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Research article

Power unit inverse Lindley distribution with different measures of uncertainty, estimation and applications

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Abstract: This paper introduced and investigated the power unit inverse Lindley distribution (PUILD), a novel two-parameter generalization of the famous unit inverse Lindley distribution. Among its notable functional properties, the corresponding probability density function can be unimodal, decreasing, increasing, or right-skewed. In addition, the hazard rate function can be increasing, U-shaped, or N-shaped. The PUILD thus takes advantage of these characteristics to gain flexibility in the analysis of unit data compared to the former unit inverse Lindley distribution, among others. From a theoretical point of view, many key measures were determined under closed-form expressions, including mode, quantiles, median, Bowley's skewness, Moor's kurtosis, coefficient of variation, index of dispersion, moments of various types, and Lorenz and Bonferroni curves. Some important measures of uncertainty were also calculated, mainly through the incomplete gamma function. In the statistical part, the estimation of the parameters involved was studied using fifteen different methods, including the maximum likelihood method. The invariant property of this approach was then used to efficiently estimate different uncertainty measures. Some simulation results were presented to support this claim. The significance of the PUILD underlying model compared to several current statistical models, including the unit inverse Lindley, exponentiated Topp-Leone, Kumaraswamy, and beta and transformed gamma models, was illustrated by two applications using real datasets.

Keywords: Shannon entropy; Rényi entropy; exponential entropy; Havrda and Charvat entropy; unit

inverse Lindley distribution; extropy; weighted extropy; maximum product spacing; minimum spacing Linex distance
Mathematics Subject Classification: 62F15, 62G20, 65C60

1. Introduction

In the context of Bayesian statistics, Lindley (L) [1] explored the idea of the one-parameter L distribution (LD) as a counterexample to fiducial distributions. Reference [2] provides a comprehensive explanation of the statistical properties of the LD. In particular, it is emphasized that the LD outperforms the well-known exponential distribution in several important aspects. One of these is that by simply adjusting its unique parameter, the LD can fit data with an increasing failure rate quite efficiently. Logically, exploring more options and possibilities for this distribution would increase its modeling flexibility. Over the last few decades, researchers have therefore proposed several extensions of the LD under different circumstances.

Some mathematical ingredients that define the LD are now recalled. First, the LD can be viewed as a combination of exponential (β) and gamma ($2, \beta$) distributions. It has the cumulative distribution function (CDF) and probability density function (PDF) given by the following equations:

$$G(w;\beta) = \left(1 + \frac{w}{1+\beta}\right) e^{-\beta w}, \quad w > 0\ , \beta > 0, \tag{1.1}$$

with $G(w;\beta) = 0$ for $w \leq 0$, and

$$g(w;\beta) = \frac{\beta^2}{1+\beta}(1 + w)e^{-\beta w}, \quad w > 0,\ \beta > 0, \tag{1.2}$$

with $g(w;\beta) = 0$ for $w \leq 0$, respectively.

The following are some of the most important generalizations of this distribution: discrete Poisson-LD [3], zero-truncated Poisson-LD [4], three-parameter generalization of the LD [5], generalized LD [6], weighted LD [7], extended LD [8], exponential-Poisson-LD [9], special two-parameter LD [10], power LD [11], novel weighted LD [12], type I half logistic LD [13], and alpha power transformed power LD [14]. As evidenced by these works, extensions of the LD are still a hot topic, and efforts to construct flexible models based on it continue.

As suggested in reference [15], the inverse LD (ILD) is obtained by using the transformation $Y = 1/W$, where W denotes a random variable with the LD. After standard manipulations of the CDF and PDF in Eqs (1.1) and (1.2), respectively, the CDF and PDF of Y are as follows:

$$G(y;\beta) = \left(1 + \frac{\beta}{(1+\beta)y}\right) e^{-\frac{\beta}{y}}, \quad y > 0,\ \beta > 0, \tag{1.3}$$

with $G(y;\beta) = 0$ for $y \leq 0$, and

$$g(y;\beta) = \frac{\beta^2}{1+\beta} y^{-3}(1 + y)e^{-\frac{\beta}{y}}, \quad y > 0,\ \beta > 0, \tag{1.4}$$

with $g(y;\beta) = 0$ for $y \leq 0$, respectively. In fact, the ILD is only one member of the family of inverse distributions. Indeed, the literature on this family of inverse random variables is extensive and includes the inverse Kumaraswamy distribution [16], inverse power LD [17], inverse Xgamma distribution [18], inverse exponentiated Weibull distribution [19], inverse power Lomax distribution [20], inverse exponentiated Lomax distribution [21], inverse Lomax-Rayleigh distribution [22], inverse Nadarajah-Haghighi distribution [23], inverse Nakagami-m distribution [24], inverse Topp-Leone distribution [25], inverse power Muth distribution [26], inverse Maxwell distribution [27], inverse unit Teissier distribution [28], and inverse power Cauchy distribution [29].

On the other hand, beyond the distributions with support $(0, \infty)$, it is necessary to create new flexible distributions capable of offering models adapted to the analysis of datasets with values in $[0, 1]$. Indeed, many disciplines, including medical, actuarial, and financial sciences, are in need of such "unit distributions". A classic approach is to modify distributions with support $(0, \infty)$ to fit the unit interval $[0, 1]$. The resulting distributions generally provide more flexibility throughout $[0, 1]$ without changing the properties of the base distribution. With this in mind, many unit distributions have been created. For example, there is the unit Birnbaum-Saunders distribution [30], unit Weibull distribution [31–33], unit Gompertz distribution [34], unit LD [35], unit inverse Gaussian distribution [36], unit Burr XII distribution [37], unit exponentiated half-logistic distribution [38], unit half-logistic geometric distribution [39], unit omega distribution [40], unit power Burr X distribution [41], unit inverse exponentiated Weibull distribution [42], and generalized unit half-logistic geometric distribution [43].

With the above state of the art in mind, some mathematical elements and motivations are now developed to define the scope of this paper. Starting from a random variable Y with the ILD, we consider $X = Y/(1 + Y)$. Then, by construction, the support of X is $[0, 1]$. The distribution of X thus defines a unit distribution, which we logically call the unit ILD (UILD). After some standard developments based on Eqs (1.1) and (1.2), we can prove that it has the following CDF and PDF:

$$G(x;\beta) = \frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{x}\right)e^{-\frac{\beta}{x}}, \quad 0 < x \leq 1, \ \beta > 0, \tag{1.5}$$

with $G(x;\beta) = 0$ for $x \leq 0$ and $G(x;\beta) = 1$ for $x > 1$, and

$$g(x;\beta) = \frac{\beta^2 e^{\beta}}{1+\beta}x^{-3}e^{-\frac{\beta}{x}}, \quad 0 < x \leq 1, \ \beta > 0, \tag{1.6}$$

with $g(x;\beta) = 0$ for $x \leq 0$ or $x > 1$, respectively. Given this information, the focus of this study is on a derived two-parameter unit distribution, which we call the power unit ILD (PUILD). We are particularly interested in it for the following reasons:

i) It is very simple, with only two parameters to adjust, one being the scale parameter and the other the shape parameter.

ii) The PDF of the PUILD is characterized by being possibly unimodal, decreasing, increasing, and right-skewed. In addition, the hazard rate function (HRF) may be increasing, U-shaped, or N-shaped.

iii) It is possible to express in closed form the corresponding quantile and median.

iv) In fact, beyond quantile analysis, many important measures can be determined in closed form, including mode, moments, mean, variance, coefficient of variation, index of dispersion, inverse moments, harmonic mean, incomplete moments, and Lorenz and Bonferroni curves. All moment-type measures involve the incomplete gamma function, which is implemented in all mathematical software, making them easy to determine.

v) Thanks to the manageability of PUILD, some measures of uncertainty can be calculated, such as Shannon entropy, Rényi entropy, exponential entropy, Havrda and Charvat entropy, Arimoto entropy, Awad and Alawneh 1 entropy, Awad and Alawneh 2 entropy, extropy, and weighted extropy.

vi) Most of the existing parametric estimation approaches can be applied to the PUILD. In particular, this paper considers fifteen of them, which are the maximum likelihood, Anderson-Darling, Cramér-von-Mises, maximum product of spacings, least squares, right-tail Anderson-Darling, weighted least squares, left-tail Anderson-Darling, minimum spacing absolute distance, minimum spacing absolute-log distance, Anderson-Darling left-tail second order, Kolmogorov, minimum spacing square distance, minimum spacing square-log distance, and minimum spacing Linex distance approaches.

vii) As usual, the invariance property of maximum likelihood estimation can be used to estimate the different measures of uncertainty. The PUILD is of interest in this respect because of the simple expressions of these measures.

viii) Due to its flexible features, the PUILD is competitive in fitting unit data compared to the direct candidates. This importance is highlighted in this paper by considering several current statistical models, including the unit inverse Lindley, exponentiated Topp-Leone, Kumaraswamy, and beta and transformed gamma models, as well as two applications using real datasets.

There are nine parts to the paper. The creation of the PUILD is detailed in Section 2. Section 3 calculates some of its key measures. Section 4 is devoted to a wide panel of its uncertainty measures. Fifteen different estimation approaches are covered in Section 5. Section 6 discusses the results of the simulation. The practicality and adaptability of the PUILD are demonstrated in Section 7 through two real datasets. Finally, Section 8 presents the conclusion.

2. Formulation of the PUILD

This section introduces the mathematical basis of the PUILD. It is based on the transformation $Z = X^{1/\delta}$, where X is a random variable with the UILD. Based on Eqs (1.5) and (1.6), the CDF and PDF of the PUILD are given by

$$G(z;\beta,\delta) = \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}}, \quad 0 < z \leq 1,\ \beta,\delta > 0, \tag{2.1}$$

with $G(z;\beta,\delta) = 0$ for $z \leq 0$ and $G(z;\beta,\delta) = 1$ for $z > 1$, and

$$g(z;\beta,\delta) = \frac{\delta\beta^{2}e^{\beta}}{1+\beta}z^{-2\delta-1}e^{-\frac{\beta}{z^{\delta}}}, \quad 0 < z \leq 1,\ \beta,\delta > 0, \tag{2.2}$$

with $g(z;\beta,\delta) = 0$ for $z \leq 0$ or $z > 1$, respectively. We can check that $\lim_{z\to 0} g(z;\beta,\delta) = 0$ and $\lim_{z\to 1} g(z;\beta,\delta) = g(1;\beta,\delta) = \delta\beta^2/(1+\beta)$, which gives some indication of the limit possibilities of the corresponding model. More important aspects related to $g(z;\beta,\delta)$ will be revealed later.

In addition, the reliability function, the HRF, the reversed HRF, and the cumulative HRF are given, respectively, as follows:

$$S(z;\beta,\delta) = 1 - \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}},$$

$$h(z;\beta,\delta) = \frac{\delta\beta^2 e^{\beta} z^{-2\delta-1} e^{-\frac{\beta}{z^{\delta}}}}{1+\beta-e^{\beta}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}}},$$

$$\tau(z;\beta,\delta) = \frac{\delta\beta^2 z^{-2\delta-1}}{1+\frac{\beta}{z^{\delta}}},$$

and

$$H(z;\beta,\delta) = -\log\left[1-\frac{e^{\beta}}{(1+\beta)}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}}\right],$$

all being valid for $0 < z < 1$ and the standard complementary functions for the other values of z.

Figure 1 illustrates the plots of the two most important functions in terms of modeling significance: the PDF and the HRF.

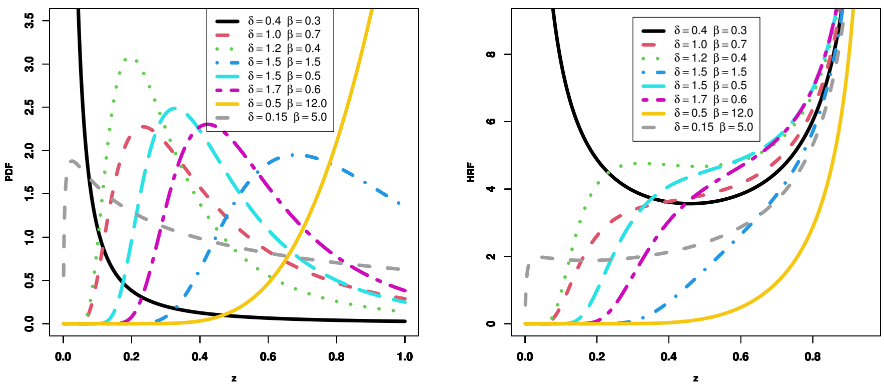

Figure 1. Plots of the PDF and HRF for the PUILD.

From this figure, it is clear that the PDF can be decreasing, increasing, unimodal, and right-skewed, and the HRF can be N-shaped, U-shaped, or increasing. This demonstrates a high degree of adaptability required for different unit data analyses.

On the other hand, the odd ratio (OR), failure rate average (FRA), and Mills ratio (MR) of the PUILD are

$$OR(z;\beta,\delta) = \left[(1+\beta)\,e^{-\beta}\left(1+\frac{\beta}{z^{\delta}}\right)^{-1}e^{\frac{\beta}{z^{\delta}}}-1\right]^{-1},$$

$$FRA(z;\beta,\delta) = \frac{-\log\left[1-\frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}}\right]}{z},$$

and

$$MR(z;\beta,\delta) = \frac{1+\beta - e^{\beta}\left(1+\frac{\beta}{z^{\delta}}\right)e^{-\frac{\beta}{z^{\delta}}}}{\delta\beta^{2}e^{\beta}z^{-2\delta-1}e^{-\frac{\beta}{z^{\delta}}}},$$

both valid for $0 < z < 1$. These expressions are quite manageable; these functions can be used for various purposes beyond those developed in this paper.

3. General properties

In this section, we examine some important measures of the PUILD. These help to understand its probabilistic properties.

3.1. Mode

If it is unique, the mode of the PUILD corresponds to the maximum point of the PDF into the support $[0, 1]$. It can be determined by equating $\frac{d\log[g(z;\beta,\delta)]}{dz}$ with 0, as follows:

$$\frac{d\log\left[g(z;\beta,\delta)\right]}{dz} = -\frac{2\delta+1}{z} + \frac{\beta\delta}{z^{\delta+1}} = 0. \tag{3.1}$$

After some reductions in complexity, Eq (3.1) becomes $-(2\delta+1)\,z^{\delta}+\beta\delta = 0$, from which we derive a solution which is given as

$$z_{*} = \left(\frac{\beta\delta}{2\delta+1}\right)^{\frac{1}{\delta}}, \tag{3.2}$$

provided that $\beta\delta \leq 2\delta+1$. Under this condition, if $z < z_{*}$, then we have $d\log\left[g(z;\beta,\delta)\right]/dz > 0$, and if $z > z_{*}$, then we have $d\log\left[g(z;\beta,\delta)\right]/dz < 0$. As a result, z_{*} is the unique mode of the PUILD.

If $\beta\delta > 2\delta+1$, the unique mode is immediately given as $z_{*} = 1$. The PUILD is thus inherently unimodal, and the closed form expression of its mode is a valuable indicator of the modeling power of the PUILD.

3.2. Quantile function

The quantile function of the PUILD is given as $Q(u;\beta,\delta) = F^{-1}(u;\beta,\delta)$, with $0 < u < 1$. It is thus calculated by inverting the CDF in Eq (2.1) as follows:

$$\left(\frac{e^{\beta}}{1+\beta}\right)\left(1+\beta(Q(u;\beta,\delta))^{-\delta}\right)e^{-\beta(Q(u;\beta,\delta))^{-\delta}} = u,$$

that provides

$$\left(1+\beta(Q(u;\beta,\delta))^{-\delta}\right)e^{-\beta(Q(u;\beta,\delta))^{-\delta}} = (1+\beta)\,e^{-\beta}u.$$

Multiplying each side of the previous equation by $-e^{-1}$ gives the following Lambert-type equation:

$$-(1+\beta(Q(u;\beta,\delta))^{-\delta})e^{-(1+\beta(Q(u;\beta,\delta))^{-\delta})} = -\,(1+\beta)\,e^{-(1+\beta)}u.$$

By introducing the negative Lambert W function of the real argument, denoted as W_{-1} (.), we find that

$$Q(u;\beta,\delta) = \left\{-\frac{1}{\beta} - \frac{1}{\beta}W_{-1}\left[-(1+\beta)\, e^{-(1+\beta)}u\right]\right\}^{\frac{-1}{\delta}}.$$

As the Lambert W function is implemented in most scientific software, we can easily manipulate this quantile function for calculation purposes. Plugging $u = 0.25$, 0.5, and 0.75 into this quantile function gives us the first, second (median), and third quantiles. Determining these quantiles facilitates statistical analysis and probabilistic modeling.

3.3. Moments

Let Z be a random variable with the PUILD. For any nonnegative integer r, since Z has a bounded support, the r_{th} moment of Z always exists and is also bounded. Let us compute it by considering its integral expression. We have

$$\mu_r' = E(Z^r) = \int_0^1 z^r g(z;\beta,\delta)dz = \frac{\delta\beta^2 e^\beta}{1+\beta}\int_0^1 z^{r-2\delta-1}e^{-\frac{\beta}{z^\delta}}dz.$$

If we apply the change of the variable $v = \frac{\beta}{z^\delta}$, we get

$$\mu_r' = \frac{\beta^{\frac{r}{\delta}}e^\beta}{1+\beta}\int_\beta^\infty v^{1-\frac{r}{\delta}}e^{-v}dv.$$

Then, by introducing the upper incomplete gamma function $\Gamma(u,t) = \int_t^\infty z^{u-1}e^{-z}dz$, with $u > 0$ and $t \geq 0$, we get

$$\mu_r' = \frac{\beta^{\frac{r}{\delta}}e^\beta}{1+\beta}\Gamma\left(2-\frac{r}{\delta},\beta\right). \tag{3.3}$$

Note that, since $\beta > 0$, this expression is valid without restriction on the parameters.

Having a closed-form expression for the moments of all orders allows a precise analytical characterization of the properties of the PUILD. It facilitates efficient computation of important measures such as those presented below.

The mean and variance of the PUILD can be calculated by inserting $r = 1$ and 2 in Eq (3.3), as follows:

$$E(Z) = \mu_1' = \frac{\beta^{\frac{1}{\delta}}e^\beta}{1+\beta}\Gamma\left(2-\frac{1}{\delta},\beta\right),$$

and

$$var(Z) = \mu_2' - {\mu_1'}^2 = \frac{\beta^{\frac{2}{\delta}}e^\beta}{1+\beta}\Gamma\left(2-\frac{2}{\delta},\beta\right) - \frac{\beta^{\frac{2}{\delta}}e^{2\beta}}{(1+\beta)^2}\left[\Gamma\left(2-\frac{1}{\delta},\beta\right)\right]^2.$$

Similarly, the corresponding skewness is given as $E\{[Z - E(Z)]^3/[var(Z)]^{3/2}\}$, the kurtosis is specified as $E\{[Z - E(Z)]^4/[var(Z)]^2\}$, the coefficient of variation (CV) is indicated as $[var(Z)]^{1/2}/E(Z)$, and the index of dispersion (ID) is expressed as $var(Z)/E(Z)$. Figure 2 shows the 3D plots of these measures for different values of β and δ.

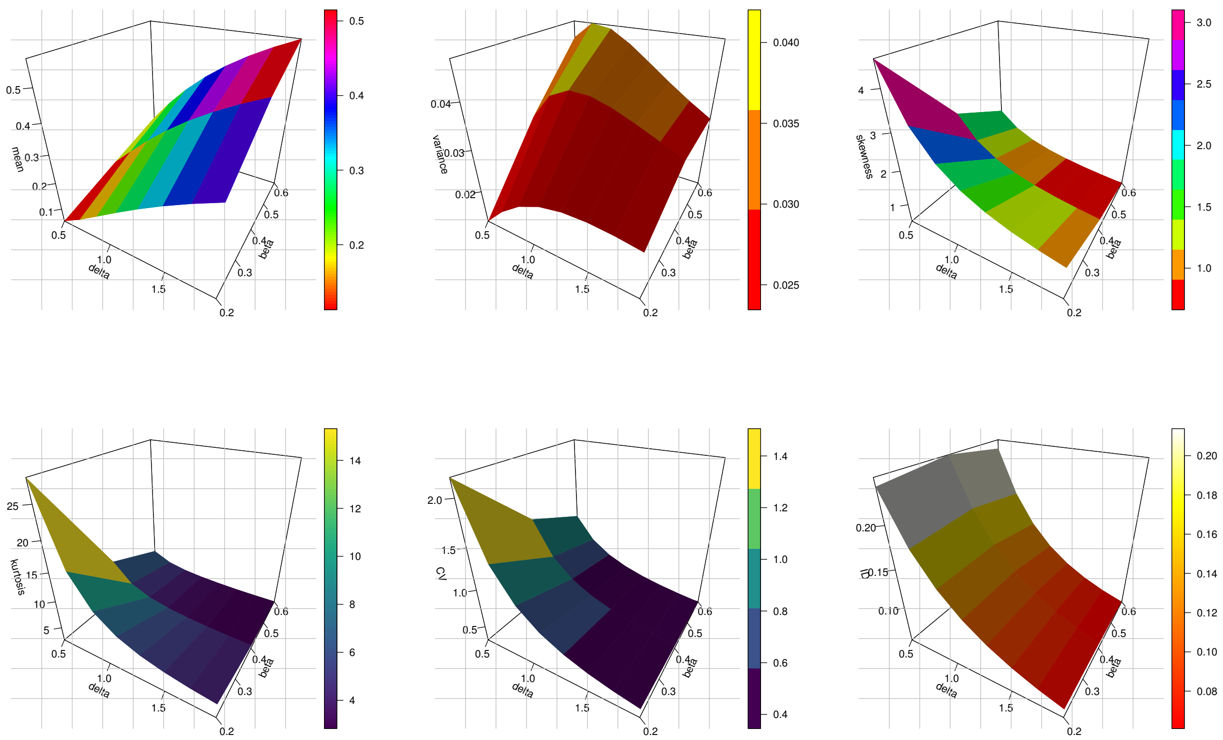

Figure 2. 3D plots of the most important moment measures for the PUILD, namely, from left to right, the mean, the variance, the skewness, the kurtosis, the CV, and the ID.

From this figure, for the values of the parameters considered, it can be seen that the skewness varies approximately from 0.5 to 4.5, indicating a wide range of possibilities. Furthermore, the corresponding kurtosis can be small or large. The PUILD thus reaches the three established kurtosis states: it can be leptokurtic, mesokurtic, and platykurtic. These facts complete the already observed shape flexibility of the PDF and HRD of the PUILD.

To complete this study of moments, for any nonnegative integer ω, let us express the ω_{th} lower incomplete moment (LIM) of Z. After some integral manipulations, we find that

$$\varrho_{\omega}(t) = E(Z^{\omega} 1_{\{Z<t\}}) = \int_0^t z^{\omega} g(z;\beta,\delta)dz = \frac{\delta\beta^2 e^{\beta}}{1+\beta}\int_0^t z^{\omega-2\delta-1} e^{-\frac{\beta}{z^{\delta}}} dz = \frac{\beta^{\frac{\omega}{\delta}} e^{\beta}}{1+\beta}\Gamma\left(2-\frac{\omega}{\delta},\beta t^{-\delta}\right).$$

By taking $t = 1$ and $\omega = r$, as expected, we refind $\varrho_{\omega}(t) = \mu'_r$.

3.4. Inverse moments

For any nonnegative integer r, the inverse r_{th} moment of Z can be calculated as follows:

$$\mu'_{-r} = E(Z^{-r}) = \int_0^1 z^{-r} g(z;\beta,\delta)dz = \frac{\delta\beta^2 e^{\beta}}{1+\beta}\int_0^1 z^{r-2\delta-1} e^{-\frac{\beta}{z^{\delta}}} dz.$$

Again, by applying $v = \frac{\beta}{z^{\delta}}$, we obtain

$$\mu'_{-r} = \frac{\beta^{\frac{-r}{\delta}} e^{\beta}}{1+\beta} \int_{\beta}^{\infty} v^{1+\frac{r}{\delta}} e^{-v} dv.$$

Then, using the incomplete gamma function, we find that

$$\mu'_{-r} = \frac{\beta^{\frac{-r}{\delta}} e^{\beta}}{1+\beta} \Gamma\left(2 + \frac{r}{\delta}, \beta\right). \tag{3.4}$$

By substituting $r = 1$ in Eq (3.4), the harmonic mean of Z can be calculated as follows:

$$\varepsilon = \frac{\beta^{\frac{-1}{\delta}} e^{\beta}}{1+\beta} \Gamma\left(2 + \frac{1}{\delta}, \beta\right).$$

These inverse moments complete the classical moment analysis of PUILD. These simple expressions show how they can be used in various probabilistic and statistical scenarios involving moments of various kinds.

3.5. Inequality curves

The Lorenz (LOR) and Bonferroni (BON) curves are essential in reliability, economics, medicine, demography, and insurance. They can also be interpreted in a unit data analysis scenario. For this reason, we express them in the context of the PUILD. The LOR and BON curves are simply calculated as

$$LOR = \frac{\varrho_1(t)}{E(Z)} = \frac{\Gamma\left(2 - \frac{1}{\delta}, \beta t^{-\delta}\right)}{\Gamma\left(2 - \frac{1}{\delta}, \beta\right)},$$

and

$$BON = \frac{LOR}{G(t; \beta, \delta)} = \frac{(1+\beta)\, \Gamma\left(2 - \frac{1}{\delta}, \beta t^{-\delta}\right) e^{\frac{\beta}{t^{\delta}}}}{\Gamma\left(2 - \frac{1}{\delta}, \beta\right) e^{\beta} \left(1 + \frac{\beta}{t^{\delta}}\right)},$$

respectively. They are easily implemented for various statistical purposes.

4. Measures of uncertainty

There are several useful measures of uncertainty for a given distribution. In this section, we examine the most famous of these in the context of PUILD. Namely, there is the Shannon entropy, the Rényi entropy, the exponential entropy, the Havrda and Charvat entropy, the Arimoto entropy, the Tsallis, Awad and Alawneh 1 entropy, the Awad and Alawneh 2 entropy, the extropy, and the weighted extropy.

4.1. Two important integrals result

The two propositions below show that some sophisticated integrals using the PDF of the PUILD can be written using the incomplete gamma function. Later, we will see how these integrals relate to the entropy measures under consideration.

Proposition 1. *Let $g(z;\beta,\delta)$ be given in Eq (2.2) and*

$$Q(\beta,\delta) = \int_0^1 g(z;\beta,\delta)\log\left[g(z;\beta,\delta)\right]dz.$$

Then, $Q(\beta,\delta)$ exists and is expressed as

$$Q(\beta,\delta) = \frac{e^{\beta}}{1+\beta}\left[(1+\beta)\,e^{-\beta}\log\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right) - \left(2+\frac{1}{\delta}\right)\left[(\beta+1)e^{-\beta}\log(\beta) - \Gamma(2,\beta)^2\right] - \Gamma(3,\beta)\right],$$

where $\Gamma^n(.,.)$ denotes the n_{th} derivative of the incomplete gamma function, that is, $\Gamma^n(u,t) = \int_t^{\infty} z^{u-1}(\log(z))^n e^{-z}dz$, with $u > 0$ and $t \geq 0$.

Proof. Thanks to Eq (2.2), we have

$$Q(\beta,\delta) = \int_0^1 g(z;\beta,\delta)\log\left[g(z;\beta,\delta)\right]dz = \frac{\delta\beta^2 e^{\beta}}{1+\beta}\int_0^1 z^{-2\delta-1}e^{-\frac{\beta}{z^{\delta}}}\log\left[\frac{\delta\beta^2 e^{\beta}}{1+\beta}z^{-2\delta-1}e^{-\frac{\beta}{z^{\delta}}}\right]dz.$$

Then, we have

$$Q(\beta,\delta) = \frac{\delta\beta^2 e^{\beta}}{1+\beta}\left[I_1 - (2\delta+1)I_2 - \beta I_3\right], \tag{4.1}$$

where

$$I_1 = \log\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)\int_0^1 z^{-2\delta-1}e^{-\frac{\beta}{z^{\delta}}}dz, \quad I_2 = \int_0^1 z^{-2\delta-1}\log(z)\,e^{-\frac{\beta}{z^{\delta}}}dz,$$

and

$$I_3 = \int_0^1 z^{-3\delta-1}e^{-\frac{\beta}{z^{\delta}}}dz.$$

For I_1, by the change of variables $v = \frac{\beta}{z^{\delta}}$, we have

$$I_1 = \frac{1}{\delta\beta^2}\log\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)\int_{\beta}^{\infty} ve^{-v}dv = \frac{(1+\beta)\,e^{-\beta}}{\delta\beta^2}\log\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)$$

and

$$I_2 = \frac{1}{\delta\beta^2}\int_{\beta}^{\infty} v\log\left(\beta^{\frac{1}{\delta}}v^{\frac{-1}{\delta}}\right)e^{-v}dv = \frac{1}{\delta^2\beta^2}\int_{\beta}^{\infty} v\left[\log(\beta) - \log(v)\right]e^{-v}dv,$$

which implies that

$$I_2 = \frac{1}{\delta^2\beta^2}\left[(\beta+1)e^{-\beta}\log(\beta) - \Gamma(2,\beta)^2\right].$$

Also, with the same technique, we get

$$I_3 = \frac{1}{\beta^3\delta}\int_{\beta}^{\infty} v^2 e^{-v}dv = \frac{\Gamma(3,\beta)}{\beta^3\delta}.$$

By inserting these expressions of I_1, I_2 and I_3 in Eq (4.1), we get

$$Q(\beta,\delta) = \frac{e^{\beta}}{1+\beta}\left[(1+\beta)\,e^{-\beta}\log\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right) - \left(2+\frac{1}{\delta}\right)\left[(\beta+1)e^{-\beta}\log(\beta) - \Gamma(2,\beta)^2\right] - \Gamma(3,\beta)\right].$$

This ends the proof of Proposition 1. □

Proposition 2. *Let* $\kappa > 0$, $\kappa \neq 1$, $g(z;\beta,\delta)$ *be given in Eq* (2.2) *and*

$$I_\kappa(\beta,\delta) = \int_0^1 g(z;\beta,\delta)^\kappa dz.$$

Then, $I_\kappa(\beta,\delta)$ *exists if, and only if,* $(2\delta+1)\kappa > 1$, *and it is expressed as*

$$I_\kappa(\beta,\delta) = \frac{1}{\delta}\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa (\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa + \frac{\kappa-1}{\delta}, \kappa\beta\right).$$

Proof. Owing to Eq (2.2), we have

$$I_\kappa(\beta,\delta) = \int_0^1 g(z;\beta,\delta)^\kappa dz = \left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa \int_0^1 z^{-2\kappa\delta-\kappa} e^{-\frac{\kappa\beta}{z^\delta}} dz.$$

By performing the change of variables $v = \frac{\kappa\beta}{z^\delta}$, we get

$$I_\kappa(\beta,\delta) = \frac{1}{\delta}\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa (\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}} \int_{\kappa\beta}^{\infty} v^{2\kappa+\frac{\kappa-1}{\delta}-1} e^{-v} dv,$$

which implies that

$$I_\kappa(\beta,\delta) = \frac{1}{\delta}\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa (\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa + \frac{\kappa-1}{\delta}, \kappa\beta\right).$$

This ends the proof of Proposition 2. □

In this study, Propositions 1 and 2 are of interest since $Q(\beta,\delta)$ and $I_\kappa(\beta,\delta)$ are the major components of various entropy measures defined in the setting of the PUILD. This is discussed in more detail in the next subsection.

4.2. Various entropy measures

As sketched in the introduction, the entropy of the PUILD can be measured in different manners. examining multiple measures of entropy for this distribution provides a comprehensive understanding of its uncertainty and complexity. This multi-faceted analysis is crucial in various fields such as information theory and machine learning. The most useful entropy measures from the literature are recalled in Table 1 for a general distribution with PDF denoted by $g(z;\beta,\delta)$. Also, we suppose that $\kappa > 0$ and $\kappa \neq 1$ are basic assumptions.

Table 1. Important entropy measures of a distribution with PDF $g(z;\beta,\delta)$ at κ.

Name of the entropy	Reference	Expression
Shannon	[44]	$S(\beta,\delta) = -\int_0^1 g(z;\beta,\delta)\log\left[g(z;\beta,\delta)\right]dz$
Rényi	[45]	$R_\kappa(\beta,\delta) = \frac{1}{1-\kappa}\log\left[\int_0^1 g(z;\beta,\delta)^\kappa dz\right]$
Exponential	[46]	$E_\kappa(\beta,\delta) = \left[\int_0^1 g(z;\beta,\delta)^\kappa dz\right]^{\frac{1}{1-\kappa}}$
Havrda and Charvat	[47]	$HC_\kappa(\beta,\delta) = \frac{1}{2^{1-\kappa}-1}\left[\int_0^1 g(z;\beta,\delta)^\kappa dz - 1\right]$
Arimoto	[48]	$A_\kappa(\beta,\delta) = \frac{\kappa}{1-\kappa}\left\{\left[\int_0^1 g(z;\beta,\delta)^\kappa dz\right]^{\frac{1}{\kappa}} - 1\right\}$
Tsallis	[49]	$T_\kappa(\beta,\delta) = \frac{1}{\kappa-1}\left[1 - \int_0^1 g(z;\beta,\delta)^\kappa dz\right]$
Awad and Alawneh 1	[50]	$AA1_\kappa(\beta,\delta) = \frac{1}{\kappa-1}\log\left\{\left[\sup_{z\in\mathbb{R}} g(z;\beta,\delta)\right]^{1-\kappa}\int_0^1 g(z;\beta,\delta)^\kappa dz\right\}$
Awad and Alawneh 2	[50]	$AA2_\kappa(\beta,\delta) = \frac{1}{2^{1-\kappa}-1}\left[\left\{\left[\sup_{z\in\mathbb{R}} g(z;\beta,\delta)\right]^{1-\kappa}\int_0^1 g(z;\beta,\delta)^\kappa dz\right\} - 1\right]$

It is assumed that $\sup_{z\in\mathbb{R}} g(z;\beta,\delta)$ is finite and well-identified for the two entropy measures proposed in [50].

Shannon entropy. Based on Table 1, Eq (2.2), and Proposition 1, the Shannon entropy of the PUILD is obtained as

$$
\begin{aligned}
S(\beta,\delta) &= -Q(\beta,\delta) \\
&= \frac{e^\beta}{1+\beta}\left[\left(2+\frac{1}{\delta}\right)\left[(\beta+1)e^{-\beta}\log(\beta) - \Gamma(2,\beta)^2\right] + \Gamma(3,\beta) - (1+\beta)\,e^{-\beta}\log\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)\right].
\end{aligned}
$$

Rényi entropy. Based on Table 1, Eq (2.2), and Proposition 2, the Rényi entropy of the PUILD can be expressed as

$$
R_\kappa(\beta,\delta) = \frac{1}{1-\kappa}\log\left[I_\kappa(\beta,\delta)\right]
$$

$$= \frac{1}{1-\kappa}\log\left\{\frac{1}{\delta}\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)^{\kappa}(\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)\right\}.$$

Exponential entropy. Based on Table 1, Eq (2.2), and Proposition 2, the exponential entropy of the PUILD is specified by

$$\begin{aligned} E_{\kappa}(\beta,\delta) &= [I_{\kappa}(\beta,\delta)]^{\frac{1}{1-\kappa}} \\ &= \left\{\frac{1}{\delta}\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)^{\kappa}(\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)\right\}^{\frac{1}{1-\kappa}}. \end{aligned}$$

Havrda and Charvát entropy. From Table 1, Eq (2.2), and Proposition 2, the Havrda and Charvát entropy of the PUILD can be expressed as

$$\begin{aligned} HC_{\kappa}(\beta,\delta) &= \frac{1}{2^{1-\kappa}-1}\left[I_{\kappa}(\beta,\delta)-1\right] \\ &= \frac{1}{2^{1-\kappa}-1}\left[\frac{1}{\delta}\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)^{\kappa}(\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)-1\right]. \end{aligned}$$

Arimoto entropy. Again, from Table 1, Eq (2.2), and Proposition 2, the Arimoto entropy of the PUILD is specified by

$$\begin{aligned} A_{\kappa}(\beta,\delta) &= \frac{\kappa}{1-\kappa}\left[I_{\kappa}(\beta,\delta)^{\frac{1}{\kappa}}-1\right] \\ &= \frac{\kappa}{1-\kappa}\left\{\left[\frac{1}{\delta}\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)^{\kappa}(\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)\right]^{\frac{1}{\kappa}}-1\right\}. \end{aligned}$$

Tsallis entropy. Based on Table 1, Eq (2.2), and Proposition 2, the Tsallis entropy of the PUILD can be expressed as

$$\begin{aligned} T_{\kappa}(\beta,\delta) &= \frac{1}{\kappa-1}\left[1-I_{\kappa}(\beta,\delta)\right] \\ &= \frac{1}{\kappa-1}\left[1-\frac{1}{\delta}\left(\frac{\delta\beta^2 e^{\beta}}{1+\beta}\right)^{\kappa}(\kappa\beta)^{-2\kappa+\frac{1-\kappa}{\delta}}\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)\right]. \end{aligned}$$

Awad and Alawneh 1 entropy. From Table 1, Eq (2.2), and Proposition 2, the Awad and Alawneh 1 entropy of the PUILD is given as

$$AA1_{\kappa}(\beta,\delta) = \frac{1}{\kappa-1}\log\left\{\left[\sup_{0<z<1} g(z;\beta,\delta)\right]^{1-\kappa} I_{\kappa}(\beta,\delta)\right\}. \tag{4.2}$$

Before going any further, we need to find $\sup_{0<z\leq 1} g(z;\beta,\delta)$. The following formula will do the necessary. For $\beta\delta < 2\delta + 1$, based on the equation (3.2), we have

$$s(\beta,\delta) = \sup_{0<z\leq 1} g(z;\beta,\delta) = g(z_*;\beta,\delta) = \left(\frac{\delta e^{\beta-\frac{1}{\delta}+2}}{(1+\beta)\beta^{\frac{1}{\delta}}}\right)\left(2+\frac{1}{\delta}\right)^{2+\frac{1}{\delta}}. \tag{4.3}$$

Otherwise, we have $z_* = 1$ and

$$s(\beta,\delta) = g(z_*;\beta,\delta) = \frac{\delta\beta^2}{1+\beta}. \tag{4.4}$$

Based on Eq (4.3) or Eq (4.4), Eq (4.2) becomes

$$AA1_\kappa(\beta,\delta) = \frac{1}{\kappa-1}\log\left\{[s(\beta,\delta)]^{1-\kappa}\frac{1}{\delta}\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa \frac{\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)}{(\kappa\beta)^{2\kappa-\frac{1-\kappa}{\delta}}}\right\}.$$

Awad and Alawneh 2 entropy. From Table 1, Eq (2.2), Proposition 2, and Eq (4.3) or Eq (4.4), the Awad and Alawneh 2 entropy of the PUILD is given as

$$\begin{aligned} AA2_\kappa &= \frac{1}{2^{1-\kappa}-1}\left[\left\{\left[\sup_{0<z<1} g(z;\beta,\delta)\right]^{1-\kappa} I_\kappa(\beta,\delta)\right\}-1\right] \\ &= \frac{1}{2^{1-\kappa}-1}\left[\left\{[s(\beta,\delta)]^{1-\kappa}\frac{1}{\delta}\left(\frac{\delta\beta^2 e^\beta}{1+\beta}\right)^\kappa \frac{\Gamma\left(2\kappa+\frac{\kappa-1}{\delta},\kappa\beta\right)}{(\kappa\beta)^{2\kappa-\frac{1-\kappa}{\delta}}}\right\}-1\right]. \end{aligned}$$

Theoretically, it is complicated to study the behavior of these entropy measures. For this reason, a numerical study is proposed in the next section.

4.3. Extropy

Lad et al. [51] introduced a new measure of uncertainty: the extropy, which can be represented as the double complement of the entropy [44]. The extropy can be used statistically to assess the accuracy of predicting distributions using the total log scoring method. The definition of the extropy of the PUILD is

$$\Phi = \frac{-1}{2}\int_0^1 g(z;\beta,\delta)^2 dz. \tag{4.5}$$

By inserting Eq (2.2) into Eq (4.5), we obtain

$$\Phi = \frac{-1}{2}\left[\frac{\delta^2\beta^4 e^{2\beta}}{(1+\beta)^2}\int_0^1 z^{-4\delta-2}e^{-\frac{2\beta}{z^\delta}}dz\right].$$

After some simplifications, we get

$$\Phi = \frac{-1}{32}\left[\frac{\delta e^{2\beta}\Gamma\left(4+\frac{1}{\delta},2\beta\right)}{(2\beta)^{\frac{1}{\delta}}(1+\beta)^2}\right].$$

Another analogue to the weighted entropy proposed in [52] is the weighted extropy. It can be expressed as

$$\Phi^w = \frac{-1}{2}\int_0^1 z g(z;\beta,\delta)^2 dz. \tag{4.6}$$

By inserting Eq (2.2) into Eq (4.6), we get

$$\Phi^{w} = \frac{-1}{32}\left[\frac{\delta e^{2\beta}\Gamma(4,2\beta)}{(1+\beta)^{2}}\right].$$

The simple expressions of Φ and Φ^{w} are an advantage of the PUILD. They allow a deeper extropy analysis without computational effort.

5. Estimation methods

In this section, we examine the conventional approaches to estimating the two parameters of the PUILD. In these estimation methods, an objective function is optimized by maximization or minimization to obtain the most appropriate estimates.

5.1. Method of maximum likelihood

This part calculates the maximum likelihood estimates (MLEs) $(\hat{\beta}_1, \hat{\delta}_1)$ of (β, δ) based on a simple random sample. To detail this procedure, assume that $z_1, z_2, \ldots, z_n$ is an observed simple random sample of size n drawn from the PUILD. Then, the log-likelihood function (L-LF) of β and δ is given by

$$\log L = n\log(\delta) + 2n\log(\beta) + n\beta - n\log(1+\beta) - (1+2\delta)\sum_{i=1}^{n}\log(z_i) - \sum_{i=1}^{n}\frac{\beta}{z_i{}^{\delta}}. \tag{5.1}$$

The desired MLEs are obtained by maximizing this L-LF. In this sense, differentiating Eq (5.1) with respect to the parameters β and δ, we obtain

$$\frac{\partial \log L}{\partial \beta} = n + \frac{2n}{\beta} - \frac{n}{1+\beta} - \sum_{i=1}^{n}\frac{1}{z_i{}^{\delta}}, \tag{5.2}$$

and

$$\frac{\partial \log L}{\partial \delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n}\log(z_i) + \beta\sum_{i=1}^{n}\frac{\log(z_i)}{z_i{}^{\delta}}. \tag{5.3}$$

Since finding the exact solution to Eqs (5.2) and (5.3) equal to 0 is difficult, we will use optimization techniques such as the Newton-Raphson approach using the Mathematica software program to maximize it.

5.2. Method of Anderson-Darling

Let $(z_{1:n}, z_{2:n}, \ldots, z_{n:n})$ represent the ordered simple random sample of $(z_1, z_2, \ldots, z_n)$. Then the Anderson-Darling estimates (ADEs) $(\hat{\beta}_2, \hat{\delta}_2)$ are obtained by minimizing the following function:

$$\begin{aligned}
A &= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[G(z_{i:n},\beta,\delta)\right] + \log\left[S(z_{n-i-1:n},\beta,\delta)\right]\right\} \\
&= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[\frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}}\right]\right. \\
&\quad \left. + \log\left[1 - \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{n-i-1:n}^{\delta}}\right)e^{-\frac{\beta}{z_{n-i-1:n}^{\delta}}}\right]\right\}.
\end{aligned}$$

5.3. Method of Cramér-von Mises

The Cramér-von Mises estimates (CVMEs) $(\hat{\beta}_3, \hat{\delta}_3)$ are determined by minimizing the following function:

$$C = \frac{1}{12n} + \sum_{i=1}^{n}\left[G(z_{i:n}, \beta, \delta) - \frac{2i-1}{2n}\right]^2$$
$$= \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{2i-1}{2n}\right]^2.$$

5.4. Method of maximum product of spacings

The method of maximum product of spacings estimates (MPSEs) $(\hat{\beta}_4, \hat{\delta}_4)$ are obtained by maximizing the following function:

$$MPS = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(\xi_{i,n}),$$

where

$$\xi_{i,n} = G(z_{i:n}, \beta, \delta) - G(z_{i-1:n}, \beta, \delta)$$
$$= \frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i-1:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i-1:n}^{\delta}}}, \tag{5.4}$$

completed by $G(z_{0:n}, \beta, \delta) = 0$ and $G(z_{n+1:n}, \beta, \delta) = 1$.

5.5. Method of ordinary least squares

The ordinary least squares estimates (OLSEs) $(\hat{\beta}_5, \hat{\delta}_5)$ are calculated by minimizing the following function:

$$V = \sum_{i=1}^{n}\left[G(z_{i:n}, \beta, \delta) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{i}{n+1}\right]^2.$$

5.6. Methods of right-tail Anderson-Darling

The right-tail ADEs (RTADEs) $(\hat{\beta}_6, \hat{\delta}_6)$ are determined by minimizing the following function:

$$R = \frac{n}{2} - 2\sum_{i=1}^{n}G(z_{i:n}, \beta, \delta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[S(z_{i:n}, \beta, \delta)\right]$$
$$= \frac{n}{2} - 2\sum_{i=1}^{n}\frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[1 - \frac{e^{\beta}}{1+\beta}\left(1 + \frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}}\right].$$

5.7. Method of weighted least squares

The weighted least squares estimates (WLSEs) $(\hat{\beta}_7, \hat{\delta}_7)$ are obtained by minimizing the following function:

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(z_{i:n}, \beta, \delta) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{e^{\beta}}{1+\beta} \left(1+\frac{\beta}{z_{i:n}^{\delta}}\right) e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{i}{n+1} \right]^2 .$$

5.8. Method of left-tail Anderson-Darling

The left-tail ADEs (LTADEs) $(\hat{\beta}_8, \hat{\delta}_8)$ are computed by minimizing the following function:

$$\begin{aligned} L &= -\frac{3}{2}n + 2\sum_{i=1}^{n} G(z_{i:n}, \beta, \delta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[G(z_{i:n}, \beta, \delta)\right] \\ &= -\frac{3}{2}n + 2\sum_{i=1}^{n} \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right) e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[\frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right) e^{-\frac{\beta}{z_{i:n}^{\delta}}}\right]. \end{aligned}$$

5.9. Method of minimum spacing absolute distance

The minimum spacing absolute distance estimates (MSADEs) $(\hat{\beta}_9, \hat{\delta}_9)$ are obtained by minimizing the following function:

$$\zeta = \sum_{i=1}^{n+1} \left| \xi_{i,n} - \frac{1}{n+1} \right|,$$

where $\xi_{i,n}$ is given in Eq (5.4).

5.10. Method of minimum spacing absolute-log distance

The minimum spacing absolute-log distance estimates (MSALDEs) $(\hat{\beta_{10}}, \hat{\delta_{10}})$ are obtained by minimizing the following function:

$$\Upsilon = \sum_{i=1}^{n+1} \left| \log(\xi_{i,n}) - \log\left(\frac{1}{n+1}\right) \right|,$$

where $\xi_{i,n}$ is given in Eq (5.4).

5.11. Method of Anderson-Darling left-tail second order

The Anderson-Darling left-tail second order estimates (ADSOEs) $(\hat{\beta_{11}}, \hat{\delta_{11}})$ are determined by minimizing the following function:

$$\begin{aligned} LTS &= 2\sum_{i=1}^{n} \log\left[G(z_{i:n}, \beta, \delta)\right] + \frac{1}{n}\sum_{i=1}^{n} \frac{(2i-1)}{G(z_{i:n}, \beta, \delta)} \\ &= 2\sum_{i=1}^{n} \log\left[\frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right) e^{-\frac{\beta}{z_{i:n}^{\delta}}}\right] + \frac{1}{n}\sum_{i=1}^{n} \frac{(2i-1)}{\frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right) e^{-\frac{\beta}{z_{i:n}^{\delta}}}}. \end{aligned}$$

5.12. Method of Kolmogorov

The Kolmogorov estimates (KEs) $(\hat{\beta_{12}}, \hat{\delta_{12}})$ are obtained by minimizing the following function:

$$\begin{aligned} KM &= \max_{i=1,\ldots,n}\left[\frac{i}{n} - G(z_{i:n},\beta,\delta), G(z_{i:n},\beta,\delta) - \frac{i-1}{n}\right] \\ &= \max_{i=1,\ldots,n}\left[\frac{i}{n} - \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}}, \frac{e^{\beta}}{1+\beta}\left(1+\frac{\beta}{z_{i:n}^{\delta}}\right)e^{-\frac{\beta}{z_{i:n}^{\delta}}} - \frac{i-1}{n}\right]. \end{aligned}$$

5.13. Method of minimum spacing square distance

The minimum spacing square distance estimates (MSSDEs) $(\hat{\beta_{13}}, \hat{\delta_{13}})$ are calculated by minimizing the following function:

$$\phi = \sum_{i=1}^{n+1}\left(\xi_{i,n} - \frac{1}{n+1}\right)^2,$$

where $\xi_{i,n}$ is given in Eq (5.4).

5.14. Method of minimum spacing square-log distance

The minimum spacing square-log distance estimates (MSSLDEs) $(\hat{\beta_{14}}, \hat{\delta_{14}})$ are obtained by minimizing the following function:

$$\delta = \sum_{i=1}^{n+1}\left[\log(\xi_{i,n}) - \log\left(\frac{1}{n+1}\right)\right]^2,$$

where $\xi_{i,n}$ is given in Eq (5.4).

5.15. Method of minimum spacing Linex distance

The minimum spacing Linex distance estimates (MSLDEs) $(\hat{\beta_{15}}, \hat{\delta_{15}})$ are determined by minimizing the following function:

$$\varDelta = \sum_{i=1}^{n+1}\left[e^{\xi_{i,n}-\frac{1}{n+1}} - \left(\xi_{i,n} - \frac{1}{n+1}\right) - 1\right],$$

where $\xi_{i,n}$ is given in Eq (5.4).

6. Numerical simulation

This section evaluates the effectiveness of the estimation techniques presented in Section 5. Simulated datasets were generated according to the proposed model, and the estimation techniques that were considered were applied to estimate the unknown parameters. The associated performance was evaluated using five different metrics described below. For $a \in \{\delta, \beta\}$, these metrics are as follows:

i) Average bias (BIAS) given as $|Bias(\hat{a})| = (1/M)\sum_{j=1}^{M}|\hat{a}_j - a|$, where j refers to the label of the considered sample, among M samples of size n,

ii) Mean squared error (MSE) indicated as $MSE = (1/M)\sum_{j=1}^{M}(\hat{a}_j - a)^2$,

iii) Mean relative error (MRE) defined as $MRE = (1/M)\sum_{j=1}^{M}|\hat{a}_j - a|/a$,

iv) Average absolute difference (D_{abs}) indicated as $D_{abs} = [1/(nM)]\sum_{j=1}^{M}\sum_{u=1}^{n}|G(x_{j,u};\beta,\delta) - G(x_{j,u};\hat{\beta}_j,\hat{\delta}_j)|$, where $x_{j,u}$ denotes the values obtained at the sample labeled j and its u_{th} component.

v) Maximum absolute difference (D_{max}) expressed as $D_{max} = (1/M)\sum_{j=1}^{M}\max_{u=1,\ldots,n}|G(x_{j,u};\beta,\delta) - G(x_{j,u};\hat{\beta}_j,\hat{\delta}_j)|$.

The purpose of the simulation study is to determine the optimal estimation strategy for the proposed model.

The simulation results are presented in Tables 2–6. Furthermore, the partial and total ranks for the estimates are given in Table 7.

The simulation results show that all parameter estimation methods for the proposed model have high accuracy and are close to the true values. The computed measures generally decrease with increasing sample size (n). KE emerges as the most effective parameter estimation method, with an overall score of 35.5 (see Table 7).

In order to have graphical benchmark, the values from Table 2 are also visualized in Figures 3–11.

In these figures, we can see the fast decay of all curves with relatively small values for n. This confirms the efficiency of the estimation methods considered in the context of the PUILD.

6.1. Numerical representation for the entropy measures

In this subsection, randomly generated datasets have been used to find different values of the estimated entropy measures derived in Section 4. These numerical values are presented in Tables 8–11. A comparative analysis of the estimated entropy values using different measures, metrics such as BIAS, MSE, and MRE, and sample sizes (n) is performed. These measures are denoted by their respective abbreviations, such as RE for Rényi entropy, ExE for exponential entropy, HCE for Havrda and Charvat entropy, ArE for Arimoto entropy, TsE for Tsallis entropy, AA1E for Awad and Alawneh 1 entropy, AA2E for Awad and Alawneh 2 entropy, ShE for Shannon entropy, EX for extropy, and WEX for weighted extropy. These numerical values are obtained by the following procedure:

i) Initialize our proposed model parameters and use them to generate random datasets.

ii) Initialize κ for the entropy measures that depend on it, then determine all their initial values, say E_0.

iii) Use the MLEs to determine the estimated entropy value, say $\hat{E}$, by the substitution method.

iv) Finally, determine the corresponding BIAS, MSE, and MRE. Then, repeat the previous steps thousands of times.

The results in Tables 8–11 show that all the estimated entropy measures tend to their initial values as the sample size increases and the other error-type measures decrease.

Table 2. Numerical values of simulation measures for $\delta = 0.7$ and $\beta = 2.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
30	BIAS($\hat{\delta}$)	0.22373[7]	0.22577[8]	0.25312[14]	0.20754[5]	0.25032[13]	0.2571[15]	0.2331[11]	0.23345[12]	0.13758[2]	0.18677[3]	0.22978[10]	0.08802[1]	0.22914[9]	0.2176[6]	0.20036[4]
	BIAS($\hat{\beta}$)	0.7744[7]	0.80526[9]	0.75617[6]	0.82699[11]	0.77464[8]	0.80846[10]	0.84149[13]	0.83473[12]	0.29217[2]	0.7215[4]	0.85207[15]	0.04434[1]	0.72321[5]	0.84278[14]	0.64726[3]
	MSE($\hat{\delta}$)	0.07932[8]	0.07826[7]	0.09935[14]	0.06438[4]	0.09656[13]	0.09952[15]	0.08196[9]	0.08382[11]	0.03691[2]	0.05693[3]	0.08258[10]	0.01303[1]	0.08619[12]	0.07251[6]	0.06923[5]
	MSE($\hat{\beta}$)	0.85507[7]	0.93048[10]	0.79375[5]	1.00678[13]	0.83857[6]	0.87698[8]	0.99586[12]	0.98746[11]	0.29807[2]	0.88098[9]	1.04622[15]	0.00824[1]	0.74237[4]	1.02031[14]	0.62886[3]
	MRE($\hat{\delta}$)	0.31961[7]	0.32252[8]	0.36159[14]	0.29648[5]	0.3576[13]	0.36728[15]	0.333[11]	0.3335[12]	0.19654[2]	0.26682[3]	0.32825[10]	0.12574[1]	0.32734[9]	0.31086[6]	0.28622[4]
	MRE($\hat{\beta}$)	0.30976[7]	0.3221[9]	0.30247[6]	0.33079[11]	0.30985[8]	0.32339[10]	0.33659[13]	0.33389[12]	0.11687[2]	0.2886[4]	0.34083[15]	0.01774[1]	0.28928[5]	0.33711[14]	0.2589[3]
	D_{abs}	0.04005[1]	0.04126[4]	0.04356[10]	0.04111[2]	0.04425[11]	0.04479[12]	0.04172[6]	0.0414[5]	0.04519[13]	0.04312[9]	0.04124[3]	0.04269[8]	0.06087[15]	0.04232[7]	0.05767[14]
	D_{max}	0.06512[3]	0.06645[4]	0.07113[12]	0.06499[2]	0.0708[11]	0.07279[13]	0.06712[8]	0.06667[5]	0.06964[10]	0.06785[9]	0.06693[6]	0.06424[1]	0.09209[15]	0.06696[7]	0.08699[14]
	$\sum Ranks$	47[4]	59[7]	81[11]	53[6]	83[12.5]	98[15]	83[12.5]	80[10]	35[2]	44[3]	84[14]	15[1]	74[8.5]	74[8.5]	50[5]
60	BIAS($\hat{\delta}$)	0.17259[5]	0.19678[11]	0.22989[15]	0.17065[4]	0.20041[13]	0.22807[14]	0.19883[12]	0.19121[9]	0.1085[2]	0.16308[3]	0.19184[10]	0.05999[1]	0.18004[7]	0.18312[8]	0.17361[6]
	BIAS($\hat{\beta}$)	0.68127[5]	0.78854[14]	0.7418[8]	0.74548[9]	0.71544[7]	0.77753[13]	0.7739[12]	0.74844[10]	0.27546[2]	0.69382[6]	0.75274[11]	0.03594[1]	0.58942[3]	0.80753[15]	0.59255[4]
	MSE($\hat{\delta}$)	0.04735[5]	0.05737[9]	0.0815[15]	0.04443[4]	0.06477[13]	0.08028[14]	0.06309[12]	0.05805[10]	0.02402[2]	0.04275[3]	0.05944[11]	0.00602[1]	0.05663[8]	0.05151[6]	0.05354[7]
	MSE($\hat{\beta}$)	0.71243[5]	0.92878[14]	0.76945[7]	0.87146[12]	0.74299[6]	0.85999[10]	0.91218[13]	0.84322[9]	0.26265[2]	0.81005[8]	0.86264[11]	0.00554[1]	0.50974[3]	0.99557[15]	0.5388[4]
	MRE($\hat{\delta}$)	0.24656[5]	0.28111[11]	0.32842[15]	0.24378[4]	0.2863[13]	0.32581[14]	0.28404[12]	0.27316[9]	0.155[2]	0.23297[3]	0.27405[10]	0.08571[1]	0.25719[7]	0.26159[8]	0.24802[6]
	MRE($\hat{\beta}$)	0.27251[5]	0.31542[14]	0.29672[8]	0.29819[9]	0.28618[7]	0.31101[13]	0.30956[12]	0.29938[10]	0.11018[2]	0.27753[6]	0.3011[11]	0.01437[1]	0.23577[3]	0.32301[15]	0.23702[4]
	D_{abs}	0.0295[2]	0.03057[5]	0.03189[13]	0.02811[1]	0.03141[11]	0.03108[9]	0.03081[6]	0.03091[8]	0.03083[7]	0.0314[10]	0.02956[3]	0.02965[4]	0.04047[14]	0.03166[12]	0.04048[15]
	D_{max}	0.04787[3]	0.04999[6]	0.05351[13]	0.04551[2]	0.05135[11]	0.05238[12]	0.05037[7]	0.05048[9]	0.04829[4]	0.05038[8]	0.04885[5]	0.04457[1]	0.06312[14]	0.05093[10]	0.06349[15]
	$\sum Ranks$	35[3]	84[11]	94[14]	45[4]	81[10]	99[15]	86[12]	74[9]	23[2]	47[5]	72[8]	11[1]	59[6]	89[13]	61[7]
100	BIAS($\hat{\delta}$)	0.14625[7]	0.16744[12]	0.19457[14]	0.14162[5]	0.185[13]	0.21524[15]	0.16618[11]	0.15543[9]	0.0977[2]	0.13859[4]	0.15888[10]	0.0488[1]	0.13599[3]	0.15167[8]	0.14603[6]
	BIAS($\hat{\beta}$)	0.59238[5]	0.68527[13]	0.68361[12]	0.67297[9]	0.67575[10]	0.75015[15]	0.66326[8]	0.62369[6]	0.27316[2]	0.64513[7]	0.70079[14]	0.03289[1]	0.48933[3]	0.67978[11]	0.52629[4]
	MSE($\hat{\delta}$)	0.03434[6]	0.04318[12]	0.0602[14]	0.02976[3]	0.05233[13]	0.07238[15]	0.04301[11]	0.0378[8]	0.02044[2]	0.03082[4]	0.03962[10]	0.0038[1]	0.03378[5]	0.03602[7]	0.03945[9]
	MSE($\hat{\beta}$)	0.56392[5]	0.7543[13]	0.69846[9]	0.74207[12]	0.66928[7]	0.84187[15]	0.68905[8]	0.62999[6]	0.25309[2]	0.73522[10]	0.78489[14]	0.00524[1]	0.37739[3]	0.74173[11]	0.48398[4]
	MRE($\hat{\delta}$)	0.20892[7]	0.2392[12]	0.27796[14]	0.20231[5]	0.26429[13]	0.30748[15]	0.2374[11]	0.22204[9]	0.13957[2]	0.19798[4]	0.22697[10]	0.06972[1]	0.19427[3]	0.21668[8]	0.20861[6]
	MRE($\hat{\beta}$)	0.23695[5]	0.27411[13]	0.27344[12]	0.26919[9]	0.2703[10]	0.30006[15]	0.26531[8]	0.24948[6]	0.10926[2]	0.25805[7]	0.28032[14]	0.01315[1]	0.19573[3]	0.27191[11]	0.21052[4]
	D_{abs}	0.02341[2]	0.02377[4]	0.02389[5]	0.02292[1]	0.02502[8]	0.02535[10]	0.02347[3]	0.024[6]	0.02561[12]	0.02526[9]	0.02537[11]	0.02447[7]	0.03101[15]	0.02653[13]	0.03041[14]
	D_{max}	0.03833[3]	0.03924[5]	0.04063[9]	0.03709[2]	0.0422[11]	0.04366[13]	0.03885[4]	0.03946[6]	0.04025[7]	0.04062[8]	0.04131[10]	0.03704[1]	0.0492[15]	0.04265[12]	0.04866[14]
	$\sum Ranks$	40[3]	84[11]	89[13]	46[4]	85[12]	113[15]	64[9]	56[7]	31[2]	53[6]	93[14]	14[1]	50[5]	81[10]	61[8]
200	BIAS($\hat{\delta}$)	0.09648[3]	0.12482[10]	0.15103[14]	0.11384[6]	0.1425[13]	0.17269[15]	0.13156[12]	0.11484[7]	0.07786[2]	0.11322[5]	0.12492[11]	0.03359[1]	0.11157[4]	0.11946[8]	0.12256[9]
	BIAS($\hat{\beta}$)	0.42875[3]	0.5295[7]	0.61287[14]	0.54979[8]	0.59034[13]	0.63661[15]	0.55697[10]	0.50632[6]	0.25968[2]	0.55546[9]	0.58785[12]	0.02817[1]	0.437[4]	0.5818[11]	0.45147[5]
	MSE($\hat{\delta}$)	0.01468[3]	0.02419[10]	0.03546[14]	0.01995[5]	0.03181[13]	0.04731[15]	0.02608[11]	0.02091[6]	0.01268[2]	0.01992[4]	0.02357[9]	0.00198[1]	0.023[8]	0.02177[7]	0.02768[12]
	MSE($\hat{\beta}$)	0.31499[3]	0.46709[7]	0.61924[14]	0.53484[9]	0.58357[12]	0.65048[15]	0.5114[8]	0.44203[6]	0.2079[2]	0.55213[10]	0.56088[11]	0.00473[1]	0.34757[4]	0.58871[13]	0.37455[5]
	MRE($\hat{\delta}$)	0.13782[3]	0.17831[10]	0.21575[14]	0.16263[6]	0.20357[13]	0.2467[15]	0.18794[12]	0.16405[7]	0.11122[2]	0.16174[5]	0.17846[11]	0.04798[1]	0.15939[4]	0.17065[8]	0.17509[9]
	MRE($\hat{\beta}$)	0.1715[3]	0.2118[7]	0.24515[14]	0.21992[8]	0.23613[13]	0.25465[15]	0.22279[10]	0.20253[6]	0.10387[2]	0.22218[9]	0.23514[12]	0.01127[1]	0.1748[4]	0.23272[11]	0.18059[5]
	D_{abs}	0.01667[2]	0.0172[5]	0.01824[9]	0.01714[4]	0.01785[8]	0.01886[11]	0.01762[6]	0.01696[3]	0.01783[7]	0.02031[13]	0.01851[10]	0.01629[1]	0.02275[15]	0.01899[12]	0.02256[14]
	D_{max}	0.0269[2]	0.02854[6]	0.03108[11]	0.02791[4]	0.03005[8]	0.03272[13]	0.0292[7]	0.02788[3]	0.02821[5]	0.03259[12]	0.03021[9]	0.02461[1]	0.03658[15]	0.03087[10]	0.03649[14]
	$\sum Ranks$	22[2]	62[7]	104[14]	50[5]	93[13]	114[15]	76[10]	44[4]	24[3]	67[8]	85[12]	8[1]	58[6]	80[11]	73[9]
300	BIAS($\hat{\delta}$)	0.08469[3]	0.10079[10]	0.1252[14]	0.09076[4]	0.12392[13]	0.14567[15]	0.10608[12]	0.09495[6]	0.06793[2]	0.09395[5]	0.10567[11]	0.02746[1]	0.09748[7]	0.10026[8]	0.10057[9]
	BIAS($\hat{\beta}$)	0.36972[3]	0.43786[8]	0.50879[13]	0.43499[7]	0.53725[14]	0.57736[15]	0.45602[10]	0.42746[6]	0.24286[2]	0.44069[9]	0.5055[12]	0.0244[1]	0.38919[5]	0.46693[11]	0.3866[4]
	MSE($\hat{\delta}$)	0.01148[3]	0.01591[8]	0.02484[14]	0.01253[4]	0.02397[13]	0.03315[15]	0.01778[10]	0.01434[6]	0.00976[2]	0.01432[5]	0.01768[9]	0.00126[1]	0.01805[11]	0.01511[7]	0.01817[12]
	MSE($\hat{\beta}$)	0.24338[3]	0.33897[7]	0.44168[12]	0.32176[6]	0.51632[14]	0.54731[15]	0.35494[9]	0.34003[8]	0.17342[2]	0.36794[10]	0.44555[13]	0.00288[1]	0.29013[5]	0.37214[11]	0.26368[4]
	MRE($\hat{\delta}$)	0.12099[3]	0.14399[10]	0.17886[14]	0.12966[4]	0.17703[13]	0.2081[15]	0.15154[12]	0.13565[6]	0.09705[2]	0.13422[5]	0.15096[11]	0.03922[1]	0.13925[7]	0.14323[8]	0.14368[9]
	MRE($\hat{\beta}$)	0.14789[3]	0.17514[8]	0.20352[13]	0.174[7]	0.2149[14]	0.23094[15]	0.18241[10]	0.17099[6]	0.09715[2]	0.17628[9]	0.2022[12]	0.00976[1]	0.15568[5]	0.18677[11]	0.15464[4]
	D_{abs}	0.01342[1]	0.0141[5]	0.01471[7]	0.01402[4]	0.01492[8]	0.01569[12]	0.015[9]	0.01378[3]	0.01434[6]	0.01559[11]	0.01595[13]	0.01376[2]	0.01845[15]	0.01547[10]	0.0179[14]
	D_{max}	0.02182[2]	0.02339[6]	0.025[8]	0.0228[5]	0.02526[10]	0.02734[13]	0.02465[7]	0.02256[3]	0.02277[4]	0.02521[9]	0.02581[12]	0.02079[1]	0.03004[15]	0.02528[11]	0.029[14]
	$\sum Ranks$	21[2]	62[6]	95[13]	41[4]	99[14]	115[15]	79[11]	44[5]	22[3]	63[7]	93[12]	9[1]	70[8.5]	77[10]	70[8.5]
400	BIAS($\hat{\delta}$)	0.07174[3]	0.09293[11]	0.10943[13]	0.07801[4]	0.11145[14]	0.13305[15]	0.09175[10]	0.08463[6]	0.0625[2]	0.08834[8]	0.09467[12]	0.02457[1]	0.08264[5]	0.08637[7]	0.09147[9]
	BIAS($\hat{\beta}$)	0.31908[3]	0.40392[9]	0.47059[14]	0.35775[5]	0.46071[13]	0.53833[15]	0.40286[8]	0.38381[7]	0.2324[2]	0.4171[11]	0.45466[12]	0.02287[1]	0.33301[4]	0.40844[10]	0.38044[6]
	MSE($\hat{\delta}$)	0.00819[3]	0.01379[10]	0.01888[13]	0.00948[4]	0.01913[14]	0.02688[15]	0.01303[9]	0.01121[5]	0.00806[2]	0.01219[7]	0.01427[12]	0.00096[1]	0.01264[8]	0.01176[6]	0.01408[11]
	MSE($\hat{\beta}$)	0.17003[3]	0.27561[8]	0.39787[14]	0.22918[5]	0.3543[12]	0.49644[15]	0.28197[9]	0.25617[7]	0.15577[2]	0.30793[11]	0.38113[13]	0.00249[1]	0.21587[4]	0.30226[10]	0.24672[6]
	MRE($\hat{\delta}$)	0.10249[3]	0.13276[11]	0.15633[13]	0.11145[4]	0.15921[14]	0.19008[15]	0.13108[10]	0.1209[6]	0.08929[2]	0.1262[8]	0.13524[12]	0.0351[1]	0.11806[5]	0.12339[7]	0.13067[9]
	MRE($\hat{\beta}$)	0.12763[3]	0.16157[9]	0.18824[14]	0.1431[5]	0.18428[13]	0.21533[15]	0.16114[8]	0.15352[7]	0.09296[2]	0.16684[11]	0.18186[12]	0.00915[1]	0.1332[4]	0.16337[10]	0.15218[6]
	D_{abs}	0.01202[1]	0.01289[5]	0.01357[9]	0.01216[3]	0.01333[8]	0.01381[11]	0.01248[4]	0.01302[7]	0.01293[6]	0.01404[12]	0.01407[13]	0.01203[2]	0.01585[15]	0.01372[10]	0.01554[14]
	D_{max}	0.01947[2]	0.02129[7]	0.02291[12]	0.01985[3]	0.0227[9]	0.02414[13]	0.02063[5]	0.02113[6]	0.02061[4]	0.02288[10]	0.02289[11]	0.01819[1]	0.02568[15]	0.02227[8]	0.02537[14]
	$\sum Ranks$	21[2]	70[9]	102[14]	33[4]	97[12.5]	114[15]	63[7]	51[5]	22[3]	78[11]	97[12.5]	9[1]	60[6]	68[8]	75[10]

Table 3. Numerical values of simulation measures for $\delta = 0.25$ and $\beta = 0.75$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
30	BIAS($\hat{\delta}$)	0.03792[2]	0.04631[7]	0.05819[14]	0.04377[5]	0.05299[10]	0.06162[15]	0.04975[9]	0.0476[8]	0.03992[3]	0.04343[4]	0.05571[11.5]	0.0272[1]	0.05571[11.5]	0.04408[6]	0.05653[13]
	BIAS($\hat{\beta}$)	0.20078[3]	0.22619[6]	0.24363[12]	0.23866[10]	0.23765[9]	0.23982[11]	0.23231[7]	0.21986[5]	0.16668[2]	0.20972[4]	0.25183[14]	0.09301[1]	0.26527[15]	0.23274[8]	0.24674[13]
	MSE($\hat{\delta}$)	0.00219[2]	0.00339[7]	0.00555[14]	0.00291[4]	0.00433[10]	0.00612[15]	0.00397[9]	0.00366[8]	0.00287[3]	0.00308[5]	0.00528[13]	0.0014[1]	0.00479[11]	0.00312[6]	0.00498[12]
	MSE($\hat{\beta}$)	0.0644[3]	0.07884[6]	0.08607[10]	0.08968[12]	0.08623[11]	0.08533[8]	0.08135[7]	0.07377[4]	0.05803[2]	0.07457[5]	0.09151[14]	0.02381[1]	0.10724[15]	0.08546[9]	0.08994[13]
	MRE($\hat{\delta}$)	0.15167[2]	0.18523[7]	0.23277[14]	0.17509[5]	0.21197[10]	0.24649[15]	0.199[9]	0.19039[8]	0.15967[3]	0.1737[4]	0.22284[11.5]	0.10879[1]	0.22284[11.5]	0.17631[6]	0.22614[13]
	MRE($\hat{\beta}$)	0.26771[3]	0.30159[6]	0.32484[12]	0.31822[10]	0.31687[9]	0.31975[11]	0.30975[7]	0.29315[5]	0.22223[2]	0.27962[4]	0.33577[14]	0.12402[1]	0.35369[15]	0.31032[8]	0.32898[13]
	D_{abs}	0.03978[1]	0.04375[5]	0.04853[12]	0.04421[7]	0.04553[8]	0.04857[13]	0.04381[6]	0.04289[2]	0.04619[9]	0.04303[3]	0.04796[11]	0.0432[4]	0.05475[15]	0.04735[10]	0.05436[14]
	D_{max}	0.06553[1]	0.07131[6]	0.08238[12]	0.07085[4]	0.07553[10]	0.08325[13]	0.0722[7]	0.07117[5]	0.07504[8]	0.07035[3]	0.07991[11]	0.06891[2]	0.08996[15]	0.07547[9]	0.0896[14]
	$\sum Ranks$	17[2]	50[6]	100[11.5]	57[7]	77[10]	101[13]	61[8]	45[5]	32[3.5]	32[3.5]	100[11.5]	12[1]	109[15]	62[9]	105[14]
60	BIAS($\hat{\delta}$)	0.02673[2]	0.03656[8]	0.04072[12]	0.03364[4]	0.04024[11]	0.04471[14]	0.03721[9]	0.03459[6]	0.02974[3]	0.03401[5]	0.03937[10]	0.02163[1]	0.04522[15]	0.03572[7]	0.04219[13]
	BIAS($\hat{\beta}$)	0.15368[3]	0.1865[8]	0.19022[9]	0.18644[7]	0.20018[11]	0.20036[12]	0.18376[6]	0.17252[4]	0.13065[2]	0.17919[5]	0.20222[14]	0.08068[1]	0.22493[15]	0.19677[10]	0.20108[13]
	MSE($\hat{\delta}$)	0.00113[2]	0.00214[8]	0.00273[12]	0.00169[4]	0.00252[10]	0.00318[15]	0.00217[9]	0.00193[7]	0.0016[3]	0.00189[6]	0.00253[11]	0.00092[1]	0.00314[14]	0.00187[5]	0.00278[13]
	MSE($\hat{\beta}$)	0.04175[3]	0.05644[6]	0.05686[7]	0.05842[9]	0.06424[14]	0.06282[13]	0.05305[5]	0.04927[4]	0.03699[2]	0.05786[8]	0.06276[12]	0.01743[1]	0.07899[15]	0.06166[10]	0.06264[11]
	MRE($\hat{\delta}$)	0.10693[2]	0.14623[8]	0.16287[12]	0.13457[4]	0.16097[11]	0.17885[14]	0.14882[9]	0.13837[6]	0.11896[3]	0.13604[5]	0.15748[10]	0.08653[1]	0.18086[15]	0.14289[7]	0.16877[13]
	MRE($\hat{\beta}$)	0.20491[3]	0.24867[8]	0.25362[9]	0.24859[7]	0.2669[11]	0.26715[12]	0.24501[6]	0.23003[4]	0.17419[2]	0.23892[5]	0.26963[14]	0.10757[1]	0.2999[15]	0.26236[10]	0.2681[13]
	D_{abs}	0.02865[1]	0.03235[3]	0.03403[10]	0.03266[6]	0.0338[8]	0.03443[11]	0.03265[5]	0.03189[2]	0.03385[9]	0.03286[7]	0.03487[12]	0.03252[4]	0.04035[14]	0.03501[13]	0.04078[15]
	D_{max}	0.04675[1]	0.05334[5]	0.0571[11]	0.05293[4]	0.05628[9]	0.05832[13]	0.05371[7]	0.05256[3]	0.05479[8]	0.05366[6]	0.05734[12]	0.05239[2]	0.06677[14]	0.05649[10]	0.06719[15]
	$\sum Ranks$	17[2]	54[7]	82[10]	45[5]	85[11]	104[13]	56[8]	36[4]	32[3]	47[6]	95[12]	12[1]	117[15]	72[9]	106[14]
100	BIAS($\hat{\delta}$)	0.0231[2]	0.02843[6]	0.03137[10]	0.02728[4]	0.0316[11]	0.03625[15]	0.02813[5]	0.02848[7]	0.02475[3]	0.02932[9]	0.03253[12]	0.01602[1]	0.03265[13]	0.029[8]	0.03311[14]
	BIAS($\hat{\beta}$)	0.1315[3]	0.14273[5]	0.14892[7]	0.15193[8]	0.15411[9]	0.16605[14]	0.14096[4]	0.14515[6]	0.1185[2]	0.15562[10]	0.1731[15]	0.06434[1]	0.16197[12]	0.16008[11]	0.16502[13]
	MSE($\hat{\delta}$)	0.00086[2]	0.00126[5]	0.00157[11]	0.00112[4]	0.00156[10]	0.00208[15]	0.00127[6]	0.00128[7]	0.00107[3]	0.00136[9]	0.00166[12.5]	0.00051[1]	0.00166[12.5]	0.00129[8]	0.00177[14]
	MSE($\hat{\beta}$)	0.03144[3]	0.03254[4]	0.03644[7]	0.03847[8]	0.03959[9]	0.04527[14]	0.03344[5]	0.03458[6]	0.0301[2]	0.04335[12]	0.04773[15]	0.01114[1]	0.04165[10]	0.04281[11]	0.04409[13]
	MRE($\hat{\delta}$)	0.09238[2]	0.11374[6]	0.12547[10]	0.10911[4]	0.12639[11]	0.14502[15]	0.1125[5]	0.11391[7]	0.09899[3]	0.11729[9]	0.13012[12]	0.0641[1]	0.13062[13]	0.11599[8]	0.13244[14]
	MRE($\hat{\beta}$)	0.17533[3]	0.1903[5]	0.19856[7]	0.20257[8]	0.20548[9]	0.2214[14]	0.18795[4]	0.19354[6]	0.158[2]	0.20749[10]	0.2308[15]	0.08578[1]	0.21596[12]	0.21345[11]	0.22003[13]
	D_{abs}	0.02498[2]	0.02503[3]	0.02656[7]	0.02588[5]	0.02596[6]	0.02722[11]	0.02554[4]	0.02663[8]	0.02832[12]	0.02699[9]	0.02868[13]	0.02409[1]	0.03054[14]	0.02703[10]	0.03077[15]
	D_{max}	0.04055[2]	0.04137[3]	0.04456[10]	0.04216[5]	0.04346[6]	0.04638[12]	0.04211[4]	0.04374[7]	0.04578[11]	0.04428[9]	0.04702[13]	0.03867[1]	0.0502[14]	0.04408[8]	0.05067[15]
	$\sum Ranks$	19[2]	37[3.5]	69[8]	46[6]	71[9]	110[14]	37[3.5]	54[7]	38[5]	77[11]	107.5[13]	8[1]	100.5[12]	75[10]	111[15]
200	BIAS($\hat{\delta}$)	0.0173[2]	0.01938[7]	0.02284[11]	0.019[4]	0.02223[10]	0.02627[15]	0.02055[9]	0.01934[6]	0.01838[3]	0.01911[5]	0.02522[13]	0.01175[1]	0.02527[14]	0.01943[8]	0.02293[12]
	BIAS($\hat{\beta}$)	0.09413[3]	0.09871[5]	0.10837[11]	0.1018[7]	0.10854[12]	0.11908[13]	0.10176[6]	0.0986[4]	0.08773[2]	0.1022[8]	0.14097[15]	0.04869[1]	0.13008[14]	0.10478[9]	0.10707[10]
	MSE($\hat{\delta}$)	0.00047[2]	0.00059[5.5]	0.00082[11]	0.00057[3]	0.00078[10]	0.00108[15]	0.00065[9]	0.00058[4]	0.00061[7]	0.00062[8]	0.00098[13.5]	0.00029[1]	0.00098[13.5]	0.00059[5.5]	0.00091[12]
	MSE($\hat{\beta}$)	0.01471[2]	0.01633[5]	0.01863[9]	0.01774[7]	0.01979[11]	0.0231[13]	0.01716[6]	0.01534[3]	0.01628[4]	0.0195[10]	0.03311[15]	0.00672[1]	0.02761[14]	0.01843[8]	0.02024[12]
	MRE($\hat{\delta}$)	0.06921[2]	0.07753[7]	0.09134[11]	0.07601[4]	0.08891[10]	0.10508[15]	0.0822[9]	0.07736[6]	0.07351[3]	0.07643[5]	0.10088[13]	0.04701[1]	0.10108[14]	0.07774[8]	0.09172[12]
	MRE($\hat{\beta}$)	0.1255[3]	0.13161[5]	0.1445[11]	0.13574[7]	0.14471[12]	0.15878[13]	0.13568[6]	0.13146[4]	0.11697[2]	0.13626[8]	0.18796[15]	0.06491[1]	0.17344[14]	0.1397[9]	0.14275[10]
	D_{abs}	0.01803[3]	0.01795[2]	0.01849[6]	0.01827[4]	0.01852[7]	0.01943[9]	0.01842[5]	0.01853[8]	0.01976[12]	0.01957[10]	0.02149[13]	0.0176[1]	0.0237[15]	0.01974[11]	0.02287[14]
	D_{max}	0.02919[2]	0.02952[3]	0.0311[8]	0.02977[4]	0.03109[7]	0.03335[12]	0.0306[6]	0.03031[5]	0.03239[11]	0.03179[9]	0.03524[13]	0.02807[1]	0.03898[15]	0.03199[10]	0.03733[14]
	$\sum Ranks$	19[2]	39.5[3]	78[10]	40[4.5]	79[11]	105[13]	56[7]	40[4.5]	44[6]	63[8]	110.5[14]	8[1]	113.5[15]	68.5[9]	96[12]
300	BIAS($\hat{\delta}$)	0.01419[2]	0.01736[9]	0.01883[13]	0.01549[4]	0.01842[12]	0.02076[15]	0.01633[7]	0.01636[8]	0.01534[3]	0.01629[6]	0.01982[14]	0.00983[1]	0.01821[11]	0.01577[5]	0.01778[10]
	BIAS($\hat{\beta}$)	0.0731[2]	0.08699[9]	0.08814[11]	0.08227[4]	0.09008[12]	0.09593[14]	0.08242[5]	0.08312[6]	0.07557[3]	0.08758[10]	0.10944[15]	0.03688[1]	0.09063[13]	0.08574[8]	0.08526[7]
	MSE($\hat{\delta}$)	0.00031[2]	0.00048[9]	0.00056[12]	0.00038[3]	0.00054[10.5]	0.00067[15]	0.00044[8]	0.00041[5.5]	0.00041[5.5]	0.00043[7]	0.00062[14]	$2e-04$[1]	0.00057[13]	$4e-04$[4]	0.00054[10.5]
	MSE($\hat{\beta}$)	0.00848[2]	0.01219[8]	0.01264[10]	0.01114[4]	0.01321[11]	0.01472[14]	0.01129[5]	0.0111[3]	0.01187[6]	0.01322[12]	0.02008[15]	0.00412[1]	0.01441[13]	0.01218[7]	0.01225[9]
	MRE($\hat{\delta}$)	0.05677[2]	0.06944[9]	0.07531[13]	0.06196[4]	0.07367[12]	0.08303[15]	0.06531[7]	0.06545[8]	0.06137[3]	0.06515[6]	0.07927[14]	0.03933[1]	0.07285[11]	0.06308[5]	0.07112[10]
	MRE($\hat{\beta}$)	0.09747[2]	0.11599[9]	0.11753[11]	0.10969[4]	0.12011[12]	0.12791[14]	0.1099[5]	0.11082[6]	0.10076[3]	0.11678[10]	0.14592[15]	0.04918[1]	0.12084[13]	0.11432[8]	0.11368[7]
	D_{abs}	0.01447[2]	0.01529[7]	0.01535[8]	0.01484[3]	0.01519[6]	0.01601[9]	0.01493[4]	0.01504[5]	0.01637[11]	0.01677[12]	0.01742[13]	0.01433[1]	0.01906[15]	0.0163[10]	0.0188[14]
	D_{max}	0.02349[2]	0.02522[6]	0.02584[8]	0.02421[3]	0.02533[7]	0.02724[12]	0.02471[4]	0.02483[5]	0.02669[10]	0.02712[11]	0.02847[13]	0.02302[1]	0.03095[15]	0.02641[9]	0.0306[14]
	$\sum Ranks$	15[1]	64[7]	84[12]	28[3]	80.5[11]	106[14]	43[5]	44.5[6]	42.5[4]	72[9]	111[15]	21[2]	102[13]	67[8]	79.5[10]
400	BIAS($\hat{\delta}$)	0.0123[2]	0.01445[7]	0.01582[12]	0.01337[4]	0.01565[11]	0.01778[14]	0.01429[6]	0.01336[3]	0.01397[5]	0.01482[9]	0.01814[15]	0.0086[1]	0.01606[13]	0.01477[8]	0.01543[10]
	BIAS($\hat{\beta}$)	0.06611[2]	0.07122[6]	0.07645[9]	0.07072[5]	0.07554[8]	0.08205[14]	0.07194[7]	0.06821[3]	0.07008[4]	0.07952[13]	0.10105[15]	0.03501[1]	0.07876[10]	0.07925[12]	0.07879[11]
	MSE($\hat{\delta}$)	0.00023[2]	0.00032[6]	$4e-04$[12]	0.00027[3.5]	0.00039[11]	0.00049[14]	0.00032[6]	0.00027[3.5]	0.00035[8]	0.00036[9]	$5e-04$[15]	0.00016[1]	0.00044[13]	0.00032[6]	0.00037[10]
	MSE($\hat{\beta}$)	0.00715[2]	0.00794[5]	0.00928[8]	0.00783[4]	0.00904[7]	0.01058[12]	0.00843[6]	0.00727[3]	0.00986[11]	0.011[14]	0.0167[15]	0.00343[1]	0.01096[13]	0.00975[10]	0.00967[9]
	MRE($\hat{\delta}$)	0.0492[2]	0.05779[7]	0.06329[12]	0.05346[4]	0.06258[11]	0.07111[14]	0.05715[6]	0.05345[3]	0.05587[5]	0.05929[9]	0.07256[15]	0.0344[1]	0.06425[13]	0.0591[8]	0.06173[10]
	MRE($\hat{\beta}$)	0.08815[2]	0.09495[6]	0.10194[9]	0.0943[5]	0.10072[8]	0.1094[14]	0.09591[7]	0.09095[3]	0.09344[4]	0.10603[13]	0.13473[15]	0.04668[1]	0.10501[10]	0.10567[12]	0.10505[11]
	D_{abs}	0.01235[2]	0.01243[3]	0.0136[8]	0.01295[6]	0.01324[7]	0.0137[9]	0.01286[5]	0.01258[4]	0.01459[11]	0.01484[12]	0.01561[13]	0.0121[1]	0.01697[15]	0.01421[10]	0.01637[14]
	D_{max}	0.02003[2]	0.02073[4]	0.02267[8]	0.0211[5]	0.02216[7]	0.02335[10]	0.02123[6]	0.02064[3]	0.02367[11]	0.02415[12]	0.02563[13]	0.01949[1]	0.02748[15]	0.02321[9]	0.02659[14]
	$\sum Ranks$	16[2]	44[5]	80[10]	36.5[4]	70[8]	100[13]	49[6]	25.5[3]	59[7]	91[12]	116[15]	8[1]	101[14]	75[9]	89[11]

Table 4. Numerical values of simulation measures for $\delta = 1.5$ and $\beta = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
30	BIAS($\hat{\delta}$)	0.36362$^{\{4\}}$	0.41591$^{\{10\}}$	0.46651$^{\{14\}}$	0.3611$^{\{3\}}$	0.43973$^{\{13\}}$	0.47679$^{\{15\}}$	0.42017$^{\{11\}}$	0.3771$^{\{7\}}$	0.31372$^{\{2\}}$	0.37161$^{\{6\}}$	0.42351$^{\{12\}}$	0.18844$^{\{1\}}$	0.41564$^{\{9\}}$	0.3691$^{\{5\}}$	0.40653$^{\{8\}}$
	BIAS($\hat{\beta}$)	0.44791$^{\{3\}}$	0.47888$^{\{6\}}$	0.50494$^{\{11\}}$	0.48893$^{\{8\}}$	0.50806$^{\{12\}}$	0.49815$^{\{10\}}$	0.50822$^{\{13\}}$	0.46815$^{\{4\}}$	0.37554$^{\{2\}}$	0.47535$^{\{5\}}$	0.51634$^{\{15\}}$	0.26964$^{\{1\}}$	0.51083$^{\{14\}}$	0.48372$^{\{7\}}$	0.49761$^{\{9\}}$
	MSE($\hat{\delta}$)	0.21037$^{\{4\}}$	0.26991$^{\{10\}}$	0.33182$^{\{14\}}$	0.20202$^{\{3\}}$	0.29159$^{\{13\}}$	0.35861$^{\{15\}}$	0.27624$^{\{11\}}$	0.22664$^{\{7\}}$	0.17428$^{\{2\}}$	0.21641$^{\{5\}}$	0.2846$^{\{12\}}$	0.05936$^{\{1\}}$	0.26835$^{\{9\}}$	0.21794$^{\{6\}}$	0.25878$^{\{8\}}$
	MSE($\hat{\beta}$)	0.29849$^{\{3\}}$	0.33823$^{\{5\}}$	0.36564$^{\{11\}}$	0.36137$^{\{10\}}$	0.37711$^{\{13\}}$	0.3567$^{\{8\}}$	0.3741$^{\{12\}}$	0.32343$^{\{4\}}$	0.25591$^{\{2\}}$	0.35154$^{\{7\}}$	0.38344$^{\{15\}}$	0.12377$^{\{1\}}$	0.38059$^{\{14\}}$	0.35106$^{\{6\}}$	0.35683$^{\{9\}}$
	MRE($\hat{\delta}$)	0.24241$^{\{4\}}$	0.27727$^{\{10\}}$	0.31101$^{\{14\}}$	0.24073$^{\{3\}}$	0.29315$^{\{13\}}$	0.31786$^{\{15\}}$	0.28011$^{\{11\}}$	0.2514$^{\{7\}}$	0.20915$^{\{2\}}$	0.24774$^{\{6\}}$	0.28234$^{\{12\}}$	0.12563$^{\{1\}}$	0.27709$^{\{9\}}$	0.24606$^{\{5\}}$	0.27102$^{\{8\}}$
	MRE($\hat{\beta}$)	0.29861$^{\{3\}}$	0.31926$^{\{6\}}$	0.33662$^{\{11\}}$	0.32595$^{\{8\}}$	0.33871$^{\{12\}}$	0.3321$^{\{10\}}$	0.33882$^{\{13\}}$	0.3121$^{\{4\}}$	0.25036$^{\{2\}}$	0.3169$^{\{5\}}$	0.34423$^{\{15\}}$	0.17976$^{\{1\}}$	0.34056$^{\{14\}}$	0.32248$^{\{7\}}$	0.33174$^{\{9\}}$
	D_{abs}	0.03944$^{\{1\}}$	0.04453$^{\{8\}}$	0.04527$^{\{10\}}$	0.04151$^{\{3\}}$	0.04361$^{\{5\}}$	0.04561$^{\{12\}}$	0.04232$^{\{4\}}$	0.04094$^{\{2\}}$	0.04552$^{\{11\}}$	0.04852$^{\{13\}}$	0.04426$^{\{7\}}$	0.04386$^{\{6\}}$	0.05483$^{\{15\}}$	0.04475$^{\{9\}}$	0.05458$^{\{14\}}$
	D_{max}	0.06565$^{\{1\}}$	0.07283$^{\{8\}}$	0.07644$^{\{11\}}$	0.06681$^{\{2\}}$	0.0725$^{\{7\}}$	0.07674$^{\{12\}}$	0.06966$^{\{5\}}$	0.06724$^{\{3\}}$	0.07288$^{\{9\}}$	0.07755$^{\{13\}}$	0.07318$^{\{10\}}$	0.06774$^{\{4\}}$	0.0875$^{\{15\}}$	0.07206$^{\{6\}}$	0.08689$^{\{14\}}$
	$\sum Ranks$	23$^{\{2\}}$	63$^{\{8\}}$	96$^{\{12\}}$	40$^{\{5\}}$	88$^{\{11\}}$	97$^{\{13\}}$	80$^{\{10\}}$	38$^{\{4\}}$	32$^{\{3\}}$	60$^{\{7\}}$	98$^{\{14\}}$	16$^{\{1\}}$	99$^{\{15\}}$	51$^{\{6\}}$	79$^{\{9\}}$
60	BIAS($\hat{\delta}$)	0.25193$^{\{2\}}$	0.31471$^{\{8\}}$	0.35735$^{\{14\}}$	0.27976$^{\{4\}}$	0.34812$^{\{12\}}$	0.40958$^{\{15\}}$	0.32601$^{\{10\}}$	0.28841$^{\{5\}}$	0.26064$^{\{3\}}$	0.29235$^{\{7\}}$	0.32535$^{\{9\}}$	0.17636$^{\{1\}}$	0.35405$^{\{13\}}$	0.28887$^{\{6\}}$	0.34765$^{\{11\}}$
	BIAS($\hat{\beta}$)	0.34938$^{\{2\}}$	0.40316$^{\{5\}}$	0.41962$^{\{7\}}$	0.412$^{\{6\}}$	0.42457$^{\{9\}}$	0.45051$^{\{14\}}$	0.42184$^{\{8\}}$	0.38627$^{\{4\}}$	0.34945$^{\{3\}}$	0.42834$^{\{10\}}$	0.44339$^{\{12\}}$	0.24504$^{\{1\}}$	0.46935$^{\{15\}}$	0.43185$^{\{11\}}$	0.44997$^{\{13\}}$
	MSE($\hat{\delta}$)	0.10131$^{\{2\}}$	0.14878$^{\{8\}}$	0.20227$^{\{14\}}$	0.12128$^{\{4\}}$	0.19801$^{\{13\}}$	0.25959$^{\{15\}}$	0.16855$^{\{10\}}$	0.12991$^{\{5\}}$	0.12121$^{\{3\}}$	0.13012$^{\{6\}}$	0.1646$^{\{9\}}$	0.05318$^{\{1\}}$	0.19114$^{\{11\}}$	0.13018$^{\{7\}}$	0.1939$^{\{12\}}$
	MSE($\hat{\beta}$)	0.19887$^{\{2\}}$	0.24543$^{\{5\}}$	0.2617$^{\{6\}}$	0.28333$^{\{9\}}$	0.28214$^{\{8\}}$	0.30154$^{\{13\}}$	0.27339$^{\{7\}}$	0.23646$^{\{4\}}$	0.23071$^{\{3\}}$	0.29922$^{\{11\}}$	0.30099$^{\{12\}}$	0.1013$^{\{1\}}$	0.3262$^{\{15\}}$	0.29724$^{\{10\}}$	0.31345$^{\{14\}}$
	MRE($\hat{\delta}$)	0.16795$^{\{2\}}$	0.20981$^{\{8\}}$	0.23823$^{\{14\}}$	0.18651$^{\{4\}}$	0.23208$^{\{12\}}$	0.27306$^{\{15\}}$	0.21734$^{\{10\}}$	0.19227$^{\{5\}}$	0.17376$^{\{3\}}$	0.1949$^{\{7\}}$	0.2169$^{\{9\}}$	0.11757$^{\{1\}}$	0.23603$^{\{13\}}$	0.19258$^{\{6\}}$	0.23177$^{\{11\}}$
	MRE($\hat{\beta}$)	0.23292$^{\{2\}}$	0.26877$^{\{5\}}$	0.27975$^{\{7\}}$	0.27467$^{\{6\}}$	0.28305$^{\{9\}}$	0.30034$^{\{14\}}$	0.28122$^{\{8\}}$	0.25751$^{\{4\}}$	0.23297$^{\{3\}}$	0.28556$^{\{10\}}$	0.2956$^{\{12\}}$	0.16336$^{\{1\}}$	0.3129$^{\{15\}}$	0.2879$^{\{11\}}$	0.29998$^{\{13\}}$
	D_{abs}	0.02961$^{\{1\}}$	0.03051$^{\{2\}}$	0.03132$^{\{6\}}$	0.03103$^{\{5\}}$	0.03319$^{\{9\}}$	0.03268$^{\{7\}}$	0.03101$^{\{4\}}$	0.03063$^{\{3\}}$	0.03576$^{\{13\}}$	0.03372$^{\{11\}}$	0.03366$^{\{10\}}$	0.03275$^{\{8\}}$	0.03914$^{\{14\}}$	0.03378$^{\{12\}}$	0.03955$^{\{15\}}$
	D_{max}	0.04842$^{\{1\}}$	0.05047$^{\{4\}}$	0.05302$^{\{7\}}$	0.04997$^{\{2\}}$	0.05541$^{\{11\}}$	0.05606$^{\{12\}}$	0.05152$^{\{6\}}$	0.05036$^{\{3\}}$	0.0573$^{\{13\}}$	0.05469$^{\{9\}}$	0.05517$^{\{10\}}$	0.0514$^{\{5\}}$	0.06451$^{\{14\}}$	0.05445$^{\{8\}}$	0.06483$^{\{15\}}$
	$\sum Ranks$	14$^{\{1\}}$	45$^{\{6\}}$	75$^{\{10\}}$	40$^{\{4\}}$	83$^{\{11.5\}}$	105$^{\{14\}}$	63$^{\{7\}}$	33$^{\{3\}}$	44$^{\{5\}}$	71$^{\{8.5\}}$	83$^{\{11.5\}}$	19$^{\{2\}}$	110$^{\{15\}}$	71$^{\{8.5\}}$	104$^{\{13\}}$
100	BIAS($\hat{\delta}$)	0.20656$^{\{2\}}$	0.25391$^{\{8\}}$	0.30172$^{\{13\}}$	0.23433$^{\{5\}}$	0.29934$^{\{12\}}$	0.327$^{\{15\}}$	0.25672$^{\{9\}}$	0.23765$^{\{6\}}$	0.2224$^{\{3\}}$	0.24672$^{\{7\}}$	0.26161$^{\{10\}}$	0.16346$^{\{1\}}$	0.30254$^{\{14\}}$	0.23314$^{\{4\}}$	0.29396$^{\{11\}}$
	BIAS($\hat{\beta}$)	0.28605$^{\{2\}}$	0.34848$^{\{6\}}$	0.37222$^{\{10\}}$	0.35822$^{\{8\}}$	0.38959$^{\{13\}}$	0.38082$^{\{12\}}$	0.34956$^{\{7\}}$	0.32677$^{\{4\}}$	0.30141$^{\{3\}}$	0.36475$^{\{9\}}$	0.37444$^{\{11\}}$	0.22382$^{\{1\}}$	0.40691$^{\{15\}}$	0.34202$^{\{5\}}$	0.40473$^{\{14\}}$
	MSE($\hat{\delta}$)	0.06736$^{\{2\}}$	0.10079$^{\{8\}}$	0.14079$^{\{14\}}$	0.08235$^{\{3\}}$	0.13993$^{\{13\}}$	0.17078$^{\{15\}}$	0.10453$^{\{9\}}$	0.09235$^{\{6\}}$	0.08912$^{\{5\}}$	0.09267$^{\{7\}}$	0.10813$^{\{10\}}$	0.0469$^{\{1\}}$	0.13908$^{\{12\}}$	0.08462$^{\{4\}}$	0.13209$^{\{11\}}$
	MSE($\hat{\beta}$)	0.14544$^{\{2\}}$	0.19976$^{\{6\}}$	0.22305$^{\{9\}}$	0.21696$^{\{8\}}$	0.23768$^{\{13\}}$	0.23128$^{\{12\}}$	0.20607$^{\{7\}}$	0.18067$^{\{4\}}$	0.1778$^{\{3\}}$	0.22367$^{\{10\}}$	0.22523$^{\{11\}}$	0.08729$^{\{1\}}$	0.26214$^{\{15\}}$	0.1935$^{\{5\}}$	0.25864$^{\{14\}}$
	MRE($\hat{\delta}$)	0.13771$^{\{2\}}$	0.16927$^{\{8\}}$	0.20115$^{\{13\}}$	0.15622$^{\{5\}}$	0.19956$^{\{12\}}$	0.218$^{\{15\}}$	0.17115$^{\{9\}}$	0.15844$^{\{6\}}$	0.14827$^{\{3\}}$	0.16448$^{\{7\}}$	0.17441$^{\{10\}}$	0.10897$^{\{1\}}$	0.2017$^{\{14\}}$	0.15543$^{\{4\}}$	0.19598$^{\{11\}}$
	MRE($\hat{\beta}$)	0.1907$^{\{2\}}$	0.23232$^{\{6\}}$	0.24815$^{\{10\}}$	0.23882$^{\{8\}}$	0.25972$^{\{13\}}$	0.25388$^{\{12\}}$	0.23304$^{\{7\}}$	0.21785$^{\{4\}}$	0.20094$^{\{3\}}$	0.24317$^{\{9\}}$	0.24963$^{\{11\}}$	0.14921$^{\{1\}}$	0.27127$^{\{15\}}$	0.22801$^{\{5\}}$	0.26982$^{\{14\}}$
	D_{abs}	0.02235$^{\{1\}}$	0.02482$^{\{3\}}$	0.02569$^{\{6\}}$	0.02462$^{\{2\}}$	0.02622$^{\{10\}}$	0.02597$^{\{8\}}$	0.0254$^{\{5\}}$	0.02484$^{\{4\}}$	0.02732$^{\{13\}}$	0.02672$^{\{12\}}$	0.02663$^{\{11\}}$	0.02587$^{\{7\}}$	0.03105$^{\{14\}}$	0.02607$^{\{9\}}$	0.03134$^{\{15\}}$
	D_{max}	0.03675$^{\{1\}}$	0.04106$^{\{4\}}$	0.04372$^{\{10\}}$	0.04$^{\{2\}}$	0.04399$^{\{12\}}$	0.04501$^{\{13\}}$	0.04203$^{\{6\}}$	0.04065$^{\{3\}}$	0.04393$^{\{11\}}$	0.04357$^{\{8\}}$	0.04363$^{\{9\}}$	0.0414$^{\{5\}}$	0.05119$^{\{14\}}$	0.04208$^{\{7\}}$	0.05143$^{\{15\}}$
	$\sum Ranks$	14$^{\{1\}}$	49$^{\{7\}}$	85$^{\{11\}}$	41$^{\{4\}}$	98$^{\{12\}}$	102$^{\{13\}}$	59$^{\{8\}}$	37$^{\{3\}}$	44$^{\{6\}}$	69$^{\{9\}}$	83$^{\{10\}}$	18$^{\{2\}}$	113$^{\{15\}}$	43$^{\{5\}}$	105$^{\{14\}}$
200	BIAS($\hat{\delta}$)	0.15288$^{\{2\}}$	0.18956$^{\{8\}}$	0.21534$^{\{11\}}$	0.16331$^{\{3\}}$	0.21778$^{\{12\}}$	0.26177$^{\{15\}}$	0.1897$^{\{9\}}$	0.1732$^{\{4\}}$	0.17595$^{\{5\}}$	0.18315$^{\{7\}}$	0.20027$^{\{10\}}$	0.13041$^{\{1\}}$	0.2255$^{\{14\}}$	0.18271$^{\{6\}}$	0.22008$^{\{13\}}$
	BIAS($\hat{\beta}$)	0.2161$^{\{2\}}$	0.25397$^{\{6\}}$	0.27965$^{\{10\}}$	0.24415$^{\{4\}}$	0.28824$^{\{11\}}$	0.32319$^{\{15\}}$	0.25894$^{\{7\}}$	0.23866$^{\{3\}}$	0.24499$^{\{5\}}$	0.27244$^{\{9\}}$	0.29863$^{\{12\}}$	0.1761$^{\{1\}}$	0.32229$^{\{14\}}$	0.26751$^{\{8\}}$	0.31447$^{\{13\}}$
	MSE($\hat{\delta}$)	0.03871$^{\{2\}}$	0.05626$^{\{8\}}$	0.07407$^{\{12\}}$	0.0409$^{\{3\}}$	0.07383$^{\{11\}}$	0.10743$^{\{15\}}$	0.05719$^{\{9\}}$	0.04689$^{\{4\}}$	0.0535$^{\{7\}}$	0.05235$^{\{6\}}$	0.06256$^{\{10\}}$	0.02939$^{\{1\}}$	0.07872$^{\{14\}}$	0.05169$^{\{5\}}$	0.07812$^{\{13\}}$
	MSE($\hat{\beta}$)	0.08188$^{\{2\}}$	0.11144$^{\{5\}}$	0.13504$^{\{10\}}$	0.10336$^{\{4\}}$	0.14447$^{\{11\}}$	0.17092$^{\{14\}}$	0.11355$^{\{6\}}$	0.09586$^{\{3\}}$	0.11854$^{\{7\}}$	0.13004$^{\{9\}}$	0.15599$^{\{12\}}$	0.0546$^{\{1\}}$	0.17622$^{\{15\}}$	0.12213$^{\{8\}}$	0.16875$^{\{13\}}$
	MRE($\hat{\delta}$)	0.10192$^{\{2\}}$	0.12638$^{\{8\}}$	0.14356$^{\{11\}}$	0.10888$^{\{3\}}$	0.14519$^{\{12\}}$	0.17452$^{\{15\}}$	0.12646$^{\{9\}}$	0.11547$^{\{4\}}$	0.1173$^{\{5\}}$	0.1221$^{\{7\}}$	0.13351$^{\{10\}}$	0.08694$^{\{1\}}$	0.15034$^{\{14\}}$	0.1218$^{\{6\}}$	0.14672$^{\{13\}}$
	MRE($\hat{\beta}$)	0.14407$^{\{2\}}$	0.16931$^{\{6\}}$	0.18643$^{\{10\}}$	0.16277$^{\{4\}}$	0.19216$^{\{11\}}$	0.21546$^{\{15\}}$	0.17263$^{\{7\}}$	0.1591$^{\{3\}}$	0.16333$^{\{5\}}$	0.18163$^{\{9\}}$	0.19909$^{\{12\}}$	0.1174$^{\{1\}}$	0.21486$^{\{14\}}$	0.17834$^{\{8\}}$	0.20965$^{\{13\}}$
	D_{abs}	0.01708$^{\{1\}}$	0.01778$^{\{4\}}$	0.01841$^{\{7\}}$	0.01749$^{\{2.5\}}$	0.01868$^{\{8\}}$	0.01953$^{\{9\}}$	0.01788$^{\{6\}}$	0.01749$^{\{2.5\}}$	0.02059$^{\{13\}}$	0.02042$^{\{12\}}$	0.02035$^{\{11\}}$	0.01781$^{\{5\}}$	0.02273$^{\{14\}}$	0.01969$^{\{10\}}$	0.02289$^{\{15\}}$
	D_{max}	0.02783$^{\{1\}}$	0.02966$^{\{5\}}$	0.03114$^{\{7\}}$	0.02835$^{\{2\}}$	0.03144$^{\{8\}}$	0.03378$^{\{13\}}$	0.02968$^{\{6\}}$	0.02876$^{\{4\}}$	0.03344$^{\{12\}}$	0.03311$^{\{10\}}$	0.03321$^{\{11\}}$	0.02872$^{\{3\}}$	0.03777$^{\{15\}}$	0.03208$^{\{9\}}$	0.03745$^{\{14\}}$
	$\sum Ranks$	14$^{\{1.5\}}$	50$^{\{5\}}$	78$^{\{10\}}$	25.5$^{\{3\}}$	84$^{\{11\}}$	111$^{\{14\}}$	59$^{\{6.5\}}$	27.5$^{\{4\}}$	59$^{\{6.5\}}$	69$^{\{9\}}$	88$^{\{12\}}$	14$^{\{1.5\}}$	114$^{\{15\}}$	60$^{\{8\}}$	107$^{\{13\}}$
300	BIAS($\hat{\delta}$)	0.12756$^{\{2\}}$	0.14717$^{\{6\}}$	0.17874$^{\{12\}}$	0.13713$^{\{3\}}$	0.17341$^{\{11\}}$	0.21053$^{\{15\}}$	0.16111$^{\{9\}}$	0.13772$^{\{4\}}$	0.1435$^{\{5\}}$	0.15587$^{\{8\}}$	0.16547$^{\{10\}}$	0.11423$^{\{1\}}$	0.19156$^{\{14\}}$	0.14743$^{\{7\}}$	0.18039$^{\{13\}}$
	BIAS($\hat{\beta}$)	0.17764$^{\{2\}}$	0.19923$^{\{4\}}$	0.22686$^{\{9\}}$	0.20548$^{\{6\}}$	0.22944$^{\{10\}}$	0.26697$^{\{13\}}$	0.21258$^{\{7\}}$	0.18859$^{\{3\}}$	0.20381$^{\{5\}}$	0.23225$^{\{11\}}$	0.25014$^{\{12\}}$	0.14872$^{\{1\}}$	0.27192$^{\{15\}}$	0.21783$^{\{8\}}$	0.2676$^{\{14\}}$
	MSE($\hat{\delta}$)	0.02609$^{\{2\}}$	0.03343$^{\{5\}}$	0.04979$^{\{12\}}$	0.0295$^{\{3\}}$	0.0464$^{\{11\}}$	0.06938$^{\{15\}}$	0.04051$^{\{9\}}$	0.03057$^{\{4\}}$	0.03482$^{\{7\}}$	0.03819$^{\{8\}}$	0.0425$^{\{10\}}$	0.02264$^{\{1\}}$	0.05705$^{\{14\}}$	0.03431$^{\{6\}}$	0.05061$^{\{13\}}$
	MSE($\hat{\beta}$)	0.05098$^{\{2\}}$	0.06578$^{\{4\}}$	0.09046$^{\{10\}}$	0.0717$^{\{5\}}$	0.08987$^{\{9\}}$	0.12315$^{\{13\}}$	0.07547$^{\{6\}}$	0.06013$^{\{3\}}$	0.07747$^{\{7\}}$	0.09348$^{\{11\}}$	0.10805$^{\{12\}}$	0.04003$^{\{1\}}$	0.12352$^{\{14\}}$	0.08226$^{\{8\}}$	0.1243$^{\{15\}}$
	MRE($\hat{\delta}$)	0.08504$^{\{2\}}$	0.09811$^{\{6\}}$	0.11916$^{\{12\}}$	0.09142$^{\{3\}}$	0.1156$^{\{11\}}$	0.14035$^{\{15\}}$	0.10741$^{\{9\}}$	0.09181$^{\{4\}}$	0.09567$^{\{5\}}$	0.10392$^{\{8\}}$	0.11031$^{\{10\}}$	0.07615$^{\{1\}}$	0.1277$^{\{14\}}$	0.09829$^{\{7\}}$	0.12026$^{\{13\}}$
	MRE($\hat{\beta}$)	0.11842$^{\{2\}}$	0.13282$^{\{4\}}$	0.15124$^{\{9\}}$	0.13699$^{\{6\}}$	0.15296$^{\{10\}}$	0.17798$^{\{13\}}$	0.14172$^{\{7\}}$	0.12573$^{\{3\}}$	0.13587$^{\{5\}}$	0.15484$^{\{11\}}$	0.16676$^{\{12\}}$	0.09915$^{\{1\}}$	0.18128$^{\{15\}}$	0.14522$^{\{8\}}$	0.1784$^{\{14\}}$
	D_{abs}	0.01393$^{\{1\}}$	0.01439$^{\{3\}}$	0.01544$^{\{8\}}$	0.01449$^{\{4\}}$	0.01487$^{\{6\}}$	0.01613$^{\{9\}}$	0.0152$^{\{7\}}$	0.01395$^{\{2\}}$	0.0174$^{\{13\}}$	0.01678$^{\{12\}}$	0.01636$^{\{11\}}$	0.01461$^{\{5\}}$	0.01959$^{\{15\}}$	0.0163$^{\{10\}}$	0.01954$^{\{14\}}$
	D_{max}	0.02275$^{\{1\}}$	0.02384$^{\{5\}}$	0.02607$^{\{8\}}$	0.02357$^{\{3\}}$	0.02515$^{\{6\}}$	0.0278$^{\{12\}}$	0.02531$^{\{7\}}$	0.02298$^{\{2\}}$	0.02812$^{\{13\}}$	0.02723$^{\{11\}}$	0.02671$^{\{10\}}$	0.02372$^{\{4\}}$	0.03222$^{\{15\}}$	0.02651$^{\{9\}}$	0.03184$^{\{14\}}$
	$\sum Ranks$	14$^{\{1\}}$	37$^{\{5\}}$	80$^{\{10.5\}}$	33$^{\{4\}}$	74$^{\{9\}}$	105$^{\{13\}}$	61$^{\{7\}}$	25$^{\{3\}}$	60$^{\{6\}}$	80$^{\{10.5\}}$	87$^{\{12\}}$	15$^{\{2\}}$	116$^{\{15\}}$	63$^{\{8\}}$	110$^{\{14\}}$
400	BIAS($\hat{\delta}$)	0.10866$^{\{2\}}$	0.1345$^{\{9\}}$	0.15044$^{\{11\}}$	0.11647$^{\{3\}}$	0.15668$^{\{13\}}$	0.18648$^{\{15\}}$	0.13412$^{\{8\}}$	0.11906$^{\{4\}}$	0.13407$^{\{7\}}$	0.13152$^{\{6\}}$	0.1481$^{\{10\}}$	0.10545$^{\{1\}}$	0.16093$^{\{14\}}$	0.12007$^{\{5\}}$	0.15558$^{\{12\}}$
	BIAS($\hat{\beta}$)	0.15352$^{\{2\}}$	0.18616$^{\{8\}}$	0.19179$^{\{9\}}$	0.16571$^{\{4\}}$	0.20261$^{\{11\}}$	0.23735$^{\{15\}}$	0.18354$^{\{6\}}$	0.16043$^{\{3\}}$	0.18538$^{\{7\}}$	0.19245$^{\{10\}}$	0.22086$^{\{12\}}$	0.1359$^{\{1\}}$	0.2328$^{\{14\}}$	0.17147$^{\{5\}}$	0.22366$^{\{13\}}$
	MSE($\hat{\delta}$)	0.01904$^{\{1\}}$	0.02909$^{\{8\}}$	0.0361$^{\{11\}}$	0.02075$^{\{3\}}$	0.03897$^{\{12\}}$	0.05496$^{\{15\}}$	0.02796$^{\{7\}}$	0.02201$^{\{4\}}$	0.03078$^{\{9\}}$	0.02632$^{\{6\}}$	0.03496$^{\{10\}}$	0.01912$^{\{2\}}$	0.0414$^{\{14\}}$	0.02291$^{\{5\}}$	0.03957$^{\{13\}}$
	MSE($\hat{\beta}$)	0.03755$^{\{2\}}$	0.05973$^{\{7\}}$	0.06171$^{\{9\}}$	0.04387$^{\{4\}}$	0.07077$^{\{11\}}$	0.09576$^{\{15\}}$	0.05488$^{\{6\}}$	0.04111$^{\{3\}}$	0.06503$^{\{10\}}$	0.06059$^{\{8\}}$	0.08705$^{\{13\}}$	0.03291$^{\{1\}}$	0.09212$^{\{14\}}$	0.04832$^{\{5\}}$	0.08653$^{\{12\}}$
	MRE($\hat{\delta}$)	0.07244$^{\{2\}}$	0.08967$^{\{9\}}$	0.10029$^{\{11\}}$	0.07765$^{\{3\}}$	0.10445$^{\{13\}}$	0.12432$^{\{15\}}$	0.08942$^{\{8\}}$	0.07938$^{\{4\}}$	0.08938$^{\{7\}}$	0.08768$^{\{6\}}$	0.09874$^{\{10\}}$	0.0703$^{\{1\}}$	0.10729$^{\{14\}}$	0.08004$^{\{5\}}$	0.10372$^{\{12\}}$
	MRE($\hat{\beta}$)	0.10235$^{\{2\}}$	0.1241$^{\{8\}}$	0.12786$^{\{9\}}$	0.11048$^{\{4\}}$	0.13507$^{\{11\}}$	0.15823$^{\{15\}}$	0.12236$^{\{6\}}$	0.10695$^{\{3\}}$	0.12359$^{\{7\}}$	0.1283$^{\{10\}}$	0.14724$^{\{12\}}$	0.0906$^{\{1\}}$	0.1552$^{\{14\}}$	0.11431$^{\{5\}}$	0.1491$^{\{13\}}$
	D_{abs}	0.01246$^{\{3\}}$	0.01304$^{\{6\}}$	0.0132$^{\{7\}}$	0.01242$^{\{1.5\}}$	0.0134$^{\{8\}}$	0.0142$^{\{10\}}$	0.01274$^{\{4\}}$	0.01242$^{\{1.5\}}$	0.01486$^{\{13\}}$	0.01465$^{\{11\}}$	0.01477$^{\{12\}}$	0.01284$^{\{5\}}$	0.01692$^{\{15\}}$	0.01356$^{\{9\}}$	0.01666$^{\{14\}}$
	D_{max}	0.0202$^{\{1\}}$	0.02154$^{\{6\}}$	0.02231$^{\{8\}}$	0.02028$^{\{2\}}$	0.02261$^{\{9\}}$	0.02465$^{\{13\}}$	0.02112$^{\{5\}}$	0.02043$^{\{3\}}$	0.02406$^{\{11\}}$	0.02377$^{\{10\}}$	0.02409$^{\{12\}}$	0.02086$^{\{4\}}$	0.02768$^{\{15\}}$	0.02207$^{\{7\}}$	0.02724$^{\{14\}}$
	$\sum Ranks$	15$^{\{1\}}$	61$^{\{7\}}$	75$^{\{10\}}$	24.5$^{\{3\}}$	88$^{\{11\}}$	113$^{\{14\}}$	50$^{\{6\}}$	25.5$^{\{4\}}$	71$^{\{9\}}$	67$^{\{8\}}$	91$^{\{12\}}$	16$^{\{2\}}$	114$^{\{15\}}$	46$^{\{5\}}$	103$^{\{13\}}$

Table 5. Numerical values of simulation measures for $\delta = 0.5$ and $\beta = 2.0$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
30	BIAS($\hat{\delta}$)	0.13624[4]	0.15298[10]	0.16819[14]	0.13657[5]	0.15175[8]	0.18055[15]	0.15251[9]	0.14595[6]	0.09288[2]	0.12666[3]	0.15422[11]	0.05648[1]	0.1601[12]	0.14753[7]	0.16209[13]
	BIAS($\hat{\beta}$)	0.57765[4]	0.65842[12]	0.63719[9]	0.64979[10]	0.60867[7]	0.68047[14]	0.65627[11]	0.62937[8]	0.25245[2]	0.57701[3]	0.66039[13]	0.04775[1]	0.5962[6]	0.69113[15]	0.58347[5]
	MSE($\hat{\delta}$)	0.02921[5]	0.03635[10]	0.04402[14]	0.02809[4]	0.03633[9]	0.04928[15]	0.03627[8]	0.0338[6]	0.01634[2]	0.02598[3]	0.03818[11]	0.00526[1]	0.04053[12]	0.03407[7]	0.0423[13]
	MSE($\hat{\beta}$)	0.49086[3]	0.62358[11]	0.56339[7]	0.62516[12]	0.53029[6]	0.6451[14]	0.61704[10]	0.58229[9]	0.2075[2]	0.56556[8]	0.6369[13]	0.01233[1]	0.51078[5]	0.68808[15]	0.49454[4]
	MRE($\hat{\delta}$)	0.27249[4]	0.30596[10]	0.33638[14]	0.27314[5]	0.3035[8]	0.36111[15]	0.30502[9]	0.29189[6]	0.18576[2]	0.25331[3]	0.30843[11]	0.11296[1]	0.32019[12]	0.29507[7]	0.32419[13]
	MRE($\hat{\beta}$)	0.28882[4]	0.32921[12]	0.31859[9]	0.3249[10]	0.30434[7]	0.34024[14]	0.32814[11]	0.31469[8]	0.12623[2]	0.28851[3]	0.33019[13]	0.02388[1]	0.2981[6]	0.34556[15]	0.29173[5]
	D_{abs}	0.03768[1]	0.04098[5]	0.04468[12]	0.04059[2]	0.04585[13]	0.04352[10]	0.04069[3]	0.04313[9]	0.0425[7]	0.04216[6]	0.04287[8]	0.04096[4]	0.06514[15]	0.04419[11]	0.06029[14]
	D_{max}	0.06199[1]	0.06671[5]	0.07388[13]	0.06517[3]	0.07369[12]	0.07286[11]	0.06621[4]	0.07003[8]	0.06696[6]	0.06706[7]	0.07019[9]	0.0623[2]	0.10106[15]	0.07066[10]	0.09345[14]
	$\sum Ranks$	26[3]	75[9]	92[14]	51[5]	70[8]	108[15]	65[7]	60[6]	25[2]	36[4]	89[13]	12[1]	83[11]	87[12]	81[10]
60	BIAS($\hat{\delta}$)	0.10659[3]	0.12172[9]	0.14371[14]	0.10716[4]	0.13619[13]	0.16304[15]	0.13084[11]	0.11699[8]	0.07293[2]	0.10752[5]	0.12531[10]	0.04248[1]	0.13535[12]	0.11105[6]	0.11259[7]
	BIAS($\hat{\beta}$)	0.50725[5]	0.55301[7]	0.6035[12]	0.58515[9]	0.5899[11]	0.64435[15]	0.60784[14]	0.58714[10]	0.23162[2]	0.5467[6]	0.60699[13]	0.03925[1]	0.49202[4]	0.56928[8]	0.4335[3]
	MSE($\hat{\delta}$)	0.01774[4]	0.02332[9]	0.03276[14]	0.01729[3]	0.0283[12]	0.04047[15]	0.02684[11]	0.0214[7]	0.01089[2]	0.01838[5]	0.02516[10]	0.00304[1]	0.03231[13]	0.02003[6]	0.02258[8]
	MSE($\hat{\beta}$)	0.41751[5]	0.47226[6]	0.53147[10]	0.56899[14]	0.50961[8]	0.58852[15]	0.55385[12]	0.54235[11]	0.18176[2]	0.50391[7]	0.56276[13]	0.00761[1]	0.36579[4]	0.51548[9]	0.28438[3]
	MRE($\hat{\delta}$)	0.21318[3]	0.24344[9]	0.28743[14]	0.21432[4]	0.27239[13]	0.32609[15]	0.26168[11]	0.23398[8]	0.14585[2]	0.21505[5]	0.25063[10]	0.08496[1]	0.27071[12]	0.22211[6]	0.22518[7]
	MRE($\hat{\beta}$)	0.25363[5]	0.27651[7]	0.30175[12]	0.29257[9]	0.29495[11]	0.32217[15]	0.30392[14]	0.29357[10]	0.11581[2]	0.27335[6]	0.30349[13]	0.01962[1]	0.24601[4]	0.28464[8]	0.21675[3]
	D_{abs}	0.02879[1]	0.02974[2]	0.03169[9]	0.03041[4]	0.03128[7]	0.03293[13]	0.03119[6]	0.03093[5]	0.03153[8]	0.03238[12]	0.03207[10]	0.03029[3]	0.04427[15]	0.03231[11]	0.04117[14]
	D_{max}	0.04743[2]	0.04934[4]	0.05375[12]	0.0493[3]	0.05256[10]	0.0566[13]	0.05167[7]	0.05037[6]	0.04949[5]	0.05255[9]	0.05272[11]	0.04604[1]	0.06992[15]	0.05183[8]	0.06489[14]
	$\sum Ranks$	28[3]	53[5]	97[14]	50[4]	85[11]	116[15]	86[12]	65[9]	25[2]	55[6]	90[13]	10[1]	79[10]	62[8]	59[7]
100	BIAS($\hat{\delta}$)	0.08286[3]	0.10295[11]	0.11725[14]	0.08874[4]	0.11484[13]	0.14099[15]	0.10506[12]	0.09668[9]	0.06209[2]	0.09521[8]	0.10061[10]	0.0325[1]	0.09485[7]	0.09062[6]	0.09056[5]
	BIAS($\hat{\beta}$)	0.41544[5]	0.51195[9]	0.51848[11]	0.50035[7]	0.55605[14]	0.57985[15]	0.51445[10]	0.48798[6]	0.22274[2]	0.52223[12]	0.52272[13]	0.03493[1]	0.36852[3]	0.5066[8]	0.392[4]
	MSE($\hat{\delta}$)	0.01073[3]	0.01683[10]	0.02226[14]	0.01217[4]	0.02079[13]	0.03092[15]	0.01737[12]	0.01473[7]	0.00763[2]	0.01427[6]	0.01587[9]	0.00182[1]	0.0169[11]	0.0124[5]	0.01525[8]
	MSE($\hat{\beta}$)	0.29348[5]	0.42649[9]	0.43143[11]	0.42826[10]	0.49967[14]	0.51432[15]	0.42161[8]	0.38818[6]	0.15605[2]	0.47725[13]	0.43798[12]	0.0066[1]	0.21342[3]	0.41474[7]	0.23478[4]
	MRE($\hat{\delta}$)	0.16573[3]	0.2059[11]	0.23451[14]	0.17748[4]	0.22968[13]	0.28198[15]	0.21011[12]	0.19336[9]	0.12419[2]	0.19042[8]	0.20121[10]	0.065[1]	0.18971[7]	0.18125[6]	0.18112[5]
	MRE($\hat{\beta}$)	0.20772[5]	0.25598[9]	0.25924[11]	0.25018[7]	0.27802[14]	0.28993[15]	0.25722[10]	0.24399[6]	0.11137[2]	0.26111[12]	0.26136[13]	0.01746[1]	0.18426[3]	0.2533[8]	0.196[4]
	D_{abs}	0.02231[1]	0.02454[6]	0.02511[9]	0.02269[3]	0.02562[10]	0.02595[12]	0.02449[5]	0.02497[7]	0.02396[4]	0.02574[11]	0.02509[8]	0.02261[2]	0.03104[14]	0.02611[13]	0.03167[15]
	D_{max}	0.03664[2]	0.04054[5.5]	0.0426[11]	0.03686[3]	0.04306[12]	0.04533[13]	0.04054[5.5]	0.04099[7]	0.03813[4]	0.04213[10]	0.04108[8]	0.03463[1]	0.0495[14]	0.04206[9]	0.05085[15]
	$\sum Ranks$	27[3]	70.5[9]	95[13]	42[4]	103[14]	115[15]	74.5[10]	57[5]	20[2]	80[11]	83[12]	9[1]	62[7.5]	62[7.5]	60[6]
200	BIAS($\hat{\delta}$)	0.06116[3]	0.07841[11]	0.08834[13]	0.06582[5]	0.08967[14]	0.10318[15]	0.07657[10]	0.0701[7]	0.05022[2]	0.07169[8]	0.0803[12]	0.02285[1]	0.06434[4]	0.07255[9]	0.06584[6]
	BIAS($\hat{\beta}$)	0.32425[5]	0.40338[10]	0.42002[12]	0.37519[7]	0.44639[13]	0.47646[15]	0.38899[8]	0.36465[6]	0.20622[2]	0.39983[9]	0.45097[14]	0.02978[1]	0.309[3]	0.40885[11]	0.32061[4]
	MSE($\hat{\delta}$)	0.00603[3]	0.0094[11]	0.01227[14]	0.00684[4]	0.0121[13]	0.01636[15]	0.00916[10]	0.00762[6]	0.00531[2]	0.00804[8]	0.0099[12]	0.00084[1]	0.00725[5]	0.00822[9]	0.00789[7]
	MSE($\hat{\beta}$)	0.18213[4]	0.27797[9]	0.29886[12]	0.25897[8]	0.34454[14]	0.37648[15]	0.2585[7]	0.23044[6]	0.13188[2]	0.2854[10]	0.34418[13]	0.00461[1]	0.1536[3]	0.29853[11]	0.18333[5]
	MRE($\hat{\delta}$)	0.12232[3]	0.15682[11]	0.17668[13]	0.13164[5]	0.17934[14]	0.20636[15]	0.15314[10]	0.1402[7]	0.10044[2]	0.14338[8]	0.16059[12]	0.0457[1]	0.12868[4]	0.14509[9]	0.13169[6]
	MRE($\hat{\beta}$)	0.16212[5]	0.20169[10]	0.21001[12]	0.18759[7]	0.2232[13]	0.23823[15]	0.1945[8]	0.18232[6]	0.10311[2]	0.19991[9]	0.22548[14]	0.01489[1]	0.1545[3]	0.20442[11]	0.16031[4]
	D_{abs}	0.01724[2]	0.01806[7]	0.01844[10]	0.01741[3]	0.01838[8]	0.01843[9]	0.01779[4]	0.01797[6]	0.01789[5]	0.02027[13]	0.01959[12]	0.01648[1]	0.02114[14]	0.0194[11]	0.02249[15]
	D_{max}	0.02791[2]	0.0299[7]	0.03118[9]	0.02827[3]	0.03116[8]	0.03221[12]	0.02954[6]	0.0293[5]	0.02849[4]	0.03291[13]	0.03189[11]	0.02515[1]	0.03433[14]	0.03138[10]	0.03655[15]
	$\sum Ranks$	27[3]	76[9]	95[12]	42[4]	97[13]	111[15]	63[8]	49[5]	21[2]	78[10]	100[14]	8[1]	50[6]	81[11]	62[7]
300	BIAS($\hat{\delta}$)	0.04976[3]	0.063[10]	0.07722[14]	0.05537[5]	0.07222[13]	0.08992[15]	0.06346[11]	0.05881[7]	0.04405[2]	0.06157[9]	0.06661[12]	0.01923[1]	0.05374[4]	0.05954[8]	0.05577[6]
	BIAS($\hat{\beta}$)	0.26113[3]	0.32348[9]	0.37643[14]	0.30537[6]	0.36663[12]	0.42261[15]	0.32101[8]	0.30553[7]	0.18919[2]	0.33927[11]	0.37228[13]	0.02946[1]	0.27025[5]	0.32572[10]	0.2684[4]
	MSE($\hat{\delta}$)	0.00405[3]	0.00614[10]	0.00941[14]	0.00478[4]	0.00824[13]	0.01248[15]	0.0063[11]	0.00566[7]	0.00375[2]	0.00597[9]	0.00674[12]	0.00063[1]	0.00503[5]	0.00548[6]	0.00588[8]
	MSE($\hat{\beta}$)	0.12154[3]	0.17435[8]	0.24526[14]	0.16177[7]	0.2386[13]	0.30176[15]	0.17584[9]	0.15982[6]	0.10377[2]	0.20948[11]	0.23664[12]	0.00414[1]	0.13235[4]	0.18403[10]	0.14778[5]
	MRE($\hat{\delta}$)	0.09951[3]	0.12601[10]	0.15444[14]	0.11075[5]	0.14445[13]	0.17985[15]	0.12692[11]	0.11761[7]	0.0881[2]	0.12314[9]	0.13323[12]	0.03846[1]	0.10748[4]	0.11909[8]	0.11153[6]
	MRE($\hat{\beta}$)	0.13057[3]	0.16174[9]	0.18822[14]	0.15269[6]	0.18332[12]	0.2113[15]	0.1605[8]	0.15276[7]	0.09459[2]	0.16963[11]	0.18614[13]	0.01473[1]	0.13513[5]	0.16286[10]	0.1342[4]
	D_{abs}	0.01433[2]	0.01475[5.5]	0.0156[10]	0.01439[3]	0.01527[7]	0.01648[12]	0.01475[5.5]	0.01471[4]	0.01548[8]	0.01669[13]	0.01557[9]	0.01377[1]	0.01746[14]	0.01569[11]	0.01762[15]
	D_{max}	0.02313[2]	0.02441[5]	0.02642[11]	0.0234[3]	0.02566[10]	0.02851[14]	0.02444[6]	0.02405[4]	0.02467[7]	0.02709[12]	0.02545[8]	0.02099[1]	0.02842[13]	0.02547[9]	0.02869[15]
	$\sum Ranks$	22[2]	66.5[8]	105[14]	39[4]	93[13]	116[15]	69.5[9]	49[5]	27[3]	85[11]	91[12]	8[1]	54[6]	72[10]	63[7]
400	BIAS($\hat{\delta}$)	0.04586[3]	0.05457[10]	0.06718[14]	0.04629[4]	0.06491[13]	0.07564[15]	0.05516[11]	0.05045[8]	0.04056[2]	0.05384[9]	0.05857[12]	0.01681[1]	0.04911[5]	0.0504[7]	0.04983[6]
	BIAS($\hat{\beta}$)	0.24121[4]	0.2717[9]	0.32822[14]	0.25116[6]	0.31726[12]	0.35759[15]	0.2792[10]	0.25723[7]	0.17791[2]	0.29726[11]	0.3189[13]	0.02741[1]	0.2501[5]	0.27113[8]	0.23703[3]
	MSE($\hat{\delta}$)	0.00335[4]	0.00479[10]	0.00713[14]	0.00334[3]	0.00654[13]	0.00909[15]	0.0047[9]	0.00393[5]	0.00319[2]	0.00461[8]	0.00534[12]	0.00051[1]	0.00433[7]	0.00397[6]	0.00484[11]
	MSE($\hat{\beta}$)	0.09651[3]	0.12678[8]	0.18883[14]	0.10551[4]	0.17475[13]	0.22525[15]	0.12978[10]	0.11072[5]	0.08312[2]	0.15568[11]	0.17447[12]	0.00395[1]	0.12704[9]	0.12604[7]	0.11073[6]
	MRE($\hat{\delta}$)	0.09172[3]	0.10915[10]	0.13437[14]	0.09257[4]	0.12982[13]	0.15127[15]	0.11032[11]	0.10089[8]	0.08112[2]	0.10768[9]	0.11714[12]	0.03363[1]	0.09822[5]	0.10079[7]	0.09966[6]
	MRE($\hat{\beta}$)	0.12061[4]	0.13585[9]	0.16411[14]	0.12558[6]	0.15863[12]	0.1788[15]	0.1396[10]	0.12862[7]	0.08896[2]	0.14863[11]	0.15945[13]	0.01371[1]	0.12505[5]	0.13556[8]	0.11852[3]
	D_{abs}	0.01235[3]	0.01251[4]	0.01316[7]	0.01191[2]	0.0133[10]	0.01357[11]	0.01283[6]	0.01262[5]	0.01327[9]	0.01469[13]	0.01384[12]	0.0118[1]	0.01612[15]	0.01319[8]	0.01595[14]
	D_{max}	0.02003[3]	0.02068[5]	0.02249[10]	0.0194[2]	0.02245[9]	0.02345[12]	0.0212[6]	0.02064[4]	0.02127[7]	0.02382[13]	0.02263[11]	0.01806[1]	0.02613[15]	0.0215[8]	0.02592[14]
	$\sum Ranks$	27[2]	65[8]	101[14]	31[4]	95[12]	113[15]	73[10]	49[5]	28[3]	85[11]	97[13]	8[1]	66[9]	59[6]	63[7]

Table 6. Numerical values of simulation measures for $\delta = 2.5$ and $\beta = 0.4$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
30	BIAS($\hat{\delta}$)	$0.24947^{\{2\}}$	$0.39247^{\{8\}}$	$0.45109^{\{13\}}$	$0.34898^{\{4\}}$	$0.4092^{\{10\}}$	$0.51093^{\{15\}}$	$0.3875^{\{7\}}$	$0.40032^{\{9\}}$	$0.2568^{\{3\}}$	$0.36632^{\{6\}}$	$0.47352^{\{14\}}$	$0.0773^{\{1\}}$	$0.42866^{\{12\}}$	$0.3597^{\{5\}}$	$0.40978^{\{11\}}$
	BIAS($\hat{\beta}$)	$0.10849^{\{3\}}$	$0.12312^{\{5\}}$	$0.12419^{\{6\}}$	$0.12427^{\{7\}}$	$0.12917^{\{11\}}$	$0.13824^{\{12\}}$	$0.12135^{\{4\}}$	$0.12589^{\{8\}}$	$0.10086^{\{2\}}$	$0.12906^{\{10\}}$	$0.14169^{\{15\}}$	$0.05665^{\{1\}}$	$0.14068^{\{13\}}$	$0.12644^{\{9\}}$	$0.14073^{\{14\}}$
	MSE($\hat{\delta}$)	$0.1048^{\{2\}}$	$0.25062^{\{8\}}$	$0.35457^{\{13\}}$	$0.19139^{\{4\}}$	$0.26462^{\{10\}}$	$0.44465^{\{15\}}$	$0.24711^{\{7\}}$	$0.26728^{\{11\}}$	$0.13483^{\{3\}}$	$0.20392^{\{6\}}$	$0.39226^{\{14\}}$	$0.02074^{\{1\}}$	$0.27898^{\{12\}}$	$0.19815^{\{5\}}$	$0.25293^{\{9\}}$
	MSE($\hat{\beta}$)	$0.02079^{\{3\}}$	$0.02301^{\{6\}}$	$0.02292^{\{5\}}$	$0.02435^{\{8\}}$	$0.0252^{\{10\}}$	$0.02791^{\{12\}}$	$0.02257^{\{4\}}$	$0.02416^{\{7\}}$	$0.01873^{\{2\}}$	$0.02604^{\{11\}}$	$0.02917^{\{13\}}$	$0.00597^{\{1\}}$	$0.03018^{\{15\}}$	$0.02461^{\{9\}}$	$0.02959^{\{14\}}$
	MRE($\hat{\delta}$)	$0.09979^{\{2\}}$	$0.15699^{\{8\}}$	$0.18043^{\{13\}}$	$0.13959^{\{4\}}$	$0.16368^{\{10\}}$	$0.20437^{\{15\}}$	$0.155^{\{7\}}$	$0.16013^{\{9\}}$	$0.10272^{\{3\}}$	$0.14653^{\{6\}}$	$0.18941^{\{14\}}$	$0.03092^{\{1\}}$	$0.17146^{\{12\}}$	$0.14388^{\{5\}}$	$0.16391^{\{11\}}$
	MRE($\hat{\beta}$)	$0.27123^{\{3\}}$	$0.3078^{\{5\}}$	$0.31047^{\{6\}}$	$0.31068^{\{7\}}$	$0.32292^{\{11\}}$	$0.34559^{\{12\}}$	$0.30337^{\{4\}}$	$0.31471^{\{8\}}$	$0.25215^{\{2\}}$	$0.32265^{\{10\}}$	$0.35423^{\{15\}}$	$0.14162^{\{1\}}$	$0.3517^{\{13\}}$	$0.31611^{\{9\}}$	$0.35182^{\{14\}}$
	D_{abs}	$0.04159^{\{1\}}$	$0.0451^{\{3\}}$	$0.0482^{\{9\}}$	$0.0454^{\{5\}}$	$0.04619^{\{6\}}$	$0.0507^{\{13\}}$	$0.04517^{\{4\}}$	$0.04721^{\{8\}}$	$0.04986^{\{11.5\}}$	$0.04986^{\{11.5\}}$	$0.04865^{\{10\}}$	$0.04264^{\{2\}}$	$0.05413^{\{15\}}$	$0.0466^{\{7\}}$	$0.05336^{\{14\}}$
	D_{max}	$0.066^{\{2\}}$	$0.07344^{\{4\}}$	$0.08102^{\{12\}}$	$0.07193^{\{3\}}$	$0.076^{\{7\}}$	$0.08624^{\{14\}}$	$0.07378^{\{5\}}$	$0.07752^{\{9\}}$	$0.07682^{\{8\}}$	$0.07921^{\{10\}}$	$0.08099^{\{11\}}$	$0.06259^{\{1\}}$	$0.08762^{\{15\}}$	$0.07444^{\{6\}}$	$0.0855^{\{13\}}$
	$\sum Ranks$	$18^{\{2\}}$	$47^{\{6\}}$	$77^{\{11\}}$	$42^{\{4.5\}}$	$75^{\{10\}}$	$108^{\{15\}}$	$42^{\{4.5\}}$	$69^{\{8\}}$	$34.5^{\{3\}}$	$70.5^{\{9\}}$	$106^{\{13\}}$	$9^{\{1\}}$	$107^{\{14\}}$	$55^{\{7\}}$	$100^{\{12\}}$
60	BIAS($\hat{\delta}$)	$0.20453^{\{2\}}$	$0.2779^{\{6\}}$	$0.29563^{\{10\}}$	$0.27622^{\{5\}}$	$0.31346^{\{11\}}$	$0.37466^{\{15\}}$	$0.28173^{\{7\}}$	$0.28733^{\{9\}}$	$0.24988^{\{3\}}$	$0.28363^{\{8\}}$	$0.35262^{\{14\}}$	$0.07199^{\{1\}}$	$0.33443^{\{13\}}$	$0.27509^{\{4\}}$	$0.32222^{\{12\}}$
	BIAS($\hat{\beta}$)	$0.08487^{\{2\}}$	$0.09474^{\{5\}}$	$0.09425^{\{3\}}$	$0.10053^{\{9\}}$	$0.09961^{\{8\}}$	$0.10752^{\{12\}}$	$0.09656^{\{7\}}$	$0.09498^{\{6\}}$	$0.09445^{\{4\}}$	$0.10474^{\{11\}}$	$0.11801^{\{13\}}$	$0.04214^{\{1\}}$	$0.11894^{\{14\}}$	$0.10253^{\{10\}}$	$0.1193^{\{15\}}$
	MSE($\hat{\delta}$)	$0.07403^{\{2\}}$	$0.12349^{\{6\}}$	$0.14155^{\{10\}}$	$0.11748^{\{4\}}$	$0.15224^{\{11\}}$	$0.22331^{\{15\}}$	$0.12956^{\{8\}}$	$0.13518^{\{9\}}$	$0.11795^{\{5\}}$	$0.1257^{\{7\}}$	$0.20342^{\{14\}}$	$0.01871^{\{1\}}$	$0.17116^{\{13\}}$	$0.1157^{\{3\}}$	$0.15442^{\{12\}}$
	MSE($\hat{\beta}$)	$0.01313^{\{2\}}$	$0.01465^{\{5\}}$	$0.0141^{\{3\}}$	$0.01638^{\{9\}}$	$0.01589^{\{8\}}$	$0.01792^{\{11\}}$	$0.01554^{\{7\}}$	$0.01446^{\{4\}}$	$0.01553^{\{6\}}$	$0.01793^{\{12\}}$	$0.0212^{\{13\}}$	$0.00382^{\{1\}}$	$0.02299^{\{15\}}$	$0.01724^{\{10\}}$	$0.02298^{\{14\}}$
	MRE($\hat{\delta}$)	$0.08181^{\{2\}}$	$0.11116^{\{6\}}$	$0.11825^{\{10\}}$	$0.11049^{\{5\}}$	$0.12538^{\{11\}}$	$0.14986^{\{15\}}$	$0.11269^{\{7\}}$	$0.11493^{\{9\}}$	$0.09995^{\{3\}}$	$0.11345^{\{8\}}$	$0.14105^{\{14\}}$	$0.0288^{\{1\}}$	$0.13377^{\{13\}}$	$0.11003^{\{4\}}$	$0.12889^{\{12\}}$
	MRE($\hat{\beta}$)	$0.21217^{\{2\}}$	$0.23684^{\{5\}}$	$0.23563^{\{3\}}$	$0.25133^{\{9\}}$	$0.24903^{\{8\}}$	$0.26879^{\{12\}}$	$0.24139^{\{7\}}$	$0.23744^{\{6\}}$	$0.23612^{\{4\}}$	$0.26185^{\{11\}}$	$0.29503^{\{13\}}$	$0.10535^{\{1\}}$	$0.29735^{\{14\}}$	$0.25632^{\{10\}}$	$0.29825^{\{15\}}$
	D_{abs}	$0.03306^{\{3\}}$	$0.0336^{\{6\}}$	$0.03402^{\{7\}}$	$0.03258^{\{2\}}$	$0.03344^{\{5\}}$	$0.03621^{\{9\}}$	$0.03435^{\{8\}}$	$0.03312^{\{4\}}$	$0.03754^{\{12\}}$	$0.03649^{\{10\}}$	$0.03851^{\{13\}}$	$0.02949^{\{1\}}$	$0.03966^{\{14\}}$	$0.0365^{\{11\}}$	$0.04037^{\{15\}}$
	D_{max}	$0.05248^{\{2\}}$	$0.05501^{\{5\}}$	$0.05639^{\{8\}}$	$0.05259^{\{3\}}$	$0.05553^{\{6\}}$	$0.06191^{\{12\}}$	$0.05603^{\{7\}}$	$0.05434^{\{4\}}$	$0.05941^{\{11\}}$	$0.05857^{\{10\}}$	$0.06353^{\{13\}}$	$0.04401^{\{1\}}$	$0.06477^{\{14\}}$	$0.05849^{\{9\}}$	$0.06561^{\{15\}}$
	$\sum Ranks$	$17^{\{2\}}$	$44^{\{3\}}$	$54^{\{7\}}$	$46^{\{4\}}$	$68^{\{10\}}$	$101^{\{12\}}$	$58^{\{8\}}$	$51^{\{6\}}$	$48^{\{5\}}$	$77^{\{11\}}$	$107^{\{13\}}$	$8^{\{1\}}$	$110^{\{14.5\}}$	$61^{\{9\}}$	$110^{\{14.5\}}$
100	BIAS($\hat{\delta}$)	$0.16869^{\{2\}}$	$0.21955^{\{6\}}$	$0.25099^{\{11\}}$	$0.20658^{\{4\}}$	$0.23978^{\{10\}}$	$0.26936^{\{13\}}$	$0.22407^{\{7\}}$	$0.21601^{\{5\}}$	$0.20488^{\{3\}}$	$0.23686^{\{9\}}$	$0.2921^{\{15\}}$	$0.06504^{\{1\}}$	$0.27867^{\{14\}}$	$0.23071^{\{8\}}$	$0.26579^{\{12\}}$
	BIAS($\hat{\beta}$)	$0.06441^{\{2\}}$	$0.0751^{\{5\}}$	$0.08048^{\{8\}}$	$0.07758^{\{6\}}$	$0.08158^{\{9\}}$	$0.08422^{\{10\}}$	$0.07438^{\{4\}}$	$0.07323^{\{3\}}$	$0.07816^{\{7\}}$	$0.0888^{\{12\}}$	$0.10363^{\{15\}}$	$0.03587^{\{1\}}$	$0.10299^{\{14\}}$	$0.08464^{\{11\}}$	$0.09863^{\{13\}}$
	MSE($\hat{\delta}$)	$0.04708^{\{2\}}$	$0.07537^{\{5\}}$	$0.09757^{\{11\}}$	$0.0662^{\{3\}}$	$0.08954^{\{10\}}$	$0.11225^{\{13\}}$	$0.07976^{\{8\}}$	$0.07321^{\{4\}}$	$0.07788^{\{6\}}$	$0.08763^{\{9\}}$	$0.13241^{\{15\}}$	$0.01303^{\{1\}}$	$0.11883^{\{14\}}$	$0.07847^{\{7\}}$	$0.10997^{\{12\}}$
	MSE($\hat{\beta}$)	$0.00749^{\{2\}}$	$0.00915^{\{4\}}$	$0.0102^{\{6\}}$	$0.01024^{\{7\}}$	$0.0109^{\{8\}}$	$0.01143^{\{9\}}$	$0.00918^{\{5\}}$	$0.00855^{\{3\}}$	$0.01159^{\{11\}}$	$0.0135^{\{12\}}$	$0.01684^{\{13\}}$	$0.00259^{\{1\}}$	$0.01761^{\{15\}}$	$0.01144^{\{10\}}$	$0.01689^{\{14\}}$
	MRE($\hat{\delta}$)	$0.06747^{\{2\}}$	$0.08782^{\{6\}}$	$0.10039^{\{11\}}$	$0.08263^{\{4\}}$	$0.09591^{\{10\}}$	$0.10774^{\{13\}}$	$0.08963^{\{7\}}$	$0.0864^{\{5\}}$	$0.08195^{\{3\}}$	$0.09474^{\{9\}}$	$0.11684^{\{15\}}$	$0.02602^{\{1\}}$	$0.11147^{\{14\}}$	$0.09228^{\{8\}}$	$0.10632^{\{12\}}$
	MRE($\hat{\beta}$)	$0.16103^{\{2\}}$	$0.18776^{\{5\}}$	$0.20121^{\{8\}}$	$0.19396^{\{6\}}$	$0.20394^{\{9\}}$	$0.21056^{\{10\}}$	$0.18594^{\{4\}}$	$0.18308^{\{3\}}$	$0.1954^{\{7\}}$	$0.22201^{\{12\}}$	$0.25908^{\{15\}}$	$0.08969^{\{1\}}$	$0.25746^{\{14\}}$	$0.2116^{\{11\}}$	$0.24657^{\{13\}}$
	D_{abs}	$0.02344^{\{1\}}$	$0.0258^{\{4\}}$	$0.02729^{\{8\}}$	$0.02632^{\{5\}}$	$0.02689^{\{7\}}$	$0.02795^{\{9\}}$	$0.02543^{\{3\}}$	$0.02678^{\{6\}}$	$0.03037^{\{13\}}$	$0.02968^{\{12\}}$	$0.02967^{\{11\}}$	$0.02442^{\{2\}}$	$0.03344^{\{15\}}$	$0.02859^{\{10\}}$	$0.03216^{\{14\}}$
	D_{max}	$0.03772^{\{2\}}$	$0.04225^{\{4\}}$	$0.04553^{\{8\}}$	$0.04245^{\{5\}}$	$0.0444^{\{7\}}$	$0.04705^{\{10\}}$	$0.04196^{\{3\}}$	$0.04374^{\{6\}}$	$0.0487^{\{12\}}$	$0.04793^{\{11\}}$	$0.04912^{\{13\}}$	$0.03646^{\{1\}}$	$0.05452^{\{15\}}$	$0.04636^{\{9\}}$	$0.05237^{\{14\}}$
	$\sum Ranks$	$15^{\{2\}}$	$39^{\{4\}}$	$71^{\{9\}}$	$40^{\{5\}}$	$70^{\{8\}}$	$87^{\{12\}}$	$41^{\{6\}}$	$35^{\{3\}}$	$62^{\{7\}}$	$86^{\{11\}}$	$112^{\{14\}}$	$9^{\{1\}}$	$115^{\{15\}}$	$74^{\{10\}}$	$104^{\{13\}}$
200	BIAS($\hat{\delta}$)	$0.12568^{\{2\}}$	$0.15802^{\{6\}}$	$0.16754^{\{10\}}$	$0.14131^{\{3\}}$	$0.17205^{\{11\}}$	$0.19869^{\{12\}}$	$0.1585^{\{8\}}$	$0.15009^{\{5\}}$	$0.14533^{\{4\}}$	$0.16686^{\{9\}}$	$0.215^{\{15\}}$	$0.05087^{\{1\}}$	$0.20091^{\{13\}}$	$0.15808^{\{7\}}$	$0.20236^{\{14\}}$
	BIAS($\hat{\beta}$)	$0.04788^{\{2\}}$	$0.05346^{\{5\}}$	$0.05363^{\{6\}}$	$0.0502^{\{3\}}$	$0.05533^{\{9\}}$	$0.06203^{\{12\}}$	$0.05404^{\{7\}}$	$0.05068^{\{4\}}$	$0.05485^{\{8\}}$	$0.06093^{\{11\}}$	$0.07871^{\{15\}}$	$0.02619^{\{1\}}$	$0.07197^{\{13\}}$	$0.05806^{\{10\}}$	$0.07216^{\{14\}}$
	MSE($\hat{\delta}$)	$0.02442^{\{2\}}$	$0.0386^{\{5\}}$	$0.04604^{\{11\}}$	$0.03106^{\{3\}}$	$0.04597^{\{10\}}$	$0.06064^{\{13\}}$	$0.03893^{\{6\}}$	$0.03567^{\{4\}}$	$0.03895^{\{7\}}$	$0.04485^{\{9\}}$	$0.06901^{\{15\}}$	$0.00783^{\{1\}}$	$0.0603^{\{12\}}$	$0.03976^{\{8\}}$	$0.06373^{\{14\}}$
	MSE($\hat{\beta}$)	$0.00378^{\{2\}}$	$0.00458^{\{5\}}$	$0.00482^{\{7\}}$	$0.00408^{\{3\}}$	$0.00482^{\{7\}}$	$0.00624^{\{12\}}$	$0.00482^{\{7\}}$	$0.00409^{\{4\}}$	$0.00537^{\{9\}}$	$0.00623^{\{11\}}$	$0.00978^{\{15\}}$	$0.00135^{\{1\}}$	$0.00858^{\{13\}}$	$0.00572^{\{10\}}$	$0.00898^{\{14\}}$
	MRE($\hat{\delta}$)	$0.05027^{\{2\}}$	$0.06321^{\{6\}}$	$0.06702^{\{10\}}$	$0.05652^{\{3\}}$	$0.06882^{\{11\}}$	$0.07948^{\{12\}}$	$0.0634^{\{8\}}$	$0.06003^{\{5\}}$	$0.05813^{\{4\}}$	$0.06675^{\{9\}}$	$0.086^{\{15\}}$	$0.02035^{\{1\}}$	$0.08036^{\{13\}}$	$0.06323^{\{7\}}$	$0.08094^{\{14\}}$
	MRE($\hat{\beta}$)	$0.1197^{\{2\}}$	$0.13366^{\{5\}}$	$0.13408^{\{6\}}$	$0.12549^{\{3\}}$	$0.13833^{\{9\}}$	$0.15508^{\{12\}}$	$0.13511^{\{7\}}$	$0.12671^{\{4\}}$	$0.13713^{\{8\}}$	$0.15232^{\{11\}}$	$0.19678^{\{15\}}$	$0.06548^{\{1\}}$	$0.17993^{\{13\}}$	$0.14516^{\{10\}}$	$0.1804^{\{14\}}$
	D_{abs}	$0.01719^{\{2\}}$	$0.01841^{\{5\}}$	$0.01851^{\{6\}}$	$0.0174^{\{3\}}$	$0.01866^{\{7\}}$	$0.02028^{\{9\}}$	$0.01868^{\{8\}}$	$0.01825^{\{4\}}$	$0.02193^{\{12\}}$	$0.02128^{\{11\}}$	$0.02281^{\{13\}}$	$0.0171^{\{1\}}$	$0.02382^{\{15\}}$	$0.02076^{\{10\}}$	$0.02362^{\{14\}}$
	D_{max}	$0.02788^{\{2\}}$	$0.03027^{\{5\}}$	$0.03093^{\{7\}}$	$0.02818^{\{3\}}$	$0.03111^{\{8\}}$	$0.03422^{\{10\}}$	$0.03083^{\{6\}}$	$0.02987^{\{4\}}$	$0.03496^{\{12\}}$	$0.03435^{\{11\}}$	$0.03748^{\{13\}}$	$0.02581^{\{1\}}$	$0.03887^{\{15\}}$	$0.03338^{\{9\}}$	$0.03869^{\{14\}}$
	$\sum Ranks$	$16^{\{2\}}$	$42^{\{5\}}$	$63^{\{7\}}$	$24^{\{3\}}$	$72^{\{10\}}$	$92^{\{12\}}$	$57^{\{6\}}$	$34^{\{4\}}$	$64^{\{8\}}$	$82^{\{11\}}$	$116^{\{15\}}$	$8^{\{1\}}$	$107^{\{13\}}$	$71^{\{9\}}$	$112^{\{14\}}$
300	BIAS($\hat{\delta}$)	$0.10138^{\{2\}}$	$0.12388^{\{6\}}$	$0.13609^{\{10\}}$	$0.11995^{\{4\}}$	$0.13703^{\{11\}}$	$0.15219^{\{12\}}$	$0.11897^{\{3\}}$	$0.12185^{\{5\}}$	$0.13147^{\{8\}}$	$0.13467^{\{9\}}$	$0.17695^{\{15\}}$	$0.04844^{\{1\}}$	$0.1656^{\{14\}}$	$0.12572^{\{7\}}$	$0.15695^{\{13\}}$
	BIAS($\hat{\beta}$)	$0.03889^{\{2\}}$	$0.04166^{\{4\}}$	$0.04473^{\{8\}}$	$0.04246^{\{6\}}$	$0.04436^{\{7\}}$	$0.04875^{\{12\}}$	$0.04023^{\{3\}}$	$0.04196^{\{5\}}$	$0.04867^{\{11\}}$	$0.0481^{\{10\}}$	$0.06487^{\{15\}}$	$0.02362^{\{1\}}$	$0.05683^{\{14\}}$	$0.04507^{\{9\}}$	$0.05475^{\{13\}}$
	MSE($\hat{\delta}$)	$0.01642^{\{2\}}$	$0.02414^{\{6\}}$	$0.02962^{\{10\}}$	$0.0216^{\{3\}}$	$0.02872^{\{9\}}$	$0.03784^{\{12\}}$	$0.02234^{\{4\}}$	$0.02409^{\{5\}}$	$0.03044^{\{11\}}$	$0.02838^{\{8\}}$	$0.04928^{\{15\}}$	$0.00716^{\{1\}}$	$0.04261^{\{14\}}$	$0.02478^{\{7\}}$	$0.03848^{\{13\}}$
	MSE($\hat{\beta}$)	$0.00251^{\{2\}}$	$0.00274^{\{4\}}$	$0.00319^{\{8\}}$	$0.00286^{\{6\}}$	$0.00308^{\{7\}}$	$0.0039^{\{11\}}$	$0.00252^{\{3\}}$	$0.00285^{\{5\}}$	$0.00411^{\{12\}}$	$0.00379^{\{10\}}$	$0.00722^{\{15\}}$	$0.00119^{\{1\}}$	$0.00549^{\{14\}}$	$0.00329^{\{9\}}$	$0.00507^{\{13\}}$
	MRE($\hat{\delta}$)	$0.04055^{\{2\}}$	$0.04955^{\{6\}}$	$0.05444^{\{10\}}$	$0.04798^{\{4\}}$	$0.05481^{\{11\}}$	$0.06088^{\{12\}}$	$0.04759^{\{3\}}$	$0.04874^{\{5\}}$	$0.05259^{\{8\}}$	$0.05387^{\{9\}}$	$0.07078^{\{15\}}$	$0.01938^{\{1\}}$	$0.06624^{\{14\}}$	$0.05029^{\{7\}}$	$0.06278^{\{13\}}$
	MRE($\hat{\beta}$)	$0.09723^{\{2\}}$	$0.10415^{\{4\}}$	$0.11183^{\{8\}}$	$0.10614^{\{6\}}$	$0.11091^{\{7\}}$	$0.12186^{\{12\}}$	$0.10057^{\{3\}}$	$0.10489^{\{5\}}$	$0.12168^{\{11\}}$	$0.12025^{\{10\}}$	$0.16218^{\{15\}}$	$0.05905^{\{1\}}$	$0.14207^{\{14\}}$	$0.11268^{\{9\}}$	$0.13687^{\{13\}}$
	D_{abs}	$0.01417^{\{1\}}$	$0.01453^{\{2\}}$	$0.01543^{\{7\}}$	$0.01481^{\{5\}}$	$0.01544^{\{8\}}$	$0.01573^{\{9\}}$	$0.01455^{\{3\}}$	$0.0152^{\{6\}}$	$0.01852^{\{13\}}$	$0.01696^{\{11\}}$	$0.0176^{\{12\}}$	$0.0147^{\{4\}}$	$0.01974^{\{15\}}$	$0.01602^{\{10\}}$	$0.01923^{\{14\}}$
	D_{max}	$0.02287^{\{2\}}$	$0.02379^{\{4\}}$	$0.02546^{\{7\}}$	$0.02403^{\{5\}}$	$0.02569^{\{8\}}$	$0.0263^{\{10\}}$	$0.02377^{\{3\}}$	$0.02487^{\{6\}}$	$0.02963^{\{13\}}$	$0.02753^{\{11\}}$	$0.02918^{\{12\}}$	$0.02235^{\{1\}}$	$0.03243^{\{15\}}$	$0.02583^{\{9\}}$	$0.03132^{\{14\}}$
	$\sum Ranks$	$15^{\{2\}}$	$36^{\{4\}}$	$68^{\{8.5\}}$	$39^{\{5\}}$	$68^{\{8.5\}}$	$90^{\{12\}}$	$25^{\{3\}}$	$42^{\{6\}}$	$87^{\{11\}}$	$78^{\{10\}}$	$114^{\{14.5\}}$	$11^{\{1\}}$	$114^{\{14.5\}}$	$67^{\{7\}}$	$106^{\{13\}}$
400	BIAS($\hat{\delta}$)	$0.08539^{\{2\}}$	$0.10758^{\{5\}}$	$0.12255^{\{11\}}$	$0.10162^{\{3\}}$	$0.12057^{\{10\}}$	$0.13935^{\{12\}}$	$0.10965^{\{6\}}$	$0.11168^{\{7\}}$	$0.10747^{\{4\}}$	$0.11634^{\{9\}}$	$0.15238^{\{15\}}$	$0.03778^{\{1\}}$	$0.14254^{\{13\}}$	$0.11229^{\{8\}}$	$0.14255^{\{14\}}$
	BIAS($\hat{\beta}$)	$0.03288^{\{2\}}$	$0.03624^{\{4\}}$	$0.04047^{\{10\}}$	$0.03533^{\{3\}}$	$0.03953^{\{9\}}$	$0.04295^{\{12\}}$	$0.03632^{\{5\}}$	$0.03807^{\{6\}}$	$0.03897^{\{7\}}$	$0.04102^{\{11\}}$	$0.05635^{\{15\}}$	$0.01866^{\{1\}}$	$0.0513^{\{14\}}$	$0.03935^{\{8\}}$	$0.04946^{\{13\}}$
	MSE($\hat{\delta}$)	$0.01164^{\{2\}}$	$0.01815^{\{4\}}$	$0.02321^{\{11\}}$	$0.01641^{\{3\}}$	$0.02278^{\{10\}}$	$0.02969^{\{12\}}$	$0.01878^{\{5\}}$	$0.01988^{\{7\}}$	$0.02066^{\{8\}}$	$0.02178^{\{9\}}$	$0.03737^{\{15\}}$	$0.00425^{\{1\}}$	$0.03212^{\{14\}}$	$0.01965^{\{6\}}$	$0.03198^{\{13\}}$
	MSE($\hat{\beta}$)	$0.00175^{\{2\}}$	$0.00205^{\{4\}}$	$0.00259^{\{9\}}$	$0.00201^{\{3\}}$	$0.0025^{\{8\}}$	$0.00289^{\{12\}}$	$0.00208^{\{5\}}$	$0.00229^{\{6\}}$	$0.00265^{\{10\}}$	$0.00283^{\{11\}}$	$0.0054^{\{15\}}$	$0.00073^{\{1\}}$	$0.00429^{\{14\}}$	$0.00241^{\{7\}}$	$0.00393^{\{13\}}$
	MRE($\hat{\delta}$)	$0.03416^{\{2\}}$	$0.04303^{\{5\}}$	$0.04902^{\{11\}}$	$0.04065^{\{3\}}$	$0.04823^{\{10\}}$	$0.05574^{\{12\}}$	$0.04386^{\{6\}}$	$0.04467^{\{7\}}$	$0.04299^{\{4\}}$	$0.04654^{\{9\}}$	$0.06095^{\{15\}}$	$0.01511^{\{1\}}$	$0.05701^{\{13\}}$	$0.04492^{\{8\}}$	$0.05702^{\{14\}}$
	MRE($\hat{\beta}$)	$0.08219^{\{2\}}$	$0.09059^{\{4\}}$	$0.10117^{\{10\}}$	$0.08833^{\{3\}}$	$0.09881^{\{9\}}$	$0.10736^{\{12\}}$	$0.0908^{\{5\}}$	$0.09518^{\{6\}}$	$0.09744^{\{7\}}$	$0.10255^{\{11\}}$	$0.14089^{\{15\}}$	$0.04664^{\{1\}}$	$0.12824^{\{14\}}$	$0.09839^{\{8\}}$	$0.12364^{\{13\}}$
	D_{abs}	$0.01182^{\{1\}}$	$0.01268^{\{4\}}$	$0.01351^{\{7.5\}}$	$0.01208^{\{3\}}$	$0.01351^{\{7.5\}}$	$0.01356^{\{9\}}$	$0.01291^{\{5\}}$	$0.0134^{\{6\}}$	$0.0148^{\{11\}}$	$0.01494^{\{12\}}$	$0.016^{\{13\}}$	$0.01204^{\{2\}}$	$0.01733^{\{15\}}$	$0.01386^{\{10\}}$	$0.01687^{\{14\}}$
	D_{max}	$0.01903^{\{2\}}$	$0.02082^{\{4\}}$	$0.02243^{\{8\}}$	$0.01969^{\{3\}}$	$0.02228^{\{7\}}$	$0.02295^{\{10\}}$	$0.02117^{\{5\}}$	$0.02188^{\{6\}}$	$0.02381^{\{11\}}$	$0.02425^{\{12\}}$	$0.02642^{\{13\}}$	$0.01826^{\{1\}}$	$0.02832^{\{15\}}$	$0.02247^{\{9\}}$	$0.02769^{\{14\}}$
	$\sum Ranks$	$15^{\{2\}}$	$34^{\{4\}}$	$77.5^{\{10\}}$	$24^{\{3\}}$	$70.5^{\{9\}}$	$91^{\{12\}}$	$42^{\{5\}}$	$51^{\{6\}}$	$62^{\{7\}}$	$84^{\{11\}}$	$116^{\{15\}}$	$9^{\{1\}}$	$112^{\{14\}}$	$64^{\{8\}}$	$108^{\{13\}}$

Table 7. Partial and overall ranks of all the methods of estimation for various values of the parameters.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSD	MSSLD	MSLND
$\delta = 0.7$, $\beta = 2.5$	30	4.0	7.0	11.0	6.0	12.5	15.0	12.5	10.0	2.0	3.0	14.0	1.0	8.5	8.5	5.0
	60	3.0	11.0	14.0	4.0	10.0	15.0	12.0	9.0	2.0	5.0	8.0	1.0	6.0	13.0	7.0
	100	3.0	11.0	13.0	4.0	12.0	15.0	9.0	7.0	2.0	6.0	14.0	1.0	5.0	10.0	8.0
	200	2.0	7.0	14.0	5.0	13.0	15.0	10.0	4.0	3.0	8.0	12.0	1.0	6.0	11.0	9.0
	300	2.0	6.0	13.0	4.0	14.0	15.0	11.0	5.0	3.0	7.0	12.0	1.0	8.5	10.0	8.5
	400	2.0	9.0	14.0	4.0	12.5	15.0	7.0	5.0	3.0	11.0	12.5	1.0	6.0	8.0	10.0
$\delta = 0.25$, $\beta = 0.75$	30	2.0	6.0	11.5	7.0	10.0	13.0	8.0	5.0	3.5	3.5	11.5	1.0	15.0	9.0	14.0
	60	2.0	7.0	10.0	5.0	11.0	13.0	8.0	4.0	3.0	6.0	12.0	1.0	15.0	9.0	14.0
	100	2.0	3.5	8.0	6.0	9.0	14.0	3.5	7.0	5.0	11.0	13.0	1.0	12.0	10.0	15.0
	200	2.0	3.0	10.0	4.5	11.0	13.0	7.0	4.5	6.0	8.0	14.0	1.0	15.0	9.0	12.0
	300	1.0	7.0	12.0	3.0	11.0	14.0	5.0	6.0	4.0	9.0	15.0	2.0	13.0	8.0	10.0
	400	2.0	5.0	10.0	4.0	8.0	13.0	6.0	3.0	7.0	12.0	15.0	1.0	14.0	9.0	11.0
$\delta = 1.5$, $\beta = 1.5$	30	2.0	8.0	12.0	5.0	11.0	13.0	10.0	4.0	3.0	7.0	14.0	1.0	15.0	6.0	9.0
	60	1.0	6.0	10.0	4.0	11.5	14.0	7.0	3.0	5.0	8.5	11.5	2.0	15.0	8.5	13.0
	100	1.0	7.0	11.0	4.0	12.0	13.0	8.0	3.0	6.0	9.0	10.0	2.0	15.0	5.0	14.0
	200	1.5	5.0	10.0	3.0	11.0	14.0	6.5	4.0	6.5	9.0	12.0	1.5	15.0	8.0	13.0
	300	1.0	5.0	10.5	4.0	9.0	13.0	7.0	3.0	6.0	10.5	12.0	2.0	15.0	8.0	14.0
	400	1.0	7.0	10.0	3.0	11.0	14.0	6.0	4.0	9.0	8.0	12.0	2.0	15.0	5.0	13.0
$\delta = 0.5$, $\beta = 2.0$	30	3.0	9.0	14.0	5.0	8.0	15.0	7.0	6.0	2.0	4.0	13.0	1.0	11.0	12.0	10.0
	60	3.0	5.0	14.0	4.0	11.0	15.0	12.0	9.0	2.0	6.0	13.0	1.0	10.0	8.0	7.0
	100	3.0	9.0	13.0	4.0	14.0	15.0	10.0	5.0	2.0	11.0	12.0	1.0	7.5	7.5	6.0
	200	3.0	9.0	12.0	4.0	13.0	15.0	8.0	5.0	2.0	10.0	14.0	1.0	6.0	11.0	7.0
	300	2.0	8.0	14.0	4.0	13.0	15.0	9.0	5.0	3.0	11.0	12.0	1.0	6.0	10.0	7.0
	400	2.0	8.0	14.0	4.0	12.0	15.0	10.0	5.0	3.0	11.0	13.0	1.0	9.0	6.0	7.0
$\delta = 2.5$, $\beta = 0.4$	30	2.0	6.0	11.0	4.5	10.0	15.0	4.5	8.0	3.0	9.0	13.0	1.0	14.0	7.0	12.0
	60	2.0	3.0	7.0	4.0	10.0	12.0	8.0	6.0	5.0	11.0	13.0	1.0	14.5	9.0	14.5
	100	2.0	4.0	9.0	5.0	8.0	12.0	6.0	3.0	7.0	11.0	14.0	1.0	15.0	10.0	13.0
	200	2.0	5.0	7.0	3.0	10.0	12.0	6.0	4.0	8.0	11.0	15.0	1.0	13.0	9.0	14.0
	300	2.0	4.0	8.5	5.0	8.5	12.0	3.0	6.0	11.0	10.0	14.5	1.0	14.5	7.0	13.0
	400	2.0	4.0	10.0	3.0	9.0	12.0	5.0	6.0	7.0	11.0	15.0	1.0	14.0	8.0	13.0
$\sum$ Ranks		62.5	194.5	337.5	129.0	326.0	416.0	232.0	158.5	134.0	257.5	386.0	35.5	348.5	259.5	323.0
Overall Rank		2	6	12	3	11	15	7	5	4	8	14	1	13	9	10

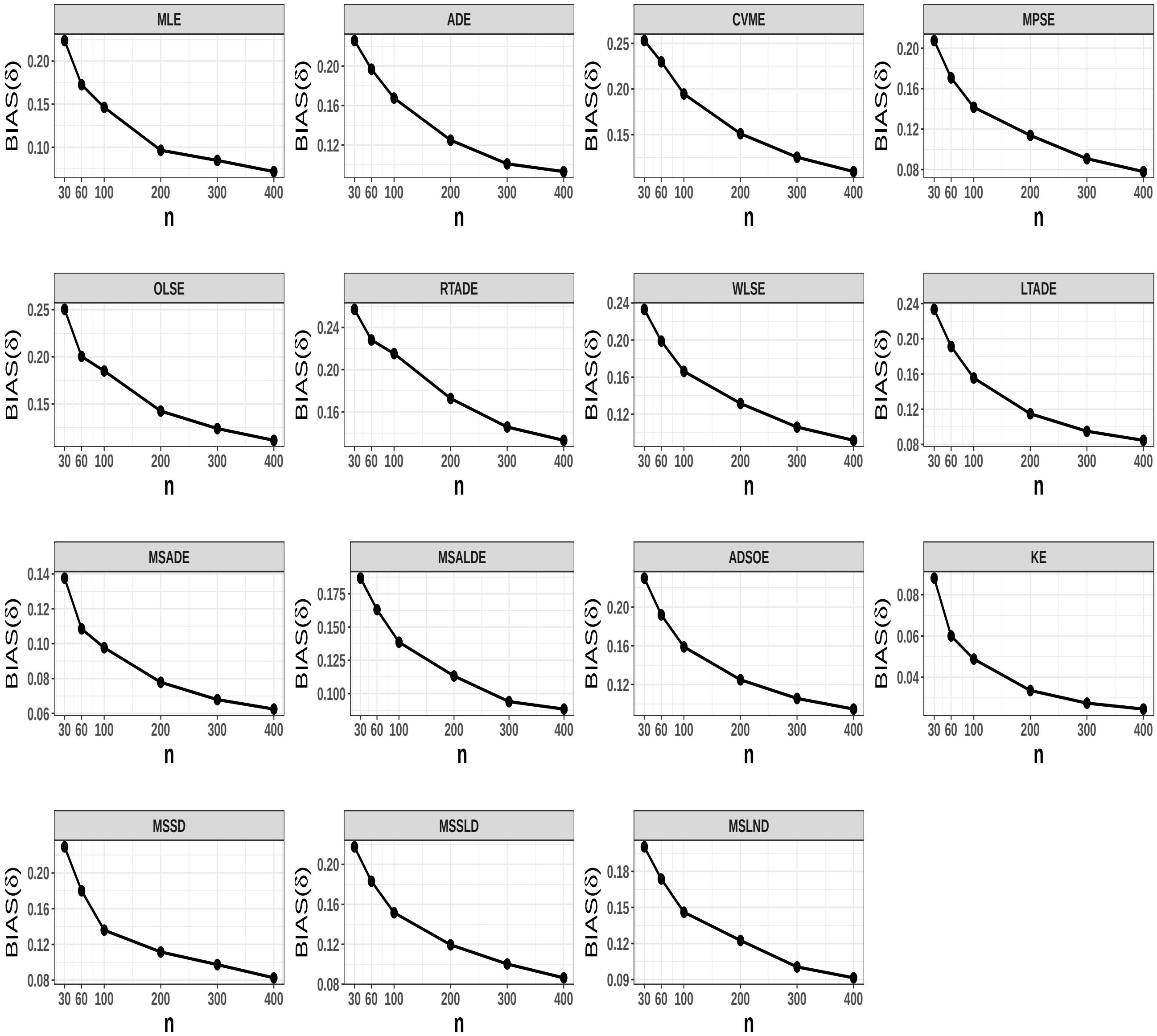

Figure 3. Graphical representations for the BIAS values of $\hat{\delta}$ presented in Table 2.

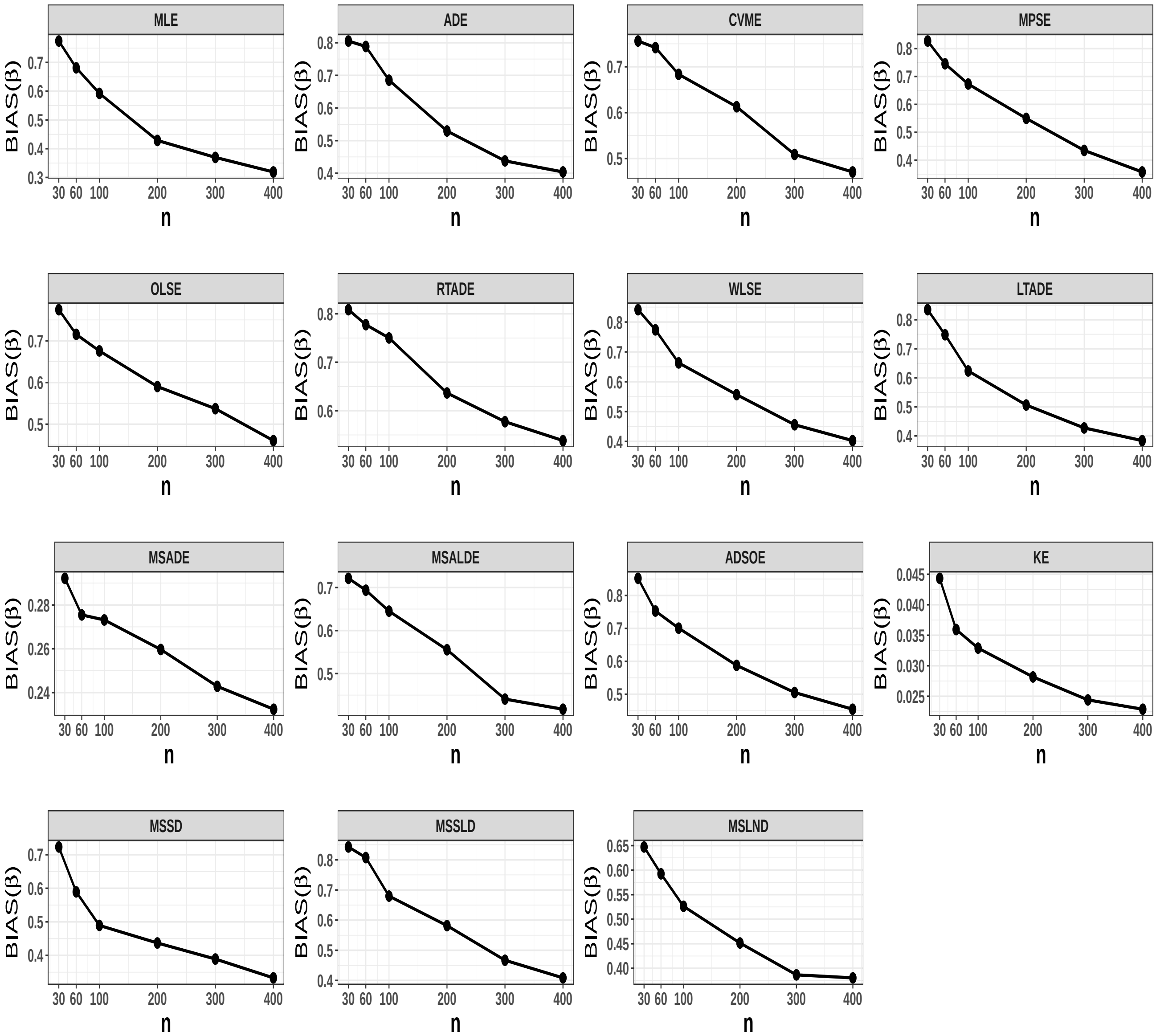

Figure 4. Graphical representations for the BIAS values of $\hat{\beta}$ presented in Table 2.

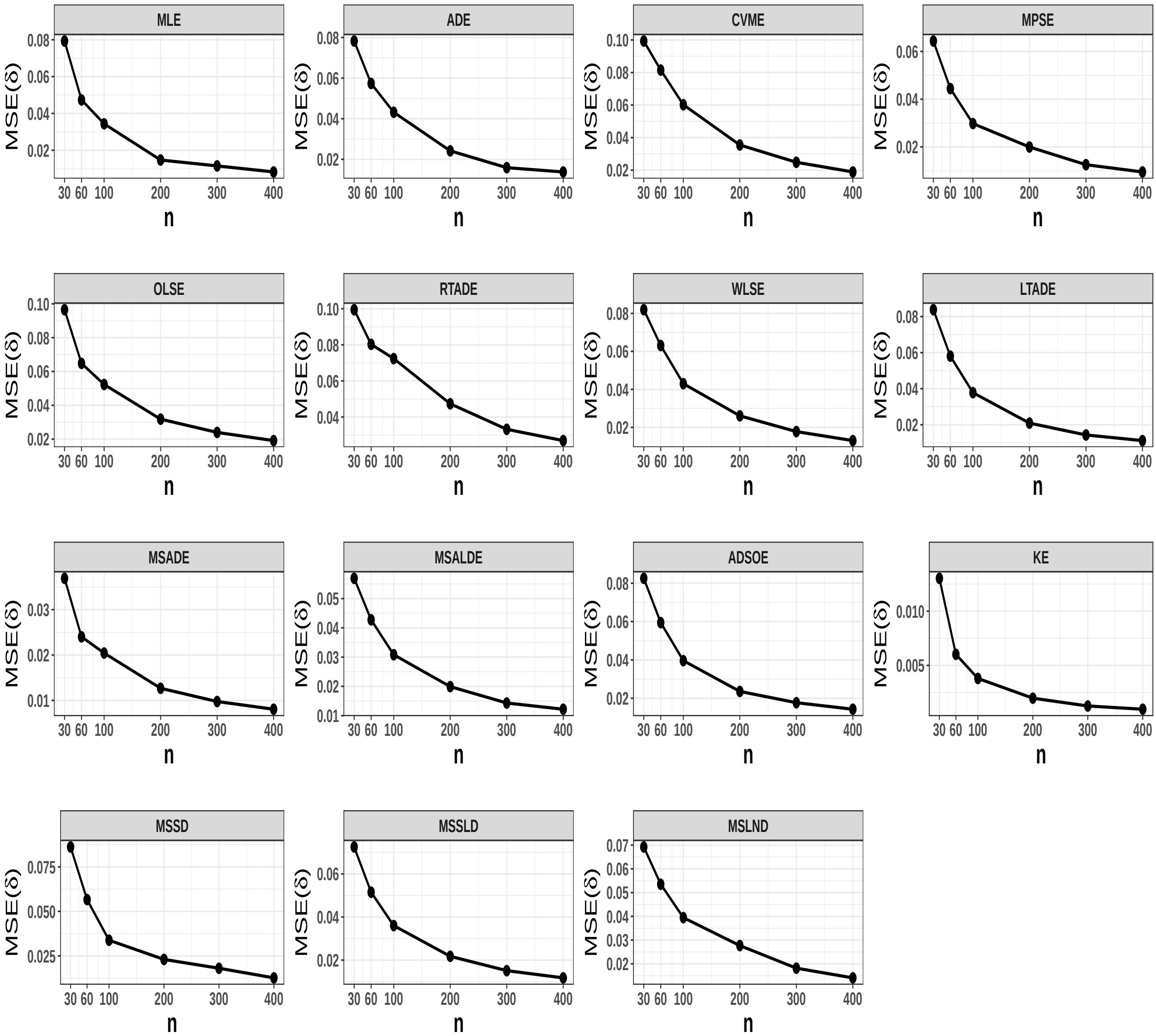

Figure 5. Graphical representations for the MSE values of $\hat{\delta}$ presented in Table 2.

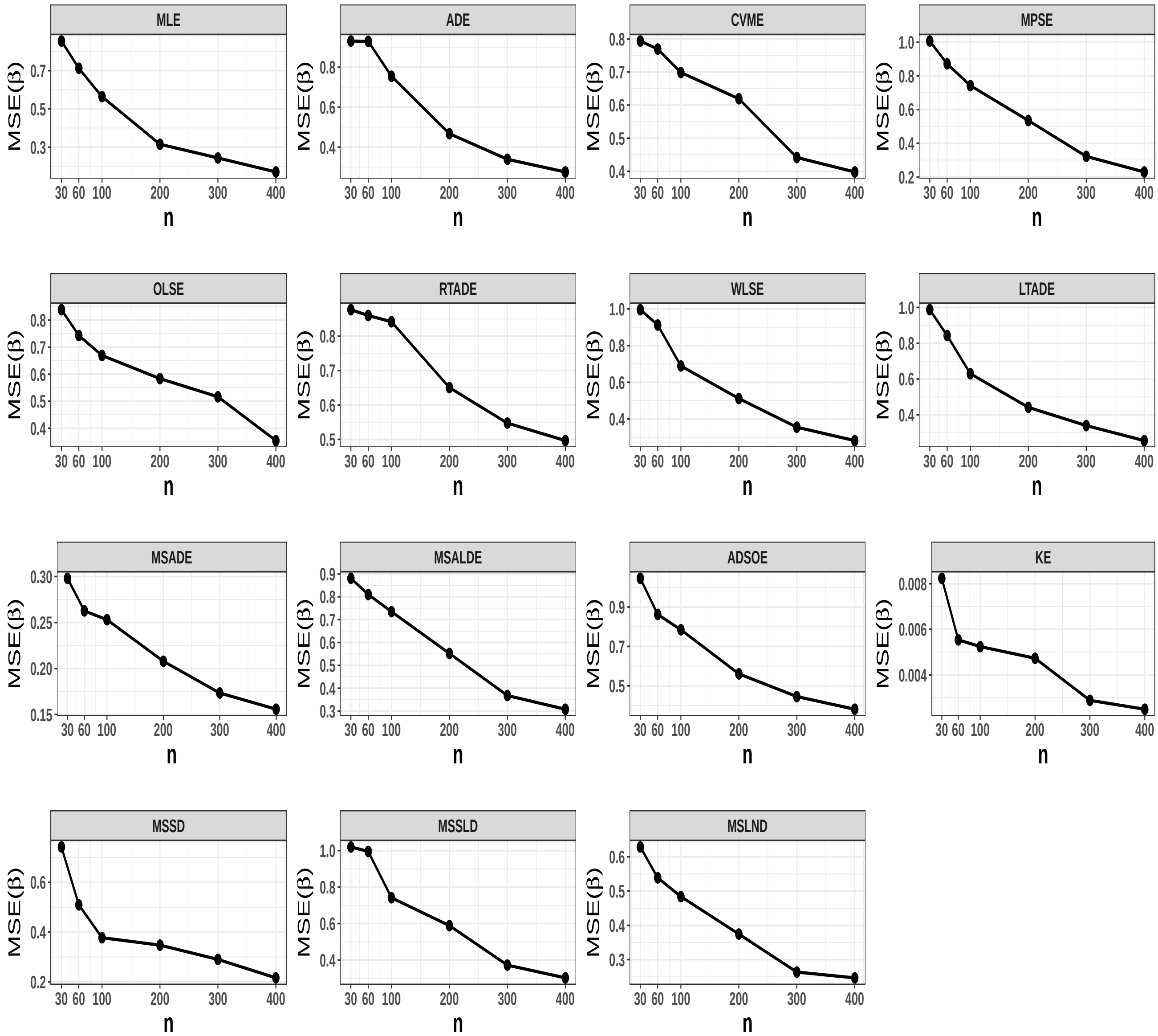

Figure 6. Graphical representations for the MSE values of $\hat{\beta}$ presented in Table 2.

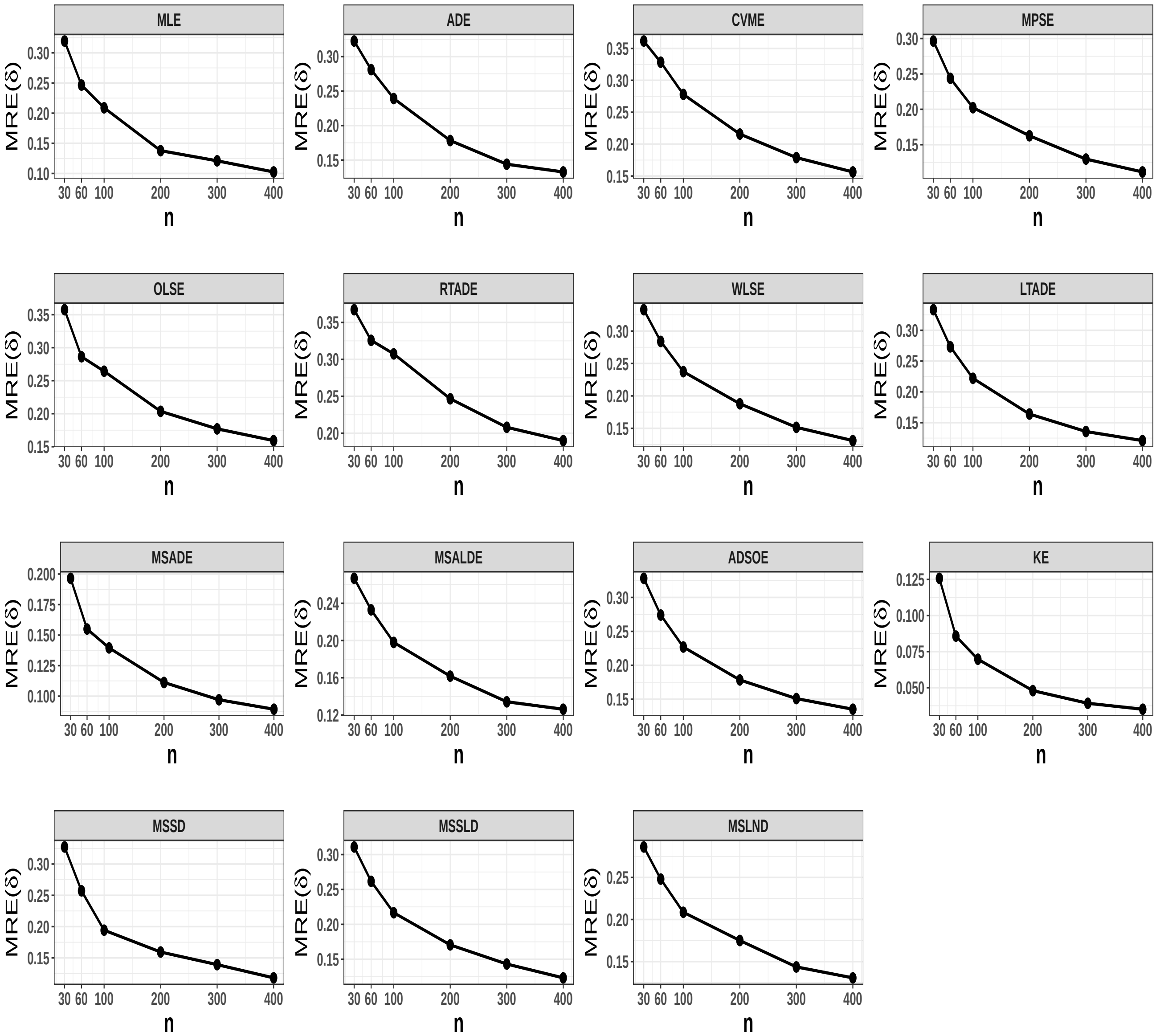

Figure 7. Graphical representations for the MRE values of $\hat{\delta}$ presented in Table 2.

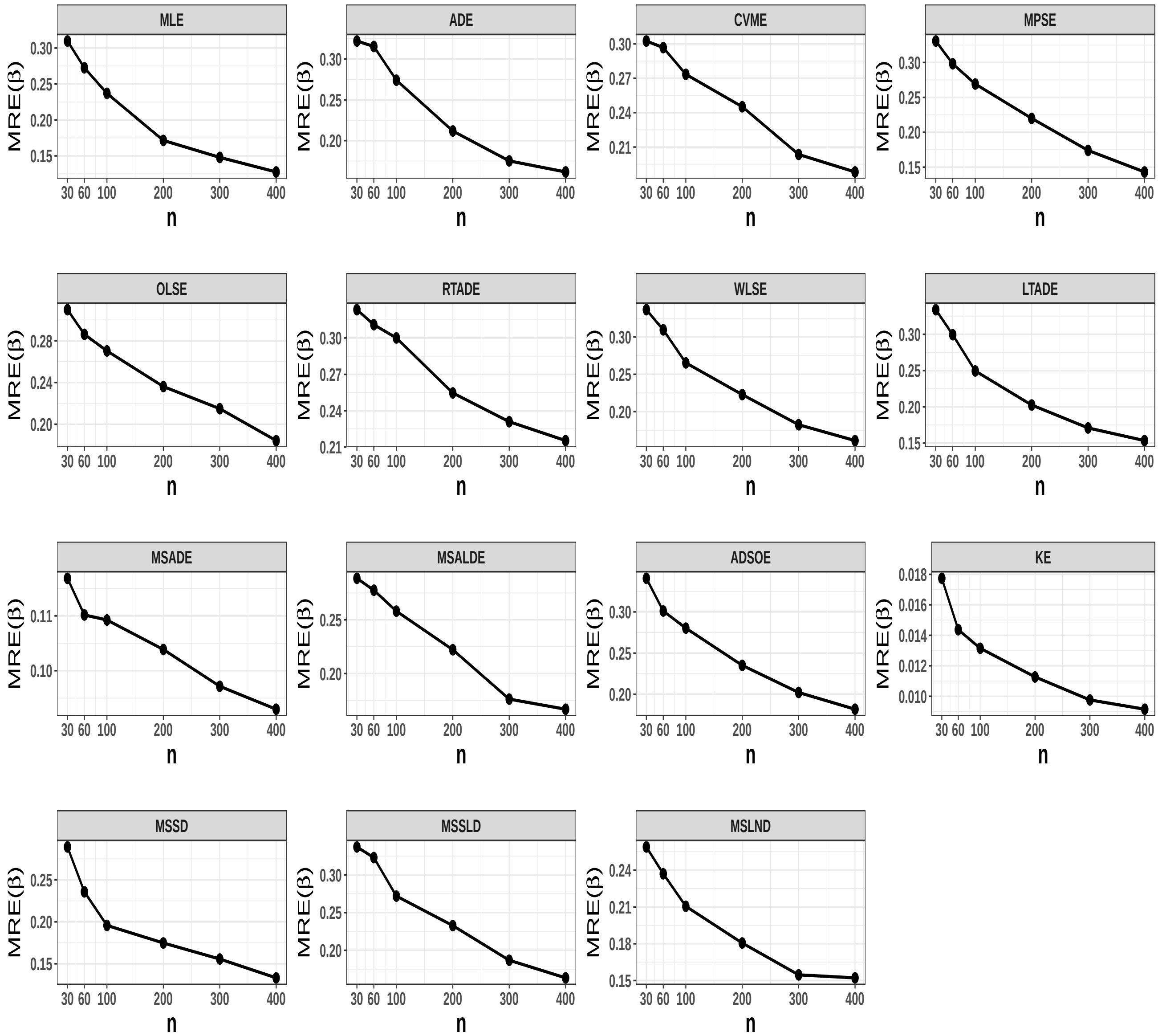

Figure 8. Graphical representations for the MRE values of $\hat{\beta}$ presented in Table 2.

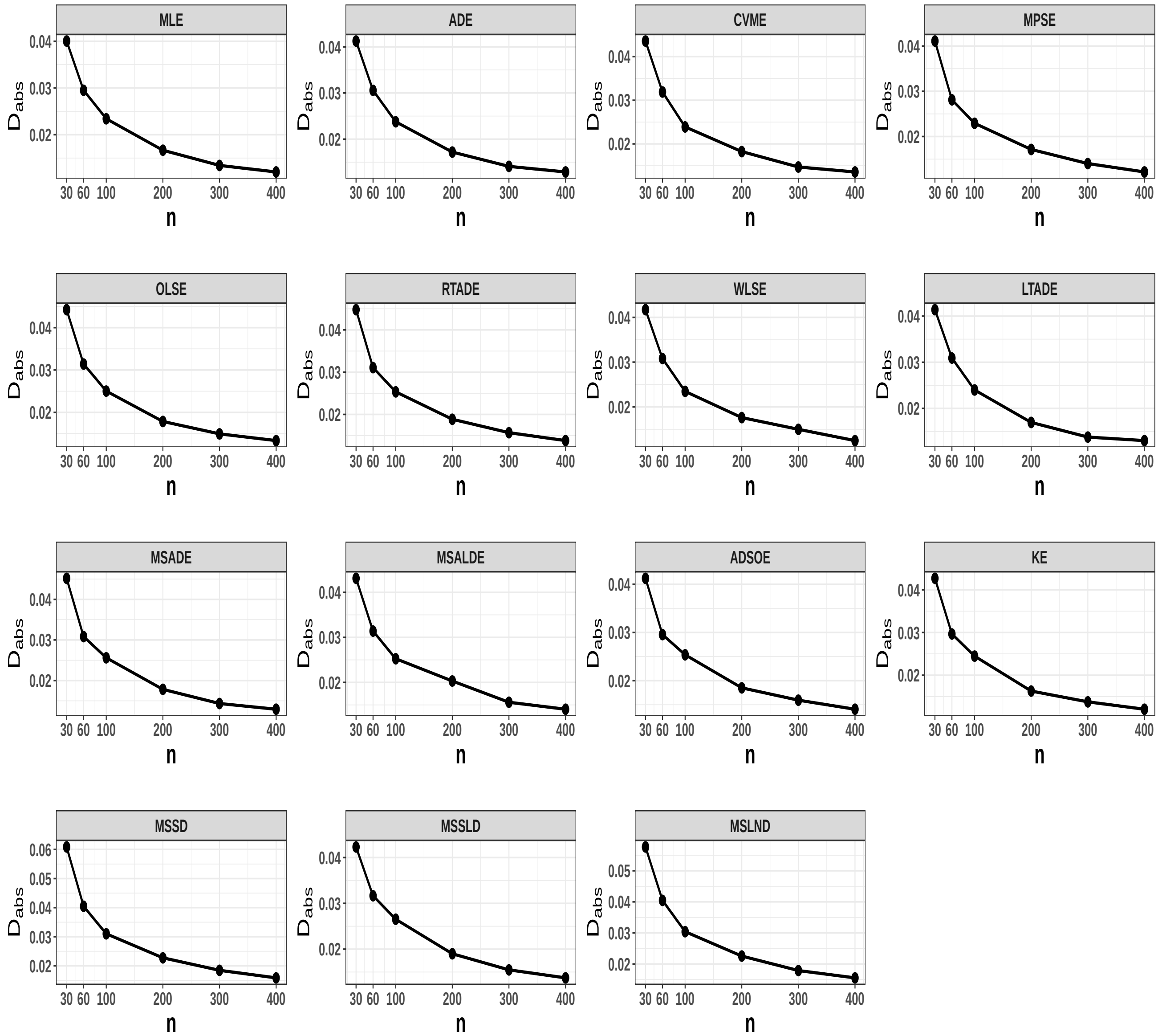

Figure 9. Graphical representations for the D_{abs} values presented in Table 2.

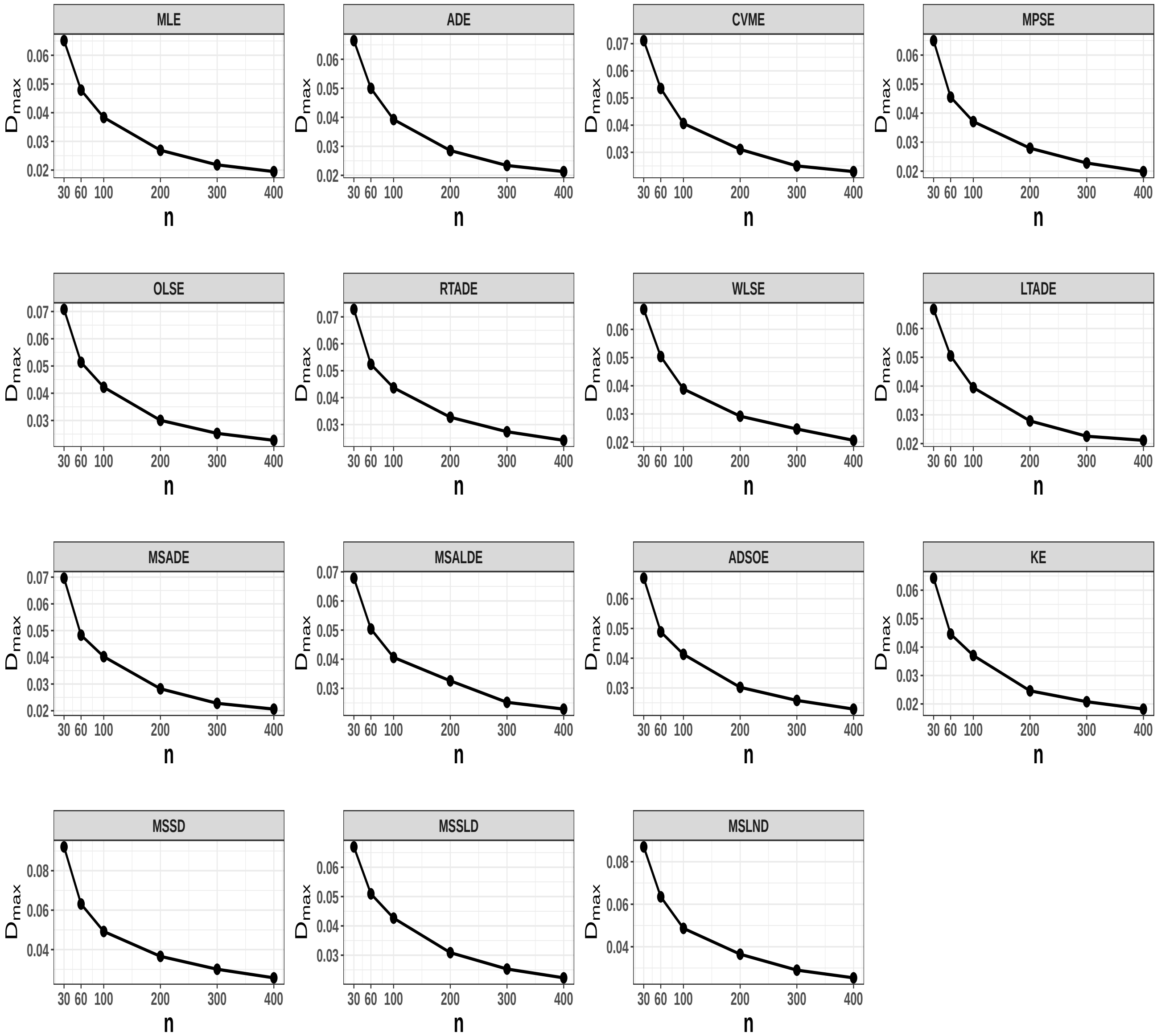

Figure 10. Graphical representations for the D_{max} values presented in Table 2.

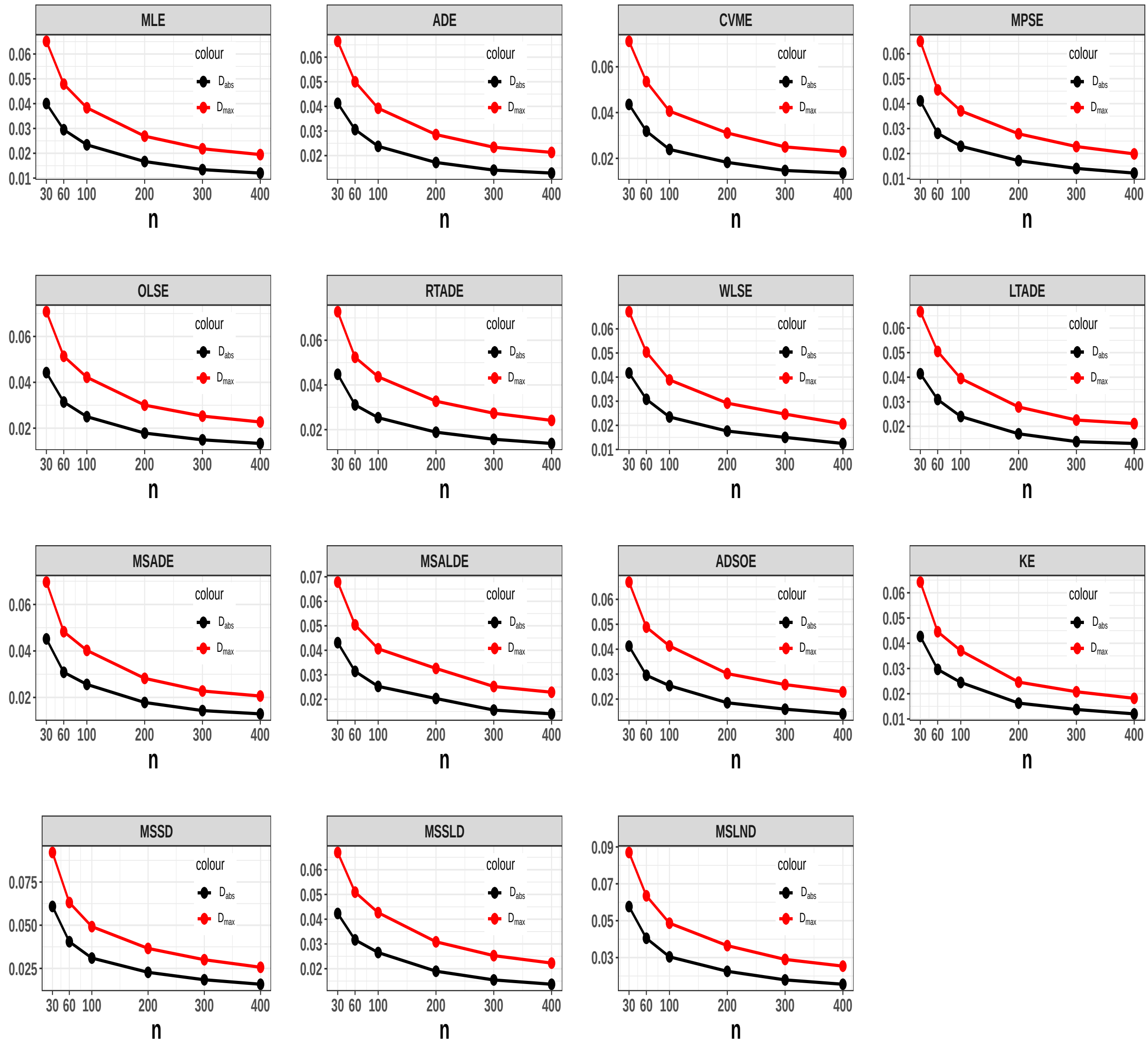

Figure 11. Comparison between the D_{abs} and D_{max} values presented in Table 2.

Table 8. Numerical values of the different entropy measures and their BIAS, MSE, and MRE for $\delta = 1.5$ and $\beta = 2.0$.

n	Measure	RE	ExE	HCE	ArE	TsE	AA1E	AA2E	ShE	DEX	WEX
		(-0.49641)	(0.60871)	(-0.53065)	(-0.39129)	(-0.43960)	(-0.29722)	(0.38680)	(-0.56988)	(-0.94070)	(-0.73958)
20	$\hat{E}$	-0.53095	0.59059	-0.56088	-0.40941	-0.46465	-0.34499	0.45684	-0.60354	-0.97991	-0.76199
	BIAS	0.07837	0.04604	0.07246	0.04604	0.06003	0.05431	0.07916	0.08133	0.08408	0.08889
	MSE	0.00993	0.00330	0.00831	0.00330	0.00570	0.00859	0.01919	0.01077	0.01236	0.01358
	MRE	0.15788	0.07564	0.13656	0.11767	0.13656	0.18274	0.20465	0.14272	0.08938	0.12019
60	$\hat{E}$	-0.51151	0.60061	-0.54401	-0.39939	-0.45067	-0.33185	0.43748	-0.58617	-0.96029	-0.75437
	BIAS	0.04763	0.02850	0.04447	0.02850	0.03684	0.04169	0.06052	0.04871	0.04965	0.05278
	MSE	0.00366	0.00128	0.00315	0.00128	0.00216	0.00597	0.01312	0.00382	0.00409	0.00461
	MRE	0.09595	0.04683	0.08379	0.07285	0.08379	0.14026	0.15646	0.08548	0.05278	0.07137
100	$\hat{E}$	-0.50365	0.60492	-0.53699	-0.39508	-0.44486	-0.32447	0.42653	-0.57848	-0.95201	-0.74854
	BIAS	0.03577	0.02159	0.03354	0.02159	0.02778	0.03496	0.05049	0.03662	0.03749	0.04155
	MSE	0.00203	0.00073	0.00178	0.00073	0.00122	0.00432	0.00937	0.00212	0.00225	0.00277
	MRE	0.07206	0.03546	0.06320	0.05517	0.06320	0.11763	0.13054	0.06426	0.03985	0.05618
150	$\hat{E}$	-0.50062	0.60654	-0.53431	-0.39346	-0.44263	-0.31955	0.41918	-0.57587	-0.94935	-0.74777
	BIAS	0.02875	0.01742	0.02701	0.01742	0.02237	0.02987	0.04291	0.02945	0.03022	0.03410
	MSE	0.00131	0.00048	0.00115	0.00048	0.00079	0.00307	0.00658	0.00136	0.00147	0.00187
	MRE	0.05791	0.02861	0.05090	0.04451	0.05090	0.10051	0.11094	0.05167	0.03213	0.04610
200	$\hat{E}$	-0.49922	0.60731	-0.53305	-0.39269	-0.44159	-0.31498	0.41241	-0.57418	-0.94720	-0.74586
	BIAS	0.02534	0.01537	0.02382	0.01537	0.01973	0.02508	0.03583	0.02571	0.02619	0.02975
	MSE	0.00100	0.00037	0.00088	0.00037	0.00061	0.00205	0.00433	0.00104	0.00110	0.00142
	MRE	0.05105	0.02525	0.04489	0.03928	0.04489	0.08439	0.09264	0.04512	0.02784	0.04023
250	$\hat{E}$	-0.49872	0.60755	-0.53263	-0.39245	-0.44125	-0.31227	0.40843	-0.57363	-0.94642	-0.74544
	BIAS	0.02237	0.01358	0.02104	0.01358	0.01743	0.02194	0.03125	0.02259	0.02298	0.02633
	MSE	0.00078	0.00029	0.00069	0.00029	0.00047	0.00152	0.00317	0.00080	0.00085	0.00111
	MRE	0.04507	0.02231	0.03965	0.03471	0.03965	0.07382	0.08078	0.03964	0.02443	0.03560
300	$\hat{E}$	-0.49867	0.60754	-0.53262	-0.39246	-0.44124	-0.30972	0.40472	-0.57329	-0.94570	-0.74456
	BIAS	0.02057	0.01249	0.01934	0.01249	0.01602	0.01937	0.02751	0.02078	0.02112	0.02426
	MSE	0.00066	0.00024	0.00059	0.00024	0.00040	0.00114	0.00237	0.00068	0.00071	0.00094
	MRE	0.04143	0.02051	0.03645	0.03191	0.03645	0.06516	0.07111	0.03647	0.02246	0.03281
400	$\hat{E}$	-0.49783	0.60800	-0.53187	-0.39200	-0.44061	-0.30769	0.40175	-0.57248	-0.94480	-0.74409
	BIAS	0.01785	0.01084	0.01679	0.01084	0.01391	0.01677	0.02374	0.01791	0.01813	0.02087
	MSE	0.00050	0.00019	0.00045	0.00019	0.00031	0.00078	0.00159	0.00051	0.00053	0.00070
	MRE	0.03595	0.01782	0.03165	0.02772	0.03165	0.05643	0.06138	0.03143	0.01927	0.02822

Table 9. Numerical values of the different entropy measures and their BIAS, MSE, and MRE for $\delta = 0.5$ and $\beta = 1.5$.

n	Measure	RE	ExE	HCE	ArE	TsE	AA1E	AA2E	ShE	DEX	WEX
		(-0.06783)	(0.93442)	(-0.08051)	(-0.06558)	(-0.06670)	(-0.55698)	(0.77529)	(-0.11870)	(-0.61333)	(-0.19500)
20	$\hat{E}$	-0.08600	0.91826	-0.10119	-0.08174	-0.08383	-0.57359	0.81240	-0.14688	-0.64206	-0.20407
	BIAS	0.03295	0.03015	0.03804	0.03015	0.03152	0.13286	0.21409	0.05684	0.06063	0.02178
	MSE	0.00181	0.00148	0.00238	0.00148	0.00163	0.02637	0.06957	0.00541	0.00640	0.00083
	MRE	0.48578	0.03227	0.47257	0.45978	0.47257	0.23853	0.27615	0.47882	0.09886	0.11170
60	$\hat{E}$	-0.07058	0.93218	-0.08350	-0.06782	-0.06918	-0.54615	0.76306	-0.12237	-0.61731	-0.19918
	BIAS	0.01987	0.01844	0.02311	0.01844	0.01914	0.08949	0.14217	0.03542	0.03742	0.01242
	MSE	0.00071	0.00060	0.00095	0.00060	0.00065	0.01272	0.03216	0.00224	0.00257	0.00026
	MRE	0.29296	0.01973	0.28701	0.28119	0.28701	0.16067	0.18337	0.29838	0.06101	0.06367
100	$\hat{E}$	-0.07079	0.93192	-0.08380	-0.06808	-0.06942	-0.55144	0.77021	-0.12307	-0.61799	-0.19801
	BIAS	0.01830	0.01700	0.02129	0.01700	0.01764	0.07792	0.12400	0.03260	0.03419	0.00970
	MSE	0.00057	0.00049	0.00077	0.00049	0.00053	0.00945	0.02392	0.00179	0.00197	0.00015
	MRE	0.26984	0.01819	0.26448	0.25923	0.26448	0.13990	0.15994	0.27467	0.05574	0.04975
150	$\hat{E}$	-0.06964	0.93291	-0.08251	-0.06709	-0.06835	-0.55277	0.77125	-0.12140	-0.61630	-0.19696
	BIAS	0.01545	0.01438	0.01799	0.01438	0.01490	0.06590	0.10492	0.02758	0.02884	0.00791
	MSE	0.00039	0.00034	0.00053	0.00034	0.00036	0.00675	0.01708	0.00124	0.00136	0.00010
	MRE	0.22776	0.01539	0.22349	0.21929	0.22349	0.11832	0.13533	0.23232	0.04703	0.04055
200	$\hat{E}$	-0.06936	0.93313	-0.08220	-0.06687	-0.06810	-0.55416	0.77283	-0.12100	-0.61582	-0.19650
	BIAS	0.01350	0.01258	0.01573	0.01258	0.01303	0.05742	0.09143	0.02413	0.02521	0.00679
	MSE	0.00029	0.00025	0.00039	0.00025	0.00027	0.00513	0.01298	0.00092	0.00101	0.00007
	MRE	0.19902	0.01346	0.19539	0.19182	0.19539	0.10308	0.11793	0.20332	0.04110	0.03481
250	$\hat{E}$	-0.06880	0.93362	-0.08157	-0.06638	-0.06758	-0.55348	0.77135	-0.12007	-0.61481	-0.19632
	BIAS	0.01209	0.01127	0.01409	0.01127	0.01167	0.05144	0.08187	0.02162	0.02254	0.00606
	MSE	0.00023	0.00020	0.00031	0.00020	0.00021	0.00412	0.01041	0.00073	0.00079	0.00006
	MRE	0.17817	0.01206	0.17501	0.17189	0.17501	0.09235	0.10559	0.18216	0.03675	0.03107
300	$\hat{E}$	-0.06849	0.93389	-0.08122	-0.06611	-0.06728	-0.55377	0.77148	-0.11958	-0.61429	-0.19608
	BIAS	0.01080	0.01008	0.01259	0.01008	0.01043	0.04597	0.07317	0.01930	0.02008	0.00549
	MSE	0.00018	0.00016	0.00024	0.00016	0.00017	0.00331	0.00836	0.00057	0.00062	0.00005
	MRE	0.15919	0.01079	0.15641	0.15367	0.15641	0.08253	0.09437	0.16256	0.03273	0.02814
400	$\hat{E}$	-0.06779	0.93451	-0.08043	-0.06549	-0.06663	-0.55280	0.76960	-0.11845	-0.61313	-0.19585
	BIAS	0.00922	0.00861	0.01075	0.00861	0.00891	0.03952	0.06287	0.01649	0.01714	0.00468
	MSE	0.00013	0.00011	0.00018	0.00011	0.00012	0.00246	0.00620	0.00042	0.00045	0.00003
	MRE	0.13587	0.00921	0.13356	0.13128	0.13356	0.07096	0.08109	0.13893	0.02795	0.02402

Table 10. Numerical values of the different entropy measures and their BIAS, MSE, and MRE for $\delta = 2.5$ and $\beta = 0.75$.

n	Measure	RE (-0.50778)	ExE (0.60183)	HCE (-0.54131)	ArE (-0.39817)	TsE (-0.44844)	AA1E (-0.41510)	AA2E (0.55685)	ShE (-0.56943)	DEX (-0.94546)	WEX (-0.64094)
20	$\hat{E}$	-0.53963	0.58474	-0.56950	-0.41526	-0.47179	-0.42231	0.56992	-0.60692	-0.99220	-0.68195
	BIAS	0.06486	0.03768	0.05965	0.03768	0.04941	0.06592	0.09848	0.07228	0.08395	0.06627
	MSE	0.00716	0.00232	0.00593	0.00232	0.00407	0.00626	0.01412	0.00905	0.01324	0.00793
	MRE	0.12773	0.06261	0.11019	0.09464	0.11019	0.15882	0.17685	0.12693	0.08879	0.10339
60	$\hat{E}$	-0.51075	0.60061	-0.54366	-0.39939	-0.45038	-0.39987	0.53536	-0.57260	-0.94830	-0.65192
	BIAS	0.03416	0.02045	0.03190	0.02045	0.02642	0.04356	0.06430	0.03772	0.04263	0.03160
	MSE	0.00191	0.00068	0.00165	0.00068	0.00114	0.00303	0.00661	0.00240	0.00323	0.00169
	MRE	0.06728	0.03398	0.05893	0.05135	0.05893	0.10495	0.11547	0.06624	0.04509	0.04930
100	$\hat{E}$	-0.50817	0.60195	-0.54141	-0.39805	-0.44852	-0.40209	0.53840	-0.56973	-0.94512	-0.64743
	BIAS	0.02656	0.01595	0.02484	0.01595	0.02058	0.03806	0.05627	0.02990	0.03474	0.02354
	MSE	0.00118	0.00042	0.00102	0.00042	0.00070	0.00235	0.00513	0.00154	0.00218	0.00093
	MRE	0.05231	0.02650	0.04589	0.04005	0.04589	0.09169	0.10105	0.05251	0.03675	0.03672
150	$\hat{E}$	-0.50866	0.60156	-0.54195	-0.39844	-0.44896	-0.40642	0.54463	-0.57036	-0.94611	-0.64604
	BIAS	0.02290	0.01375	0.02142	0.01375	0.01774	0.03357	0.04972	0.02612	0.03078	0.01965
	MSE	0.00085	0.00031	0.00074	0.00031	0.00051	0.00177	0.00388	0.00113	0.00162	0.00064
	MRE	0.04510	0.02284	0.03956	0.03452	0.03956	0.08087	0.08930	0.04588	0.03256	0.03065
200	$\hat{E}$	-0.50805	0.60185	-0.54143	-0.39815	-0.44853	-0.40717	0.54558	-0.56956	-0.94505	-0.64468
	BIAS	0.01964	0.01180	0.01838	0.01180	0.01523	0.02931	0.04342	0.02235	0.02632	0.01691
	MSE	0.00061	0.00022	0.00054	0.00022	0.00037	0.00134	0.00294	0.00081	0.00115	0.00046
	MRE	0.03868	0.01961	0.03395	0.02965	0.03395	0.07061	0.07797	0.03925	0.02783	0.02639
250	$\hat{E}$	-0.50768	0.60204	-0.54111	-0.39796	-0.44827	-0.40773	0.54631	-0.56909	-0.94448	-0.64387
	BIAS	0.01792	0.01078	0.01678	0.01078	0.01390	0.02639	0.03910	0.02037	0.02394	0.01534
	MSE	0.00051	0.00018	0.00044	0.00018	0.00030	0.00108	0.00237	0.00066	0.00093	0.00038
	MRE	0.03529	0.01791	0.03099	0.02707	0.03099	0.06359	0.07022	0.03578	0.02532	0.02394
300	$\hat{E}$	-0.50732	0.60223	-0.54079	-0.39777	-0.44801	-0.40808	0.54678	-0.56866	-0.94398	-0.64321
	BIAS	0.01623	0.00976	0.01520	0.00976	0.01259	0.02435	0.03607	0.01864	0.02221	0.01370
	MSE	0.00042	0.00015	0.00037	0.00015	0.00025	0.00091	0.00200	0.00056	0.00078	0.00030
	MRE	0.03196	0.01623	0.02807	0.02452	0.02807	0.05866	0.06477	0.03274	0.02349	0.02138
400	$\hat{E}$	-0.50687	0.60247	-0.54040	-0.39753	-0.44768	-0.40877	0.54769	-0.56812	-0.94337	-0.64228
	BIAS	0.01396	0.00840	0.01307	0.00840	0.01083	0.02063	0.03056	0.01594	0.01885	0.01196
	MSE	0.00031	0.00011	0.00027	0.00011	0.00019	0.00065	0.00142	0.00040	0.00055	0.00023
	MRE	0.02748	0.01396	0.02415	0.02111	0.02415	0.04969	0.05487	0.02800	0.01994	0.01866

Table 11. Numerical values of the different entropy measures and their BIAS, MSE, and MRE for $\delta = 0.9$ and $\beta = 0.5$.

n	Measure	RE	ExE	HCE	ArE	TsE	AA1E	AA2E	ShE	DEX	WEX
		(-0.24470)	(0.78294)	(-0.27803)	(-0.21706)	(-0.23032)	(-0.93457)	(1.43805)	(-0.42447)	(-0.96243)	(-0.20000)
20	$\hat{E}$	-0.22848	0.79765	-0.25933	-0.20235	-0.21483	-0.89872	1.37489	-0.39453	-0.93073	-0.20048
	BIAS	0.06023	0.04755	0.06458	0.04755	0.05350	0.08863	0.16768	0.10001	0.12505	0.01564
	MSE	0.00513	0.00318	0.00587	0.00318	0.00403	0.01248	0.04391	0.01414	0.02254	0.00041
	MRE	0.24614	0.06073	0.23228	0.21907	0.23228	0.09483	0.11660	0.23562	0.12993	0.07820
60	$\hat{E}$	-0.22444	0.79973	-0.25575	-0.20027	-0.21187	-0.90532	1.38427	-0.39031	-0.92289	-0.19737
	BIAS	0.03918	0.03122	0.04221	0.03122	0.03497	0.05784	0.10973	0.06527	0.08070	0.00931
	MSE	0.00233	0.00150	0.00272	0.00150	0.00187	0.00543	0.01921	0.00653	0.00994	0.00013
	MRE	0.16010	0.03987	0.15182	0.14382	0.15182	0.06189	0.07631	0.15377	0.08385	0.04657
100	$\hat{E}$	-0.22910	0.79586	-0.26088	-0.20414	-0.21612	-0.91192	1.39608	-0.39810	-0.93166	-0.19806
	BIAS	0.03308	0.02624	0.03556	0.02624	0.02946	0.04549	0.08654	0.05470	0.06766	0.00776
	MSE	0.00179	0.00113	0.00207	0.00113	0.00142	0.00351	0.01253	0.00486	0.00735	0.00010
	MRE	0.13519	0.03351	0.12789	0.12087	0.12789	0.04868	0.06018	0.12887	0.07030	0.03881
150	$\hat{E}$	-0.23449	0.79145	-0.26676	-0.20855	-0.22099	-0.91935	1.40990	-0.40710	-0.94232	-0.19880
	BIAS	0.02881	0.02270	0.03087	0.02270	0.02557	0.03868	0.07394	0.04755	0.05926	0.00679
	MSE	0.00132	0.00082	0.00151	0.00082	0.00104	0.00245	0.00889	0.00358	0.00556	0.00007
	MRE	0.11772	0.02900	0.11102	0.10459	0.11102	0.04138	0.05141	0.11203	0.06158	0.03394
200	$\hat{E}$	-0.23524	0.79074	-0.26765	-0.20926	-0.22173	-0.92085	1.41249	-0.40845	-0.94356	-0.19891
	BIAS	0.02538	0.02001	0.02721	0.02001	0.02254	0.03402	0.06510	0.04188	0.05223	0.00608
	MSE	0.00098	0.00061	0.00113	0.00061	0.00078	0.00184	0.00669	0.00268	0.00415	0.00006
	MRE	0.10374	0.02556	0.09785	0.09220	0.09785	0.03640	0.04527	0.09866	0.05426	0.03038
250	$\hat{E}$	-0.23646	0.78972	-0.26900	-0.21028	-0.22285	-0.92295	1.41637	-0.41058	-0.94608	-0.19896
	BIAS	0.02336	0.01841	0.02504	0.01841	0.02074	0.03051	0.05844	0.03856	0.04798	0.00550
	MSE	0.00082	0.00051	0.00095	0.00051	0.00065	0.00149	0.00544	0.00224	0.00346	0.00005
	MRE	0.09548	0.02352	0.09005	0.08483	0.09005	0.03265	0.04064	0.09083	0.04986	0.02752
300	$\hat{E}$	-0.23566	0.79028	-0.26819	-0.20972	-0.22217	-0.92243	1.41526	-0.40936	-0.94439	-0.19879
	BIAS	0.02066	0.01630	0.02215	0.01630	0.01835	0.02713	0.05194	0.03399	0.04211	0.00489
	MSE	0.00066	0.00041	0.00076	0.00041	0.00052	0.00124	0.00450	0.00181	0.00278	0.00004
	MRE	0.08442	0.02082	0.07967	0.07511	0.07967	0.02903	0.03612	0.08008	0.04375	0.02447
400	$\hat{E}$	-0.23764	0.78865	-0.27036	-0.21135	-0.22397	-0.92502	1.42006	-0.41266	-0.94823	-0.19911
	BIAS	0.01779	0.01402	0.01906	0.01402	0.01579	0.02303	0.04416	0.02926	0.03636	0.00412
	MSE	0.00048	0.00030	0.00055	0.00030	0.00038	0.00085	0.00310	0.00129	0.00200	0.00003
	MRE	0.07271	0.01790	0.06856	0.06457	0.06856	0.02464	0.03071	0.06893	0.03777	0.02061

In order to have graphical benchmark, the values from Table 8 are also visualized in Figures 12–14.

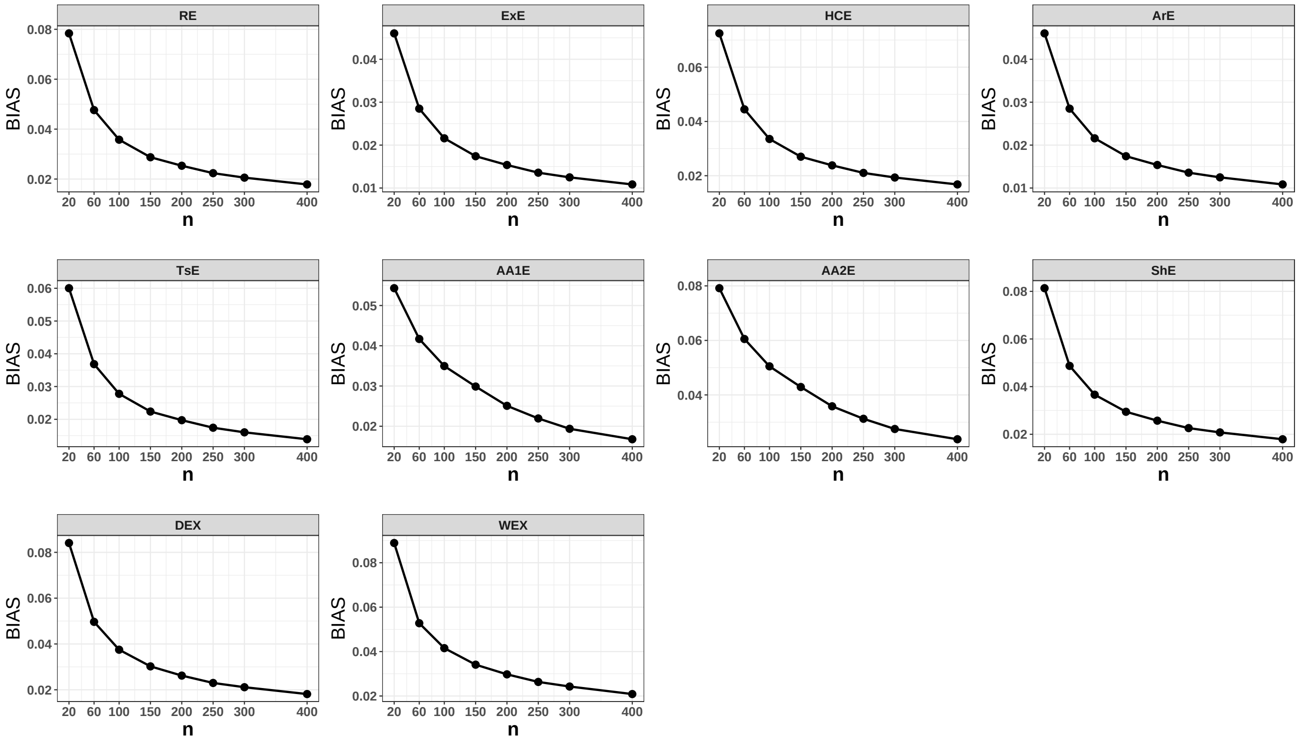

Figure 12. Graphical representation for the BIAS values presented in Table 8.

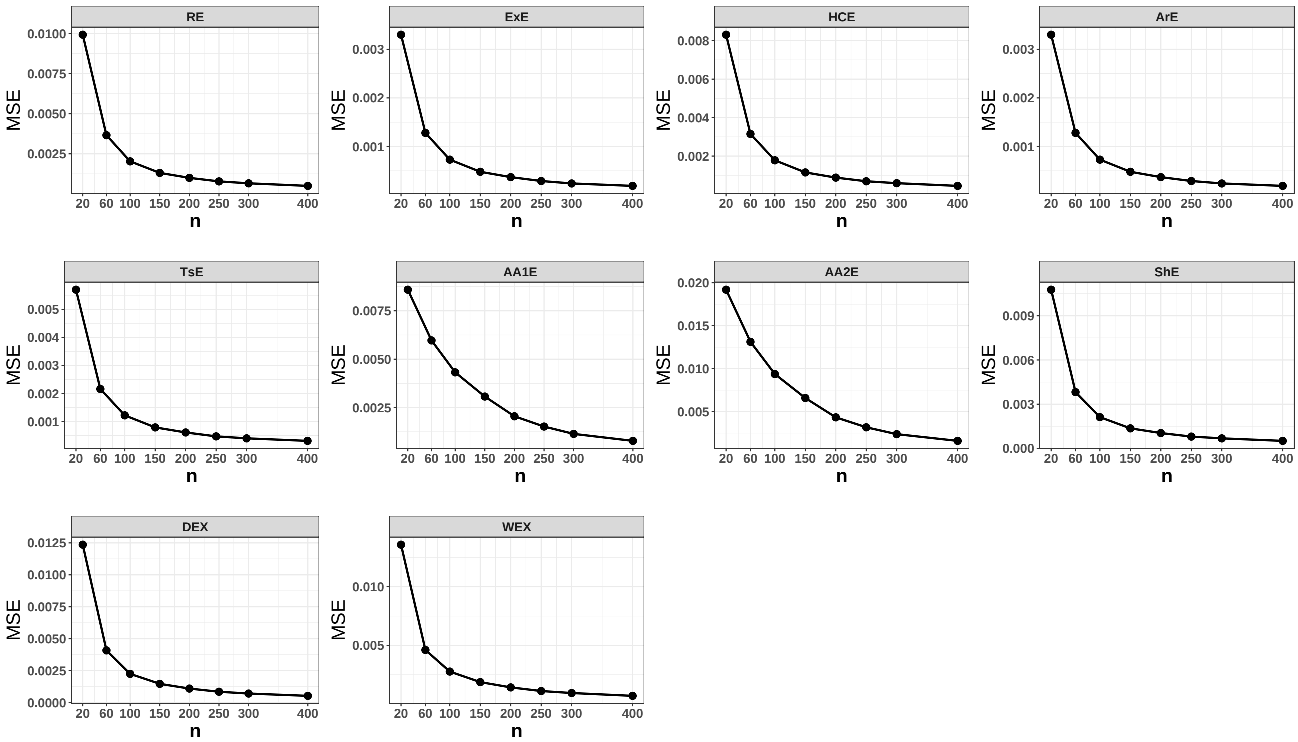

Figure 13. Graphical representation for the MSE values presented in Table 8.

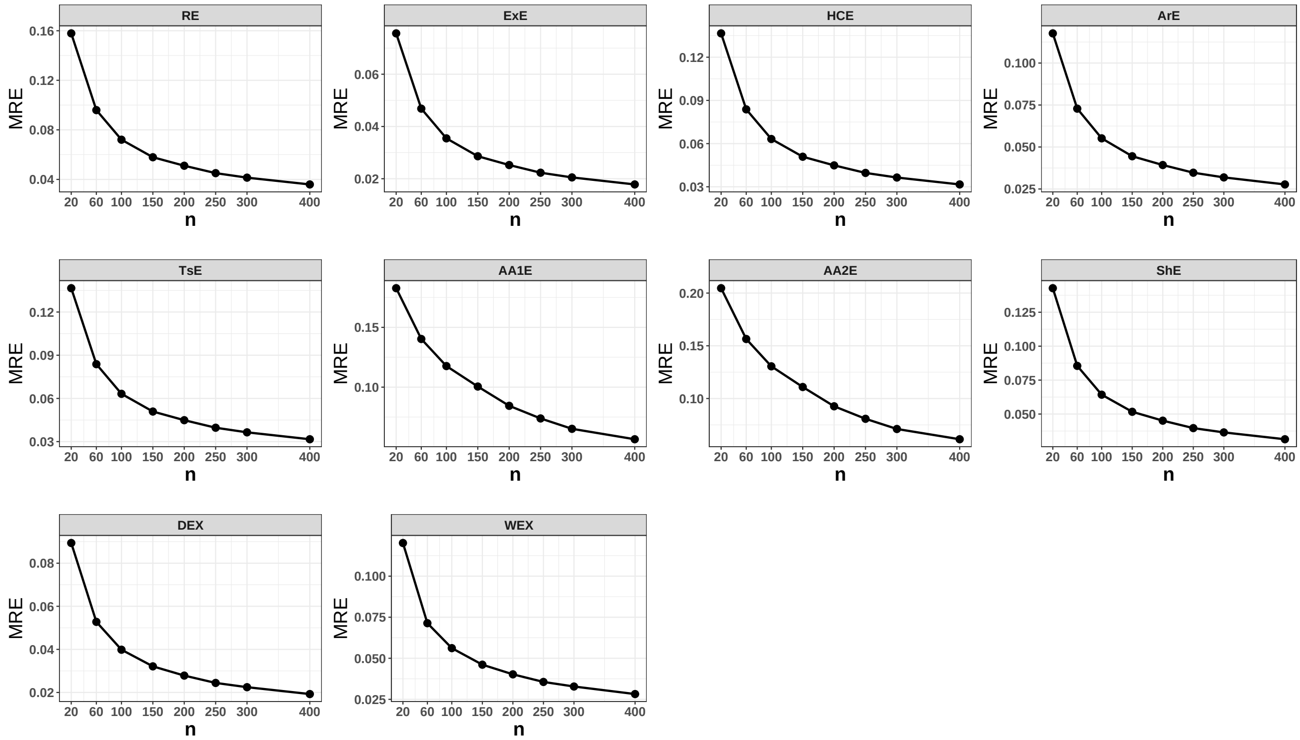

Figure 14. Graphical representation for the MRE values presented in Table 8.

In these figures, we can observe the fast decay of all curves with relatively small values for n. This confirms the efficiency of the estimation of the entropy measures considered in the context of the PUILD.

7. Real data analysis

In this section, we illustrate the importance and adaptability of the underlying PUILD model for fitting real-world unit data across different disciplines. In fact, two real datasets are considered and described below.

The first dataset contains failure rates for twenty mechanical parts. It was studied by Murthy et al. [53]. The corresponding values are 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, and 0.485.

The second dataset was studied by Krishna et al. [54]. It is about the highest flood level (measured in millions of cubic feet per second) that occurred at Harrisburg, Pennsylvania, on the Susquehanna River during twenty years spanning from 1890 to 1969. The corresponding values are 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.296, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, and 0.265.

Some graphical representations of these two datasets are shown in Figures 15 and 16, respectively. These include the histograms, kernel density estimates, violin plots, box plots, total time on test plots, and quantile-quantile (QQ) plots.

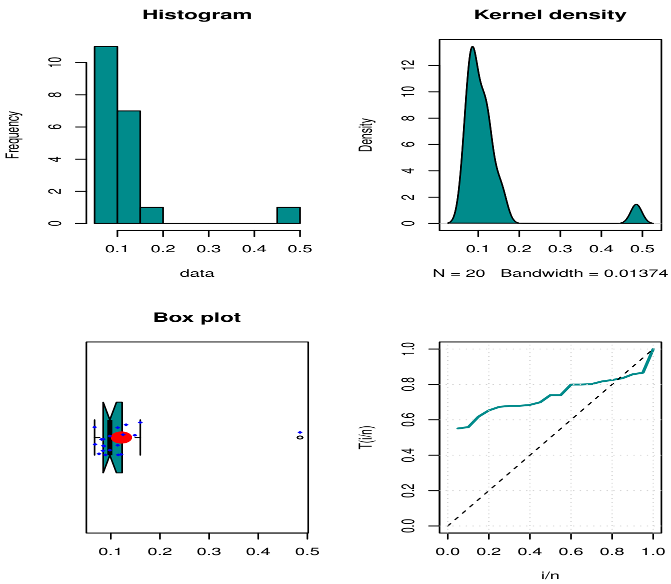

Figure 15. Some nonparametric plots for the first dataset.

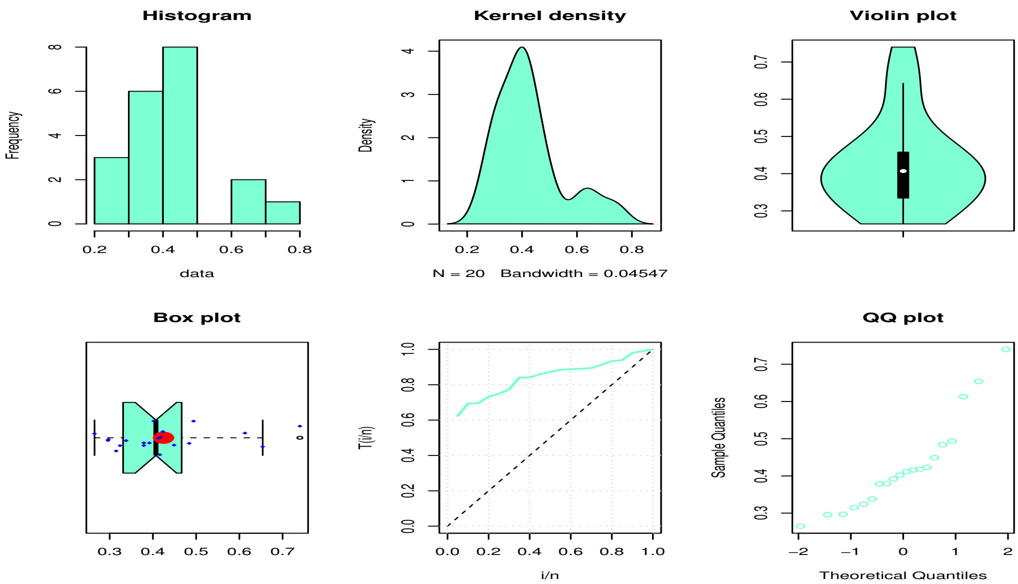

Figure 16. Some nonparametric plots for the second dataset.

These figures show that the first dataset is mainly right-skewed with some outliers and has an increasing HRF and that the second dataset is almost symmetrical also with an increasing HRF. These characteristics can be handled by the PUILD model as developed in the theoretical results.

The models compared with the PUILD model are derived from the UIL distribution, the exponentiated Topp-Leone (ETL) distribution [55], the Kumaraswamy (Km) distribution, the beta (Be) distribution, and the transformed gamma (TrG) distribution [56]. Preliminary tests show that our proposed model performs very well when compared to the others, which are known for their ability to fit the real datasets considered, using the maximum likelihood method. All the relevant parameter estimates and standard errors (SEs) for the two real datasets are presented in Tables 12 and 13, respectively.

To support this claim, we use a variety of information criteria (ICs), including Akaike IC (Aic), corrected Akaike IC (Caic), Bayesian IC (Bic), and Hannan-Quinn IC (Hqic), to determine which model is most appropriate for fitting the two datasets. We also consider the goodness of fit metrics, including Anderson-Darling (A), Cramér-von Mises (W), and Kolmogorov-Smirnov (KS) with its p-value (KSp). The main novelty of this part is that we use new measures of uncertainty to compare the models, namely, ShE, DEX, and WEX. It is known that the model with less uncertainty information is the best.

All compared measures for the two datasets are presented in Tables 14 and 15, respectively.

Plots of the estimated CDFs and histograms with the estimated PDFs of all compared models for the two datasets are shown in Figures 17 and 18, respectively.

Clearly, the estimated curves fit the empirical objects very well.

The behavior of the L-LF with our proposed model estimates is also shown in Figures 19 and 20, respectively.

As expected, the uniqueness of these estimates is confirmed, as shown by the unique red point.

Table 12. Estimated parameters with the SEs of all the compared models for the first dataset.

Model	$\hat{\delta}$	SE($\hat{\delta}$)	$\hat{\beta}$	SE($\hat{\beta}$)
PUIL	2.4144	0.4321	0.0068	0.0072
UIL	0.2045	0.0326		
ETL	1.7370	0.2896	9.7115	3.8780
Km	1.5878	0.2444	21.8673	10.2082
Be	3.1127	0.9368	21.8246	7.0422
TrG	14.6813	2.3213		

Table 13. Estimated parameters with the SEs of all the compared models for the second dataset.

Model	$\hat{\delta}$	SE($\hat{\delta}$)	$\hat{\beta}$	SE($\hat{\beta}$)
PUIL	2.9709	0.5340	0.1091	0.0645
UIL	0.9867	0.1666		
ETL	4.6858	0.9595	4.1306	1.5083
Km	3.4039	0.6073	12.0731	5.4978
Be	6.9757	2.1638	9.3522	2.9276
TrG	3.4438	0.54452		

Table 14. Numerical values for analyzing the first dataset.

Model	Aic	Caic	Bic	Hqic	A	W	KS	KSp	ShE	DEX	WEX
PUIL	-71.5426	-70.8367	-69.5511	-71.1538	0.4162	0.0500	0.1259	0.9092	-1.9713	-4.6321	-0.4527
UIL	-57.9514	-57.7292	-56.9557	-57.7570	2.6972	0.5309	0.3036	0.05012	-0.9807	-1.9026	-0.1944
ETL	48.2272	-47.5213	-46.2358	-47.8385	2.6147	0.4524	0.2641	0.1229	-1.2521	-2.0067	-0.2121
Km	-47.2969	-46.5910	-45.3054	-46.9081	2.6889	0.4681	0.2627	0.1265	-1.2290	-1.9560	-0.2031
Be	-51.7626	-51.0567	-49.7711	-51.3738	2.2611	0.3727	0.2538	0.1521	-1.3941	-2.3467	-0.2561
TrG	-51.8497	-51.6275	-50.8540	-51.6553	2.5040	0.4327	0.2709	0.1062	-1.2456	-2.0362	-0.2010

Table 15. Numerical values for analyzing the second dataset.

Model	Aic	Caic	Bic	Hqic	A	W	KS	KSp	ShE	DEX	WEX
PUIL	-30.0341	-29.3282	-28.0427	-29.6454	0.2522	0.0425	0.1226	0.9247	-0.8583	-1.4576	-0.5632
UIL	-16.4854	-16.2631	-15.4896	-16.2910	2.4384	0.4656	0.3009	0.0535	-0.2021	-0.6580	-0.2946
ETL	-23.6156	-22.9097	-21.6241	-23.2268	0.8845	0.1553	0.2110	0.3353	-0.6532	-1.0934	-0.4659
Km	-22.0935	-21.3876	-20.1020	-21.7047	1.0040	0.17602	0.2151	0.3132	-0.6031	-1.0336	-0.4439
Be	-24.6329	-23.9270	-22.6414	-24.2441	0.7991	0.1345	0.2038	0.3771	-0.7158	-1.1654	-0.4923
TrG	-15.3907	-15.1684	-14.3949	-15.1963	2.1343	0.3959	0.2905	0.0684	-0.2401	-0.6892	-0.2587

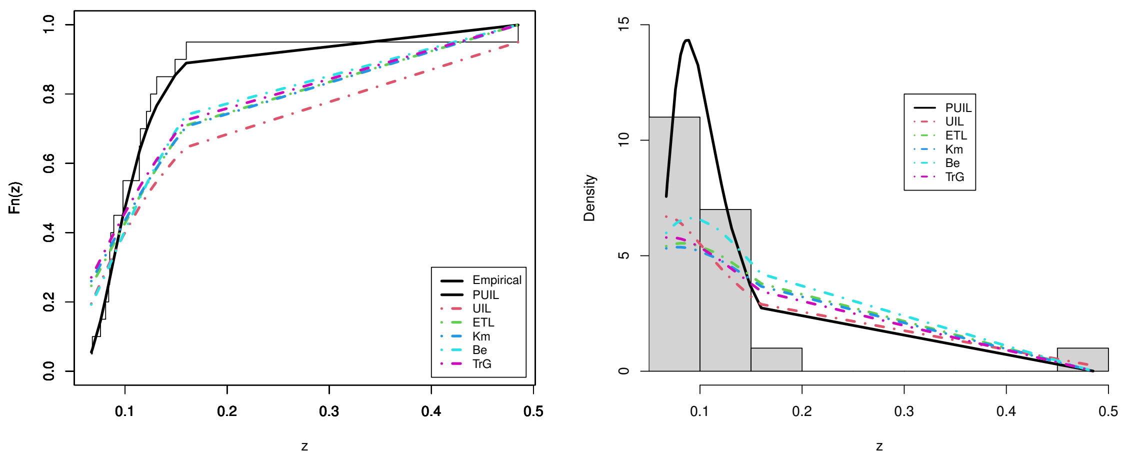

Figure 17. Estimated CDFs and PDFs for the first dataset.

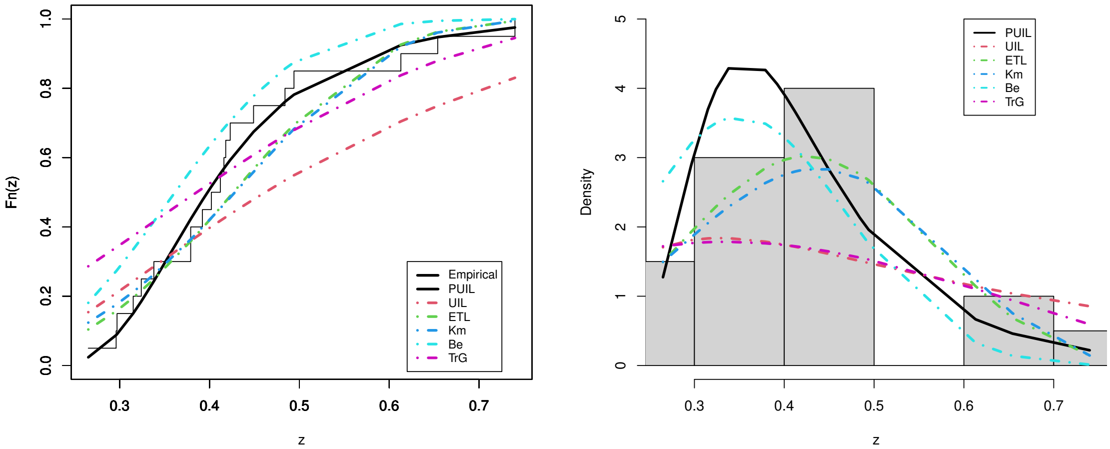

Figure 18. Estimated CDFs and PDFs for the second dataset.

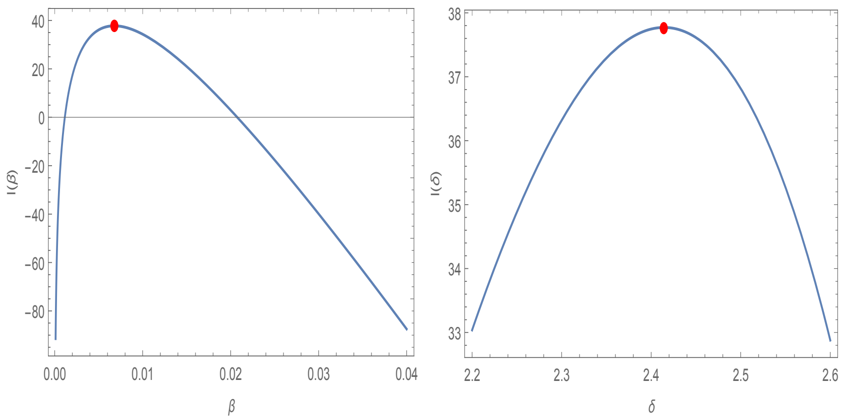

Figure 19. Profile plots of the L-LF for the first dataset.

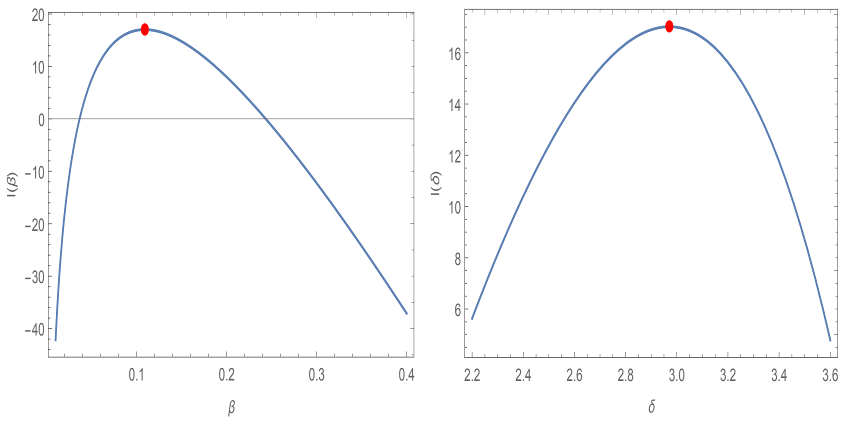

Figure 20. Profile plots of the L-LF for the second dataset.

8. Concluding remarks

This article focuses on the PUILD, which is presented as a new valuable generalization of the UILD. An analysis showed that it has a PDF that can be unimodal, decreasing, increasing, or right-skewed. On the other hand, the HRF can be U-shaped, N-shaped, or increasing. The mode, quantiles, median, skewness, moments, variance, coefficient of variation, index of dispersion, harmonic mean, incomplete moments, inverse moments, and Lorenz and Bonferroni curves are among the many measures calculated in closed form. ShE, RE, ExE, HCE, ArE, TsE, AA1E, AA2E, Ex and WEX are the uncertainty measures computed. The incomplete gamma function was a key mathematical tool in this context. Methods such as maximum likelihood, Anderson-Darling, Cramér-von-Mises, least squares, right-tail Anderson-Darling, weighted least squares, left-tail Anderson-Darling, minimum spacing absolute distance, minimum spacing absolute-logarithmic distance, Anderson-Darling left-tail second order, Kolmogorov, minimum spacing square distance, minimum spacing square-logarithmic distance, and minimum spacing Linex distance were used. The invariance property of the MLEs was also used to estimate the different uncertainty measures. A simulation study validates these methods. The significance of the model associated with the PUILD compared to various current statistical models, including the UIL, exponentiated Topp-Leone, Kumaraswamy, and beta and transformed gamma models, is illustrated by two applications using real datasets.

Author contributions

Ahmed M. Gemeay, Najwan Alsadat, Christophe Chesneau, Mohammed Elgarhy: Writing – original draft, Formal analysis, Validation, Writing – review & editing. The authors contributed equally to this work. All the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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