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Research article

Sensitivity analysis of a non-Markovian feedback retrial queue, reneging, delayed repair with working vacation subject to server breakdown

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Abstract: This study investigated the steady-state characteristics of a non-Markovian feedback retrial queue with reneging, delayed repair, and working vacation. In this scenario, we assumed that consumers arrive through Poisson processes and the server provides service to consumers during both regular and working vacation periods. However, it is subject to breakdowns at any moment, resulting in a service interruption for a random duration. Additionally, the concept of delay time was also presented. The consumer that is dissatisfied with the service may re-enter the orbit to receive another service; this individual is considered a feedback consumer. The server will go on a working vacation if the orbit is empty after successfully serving a satisfied consumer. By utilizing the supplementary variable technique (SVT), we examined the steady-state probability generating function of the system and orbit sizes. Finally, numerical outcomes and a sensitivity analysis were given to verify the analytical findings of important performance indicators.

Keywords: retrial queue; working vacation; server breakdown; feedback; delayed repair; supplementary variable technique
Mathematics Subject Classification: 60K25, 68M20, 90B22

1. Introduction

Retrial queues (RQs) with unreliable servers have been extensively investigated due to their wide range of applications in domains such as consumer service centers, computer and communications networks, and systems for production. Retrial queues can serve as an indication of consumer service demands. When consumers find an inaccessible server, they have the option to join a retry group, referred to as an orbit, and make a request for their desired services at another moment in time. To access survey papers, bibliographic information, and books, readers are referred to Falin [1], Artalejo [2], Falin and Templeton [3], and the references provided in these sources. Boussaha et al. [4] explored feedback retrial queues and orbit search with the M/G/1 queueing system. Atencia et al. [5] examined a non-Markovian retrial queue and discussed the customer's sojourn time in the server, system, and orbit. Jeganathan et al. [6] conducted an analysis of asynchronous multiple vacations using a multi-server retrial queueing inventory system. Micheal and Indhira [7] have analyzed a retrial queueing model with two-phase service under Bernoulli working vacations (BWV). They introduced adaptive neuro-fuzzy inference system (ANFIS) computation and cost optimization of nonlinear metaheuristics to validate their model and also compared the results with other methods, such as artificial bee colonies, genetic algorithms, and particle swarm optimization.

Queuing situations where idle servers may engage in vacations can be discovered in IT networks, machinery, and manufacturing systems, among several other domains. During a working vacation (WV) time, the server delivers its service to consumers at a slow rate, but during an ordinary vacation period, the server completely stops its service to consumers. Servi and Finn [8] presented a single-srever Markovian queueing system with WV. Wu and Takagi [9] expanded the Markovian queue with working vacation into the non-Markovian queue with WV. Gao et al. [10] introduced an M/G/1 retrial queue model that takes into account retrial times of a general nature, WV, and vacation interruptions. Rajadurai [11] examined the non-Markovian retrial queue and considered the three consumer-type categories, which are positive, negative, and priority consumers under WV policy. Yang et al. [12] presented a retrial queue model with WV and examined the occurrence of server failures at the start of service. Li et al. [13] investigated the M/G/1 type retrial queueing model, which incorporates single working vacation under Bernoulli schedule. Bouchentouf et al. [14] explored Markovian queues with finite capacity and differentiated working vacation (DWV). Jain et al. performed a study on Markovian queues with disaster failure and MWVs [15]. Murugan and Keerthana [16] investigated a single-server retrial queueing model that incorporates G-queue and MWV concepts. Bharathy and Saravanarajan [17] examined the unreliable retrial queue and two essential services using WV. Chen et al. [18] investigated random working vacation and improved service efficiency vacation policies using an M/G/1 queueing model.

In reality, we frequently encounter situations in which servers fail but may be repaired. Furthermore, it is thought that server failures are the most common reason for service interruptions. Similarly, there are numerous instances that take place in the domains of digital networks, manufacturing systems, production control, and other related areas. Limited maintenance capabilities and service station breakdowns can have a significant impact on system performance. Therefore, queueing systems with unreliable service stations are worth investigating from a performance prediction perspective. Rajadurai et al. [19] explored non-Markovian type retrial queue with multiple WV incorporating server breakdown. Varalakshmi et al. [20] analyzed immediate feedback single server queues with working vacations and server breakdowns and used the supplementary variable technique to find steady-state results. Rajadurai et al. [21] investigated the cost optimization technique of a retrial queue with K-phase optional type of service with multiple working vacations subject to server breakdown. Gao et al. [22] investigated a retrial queue with active and passive breakdowns and delayed repairs. Ke et al. [23] studied the feedback retrial queue and balking, including the server breakdown. In addition, the author used the Probabilistic Global Search Lausanne approach to solve the optimization problem. Liu et al. [24] presented a Markovian queue that incorporates preemptive priority and WV interruption. Recently, a multitude of authors have extensively examined the concept of delayed repair from various perspectives [25–30].

Many queueing scenarios involve consumers being serviced repeatedly for a specific purpose. If a consumer is dissatisfied with the service, they can attempt it multiple times until it is successfully completed. These queueing models are used in stochastic modeling of various real-world, such as in data transmission and packet switching networks. Sharma [31] presented Bernoulli feedback retrial queue with modified vacation subject to random breakdowns. Chang et al. [32] investigated unreliable retrial queue, which included impatient and feedback customers. The author also examined the methods used to perform the optimization tasks, including the Nelder-Mead simplex direct search method, the pattern search method, and the quasi-Newton approach. Ayyappan et al. [33] analyzed the concept of a single server queue and considered the customer priority type, reneging, and immediate feedback under WV. Abdollahi et al. [34] investigated single-server retrial queues, first essential, and k-phase optional services that incorporate server vacation and feedback policies. Jain and Kaur [35] explored a bulk arrival retrial model incorporating optional service and Bernoulli feedback and utilized the maximum entropy principle (MEP) to find steady state probability and waiting time; the Quasi Newton method was also used to find the optimal cost. Wang et al. [36] studied the machine learning-based compressed sensing channel estimation method for wireless communications, which is important for industrial internet of things (IIoT) uses. They explored a distributed compressed sensing-based method for the sparse correlation between channels in multiple-input multiple-output filter bank multicarrier (MIMO-FBMC) with offset quadrature amplitude modulation (OQAM) systems.

Supplementary variables, representing elapsed and remaining times, determine the forward and backward Chapman-Kolmogorov equations that govern the relevant model. To manage the elapsed time, we can introduce supplementary variables that match each non-exponential governing random variable. This procedure transforms the non-Markovian process into a Markovian process by acquiring all essential information, ensuring that its present state alone determines its future. Numerous queue theorists have employed the supplementary variable technique, a widely recognized methodology in queueing theory, to address non-Markovian congestion issues in both everyday and industrial contexts. Using the approach of SVTs, Cox (1955) has examined a non-Markovian approach [37]. Jain et al. [38] surveyed the supplemental variable approach for studying M/G/1 queues with service interruptions caused by vacation or server breakdowns. Deepa and Azhagappan [39] used the SVT method to analyze the bulk arrival queue with optional second-phase service, including the optional re-service concept. Huang [40] presented an analysis of a batch arrival queue and service pattern in an optional phase with a randomized vacation concept using the SVT method.

This study presents a novel approach to modeling an unreliable server in a single arrival feedback retrial queue with reneging, and delayed repair under working vacation. The current study enhances the previous research conducted by Madhu Jain [41] by incorporating the novel concepts of (1) single arrival, (2) reneging, and (3) delayed repair during busy and WV periods. To the authors, best knowledge, there is no existing literature in the field of the queueing model that discusses non-Markovian queues incorporating the concepts of general retry times, single arrival, feedback, reneging, and delayed repair in both RS and WV circumstances. Our study specifically considers the problem of an unreliable service in the RS and WV modes. The fundamental motivation for developing the suggested generalized model is its potential application in real-world queueing scenarios such as communication networks and consumer care.

This article's next sections are organized as follows: Section 2 provides a detailed explanation of the system model and includes a real-life example. Section 3 presents the steady-state probabilities. Section 4 discusses the performance characteristics of the model. Section 5 examines special cases. Section 6 presents an analysis of cost optimization. Section 7 presents numerical examples that demonstrate the effects of different system performance factors. Finally, Section 8 provides the article's conclusion.

2. Formulation of the model

In this section, we consider an M/G/1 feedback retrial queue with reneging, delayed repair, and WV subject to server breakdown. The following is an explanation of our proposed model (Figure 1), provided by

- **Consumers arrival process**: The consumers arrive at the queueing system following a Poisson process, with rates of λ in the busy state and λ_v in the working vacation state.
- **The retrial rule**: If the arriving primary consumer discovers that the server is available, the consumer starts his service instantly. However, the consumer has the option to join the orbit's retry group if the server is busy, or working vacation, or breakdown. They attempt to make repeated service requests at random times based on the "First In-First Out" discipline; this indicates that only the single customer at the head of the orbit queue can access the server. In the event that the primary customer comes first, the retrial customer is able to cancel their request and proceed to either rejoin the retrial queue with probability (prob.) r or exit the system with prob. $(1 - r)$. This is the behavior of the retrial consumer, also known as reneging. It was considered that the inter-retrial time is indicated by the distribution function $A(\mathrm{t})$, while the Laplace Stieltjes Transform (LST) is denoted as $A^*(\hat{\upsilon})$.
- **Working vacation policy**: When the orbit appears to be empty, the server commences its working vacation (WV), and the vacation time follows an exponential distribution with the characteristic parameter θ. When primary consumers arrive during a working vacation, the server delivers service at a slow rate. If there is any consumer present in the system during the slow speed service completion in the vacation state, the server will stop the vacation and resume the regular busy period, causing an interruption of the vacation. When a vacation period ends and there are still consumers in orbit, the server starts up normally. Otherwise, the server will take another vacation. The service time takes the form of a distribution function $N_v(\mathrm{t})$ during the WV period and its LST is represented as $N_v^*(\hat{\upsilon})$.
- **Regular service process**: Upon the arrival of a new consumer or a retry consumer at the service station, if the server is free, it immediately commences rendering regular service to the consumers. The service time is denoted by the general distribution and its function $N_b(\mathrm{t})$ and LST are represented as $N_b^*(\hat{\upsilon})$ and $E(N_b)$, $E(N_b)^2$ denote the first and second moments, respectively.
- **Feedback rule**: Following the completion of each consumer's service, dissatisfied consumers i.e., customers who are not satisfied with their service, may join the retry group with a prob. of f $(0 \leq f \leq 1)$ or leave the system with a prob. of $\overline{f} = 1 - f$.
- **Server breakdown event**: In fact, the server operating both in normal service mode and in working vacation mode is vulnerable to failures occurring at any moment, resulting in an unpredictable disruption of the service time. The durations of breakdowns in both scenarios, namely the regular and working vacation states, are followed by exponential distributions with rates δ and δ_v, respectively.
- **Repair process**: The server repair task commences instantly upon the server's failure, whether it occurs during peak activity or on a working vacation. While undergoing repairs, the server temporarily stop its services until the repair process is finished. Note that the consumer is waiting for the remaining services after receiving assistance from the server during a breakdown. When a server encounters a failure, it undergoes repair. During the breakdown time, the server is unable to serve the consumers and is waiting for the repair process to begin, which we refer to as the

server's waiting time. We define the waiting period as the delay period. The delay timings are represented by the functions $D_b(\mathrm{t})$ and $D_v(\mathrm{t})$, their Laplace-Stieltjes transforms are $D_b^*(y)$ and $D_v^*(y)$, and its first two moments are assumed to be $\mathfrak{w}^{(1)}$, $\mathfrak{w}^{(2)}$, and $\mathfrak{w}_{\mathfrak{v}}{}^{(1)}$, $\mathfrak{w}_{\mathfrak{v}}{}^{(2)}$, depending on the busy and working periods, respectively. Similarly, the repair times are evaluated using the distribution functions $G_b(\mathrm{t})$ and $G_v(\mathrm{t})$, with their Laplace-Stieltjes transforms denoted as $G_b^*(y)$ and $G_v^*(y)$, and its two moments are assumed to be $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$, whereas the subsequent two moments are represented as $\mathfrak{g}_{\mathfrak{v}}{}^{(1)}$ and $\mathfrak{g}_{\mathfrak{v}}{}^{(2)}$, which correspond to the busy and working vacation states, respectively.

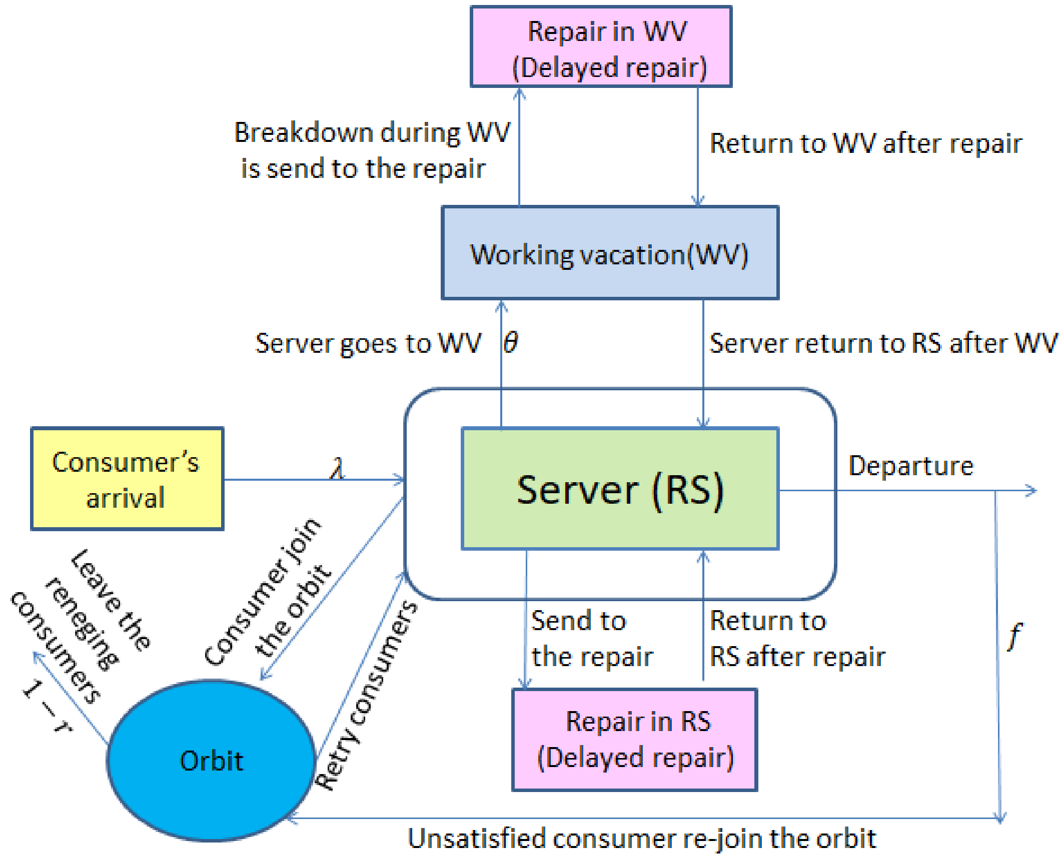

Figure 1. Diagram of our proposed model.

2.1. *Application*

Our approach has practical application in various domains like data processing, internet access, manufacturing and industrial production systems, management of inventory systems, software, and experiments. Let's take a network of telecommunications as an example. Call centers have a significant impact on several industries and businesses in the field of telecommunications. Consumers are initiating contact with call centers by engaging in conversation with a consumer care representative (CCR). An incoming voice call is taken immediately by the idle CCR. During a voice call, if the CCR is unavailable because it is busy with other calls, the call will be temporarily held in a retrial buffer (orbit) with a finite capacity. If there is space available, the call will be handled at a later time (retrial time), according to the FCFS principle. The caller attempts to retry the call from orbit, but if they do not receive a response, they may choose to cancel their effort for service and exit the system, a process known as reneging. The service may experience electronic fails during the regular service mode (breakdown RS). Upon the end of call processing, the internet service may need the CCR to provide the same service again (feedback) in the event of any failures in the previous process. If the CCR does not detect any voice calls, it will carry out a series of maintenance tasks, such as doing virus scans (WV) on the system. During the repair time, the conventional center is equipped with various components, including an automatic call distributor (ACD) and an interactive voice response (IVR) unit (referred to as the WV server). These components are capable of handling calls at a slower rate during the working vacation period. However, it is possible that this server may experience technical issues during this period (referred to as Breakdown WV). When a server experiences a failure, it is taken out of service for repairs. This results in a temporary interruption of consumer service, termed the server's waiting time (referred to as delayed (RS, WV)). We define the duration of waiting as a period of delay. This type of retrial queue, which incorporates working vacations, serves as a reliable approximation of such for telecommunication processing systems. The suggested method can be used in modern culture, particularly in healthcare systems that use telephone consultations.

3. Steady-state probabilities

The steady-state governing equations are derived using the SVT. The probability-generating function (PGF) was obtained for the server states and also for the number of consumers in the orbit and system.

We consider that $A(0) = 0$, $A(\infty) = 1$, $N_b(0) = 0$, $N_b(\infty) = 1$, $N_v(0) = 0$, $N_v(\infty) = 1$, $W_b(0) = 0$, $W_b(\infty) = 1$, $W_v(0) = 0$, $W_v(\infty) = 1$, are continuous at $\hat{\upsilon} = 0$ $G_b(0) = 0$, $G_b(\infty) = 1$ and $G_v(0) = 0$, $G_v(\infty) = 1$ are continuous at $y = 0$. We assume the hazard rate functions as $\dot{a}\ (\hat{\upsilon})$, $\beta_b(\hat{\upsilon})$, $\beta_v(\hat{\upsilon})$, $\chi_b(y)$, $\chi_v(y)$, $\gamma_b(y)$, and $\gamma_v(y)$ for retrial, regular service, slow rate service, delay repair (RS, WV) and for the maintainance (RS, WV) in that order, respectively.

$$
\begin{aligned}
&\dot{a}(\hat{\upsilon})\ d(\hat{\upsilon}) = \frac{d(A(\hat{\upsilon}))}{(1 - A(\hat{\upsilon}))}; \quad \beta_b(\hat{\upsilon})d\hat{\upsilon} = \frac{d(N_b(\hat{\upsilon}))}{(1 - N_b(\hat{\upsilon}))}; \quad \beta_v(\hat{\upsilon})d\hat{\upsilon} = \frac{d\ (N_v(\hat{\upsilon}))}{(1 - N_v(\hat{\upsilon}))};\\
&\chi_b(y)dy = \frac{d(W_b(y))}{(1 - W_b(y))}; \quad \chi_v(y)dy = \frac{d(W_v(y))}{(1 - W_v(y))}; \quad \gamma_b(y)dy = \frac{d(G_b(y))}{(1 - G(y))};\\
&\gamma_v(y)dy = \frac{d(G_v(y))}{(1 - G(y))}.
\end{aligned}
$$

In addition, let $A^0, N_b^0, N_v^0, W_b^0, W_v^0, G_b^0$and G_v^0 be the expired retrial, busy, working vacation (WV), delay to repair, and repair times shown at period t. We also assume the random variable (RV),

$$
S(\mathrm{t}) = \begin{cases}
0, & \text{server is unoccupied}\\
1, & \text{server is unoccupied and in RS mode}\\
2, & \text{server is occupied and in RS mode}\\
3, & \text{server is occupied and in lower-service mode}\\
4, & \text{server is waiting for repair in WV mode}\\
5, & \text{server is waiting for repair RS mode}\\
6, & \text{the server is undergoing maintenance WV mode}\\
7, & \text{the server is undergoing maintenance RS mode}
\end{cases}
$$

$\{S(t), C(t); \mathrm{t} \geq 0\}$ show the bivariate Markov process, where $C(t)$ is the number of consumers in the orbit at time t. The function $S(t)$ denotes the server states $(0,1,2,3,4,5,6,7)$ depending on if the server is unoccupied, RS, WV, delayed repair (WV, RS), and under repair (WV, RS). Let us assume that the limiting probabilities $P_0(\mathrm{t}) = Prob.S(t) = 0, C(t) = 0$ and the prob. densities are

$$I_{\mathrm{n}}(\hat{\upsilon}, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 1, C(t) = \mathrm{n},\ \hat{\upsilon} \leq A^0(\mathrm{t}) < \hat{\upsilon} + d\hat{\upsilon}\}$$
$$B_{b,\mathrm{n}}(\hat{\upsilon}, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 2, C(t) = \mathrm{n},\ \hat{\upsilon} \leq N_b^0(\mathrm{t}) < \hat{\upsilon} + d\hat{\upsilon}\}$$
$$\phi_{v,\mathrm{n}}(\hat{\upsilon}, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 3, C(t) = \mathrm{n},\ \hat{\upsilon} \leq N_v^0(\mathrm{t}) < \hat{\upsilon} + d\hat{\upsilon}\}$$
$$D_{b,\mathrm{n}}(\hat{\upsilon}, y, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 4, C(t) = \mathrm{n},\ y \leq W_b^0(\mathrm{t}) < y + dy/N_b^0(\mathrm{t}) = \hat{\upsilon}\}$$
$$D_{v,\mathrm{n}}(\hat{\upsilon}, y, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 5, C(t) = \mathrm{n},\ y \leq W_v^0(\mathrm{t}) < y + dy/N_v^0(\mathrm{t}) = \hat{\upsilon}\}$$
$$\check{R}_b(\hat{\upsilon}, y, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 6, C(t) = \mathrm{n},\ y \leq G_b^0(\mathrm{t}) < y + dy/N_b^0(\mathrm{t}) = \hat{\upsilon}\}$$
$$\check{R}_v(\hat{\upsilon}, y, \mathrm{t})d\hat{\upsilon} = \lim_{\mathrm{t}\to\infty} Prob.\{S(t) = 7, C(t) = \mathrm{n},\ y \leq G_v^0(\mathrm{t}) < y + dy/N_b^0(\mathrm{t}) = \hat{\upsilon}\}$$
$$\forall\ \mathrm{t} \geq 1,\ \hat{\upsilon} \geq 1,\ \mathrm{n} \geq 1.$$

The time $(\mathrm{t}_{\mathrm{n}}; \mathrm{n} = 1, 2, ...)$ represents a sequence of epochs that correspond to the completion times of WV, or the point at which the delay is resolved and the repair period concludes. A Markov chain is created by a set of random vectors $\psi_{\mathrm{n}} = \{S(\mathrm{t}n+), C(\mathrm{t}n+)\}$ forms a Markov chain that is embedded in the RQ system. It follows from Appendix A that $\{\pi_{\mathrm{n}}; \mathrm{n} \in \mathbb{N}\}$ is ergodic if and only if $\Gamma < 1$ for our system to be stable, where $\Gamma = f + r(1 - A^*(\lambda) + E(N_b)(\lambda + \lambda\delta(w^1 + g^1)))$.

3.1. Steady-state conditions

The governing equations are formulated using the supplementary variable approach.

$$(\lambda_v + \theta)P_0 = \theta P_0 + \overline{f}\left(\int_0^\infty B_{b,0}(\hat{\upsilon})\beta_b(\hat{\upsilon})d\hat{\upsilon} + \int_0^\infty \phi_{v,0}(\hat{\upsilon})\beta_v(\hat{\upsilon})d\hat{\upsilon}\right). \tag{3.1}$$

$$\frac{d}{d\hat{\upsilon}} I_{\mathrm{n}}(\hat{\upsilon}) = -[\lambda + \dot{a}(\hat{\upsilon})]I_{\mathrm{n}}(\hat{\upsilon}), \quad \mathrm{n} \geq 1. \tag{3.2}$$

$$\frac{d}{d\hat{\upsilon}} B_{b,0}(\hat{\upsilon}) = -[\lambda + \delta + \beta_b(\hat{\upsilon})]B_{b,0}(\hat{\upsilon}), \quad \mathrm{n} = 0. \tag{3.3}$$

$$\frac{d}{d\hat{\upsilon}} B_{b,\mathrm{n}}(\hat{\upsilon}) = -[\lambda + \delta + \beta_b(\hat{\upsilon})]B_{b,\mathrm{n}}(\hat{\upsilon}) + \lambda B_{b,\mathrm{n}-1}(\hat{\upsilon}) + \int_0^\infty \check{R}_{b,n}(\hat{\upsilon}, y)\gamma_b(y)dy, \quad \mathrm{n} \geq 1. \tag{3.4}$$

$$\frac{d}{d\hat{\upsilon}} \phi_{v,0}(\hat{\upsilon}) = -[\lambda_v + \delta_v + \beta_v(\hat{\upsilon})]\phi_{v,0}(\hat{\upsilon}), \quad \mathrm{n} = 0. \tag{3.5}$$

$$\frac{d}{d\hat{\upsilon}} \phi_{v,\mathrm{n}}(\hat{\upsilon}) = -[\lambda_v + \delta_v + \beta_v(\hat{\upsilon})]\phi_{v,n}(\hat{\upsilon}) + \lambda_v\phi_{v,\mathrm{n}-1}(\hat{\upsilon}) + \int_0^\infty \check{R}_{v,n}(\hat{\upsilon}, y)\gamma_v(y)dy, \quad \mathrm{n} \geq 1. \tag{3.6}$$

$$\frac{d}{dy} D_{b,0}(\hat{\upsilon}, y) = -[\lambda + \chi_b(y)]D_{b,0}(\hat{\upsilon}, y), \quad \mathrm{n} = 0. \tag{3.7}$$

$$\frac{d}{dy} D_{b,\mathrm{n}}(\hat{\upsilon}, y) = -[\lambda + \chi_b(y)]D_{b,\mathrm{n}}(\hat{\upsilon}, y) + \lambda_v D_{b,\mathrm{n}-1}(\hat{\upsilon}, y), \quad \mathrm{n} \geq 1. \tag{3.8}$$

$$\frac{d}{dy} D_{v,0}(\hat{\upsilon}, y) = -[\lambda_v + \chi_v(y)]D_{v,0}(\hat{\upsilon}, y), \quad \mathrm{n} = 0. \tag{3.9}$$

$$\frac{d}{dy} D_{v,\mathrm{n}}(\hat{\upsilon}, y) = -[\lambda_v + \chi_v(y)]D_{\mathrm{n}}(\hat{\upsilon}, y) + \lambda_v D_{v,\mathrm{n}-1}(\hat{\upsilon}, y), \quad \mathrm{n} \geq 1. \tag{3.10}$$

$$\frac{d}{dy}\check{R}_{b,0}(\hat{\upsilon},y) = -(\lambda+\gamma_b(y))\check{R}_{b,0}(\hat{\upsilon},y), \quad \mathfrak{n}=0. \tag{3.11}$$

$$\frac{d}{dy}\check{R}_{b,\mathfrak{n}}(\hat{\upsilon},y) = -[\lambda+\gamma_b(y)]\check{R}_{\mathfrak{n}}(\hat{\upsilon},y) + \lambda\hat{\upsilon}_{b,\mathfrak{n}-1}(\hat{\upsilon},y), \quad \mathfrak{n}\geq 1. \tag{3.12}$$

$$\frac{d}{dy}\check{R}_{v,0}(\hat{\upsilon},y) = -(\lambda_v+\gamma_v(y))\check{R}_{v,0}(\hat{\upsilon},y), \quad \mathfrak{n}=0. \tag{3.13}$$

$$\frac{d}{dy}\check{R}_{v,\mathfrak{n}}(\hat{\upsilon},y) = -[\lambda_v+\gamma_v(y)]\check{R}_{\mathfrak{n}}(\hat{\upsilon},y) + \lambda_v\hat{\upsilon}_{v,\mathfrak{n}-1}(\hat{\upsilon},y), \quad \mathfrak{n}\geq 1. \tag{3.14}$$

When $\hat{\upsilon}=0$ and $y=0$, the associated boundary conditions are stated as:

$$\begin{aligned} I_{\mathfrak{n}}(0) &= \overline{f}\left(\int_0^\infty B_{b,\mathfrak{n}}(\hat{\upsilon})\beta_b(\hat{\upsilon})d\hat{\upsilon} + \int_0^\infty \phi_{v,\mathfrak{n}}(\hat{\upsilon})\beta_v(\hat{\upsilon})d\hat{\upsilon}\right) \\ &\quad + f\left(\int_0^\infty B_{b,\mathfrak{n}-1}(\hat{\upsilon})\beta_b(\hat{\upsilon})d\hat{\upsilon} + \int_0^\infty \phi_{v,\mathfrak{n}-1}(\hat{\upsilon})\beta_v(\hat{\upsilon})d\hat{\upsilon}\right) \quad \mathfrak{n}\geq 1. \end{aligned} \tag{3.15}$$

$$B_{b,0}(0) = \left(\int_0^\infty I_1(\hat{\upsilon})\dot{a}(\hat{\upsilon})d\hat{\upsilon} + \lambda(1-r)\int_0^\infty I_1(\hat{\upsilon})d\hat{\upsilon} + \theta\int_0^\infty \phi_{v,0}(\hat{\upsilon})d\hat{\upsilon}\right), \mathfrak{n}=0. \tag{3.16}$$

$$\begin{aligned} B_{b,\mathfrak{n}}(0) &= \int_0^\infty I_{\mathfrak{n}+1}(\hat{\upsilon})\dot{a}(\hat{\upsilon})d\hat{\upsilon} + \lambda r\int_0^\infty I_1(\hat{\upsilon})d\hat{\upsilon} + \lambda(1-r)\int_0^\infty I_{n+1}(\hat{\upsilon})d\hat{\upsilon} \\ &\quad + \theta\int_0^\infty \phi_{v,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon}, \mathfrak{n}\geq 1. \end{aligned} \tag{3.17}$$

$$\phi_{v,\mathfrak{n}}(0) = \begin{cases} \lambda_v P_0, & \mathfrak{n}=0; \\ 0, & \mathfrak{n}\geq 1. \end{cases} \tag{3.18}$$

$$D_{b,\mathfrak{n}}(\hat{\upsilon},0) = \delta\int_0^\infty B_{b,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon}, \quad \mathfrak{n}\geq 0. \tag{3.19}$$

$$D_{v,\mathfrak{n}}(\hat{\upsilon},0) = \delta_v\int_0^\infty \phi_{v,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon}, \quad \mathfrak{n}\geq 0. \tag{3.20}$$

$$\check{R}_{b,\mathfrak{n}}(\hat{\upsilon},0) = \int_0^\infty D_{b,\mathfrak{n}}(\hat{\upsilon},y)\chi_b(y)dy, \quad \mathfrak{n}\geq 0. \tag{3.21}$$

$$\check{R}_{v,\mathfrak{n}}(\hat{\upsilon},0) = \int_0^\infty D_{v,\mathfrak{n}}(\hat{\upsilon},y)\chi_v(y)dy, \quad \mathfrak{n}\geq 0. \tag{3.22}$$

The expression for normalizing condition is given as

$$P_0 + \sum_{\mathfrak{n}=1}^\infty\int_0^\infty I_{\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon} + \sum_{\mathfrak{n}=0}^\infty\left(\begin{array}{r} \int_0^\infty B_{b,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon} + \int_0^\infty \phi_{v,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon} \\ +\int_0^\infty\int_0^\infty \check{R}_{b,\mathfrak{n}}(\hat{\upsilon},y)d\hat{\upsilon}dy + \int_0^\infty\int_0^\infty \check{R}_{v,\mathfrak{n}}(\hat{\upsilon},y)d\hat{\upsilon}dy \\ +\int_0^\infty\int_0^\infty D_{b,\mathfrak{n}}(\hat{\upsilon},y)d\hat{\upsilon}dy + \int_0^\infty\int_0^\infty D_{v,\mathfrak{n}}(\hat{\upsilon},y)d\hat{\upsilon}dy \end{array}\right) = 1. \tag{3.23}$$

3.2. Steady-state solution

In this section, we derive the equation illustrating the steady-state of the RQ model by utilizing the prob. generating functions (PGFs) approach. To solve the above equations, the PGFs are defined for $|\varsigma|\leq 1$ in the following manner:

$$
\begin{aligned}
I(\hat{\upsilon},\ \varsigma) &= \sum_{\mathfrak{n}=1}^{\infty} I_{\mathfrak{n}}(\hat{\upsilon})\ \varsigma^{\mathfrak{n}}; & I(0,\ \varsigma) &= \sum_{\mathfrak{n}=1}^{\infty} I_{\mathfrak{n}}(0)\ \varsigma^{\mathfrak{n}};\\
B_b(\hat{\upsilon},\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} B_{b,\mathfrak{n}}(\hat{\upsilon})\ \varsigma^{\mathfrak{n}}; & B_b(0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} B_{b,\mathfrak{n}}(0)\ \varsigma^{\mathfrak{n}};\\
\phi_v(\hat{\upsilon},\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \phi_{v,\mathfrak{n}}(\hat{\upsilon})\ \varsigma^{\mathfrak{n}}; & \phi_v(0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \phi_{v,\mathfrak{n}}(0)\ \varsigma^{\mathfrak{n}};\\
D_b(\hat{\upsilon},y,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} D_{b,\mathfrak{n}}(\hat{\upsilon},y)\ \varsigma^{\mathfrak{n}}; & D_b(\hat{\upsilon},0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} D_{b,\mathfrak{n}}(\hat{\upsilon},0)\ \varsigma^{\mathfrak{n}};\\
D_v(\hat{\upsilon},y,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} D_{v,\mathfrak{n}}(\hat{\upsilon},y)\ \varsigma^{\mathfrak{n}}; & D_v(\hat{\upsilon},0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} D_{v,\mathfrak{n}}(\hat{\upsilon},0)\ \varsigma^{\mathfrak{n}};\\
\check{R}_b(\hat{\upsilon},y,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \check{R}_{b,\mathfrak{n}}(\hat{\upsilon},y)\ \varsigma^{\mathfrak{n}}; & \check{R}_b(\hat{\upsilon},0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \check{R}_{b,\mathfrak{n}}(\hat{\upsilon},0)\ \varsigma^{\mathfrak{n}};\\
\check{R}_v(\hat{\upsilon},y,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \check{R}_{v,\mathfrak{n}}(\hat{\upsilon},y)\ \varsigma^{\mathfrak{n}}; & \check{R}_v(\hat{\upsilon},0,\ \varsigma) &= \sum_{\mathfrak{n}=0}^{\infty} \check{R}_{v,\mathfrak{n}}(\hat{\upsilon},0)\ \varsigma^{\mathfrak{n}}.
\end{aligned}
$$

By multiplying equations Eqs (3.1)–(3.22) with $\varsigma^{\mathfrak{n}}$, and summing over $\mathfrak{n}$ (where $\mathfrak{n} = 0,\ 1,\ 2,\ \ldots$), we obtain:

$$\frac{\partial}{\partial\hat{\upsilon}}\ I(\hat{\upsilon},\varsigma) = -[\lambda + \dot{a}(\hat{\upsilon})]\ I(\hat{\upsilon},\varsigma); \tag{3.24}$$

$$\frac{\partial}{\partial\hat{\upsilon}}\ B_b(\hat{\upsilon},\varsigma) = -[\lambda(1-\varsigma) + \delta + \beta_b(\hat{\upsilon})]\ B_b(\hat{\upsilon},\varsigma); \tag{3.25}$$

$$\frac{\partial}{\partial\hat{\upsilon}}\ \phi_v(\hat{\upsilon},\varsigma) = -[\lambda_v(1-\varsigma) + \delta_v + \beta_v(\hat{\upsilon})]\ \phi_v(\hat{\upsilon},\varsigma); \tag{3.26}$$

$$\frac{\partial}{\partial y}\ D_b(\hat{\upsilon},y,\varsigma) = -[\lambda(1-\varsigma) + \chi_b(y)]\ D_b(\hat{\upsilon},y,\varsigma); \tag{3.27}$$

$$\frac{\partial}{\partial y}\ D_v(\hat{\upsilon},y,\varsigma) = -[\lambda_v(1-\varsigma) + \chi_v(y)]\ D_v(\hat{\upsilon},y,\varsigma); \tag{3.28}$$

$$\frac{\partial}{\partial y}\ \check{R}_b(\hat{\upsilon},y,\varsigma) = -[\lambda(1-\varsigma) + \gamma_b(y)]\ \check{R}_b(\hat{\upsilon},y,\varsigma); \tag{3.29}$$

$$\frac{\partial}{\partial y}\ \check{R}_v(\hat{\upsilon},y,\varsigma) = -[\lambda_v(1-\varsigma) + \gamma_v(y)]\ \check{R}_v(\hat{\upsilon},y,\varsigma); \tag{3.30}$$

$$I(0,\varsigma) = (f\varsigma + \overline{f})\left(\int_0^\infty B_b(\hat{\upsilon},\varsigma)\beta_b(\hat{\upsilon})d\hat{\upsilon} + \int_0^\infty \phi_v(\hat{\upsilon},\varsigma)\beta_v(\hat{\upsilon})d\hat{\upsilon}\right); \tag{3.31}$$

$$\begin{aligned}B_b(0,\varsigma) = \frac{1}{\varsigma}\int_0^\infty I(\hat{\upsilon},\varsigma)\dot{a}(\hat{\upsilon})d\hat{\upsilon} + \lambda r\int_0^\infty I(\hat{\upsilon},\varsigma)d\hat{\upsilon} + \lambda\frac{(1-r)}{\varsigma}\int_0^\infty I(\hat{\upsilon},\varsigma)d\hat{\upsilon};\\ + \theta\int_0^\infty \phi_v(\hat{\upsilon},\varsigma)d\hat{\upsilon};\end{aligned} \tag{3.32}$$

$$\phi_v(0,\varsigma) = \lambda_v P_0; \tag{3.33}$$

$$D_{b,\mathfrak{n}}(\hat{\upsilon},0,\varsigma) = \delta\int_0^\infty B_{b,\mathfrak{n}}(\hat{\upsilon})d\hat{\upsilon}; \tag{3.34}$$

$$D_{v,n}(\hat{\upsilon},0,\varsigma) = \delta_v \int_0^\infty \phi_{v,n}(\hat{\upsilon}) d\hat{\upsilon}; \tag{3.35}$$

$$\check{R}_{b,n}(\hat{\upsilon},0,\varsigma) = \int_0^\infty D_{b,n}(y)\gamma(y)dy; \tag{3.36}$$

$$\check{R}_{v,n}(\hat{\upsilon},0,\varsigma) = \int_0^\infty D_{v,n}(y)\gamma(y)dy. \tag{3.37}$$

Solving the partial differential Eqs (3.24)–(3.29), we get

$$I(\hat{\upsilon},\ \varsigma) = I(0,\ \varsigma)\,[1 - A(\hat{\upsilon})]\ exp\{-\lambda\hat{\upsilon}\}; \tag{3.38}$$

$$B_b(\hat{\upsilon},\ \varsigma) = B_b(0,\ \varsigma)\,[1 - N_b(\hat{\upsilon})]\ exp\{-A_b(\varsigma)\hat{\upsilon}\}; \tag{3.39}$$

$$\phi_v(\hat{\upsilon},\ \varsigma) = \phi_v(0,\ \varsigma)\,[1 - N_v(\hat{\upsilon})]\ exp\{-A_v(\varsigma)\hat{\upsilon}\}; \tag{3.40}$$

$$\check{R}_b(\hat{\upsilon},y,\ \varsigma) = \check{R}(\hat{\upsilon},0,\ \varsigma)\,[1 - G_b(\hat{\upsilon})]\ exp\{-b(\varsigma)y\}; \tag{3.41}$$

$$\check{R}_v(\hat{\upsilon},y,\ \varsigma) = \check{R}(\hat{\upsilon},0,\ \varsigma)\,[1 - G_v(\hat{\upsilon})]\ exp\{-b_v(\varsigma)y\}; \tag{3.42}$$

$$D_b(\hat{\upsilon},y,\ \varsigma) = D_y(\hat{\upsilon},0,\ \varsigma)\,[1 - W_b(\hat{\upsilon})]\ exp\{-b(\varsigma)y\}, \tag{3.43}$$

$$D_v(\hat{\upsilon},y,\ \varsigma) = D_y(\hat{\upsilon},0,\ \varsigma)\,[1 - W_v(\hat{\upsilon})]\ exp\{-b_v(\varsigma)y\}. \tag{3.44}$$

where $A_b(\varsigma) = (\lambda(1-\varsigma) + \delta(1 - W_b^*(b(\varsigma))G_b^*(b(\varsigma)))$,
$A_v(\varsigma) = (\lambda_v(1-\varsigma) + \theta + \delta_v(1 - W_v^*(b_v(\varsigma))G_v^*(b_v(\varsigma)))$, $b(\varsigma) = \lambda\,(1-\varsigma)$, and $b_v(\varsigma) = \lambda_v\,(1-\varsigma)$.

By substituting Eqs (3.38) and (3.40) into Eq (3.32), and subsequently applying certain modifications, we obtain the following expression:

$$B_b(0,\ \varsigma) = (I(0,\ \varsigma)/\varsigma)\,(A^*(\lambda) + (1 - r + r\varsigma)\,(1 - A^*(\lambda))) + \lambda_v P_0 V(\varsigma), \tag{3.45}$$

where

$$V(\varsigma) = \frac{\theta\ [1 - N_v^*\,(A_v(\varsigma))]}{A_v(\varsigma)}.$$

Using Eqs (3.39) and (3.40) in Eq (3.31), gives

$$I(0,\ \varsigma) = (f\varsigma + \overline{f})\,(B_b(0,\ \varsigma)\ N_b^*\,(A_b(\varsigma)) + \phi_v(0,\ \varsigma)\ N_v^*\,(A_v(\varsigma))) - \lambda_v P_0. \tag{3.46}$$

Using Eq (3.39) in Eq (3.34), we get

$$D_b(\hat{\upsilon},0,\ \varsigma) = \delta B_b(0,\ \varsigma)\left(\frac{1 - N_b^*\,(A_b(\varsigma))}{A_b(\varsigma)}\right). \tag{3.47}$$

Inserting Eq (3.40) in Eq (3.35), we obtain

$$D_v(\hat{\upsilon},0,\ \varsigma) = \delta_v\lambda_v P_0\left(\frac{1 - N_v^*\,(A_v(\varsigma))}{A_v(\varsigma)}\right). \tag{3.48}$$

Inserting the Eq (3.43) in Eq (3.36), we obtain

$$\check{R}_b(\hat{\upsilon},0,\ \varsigma) = D_b(\hat{\upsilon},0,\ \varsigma)(\chi^*(b(\varsigma))). \tag{3.49}$$

Using the Eq (3.44) in Eq (3.37), gives

$$\check{R}_v(\hat{\upsilon},0,\ \varsigma)=D_v(\hat{\upsilon},0,\ \varsigma)(\chi^*(b_v(\varsigma))). \tag{3.50}$$

Using Eq (3.33) and (3.45) in Eq (3.46), we obtain

$$I(0,\ \varsigma)=\frac{\mathrm{Nu}(\varsigma)}{\mathrm{De}(\varsigma)}. \tag{3.51}$$

Where

$$\mathrm{Nu}(\varsigma)=\varsigma\lambda_v P_0\times\left\{(f\varsigma+\overline{f})\,(N_b^*\,(A_b(\varsigma))\,V(\varsigma)+N_v^*(A_v(\varsigma)))-1\right\};$$
$$\mathrm{De}(\varsigma)=\left[\varsigma-(f\varsigma+\overline{f})\,(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\,N_b^*(A_b(\varsigma))\right].$$

Using Eq (3.51) in Eq (3.45), we get

$$B_b(0,\ \varsigma)=\frac{P_0}{De(\varsigma)}\lambda_v\times\left\{(f\varsigma+\overline{f})\,((N_v^*\,(A_b(\varsigma))-1)\,(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))+\varsigma V(\varsigma))\right\}. \tag{3.52}$$

Using Eq (3.52) in Eq (3.47), we get

$$D_b(\hat{\upsilon},0,\ \varsigma)=\frac{\delta\lambda_v P_0\left(1-N_b^*\,(A_b(\varsigma))\right)}{A_b(\varsigma)\times\mathrm{De}(\varsigma)}\left\{(f\varsigma+\overline{f})\begin{pmatrix}(N_v^*\,(A_b(\varsigma))-1)\\(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\end{pmatrix}+\varsigma V(\varsigma)\right\}. \tag{3.53}$$

Using Eq (3.53) in Eq (3.49), we get

$$\check{R}_b(\hat{\upsilon},0,\ \varsigma)=\frac{\delta P_0\left(1-N_b^*\,(A_b(\varsigma))\right)W_b^*(b(\varsigma))}{A_b(\varsigma)\times\mathrm{De}(\varsigma)}\left\{\begin{matrix}(f\varsigma+\overline{f})\begin{pmatrix}(N_v^*\,(A_b(\varsigma))-1)\\((A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\\+\varsigma V(\varsigma)\end{pmatrix}\end{matrix}\right\}. \tag{3.54}$$

Using Eq (3.48) in Eq (3.50), we get

$$\check{R}_v(\hat{\upsilon},0,\ \varsigma)=\frac{\delta_v\lambda_v P_0\,(1-N_v^*\,(A_v(\varsigma)))\,W_v^*(b_v(\varsigma))}{A_v(\varsigma)}. \tag{3.55}$$

Likewise, Eq (3.33), Eq (3.48), and Eqs (3.51)–(3.55) are inserted into Eqs (3.38)–(3.44). Next, we calculate the findings for the following PGFs: $I(\hat{\upsilon},\ \varsigma)$, $B_b(\hat{\upsilon},\varsigma)$, $\phi_v(\hat{\upsilon},\varsigma)$ $D_b(\hat{\upsilon},0,\varsigma), D_v(\hat{\upsilon},0,\varsigma)$, $\check{R}_b(\hat{\upsilon},0,\varsigma)$, and $\check{R}_v(\hat{\upsilon},0,\varsigma)$.

Theorem 1. *The prob. distributions of the number of consumers in orbit and the server states have the following PGFs.*

$$I(\varsigma)=\frac{\mathrm{Nu}(\varsigma)}{\mathrm{De}(\varsigma)}; \tag{3.56}$$
$$\mathrm{Nu}(\varsigma)=\varsigma\lambda_v P_0(1-A^*(\lambda)/\lambda)\times\left\{(f\varsigma+\overline{f})\,(N_b^*\,(A_b(\varsigma))\,V(\varsigma)+N_v^*(A_v(\varsigma)))-1\right\};$$
$$\mathrm{De}(\varsigma)=\left[\varsigma-(f\varsigma+\overline{f})\,(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\,N_b^*(A_b(\varsigma))\right].$$

$$B_b(\varsigma)=\frac{P_0\lambda_v(1-N_b^*(A_b(\varsigma)))}{A_b(\varsigma)De(\varsigma)}\left\{(f\varsigma+\overline{f})\begin{pmatrix}(N_v^*(A_b(\varsigma))-1)\\(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\end{pmatrix}+\varsigma V(\varsigma)\right\};\tag{3.57}$$

$$\phi_v(\varsigma)=\{\lambda_P V_0 V(\varsigma)/\theta\};\tag{3.58}$$

$$D_b(\varsigma)=\frac{\delta\lambda P_0\left(1-N_b^*(A_b(\varsigma))\right)(1-W_b^*(b(\varsigma)))}{A_b(\varsigma)\times b(\varsigma)\times \mathrm{De}(\varsigma)}\times$$
$$\left\{(f\varsigma+\overline{f})\left((N_v^*(A_b(\varsigma))-1)(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\right)+\varsigma V(\varsigma)\right\};\tag{3.59}$$

$$D_v(\varsigma)=\delta_v\lambda_v P_0\left(\frac{(1-W_v^*(b_v(\varsigma)))(1-N_v^*A_v(\varsigma))}{b_v(\varsigma)A_v(\varsigma)}\right);\tag{3.60}$$

$$\check{R}_b(\varsigma)=\frac{\delta P_0\left(1-N_b^*(A_b(\varsigma))\right)(1-G_b^*(b(\varsigma)))W_b^*(b(\varsigma))}{b(\varsigma)\times A_b(\varsigma)\times \mathrm{De}(\varsigma)}\times$$
$$\left\{(f\varsigma+\overline{f})\left((N_v^*(A_b(\varsigma))-1)(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\right)+\varsigma V(\varsigma)\right\};\tag{3.61}$$

$$\check{R}_v(\varsigma)=\frac{\delta_v\lambda_v P_0\left(1-N_v^*(A_v(\varsigma))(1-G_v^*(N_v(\varsigma)))\right)W_v^*(N_v(\varsigma))}{b(\varsigma)A_v(\varsigma)}.\tag{3.62}$$

where

$$P_0=\frac{1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))}{\left\{\begin{array}{r}1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))\\+\lambda_v E(N_b)\begin{pmatrix}fN_v^*(\theta)+(N_v^*(\theta)-1)r(1-A^*(\lambda))+1-N_v^*(\theta)\\-(1-N_v^*(\theta))(-\lambda-\lambda\delta_v(w^1+(g^1))/\theta)\end{pmatrix}\times(1+\delta_v(w^1+g^1))\\+\lambda_v\begin{pmatrix}1-f-r(1-A^*(\lambda)\\-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))\end{pmatrix}(1-N_b^*(\theta)/\theta)\times(1+\delta_v(w_v^1+g_v^1))\\+(f+(1-N_v^*(\theta))(\lambda+\delta(w^1\lambda+(g^1\lambda))))\times(E(N_b)+1/\theta)\end{array}\right.}\tag{3.63}$$

Proof. By integrating equations (3.38) to (3.44) with respect to $\hat{\upsilon}$, we may determine the PGFs.

$$I(\varsigma)=\int_0^\infty I(\hat{\upsilon},\varsigma)d\hat{\upsilon},\ B_b(\varsigma)=\int_0^\infty B_b(\hat{\upsilon},\varsigma)d\hat{\upsilon},\ \ \phi_v(\varsigma)=\int_0^\infty \phi_v(\hat{\upsilon},\varsigma)d\hat{\upsilon},$$
$$\check{R}_b(\varsigma)=\int_0^\infty\int_0^\infty \check{R}_b(\hat{\upsilon},y,\varsigma)d\hat{\upsilon}dy,\ \ \check{R}_v(\varsigma)=\int_0^\infty\int_0^\infty \check{R}_v(\hat{\upsilon},y,\varsigma)d\hat{\upsilon}dy,$$
$$D_b(\varsigma)=\int_0^\infty\int_0^\infty D_b(\hat{\upsilon},y,\varsigma)d\hat{\upsilon}dy,\ D_v(\varsigma)=\int_0^\infty\int_0^\infty D_v(\hat{\upsilon},y,\varsigma)d\hat{\upsilon}dy$$

The prob. of the server being idle, denoted as P_0, can be calculated using the normalized condition. Therefore, by substituting $\varsigma = 1$ into Eqs (3.56) to (3.62) and following L'Hospital's rule when appropriate, we may derive $P_0+I(1)+B_b(1)+\phi_v(1)+\check{R}_b(1)+\check{R}_v(1)+D_b(1)+D_v(1)=1$. □

3.3. Corollary

Under the system stability condition $(f+r(1-A^*(\lambda)+E(N_b)(\lambda+\lambda\delta(w^1+g^1))))<1$, the server is unoccupied, down, WV, delayed repair or under repair; then, the PGF for the number of consumers in the orbit and system is denoted as $K_o(\varsigma)$ and $K_s(\varsigma)$, respectively.

$$K_o(\varsigma)=\frac{Nu_q(\varsigma)}{De_q(\varsigma)}=P_0+I(\varsigma)+B_b(\varsigma)+\phi_v(\varsigma)+\check{R}_b(\varsigma)+\check{R}_v(\varsigma)+D_b(\varsigma)+D_v(\varsigma),\tag{3.64}$$

where

$$
Nu_0(\varsigma)=\left\{\begin{array}{r}(1-N_b^*(A_b(\varsigma)))\times\left\{(f\varsigma+\overline{f})\begin{pmatrix}(N_v^*(A_b(\varsigma))-1)\\(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\end{pmatrix}+\varsigma V(\varsigma)\right\}\\+\lambda(1-\varsigma)(\varsigma(1-A^*\lambda))((f\varsigma+(1-f))N_b^*(A_b(\varsigma))V(\varsigma)+N_v^*(A_v(\varsigma))-1)\\+\lambda(1-\varsigma)(\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))N_b^*A_b(\varsigma)\\+\lambda(1-\varsigma)\left((\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))B_b^*A_b(\varsigma)V(\varsigma)\right)\\\times(\lambda_v/\theta+\delta_v/\theta)(1-G_v^*b_v(\varsigma)W_v^*b_v(\varsigma))\end{array}\right\}.
$$

$$
De_q(\varsigma)=b(\varsigma)\times+\lambda(1-\varsigma)(\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))N_b^*A_b(\varsigma).
$$

$$
K_s(\varsigma)=\frac{Nu_s(\varsigma)}{De_q(\varsigma)}=P_0+I(\varsigma)+\varsigma(B_b(\varsigma)+\phi_v(\varsigma))+\check{R}_b(\varsigma)+D_b(\varsigma)+\check{R}_v(\varsigma)+D_v(\varsigma). \tag{3.65}
$$

$$
Nu_s(\varsigma)=\left\{\begin{array}{r}\varsigma(1-N_b^*(A_b(\varsigma)))\times\left\{(f\varsigma+\overline{f})\begin{pmatrix}(N_v^*(A_b(\varsigma))-1)\\(A^*(\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))\end{pmatrix}+\varsigma V(\varsigma)\right\}\\+\lambda(1-\varsigma)(\varsigma(1-A^*\lambda))((f\varsigma+(1-f))N_b^*(A_b(\varsigma))V(\varsigma)+N_v^*(A_v(\varsigma))-1)\\+\lambda(1-\varsigma)(\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))N_b^*A_b(\varsigma)\\+\varsigma\lambda(1-\varsigma)\left((\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))N_b^*A_b(\varsigma)V(\varsigma)\right)\\\times(\lambda_v/\theta)\\+\lambda(1-\varsigma)\left((\varsigma-(f\varsigma+(1-f))(A^*\lambda)+(1-r+r\varsigma)(1-A^*(\lambda)))N_b^*A_b(\varsigma)V(\varsigma)\right)\\\times(\delta_v/\theta)(1-G_v^*b_v(\varsigma)W_v^*b_v(\varsigma))\end{array}\right\}.
$$

4. Performance characteristic

In this section, we examine the probabilities of the system states the server being unoccupied, RS, WV, delayed repair (RS, WV) and under maintenance (RS, WV). Also, we examine the average number of consumers in orbit L_q, the average number of consumers in the system L_s, mean availability S_{Av}, system failure occur $Fail_f$, average busy time $H(T_{bs})$, and average busy cycle $H(T_{bc})$ of our model.

4.1. Probabilities of the system state

The outcomes derived from Eqs (3.56)–(3.62) are obtained by substituting $\varsigma \to 1$ and thereafter applying L-Hospital's rule as appropriate.

1) The prob. of the server remaining idle during the retrial period:

$$
I(1)=\lambda_v P_0(1-A^*(\lambda))\times\left\{\frac{(f+(1-N_v^*(\theta))(\lambda+\delta(w^1\lambda+(g^1\lambda))))\times(E(N_b)+1/\theta)}{1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))}\right\}. \tag{4.1}
$$

2) The prob. that the server is in RS period.:

$$B_b(1) = \lambda_v P_0 E(N_b) \left\{ \frac{\begin{pmatrix} fN_v^*(\theta) + (N_v^*(\theta) - 1)r(1 - A^*(\lambda)) + 1 - N_v^*(\theta) \\ -(1 - N_v^*(\theta))(-\lambda - \lambda\delta_v(w^1 + (g^1))/\theta) \end{pmatrix}}{1 - f - r(1 - A^*(\lambda) - E(N_b)(\lambda + \lambda\delta(w^1 + g^1)))} \right\}. \tag{4.2}$$

3) The prob. of the server operating at a reduced service rate:

$$\phi_v(1) = \{\lambda_v P_0 (1 - N_v^*(\theta))/\theta\}. \tag{4.3}$$

4) The prob. that the server is delayed repair in RS:

$$D_b(1) = \lambda_v P_0 \delta E(N_b) \mathfrak{w}^{(1)} \times \left\{ \frac{\begin{pmatrix} fN_v^*(\theta) + (N_v^*(\theta) - 1)r(1 - A^*(\lambda)) + 1 - N_v^*(\theta) \\ -(1 - N_v^*(\theta))(-\lambda - \lambda\delta_v(w^1 + (g^1))/\theta) \end{pmatrix}}{1 - f - r(1 - A^*(\lambda) - E(N_b)(\lambda + \lambda\delta(w^1 + g^1)))} \right\}. \tag{4.4}$$

5) The prob. that the server is delayed repair in WV:

$$D_v(1) = \lambda_v P_0 \delta_v \times \left(\frac{(1 - N_v^*(\theta))}{\theta} \right) w_v^{(1)}. \tag{4.5}$$

6) The prob. that the server is undergoing maintenance in RS is determined by:

$$\check{R}_b(1) = P_0 \delta \lambda_v E(N_b) \mathfrak{g}^{(1)} \times \left\{ \frac{\begin{pmatrix} fN_v^*(\theta) + (N_v^*(\theta) - 1)r(1 - A^*(\lambda)) + 1 - N_v^*(\theta) \\ -(1 - N_v^*(\theta))(-\lambda - \lambda\delta_v(w^1 + (g^1))/\theta) \end{pmatrix}}{1 - f - r(1 - A^*(\lambda) - E(N_b)(\lambda + \lambda\delta(w^1 + g^1)))} \right\}. \tag{4.6}$$

7) The prob. that the server is undergoing maintenance in WV is determined by:

$$\check{R}_b(1) = \lambda_v P_0 \delta_v \times \left(\frac{(1 - N_v^*(\theta))}{\theta} \right) g_v^{(1)}. \tag{4.7}$$

4.2. Average orbit size and system size

i) To get the number of consumers in the orbit L_q, we differentiate Eq (3.64) with respect to ς and evaluate it at $\varsigma = 1$.

$$L_q = K_0^{'}(1) = \lim_{\varsigma \to 1} K_0^{'}(\varsigma) = P_0 \left[\frac{Nu_q^{'''}(1) De_q^{''}(1) - De_q^{'''}(1) Nu_q^{''}(1)}{3(De_q^{''}(1))^2} \right], \tag{4.8}$$

where

$$Nu_q^{''}(1) = -2\lambda \left\{ \begin{array}{r} E(N_b) \begin{pmatrix} fN_v^*(\theta) - N_v^{*'}(\theta)(\lambda_v + \delta(w^1\lambda_v + g^1\lambda_v)) + (N_v^*(\theta) - 1)r(1 - A^*(\lambda)) \\ +V(1) + V^{'}(\varsigma) \end{pmatrix} \\ +(1 - f - r(1 - A^*(\lambda)))(1 + \dfrac{1 - N_v^*\lambda_v}{\theta}) \\ +(1 - A^*(\lambda)) \begin{pmatrix} f + E(N_b)(\lambda + \delta(w^1\lambda + g^1\lambda))V(1) + V^{'}(\varsigma) \\ -(N_v^{*'}(\theta)(\lambda_v + \delta(w^1\lambda_v + g^1\lambda_v))) \end{pmatrix} \end{array} \right\}.$$

$$Nu_q^{'''}(1) = 6\lambda \left\{ \begin{array}{r} (1-f-r(1-A^*(\lambda)))E(N_b)\lambda + \delta(w^1\lambda + g^1\lambda) \\ -(1-A^*(\lambda))fE(N_b)(\lambda+\delta\lambda(w^1+g^1))(1-N_v^*(\theta)) + V^{'}(\varsigma) + N_v^{*'}(\lambda_v + \delta\lambda_v(w^1+g^1)) + \\ (1-A^*(\lambda))E(N_b)(\lambda+\delta\lambda(w^1+g^1))V^{'}(\varsigma) \\ -(1-A^*(\lambda))f + E(N_b)(\lambda+\delta\lambda(w^1+g^1))(1-N_v^*(\theta)) + V^{'}(\varsigma) + N_v^{*'}(\lambda_v+\delta\lambda_v(w^1+g^1)) \\ -E(N_b)\left(fN_v^{*'}(\theta)(\lambda_v+\delta\lambda_v(w^1+g^1)) + \begin{pmatrix} fN_v^*(\theta) \\ -N_v^{*'}(\theta)(\lambda_v+\delta\lambda_v(w^1+g^1)) \end{pmatrix} \times r(1-A^*(\lambda)) \right) \\ +(1-A^*(\lambda))V^{'}(\varsigma) + fr(1-A^*(\lambda))\left(1 - \frac{\lambda_v(1-N_v^*)(\theta)}{\theta}\right) \\ -\lambda_v(1-f-r(1-A^*(\lambda)))\begin{pmatrix} V^{'} - E(N_b)(\lambda+\delta\lambda(w^1+g^1))(1-N_v^*(\theta)) \\ -(1-N_v^*(\theta))\delta_v\lambda(w_v^1+g_v^1) \end{pmatrix} \end{array} \right\}$$

$$-3\lambda(1-A^*(\lambda))\begin{pmatrix} E(N_b)^2(\lambda+\delta\lambda(w^1+g^1))(1-N_v^*(\theta)) \\ -E(N_b)\delta\lambda^2(w^2+g^2-2w^1g^1)(1-N_v^*(\theta)) \\ +V^{''}(\varsigma) + N_v^{*''}(\theta)(\lambda_v+\delta\lambda_v(w_v^1+g_v^1)^2) + N_v^{*'}(\theta)\delta(\lambda_v^2(w_v^1+g_v^1) - 2(w^1g^1)) \end{pmatrix}$$

$$-3\lambda^2E(N_b)^2\begin{pmatrix} fN_v^*(\theta) - N_v^{*'}(\theta)(\lambda_v+\delta\lambda_v(w^1+g^1)) + (1-N_v^*(\theta)) \times r(1-A^*) \\ +(1-N_v^*(\theta)) + V^{'}(\varsigma) \end{pmatrix}$$

$$-3E(N_b)\lambda\left(N_v^{*''}(\theta)(\lambda_v+\delta\lambda_v(w_v^1+g_v^1)^2) + N_v^{*'}(\theta)\delta(\lambda_v^2(w_v^2+g_v^2) - 2(w^1g^1))\right).$$

$$De_q^{''}(1) = -2\lambda\left((1-f-r(1-A^*(\lambda))) - E(N_b)\lambda + \delta(w^1\lambda + g^1\lambda)\right).$$

$$De_q^{'''}(1) = 6\lambda fr(1-A^*(\lambda)) + 6\lambda E(N_b)(\lambda+\delta(w^1\lambda+g^1\lambda))[f + r(1-A^*(\lambda))].$$
$$-3\lambda E(N_b)^2(\lambda+\delta(w^1\lambda+g^1\lambda)) + 3\lambda\delta E(N_b)(\lambda^2(w^2+g^2) - 2w^1g^1),$$

where

$$V^{'}(\varsigma) = B^{*'}(\theta)(\lambda_v + \delta_v(w^1\lambda_v + g^1\lambda_v)) + (1-N_v^*(\theta))(\lambda_v+\delta_v(w^1\lambda_v+g^1\lambda_v))/\theta.$$
$$V^{''}(\varsigma) = -B^{*''}(\theta)(\lambda_v+\delta_v(w^1\lambda_v+g^1\lambda_v))^2 - B^{*'}\delta_v((w^2+g^2) - 2(w^1g^1)) + 2B^{*'}(\lambda_v+\delta_v(w^1\lambda_v+g^1\lambda_v))^2/\theta$$
$$+2(1-B^*(\theta))(\lambda_v+\delta_v(w^1\lambda_v+g^1\lambda_v))^2/\theta^2 - (1-B^*(\theta))\delta_v((w^2+g^2) - 2(w^1g^1))/\theta.$$

ii) By differentiating Eq (3.65) with respect to ς and evaluating it at $\varsigma = 1$, we may find the number of consumers in the system, denoted as L_s.

$$L_s = K_s^{'}(1) = \lim_{\varsigma\to 1} K_s^{'}(\varsigma) = P_0\left[\frac{Nu_s^{'''}(1)De_q^{''}(1) - De_q^{'''}(1)Nu_q^{''}(1)}{3(De_q^{''}(1))^2}\right]. \quad (4.9)$$

Where

$$Nu_s^{'''}(1) = Nu_q^{'''}(1) - \frac{6\lambda}{\theta}(1-f-r(1-A^*))(1-N_b^*(\theta))\lambda_v.$$

iii) The average time of a consumer waiting in the system $W_s = \frac{L_s}{\lambda_{eff}}$ and the average time of a consumer waiting in the orbit $W_q = \frac{L_q}{\lambda_{eff}}$ is obtained by using the Little's formula.
Where $\lambda_{eff} = \lambda(I(1) + B_b(1) + D_b(1) + R_b(1)) + \lambda_v(\phi_v(1) + D_v(1) + R_v(1))$.

4.3. Reliability measures

To enhance the reliability of a system that is susceptible to breakdowns, it is important to decide on the reliability measurements of the model. These measures offer major details into the average server availability and other relevant indices.

1) The server's mean availability (S_{Av}) is determined by

$$\begin{aligned} S_{Av} &= 1-\lim_{\varsigma\to 1}(D_b(\varsigma)+D_v(\varsigma)+\check{R}_b(\varsigma)+\check{R}_v(\varsigma)) = 1-(D_b(1)+D_v(1)+\check{R}_b(1)+\check{R}_v(1)) \\ &= 1-\left\{\begin{array}{r} P_0\delta\lambda_v E(N_b)(g^{(1)}+w^1)\times\left\{\dfrac{\begin{pmatrix} fN_v^*(\theta)+(N_v^*(\theta)-1)r(1-A^*(\lambda))+1-N_v^*(\theta) \\ -(1-N_v^*(\theta))(-\lambda-\lambda\delta_v(w^1+(g^1))/\theta)\end{pmatrix}}{1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))}\right\} \\ +\lambda_v P_0\delta_v\left(\dfrac{(1-N_v^*(\theta))}{\theta}\right)(w_v^1+g_v^1) \end{array}\right\}. \end{aligned}$$

2) The steady-state system failure occurrence in RS is as follows:

$$Fail_f = \delta*\lambda_v P_0 E(N_b)\left\{\dfrac{\begin{pmatrix} fN_v^*(\theta)+(N_v^*(\theta)-1)r(1-A^*(\lambda))+1-N_v^*(\theta) \\ -(1-N_v^*(\theta))(-\lambda-\lambda\delta_v(w^1+(g^1))/\theta)\end{pmatrix}}{1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))}\right\}.$$

4.4. Average busy time and the busy cycle

Let the average length of the busy cycle and busy period $H(T_{bs})$ and $H(T_{bc})$ be taken using the following method:

$$P_0=\frac{H(T_0)}{H(T_0)+H(T_{bs})},\ H(T_{bs})=\frac{1}{\lambda}\left(\frac{1}{P_0}-1\right),\ \text{and } H(T_{bc})=\frac{1}{(\lambda)P_0}=H(T_0)+H(T_{bs}), \tag{4.10}$$

where the length of the system in the empty state is represented by T_0, and $H(T_0)=(1/\lambda)$. By inserting Eq (3.53) in Eq (4.10), we obtain the anticipated outcome to be

$$H(T_{bs}) = \frac{1}{\lambda}\left\{\dfrac{\begin{array}{r} E(N_b)\lambda_v\begin{pmatrix} fN_v^*(\theta)+(N_v^*(\theta)-1)r(1-A^*(\lambda))+1-N_v^*(\theta) \\ -(1-N_v^*(\theta))(-\lambda-\lambda\delta_v(w^1+(g^1))/\theta)\end{pmatrix}\times(1+\delta_v(w^1+g^1)) \\ +\lambda_v\begin{pmatrix}(1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))) \\ (1-N_v^*(\theta)/\theta)\end{pmatrix}\times(1+\delta_v(w_v^1+g_v^1)) \\ +(f+(1-N_v^*(\theta))(\lambda+\delta(w^1\lambda+(g^1\lambda))))\times(E(N_b)+1/\theta)\end{array}}{1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))}\right\}.$$

$$H(T_{bc}) = \frac{\left\{\begin{array}{r} 1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1))) \\ +E(N_b)\lambda_v\begin{pmatrix} fN_v^*(\theta)+(N_v^*(\theta)-1)r(1-A^*(\lambda))+1-N_v^*(\theta) \\ -(1-N_v^*(\theta))(-\lambda-\lambda\delta_v(w^1+(g^1))/\theta)\end{pmatrix}\times(1+\delta_v(w^1+g^1)) \\ +\lambda_v\begin{pmatrix}(1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1)))) \\ (1-N_v^*(\theta)/\theta)\end{pmatrix}\times(1+\delta_v(w_v^1+g_v^1)) \\ +(f+(1-N_v^*(\theta))(\lambda+\delta(w^1\lambda+(g^1\lambda))))\times(E(N_b)+1/\theta)\end{array}\right\}}{(1-f-r(1-A^*(\lambda)-E(N_b)(\lambda+\lambda\delta(w^1+g^1))))}.$$

5. Special cases

In this section, we analyze certain specific cases of our model that coincide with the current research.

Case (i) M/G/1 model with balking consumer in multiple WV mode. Choose $\delta_v = \delta;\ r = 0;\ \lambda_v = \lambda,$ and $\zeta = 0$. In this case, we get

$$K_s(\varsigma) = \frac{P_0\left\{\begin{array}{r} b(1-\varsigma)\left[\left(\varsigma-((1-f)+f\varsigma)(A^*(\lambda)+\varsigma 1-A^*(\lambda))N_b^*(h_b(\varsigma))\right)\left(1+\varsigma\lambda\theta^{-1}V(\varsigma)\right)\right. \\ \left.+\lambda_v\varsigma\left((1-A^*(\lambda))\left(-1+((1-f)+f\varsigma)\left(V(\varsigma)N_b^*(h_b(\varsigma))+N_v^*(h_v(\varsigma))\right)\right)\right)\right] \\ +\varsigma(1-N_b^*(h_b(\varsigma)))(\varsigma V(\varsigma)+(A^*(\lambda)+\varsigma(1-A^*(\lambda)))(-1+((1-f)+f\varsigma)N_v^*(h_v(\varsigma)))) \end{array}\right\}}{b(1-\varsigma)\left(\varsigma-((1-f)+f\varsigma)(A^*(\lambda)+\varsigma(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\right)}.$$

This matches the outcome obtained by Rajadurai et al. [19].

Case (ii) No delayed repair and reneging. Let $\zeta = r = 0$, our approach can be simplified to a $M^X/G/1$ RQ with retrial feedback queue with Balking, WVs and VI. Here, $K_s(\varsigma)$ was obtained as

$$K_s(\varsigma) = \frac{P_0\left\{\begin{array}{r} bN_v\lambda(1-C(\varsigma))\left[\left(\varsigma-(f+(1-f)\varsigma)(A^*(\lambda)+V(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\right)\left(1+\varsigma\lambda_v V(\varsigma)\theta^{-1}\right)\right. \\ \left.+\lambda_v(\varsigma(1-A^*(\lambda))(-1+((1-f)+f\varsigma)(V(\varsigma)N_b^*(h_b(\varsigma))+N_v^*(h_v(\varsigma)))))\right] \\ +\lambda_v N_v\varsigma(1-N_b^*(h_b(\varsigma)))(\varsigma V(\varsigma)+(A^*(\lambda)+\varsigma(1-A^*(\lambda)))(-1+((1-f)+f\varsigma)N_v^*(h_v(\varsigma)))) \\ +\varsigma\alpha_v\lambda bV(\varsigma)\theta^{-1}(1-N_v^*(b_v\lambda_v(1-C(\varsigma))))\begin{pmatrix}\varsigma-((1-f) \\ +f\varsigma)(A^*(\lambda)+\varsigma(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\end{pmatrix} \end{array}\right\}}{bb_v\lambda(1-C(\varsigma))\left(\varsigma-((1-f)+f\varsigma)(A^*(\lambda)+\varsigma(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\right)}.$$

This matches the outcome obtained by Jain [41].

Case (iii) M/G/1 model without feedback, deleyed repair, and reneging. Choose $\delta_v = 0,\ \delta = 0,\ r = 0,$ and $f = 0$. In this case, we obtain:

$$K_s(\varsigma) = \frac{P_0\left\{\begin{array}{r} (1-\varsigma)\left[\left(\varsigma-((1-f)+f\varsigma)(A^*(\lambda)+\varsigma(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\right)\left(1+\frac{\varsigma V(\varsigma)}{\theta}\right)\right. \\ \left.+\varsigma\left((1-A^*(\lambda))\left(-1+((1-f)+f\varsigma)\left(V(\varsigma)N_b^*(h_b(\varsigma))+N_v^*(h_v(\varsigma))\right)\right)\right)\right] \\ +(1-N_b^*(h_b(\varsigma)))(\varsigma V(\varsigma)-(A^*(\lambda)+\varsigma(1-A^*(\lambda)))(1-N_v^*(h_v(\varsigma)))) \end{array}\right\}}{(1-\varsigma)\left(\varsigma-((1-f)+f\varsigma)(A^*(\lambda)+\varsigma(1-A^*(\lambda)))N_b^*(h_b(\varsigma))\right)}. \tag{5.1}$$

This matches the outcome obtained by Gao et al. [10].

6. Cost optimization

To design a retrial queueing system to perform cost analysis, the most effective course is to determine the optimal system parameters, such as the optimal mean service rate or the optimal number of servers. In this section, we discuss the optimal structure for the single-server feedback retrial queue with working vacations subject to server failure. We obtain the total expected cost function per unit time using the following definitions of cost elements (C_h, C_0, C_s, and C_a) and the cost structure:

$$\begin{aligned} TC &= C_h L_s + C_0 \frac{H(T_{bs})}{H(T_{bc})} + C_s \frac{1}{H(T_{bc})} + C_a \frac{H(T_0)}{H(T_{bc})} \\ &= C_h L_s + C_0(1 - P_0) + C_s \lambda + C_a P_0. \end{aligned}$$

Where C_h, C_0, C_s, and C_a represent the holding costs per unit of time for each customer in the system, the cost per unit of time to keep the server in and on operations, the setup cost per busy cycle, and the startup cost per unit of time for setting up the server in order to begin providing the service, respectively. For the following values of the cost elements and other parameters, such as $\lambda \& \lambda_v = 1, \dot{a} = 2, N_b = 6, N_v = 4, \theta = 2, \chi_v \& \chi_b = 5; \gamma_v \& \gamma_b = 3, f = 0.2, \delta_v \& \delta = 0.1, r = 0.2, C_h = 5, C_o = 80, C_s = 600$, and $C_a = 80$, we find the total expected cost per unit of time, $TC = 440.4952$, assuming exponential retrial times, service times, working vacation times, and repair times. Furthermore, Tables 1–3 show the impacts of $(C_h; C_o)$, $(C_o; C_a)$, and $(C_s; C_a)$ on the expected cost function, respectively. It is evident that when cost parameters increase, the expected cost function trends upward linearly.

Table 1. The effect of (C_h, C_0) on the expected cost function TC with C_s = \$600, and C_a = \$80.

(C_h, C_0)	(5,80)	(5,90)	(5,100)	(10,80)	(15,80)
TC	440.4952	444.3521	448.2090	440.9905	441.4857

Table 2. The effect of (C_0, C_a) on the expected cost function TC with C_h = \$5, and C_s = \$600.

(C_0, C_a)	(80,80)	(85,80)	(90,80)	(80,90)	(80,95)
TC	440.4952	442.4237	444.3521	446.6383	449.7099

Table 3. The effect of (C_a, C_s) on the expected cost function TC with C_h = \$5, and C_0 = \$80.

(C_a, C_s)	(80,600)	(85,600)	(90,600)	(80,650)	(80,700)
TC	440.4952	443.5668	446.6383	470.4952	500.4952

Similarly, we can conduct a sensitivity study on specific system parameters. After establishing the above-mentioned base values, one parameter can be changed at a time to calculate the appropriate objective function value. The graphs from Figures 2–4 display the effect of specific system parameters $(\dot{a}, \lambda, N_v)$ on the overall expected cost per unit of time.

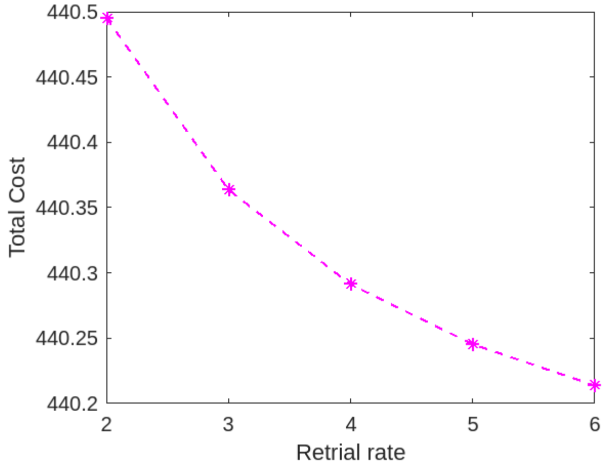

Figure 2. TC Versus $\dot{a}$.

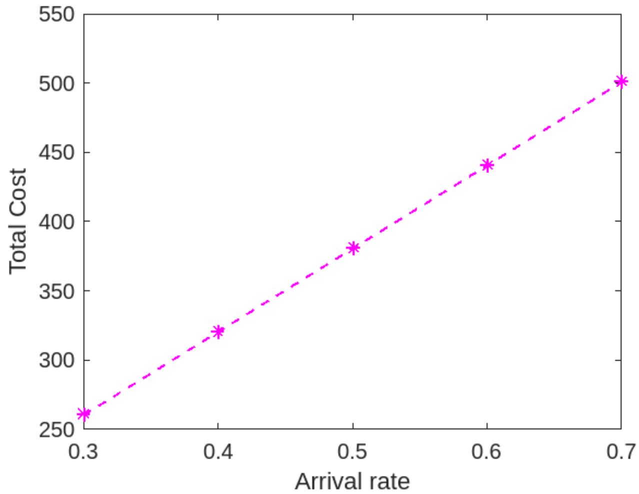

Figure 3. TC Versus $\dot{a}$.

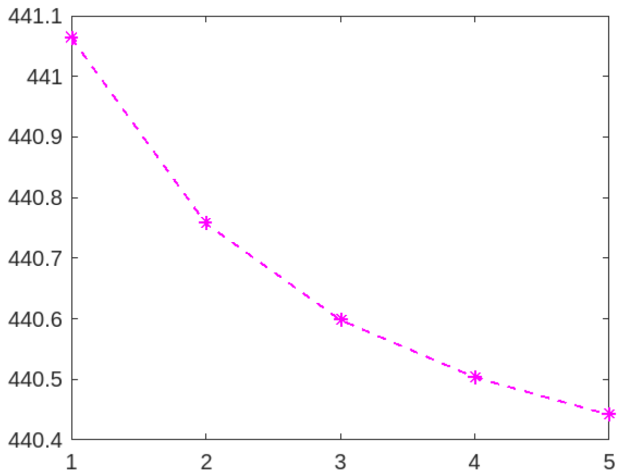

Figure 4. TC Versus N_v.

7. Numerical example

In this section, we will analyze the impact of numerous variables on the efficiency indicators of our system using various numerical demonstrations.

The examples are predicated on the assumption that all instances of retrial, RS, periods of reduced service rate, delayed repair, and maintenance times follow an exponential distribution. Consequently, the parameters are chosen with arbitrary values satisfying the stability condition. The results are visually represented using MATLAB software. It is worth noting the equation of exponential distribution $f(\hat{\upsilon}) = \nu e^{-\nu\hat{\upsilon}}, \hat{\upsilon} > 0$, Erlang-2 stage distribution $f(\hat{\upsilon}) = \nu^2\hat{\upsilon}e^{-\nu\hat{\upsilon}}, \hat{\upsilon} > 0$, and the hyper exponential distribution $f(\hat{\upsilon}) = c\nu e^{-\nu\hat{\upsilon}} + (1-c)\nu^2 e^{-\nu^2\hat{\upsilon}}, \hat{\upsilon} > 0$.

7.1. Sensitivity analysis

The data presented in Table 4 indicates that when the rate of repeated tries $\dot{a}$ increases, there is a constant decrease in the prob. of the orbit size L_q and the prob. of the server being unoccupied, while the retrial time I decreases. Meanwhile, the prob. of the server being idle, denoted as P_0, also rises. Regarding the given values $\lambda_v\&\lambda = 1$; $\theta = 3$; $N_b = 8$; $r = 0.2$; $\chi_v\&\chi_b = 4$; $\gamma_v\&\gamma_b = 4$; $\delta_v\&\delta = 0.3$; $f = 0.2$; $N_v = 4$; $c = 0.5$. The impacts of the prob. on the system's performance metrics are described, and documented in Table 5 for the given values of $\lambda_v\&\lambda = 2; \theta = 3; N_v = 4; N_b = 8; r = 0.2; \chi_v\&\chi_b = 4; \gamma_v\&\gamma_b = 4; \dot{a} = 5;\ f = 0.2; c = 0.5$. We have seen that the breakdown rate δ increases consistently as the values of the orbit size prob. L_q, the server unoccupied rate P_0, and the server idle rate during the retry period I increase. The patterns exhibited by the tables align with the expected assumptions.

Table 4. The impact of repeated attempt rate $\dot{a}$ on P_O, L_q, I.

Retrial rate	Exponential			Erlang 2 stage			Hyper Exponential		
$\dot{a}$	P_0	L_q	I	P_0	L_q	I	P_0	L_q	I
3	0.5180	0.1588	0.0711	0.5786	0.6502	0.1042	0.5053	0.0914	0.0747
4	0.5189	0.1377	0.0704	0.5799	0.5826	0.1027	0.5062	0.0736	0.0740
5	0.5195	0.1240	0.0700	0.5807	0.5372	0.1017	0.5067	0.0633	0.0736
6	0.5199	0.1143	0.0696	0.5814	0.5046	0.1009	0.507	0.0567	0.0733
7	0.5202	0.1071	0.0694	0.5819	0.4801	0.1004	0.5072	0.0520	0.0732

Table 5. The impact of breakdown rate δ on P_0, L_q, I.

Breakdown rate	Exponential			Erlang 2 stage			Hyper Exponential		
δ	P_0	L_q	I	P_0	L_q	I	P_0	L_q	I
0.20	0.5206	0.1355	0.0813	0.5938	0.4257	0.0401	0.5359	0.0879	0.0761
0.30	0.5180	0.1397	0.0780	0.5786	0.4478	0.0365	0.5341	0.0914	0.0747
0.40	0.5154	0.1439	0.0748	0.5633	0.4636	0.0329	0.5323	0.0949	0.0733
0.50	0.5128	0.1481	0.0716	0.5480	0.4744	0.0293	0.5305	0.0983	0.0719
0.60	0.5102	0.1524	0.0684	0.5326	0.4809	0.0256	0.5287	0.1017	0.0705

Specifically, in Table 6, when the lower service rate N_v escalates, the server idle rate P_0 and length of the orbit L_q also escalates, while the prob. that the server is idle during the retrial period I decreases, given the values of $\lambda_v\&\lambda = 1; \delta_v\&\delta = 0.3; N_b = 8; r = 0.2; \chi_v\&\chi_b = 4; \gamma_v\&\gamma_b = 4; \dot{a} = 5; \theta = 3; f = 0.2;\ c = 0.5$. In Table 7, when the feedback rate f rises, then the average length of the orbit Lq, the server idle during retrial period I, and server idle P_0 declines, relating to the values $\lambda_v\&\lambda = 2; \delta_v\&\delta = 0.3; N_b = 8; r = 0.2; \chi_v\&\chi_b = 4; \gamma_v\&\gamma_b = 4; \dot{a} = 5; \theta = 3;\ c = 0.5$. In Table 8, when the reneging rate r rises, the average length of the orbit Lq, the server idle during retrial period I, and the server idle P_0 decline, regarding the specified values $\lambda_v\&\lambda = 1; \delta_v\&\delta = 0.3; N_b = 8; r = 0.2; \chi_v\&\chi_b = 4; \gamma_v\&\gamma_b = 4; \dot{a} = 5; \theta = 3;\ c = 0.5$.

Table 6. The impact of working vacation period N_v on P_0, L_q, I.

Slower service rate	Exponential			Erlang 2 stage			Hyper Exponential		
N_v	P_0	L_q	I	P_0	L_q	I	P_0	L_q	I
4.5	0.5208	0.1479	0.0684	0.5798	0.6233	0.0353	0.5091	0.0834	0.0723
5.5	0.5254	0.1309	0.0639	0.5821	0.5743	0.0332	0.5148	0.0755	0.0665
6.5	0.5292	0.1184	0.0603	0.5840	0.5313	0.0314	0.5189	0.0702	0.0623
7.5	0.5322	0.1088	0.0574	0.5856	0.4935	0.0298	0.5221	0.0664	0.0592
8.5	0.5348	0.1013	0.0549	0.5870	0.4601	0.0285	0.5245	0.0636	0.0567

Table 7. The impact of feedback rate f on P_0, L_q, I.

Feedback rate	Exponential			Erlang 2 stage			Hyper Exponential		
f	P_0	L_q	I	P_0	L_q	I	P_0	L_q	I
0.2	0.5100	0.1397	0.0780	0.5786	0.4788	0.0365	0.5061	0.0914	0.0747
0.3	0.4715	0.1541	0.0966	0.5372	0.5571	0.0439	0.4662	0.1069	0.0950
0.4	0.4292	0.1703	0.1173	0.4919	0.6791	0.0520	0.4223	0.1256	0.1173
0.5	0.3825	0.1891	0.1404	0.4418	0.9021	0.0609	0.3739	0.1494	0.1420
0.6	0.3308	0.2114	0.1664	0.3864	1.4686	0.0708	0.3201	0.1821	0.1693

Table 8. The impact of reneging rate r on P_0, L_q, I.

Reneging rate	Exponential			Erlang 2 stage			Hyper Exponential		
r	P_0	L_q	I	P_0	L_q	I	P_0	L_q	I
0.2	0.4499	0.1588	0.0711	0.5786	0.6502	0.0365	0.5053	0.0914	0.0747
0.3	0.4489	0.1669	0.0729	0.5747	0.6894	0.0381	0.5036	0.0965	0.0761
0.4	0.4478	0.1756	0.0748	0.5705	0.7361	0.0399	0.5019	0.1019	0.0775
0.5	0.4467	0.1850	0.0768	0.5659	0.7927	0.0419	0.5001	0.1076	0.0789
0.6	0.4454	0.1951	0.0790	0.5608	0.8627	0.0440	0.4982	0.1135	0.0804

Figures 5–11 depict the influence of the variables $\lambda_P,\ \lambda_N,\ r,\ \dot{a},\ \theta,\ N_b,$ and N_v on the 3D graph based on system performance metrics.

In Figure 5, the server's idle rate P_0 increases when both the lower service rate N_v and the rate of retrial N_b increase. Figure 6 demonstrate that the queue length L_q lowers when the retry rate $\dot{a}$ and the rate of the working vacation N_v increase. In Figure 7, the length of the queue L_q increases as both the arrival rate λ and the breakdown rate with RS δ increase. In Figure 8, the length of the queue L_q increases as the server's rate of service N_b increases and the rate of delay ζ drops consistently. In Figure 9, the queue length L_q drops as the server's rate of service θ grows and the rate of retry $\dot{a}$ also increases. In Figure 10, the server idle rate P_0 drops as the feedback rate f and the reneging rate r grow. In Figure 11, the length of the queue L_q lowers as the server's rate of N_b and the rate of the slow service N_v constantly grow.

Figures 12–15 illustrate the impact of the variables λ_P, λ_N, r, $\dot{a}$, θ, N_b, and N_v on the 2D graph based on system performance metrics. Note that the exponential distribution is $f(\hat{\upsilon}) = \nu e^{-\nu\hat{\upsilon}}, \hat{\upsilon} > 0$, Erlang-2 stage distribution is $f(\hat{\upsilon}) = \nu^2\hat{\upsilon}e^{-\nu\hat{\upsilon}}, \hat{\upsilon} > 0$, and the hyper-exponential distribution is $f(\hat{\upsilon}) = c\nu e^{-\nu\hat{\upsilon}} + (1-c)\nu^2 e^{-\nu^2\hat{\upsilon}}, \hat{\upsilon} > 0$. Figure 12 demonstrates a positive correlation between the increase in the slow service rate N_v and the growth of the prob. of server idle rate P_0. The graph in Figure 13 illustrates that the prob. of the server being idle, denoted as P_0, decreases as the feedback rate, represented by f, increases. Figure 14 illustrates that the average size of orbit L_q exhibits an upward trend as the reneging rate r increases. Figure 15 illustrates that the average size of orbit Lq decreases as the repair rate ζ increases. The following data visualizations illustrate the influence of the attributes on the system.

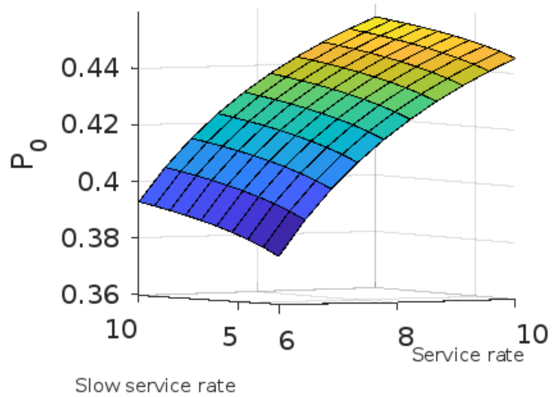

Figure 5. P_0 Versus N_b and N_v.

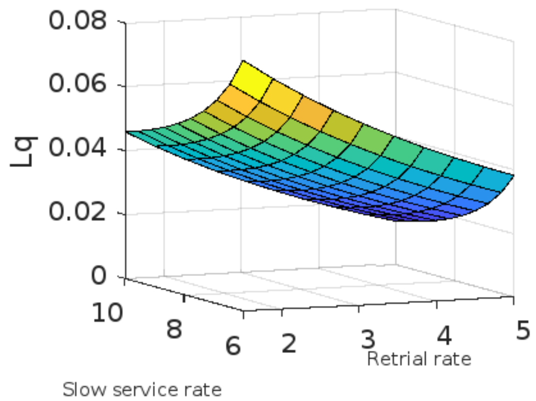

Figure 6. L_q Versus $\dot{a}$ and N_v.

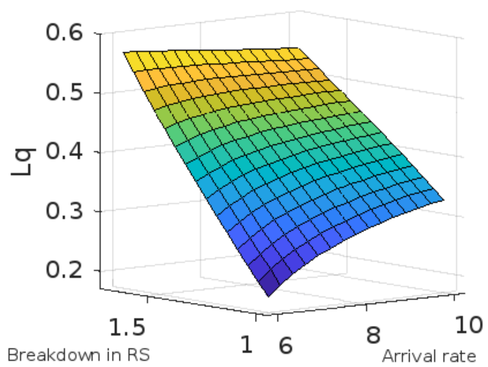

Figure 7. L_q Versus λ and δ.

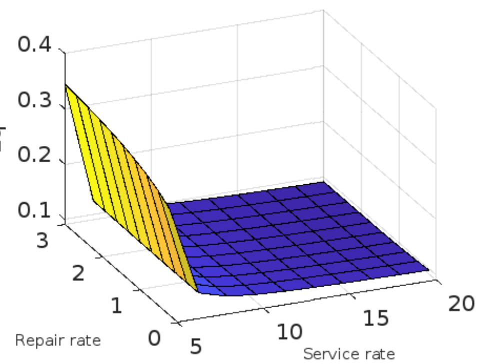

Figure 8. L_q Versus N_b and ζ.

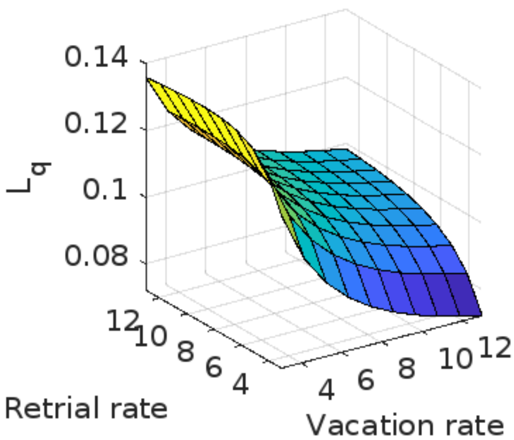

Figure 9. P_0 Versus θ and $\dot{a}$.

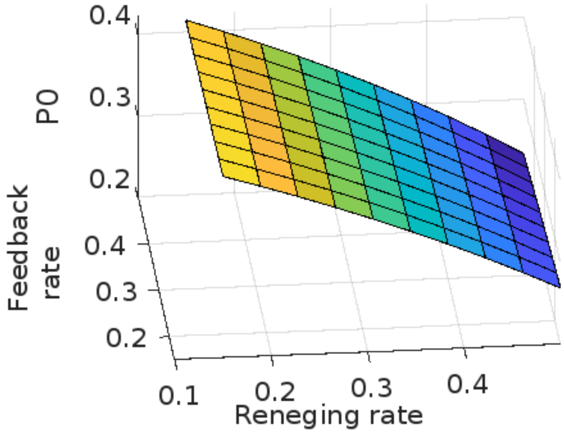

Figure 10. P_0 Versus r *and* f.

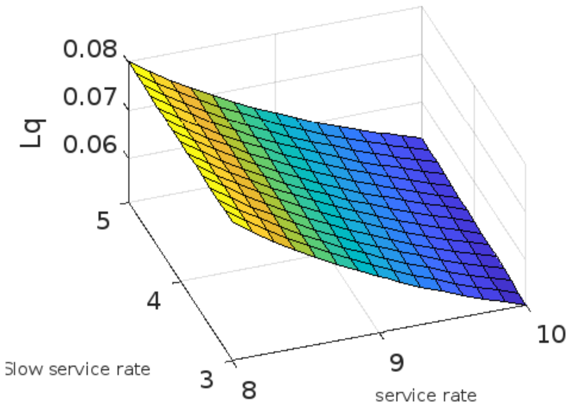

Figure 11. L_qVersus N_b and N_v.

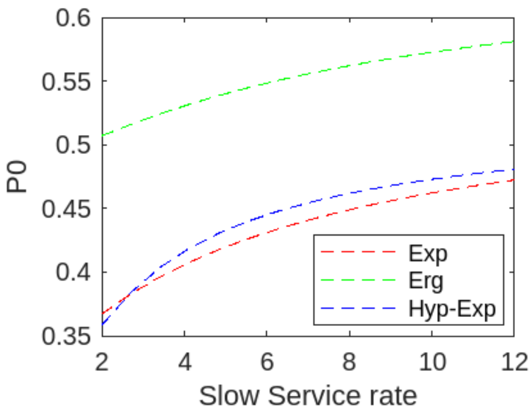

Figure 12. P_0 Versus N_v.

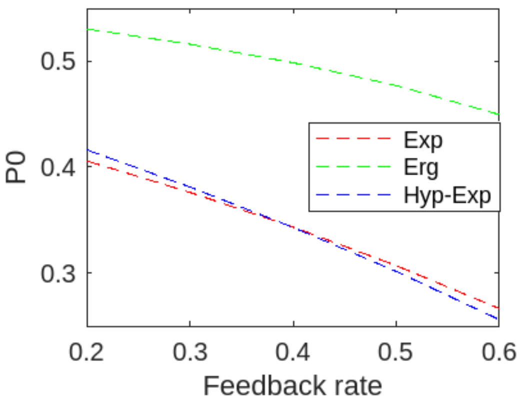

Figure 13. P_0 Versus f.

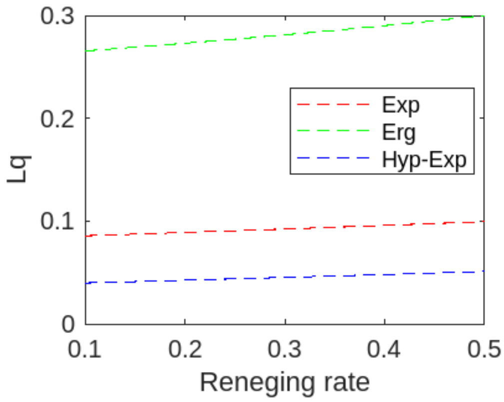

Figure 14. L_q Versus r.

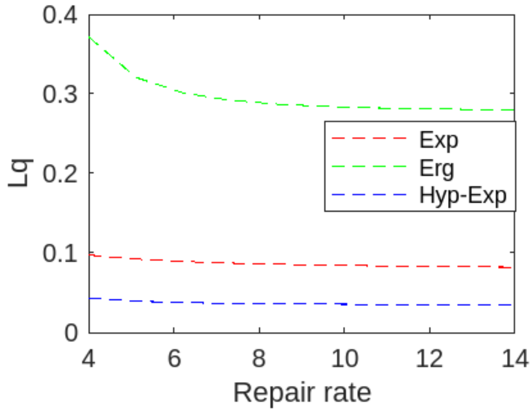

Figure 15. L_q Versus ζ.

7.2. Limitations

The proposed retrial queueing model incorporating working vacations, breakdowns, and repairs offers a comprehensive framework for analyzing service systems. However, like any model, it possesses inherent limitations and underlying assumptions that can influence the interpretation of the results.

The model frequently assumes that the retrial queue has an endless capacity. This makes analysis easier but might not accurately represent real-world situations. In reality, users may completely give up on the system if they believe that wait times are too lengthy. This may cause the model's average waiting times to increase. Usually, the model focuses on a system with just one server. This can be helpful for tractability, but in multi-server setups, complicated relationships and problems with resource allocation occur. The model may assume that service hours, arrival rates, and other related variables are fixed. However, demand and service requirements fluctuate in real-world systems. We can establish the suggested approach results in a more effective system with shorter wait times for customers, better server utilization, and less downtime from malfunctions by looking at these measures. In the proposed model, we can further extend the concepts of bulk arrival queue, optional phase service, and N-policy.

8. Conclusions

In this study, we have analyzed a non-Markovian feedback retrial queue, reneging, delayed repair, and working vacation subject to server breakdown, along with server failures that occur during service time for consumers in both normal service mode and working vacation mode. The PGFs for the number of consumers in the system during various states such as server unoccupied, occupied, on WV, delayed repair, and under repair have been obtained by using the SVT approach. Furthermore, the explicit expression for the mean queue length of both the orbit and the system was derived. We examined the probabilities of the system states and discussed some significant special cases. The utilization of numerical results in sensitivity analysis can assist decision-makers in analyzing the functioning of the system in various circumstances. Also, decision-makers and system developers in the relevant industries and service organizations will use the results of the cost analysis to help them make the best and most economical choices regarding the upgrading and capacity expansion of the service systems in consideration. The inspiration for this model comes from its widespread use in real-life systems, such as computer and telephone networks, where a single server handles messages while utilizing the working vacation policy.

Author contributions

Sundarapandiyan S.: Study design, conceptualization, methodology, formal analysis, visualization, software and writing-original draft. Nandhini S.: Study design, conceptualization, validation, methodology, formal analysis, visualization, software and supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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Appendix A

Theorem 2. *The embedded Markov−Chain* $\{\pi_{\mathfrak{n}}; \mathfrak{n} \in \mathbb{N}\}$ *is ergodic if and only if* $\Gamma < A^*(\lambda)$*, where* $\Gamma = f + r(1 - A^*(\lambda) + E(N_b)(\lambda + \lambda\delta(w^1 + g^1)))$.

Proof. To demonstrate the sufficient condition of ergodicity, it is convenient to utilize Foster's criterion (Pakes [42]) that specifies the Markov chain $\{\pi_{\mathfrak{n}}; \mathfrak{n} \in \mathbb{N}\}$ is an irreducibility, and the aperiodicity Markov-Chain is ergodic if $\forall\ \varepsilon > 0$, $\exists$ a function $h(q) \geq 0$, such that mean drift $\sigma_q = E[h(\pi_{\mathfrak{n}} + 1) - h(\pi_{\mathfrak{n}})/\pi_{\mathfrak{n}} = q] < \infty$, and $\sigma_q \leq -\varepsilon\ \forall\ q \in \mathbb{N}$. Suppose that we consider the function $h(q) = q$, then π_q can be expressed as: $\sigma_q = \begin{cases} \Gamma - 1, & q = 0; \\ \Gamma - A^{(*)}(\lambda), & q = 1, 2, 3, ... \end{cases}$

Thus, this inequality $\Gamma < A^*(\lambda)$ represents the sufficient condition for ergodicity.

The second part of the proof is carried as follows: For Markov-chain $\{\pi_{\mathfrak{n}}; \mathfrak{n} \geq 1\}$ that satisfies the Kaplan's condition $\sigma_q < \infty$, $\forall\ q \geq 0$, and $\exists\ q_0 \in \mathbb{N}$ such that $\sigma_q \geq 0\ \forall\ q \geq q_0$, as discussed in Sennot et al. [43]. Note that, from our case, obeys the Kaplan's condition since $\exists$ is a finite number k such that $m_{pq} = 0\ \forall q < p - k$, and $p > 0$, where $M = m_{pq}$ is the transition matrix of step one for $\{\pi_{\mathfrak{n}}; \mathfrak{n} \in \mathbb{N}\}$. Then, $\Gamma \geq A^*(\lambda)$ implies that non-ergodicity of the Markov-Chain. □

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