\ k = 2$, and $s = \frac{1}{2}$, we have

$$\mathcal{M}^{1,1}_{0,1}f(w) = w + \frac{1}{4}w^2.$$

Then

$$(\mathcal{M}^{1,1}_{0,1}f(w))' = 1 + \frac{1}{2}w$$

is univalent in $\mathcal{D}$. Now for $b = 4$, we get

$$G(w) = T^4 f(w) = \frac{6}{w^5}\int_0^w t^4(t+t^2)dt = w + \frac{6}{7}w^2.$$

Hence,

$$\mathcal{M}^{1,1}_{0,1}G(w) = w + \frac{3}{14}w^2 \ \text{ and } \ (\mathcal{M}^{1,1}_{0,1}G(w))' = 1 + \frac{3}{7}w \in \mathcal{Q} \cap \mathcal{K}[1,1].$$

We deduce that

$$g(w) = \frac{6}{w^6}\int_0^w \frac{1-t}{t+1}t^5 dt = -\frac{12\ln(1+w)}{w^6} - \frac{48}{w^5} - \frac{66}{w^4} - \frac{16}{w^3} - \frac{3}{w^2} + \frac{12}{5w} - 1.$$

Using Theorem 4.6, we obtain

$$\frac{1-w}{w+1} \prec_F 1 + \frac{1}{2}w$$

implies

$$-\frac{12\ln(1+w)}{w^6} - \frac{48}{w^5} - \frac{66}{w^4} - \frac{16}{w^3} - \frac{3}{w^2} + \frac{12}{5w} - 1 \prec_F 1 + \frac{3}{7}w, \quad (w \in \mathcal{D}).$$

5. Sandwich-type results

In this section, two sandwich-type results are introduced. By combining the results of Theorem 3.3 with Theorem 4.1 and we get the following sandwich-type result:

Theorem 5.1. *Let g_1 and g_2 be univalent convex functions in $\mathcal{D}$. Suppose that h_1 and h_2 are univalent convex functions in $\mathcal{D}$ with $h_1(0) = h_2(0) = 1$ and satisfy* (3.4). *Furthermore, suppose $\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)'$ is a univalent function in $\mathcal{D}$ and*

$$\frac{\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)}{w} \in \mathcal{Q} \cap \mathcal{K}[1,1].$$

If $f \in \mathcal{A}$ satisfies the following conditions:

$$\left|F_{h_1(\mathcal{D})} h_1(w)\right| \leq \left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)'\right| \leq \left|F_{h_2(\mathcal{D})} h_2(w)\right|$$
$$\Longrightarrow h_1(w) \prec_F \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)' \prec_F h_2(w), \quad (w \in \mathcal{D}),$$

then

$$\left|F_{g_1(\mathcal{D})} g_1(w)\right| \leq \left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma} f(\mathcal{D})} \frac{\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)}{w}\right| \leq \left|F_{g_2(\mathcal{D})} g_2(w)\right|$$
$$\Longrightarrow g_1(w) \prec_F \frac{\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)}{w} \prec_F g_2(w), \quad (w \in \mathcal{D}),$$

where g_1 and g_2 are the fuzzy best subordinant and the fuzzy best dominant, respectively.

Combining Theorem 3.7 with Theorem 4.5, we obtain the following sandwich-type result:

Theorem 5.2. *Let g_1 and g_2 be univalent convex functions in $\mathcal{D}$. Suppose that h_1 and h_2 are univalent convex functions in $\mathcal{D}$ with*

$$h_1(0) = h_2(0) = 1$$

and satisfy (3.4). *Furthermore, suppose that $G(w)$ is defined in* (3.9),

$$\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G(w)\right)' \in \mathcal{Q} \cap \mathcal{K}[1,1]$$

and $\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)'$ is a univalent function in $\mathcal{D}$. If $f \in \mathcal{A}$ satisfies

$$\left|F_{h_1(\mathcal{D})} h_1(w)\right| \leq \left|F_{\left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f\right)'(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)'\right| \leq \left|F_{h_2(\mathcal{D})} h_2(w)\right|$$
$$\Longrightarrow h_1(w) \prec_F \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} f(w)\right)' \prec_F h_2(w), \quad (w \in \mathcal{D}),$$

then

$$\left|F_{g_1(\mathcal{D})} g_1(w)\right| \leq \left|F_{\mathcal{M}_{\xi,\beta}^{s,\gamma} G(\mathcal{D})} \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G(w)\right)'\right| \leq \left|F_{g_2(\mathcal{D})} g_2(w)\right|$$
$$\Longrightarrow g_1(w) \prec_F \left(\mathcal{M}_{\xi,\beta}^{s,\gamma} G(w)\right)' \prec_F g_2(w), \quad (w \in \mathcal{D}),$$

where g_1 and g_2 are, respectively, the fuzzy best subordinant and the fuzzy best dominant.

6. Conclusions

In this investigation, we have derived many fuzzy differential subordinations, fuzzy differential superordinations, and sandwich results for analytic functions in the open unit disk associated with the Mittag-Leffler type Pascal distribution operator $\mathcal{M}^{s,\gamma}_{\xi,\beta}$. We used the convolution technique to define a new operator $\mathcal{M}^{s,\gamma}_{\xi,\beta}$ for analytic functions. Using the newly defined class, we have proven some important results. We have also proven the inclusion relation for this class. This study is expected to make effective contributions to the fields of geometric function theory and fuzzy set theory. It is recommended to study these results with q-calculus. This investigation will play a very important role in doing further research in the fields of fuzzy differential techniques in the modern era.

Author contributions

Madan Mohan Soren: conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing the original draft preparation, writing the review and editing, visualization, supervision, project administration.

Luminiţa-Ioana Cotîrlă: conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing the review and editing, visualization, supervision, project administration, funding acquisition.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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