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Research article

A new double inertial subgradient extragradient method for solving a non-monotone variational inequality problem in Hilbert space

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Abstract: In this paper, we introduced a new double inertial subgradient extragradient method for solving a variational inequality problem in Hilbert space. In our method, the mapping needed not to satisfy any assumption of monotonicity and two different self-adaptive step sizes were used for avoiding the need of Lipschitz constant of the mapping. The strong convergence of the proposed method was proved under some new conditions. Finally, some numerical examples were presented to illustrate the convergence of our method and compare with some related methods in the literature.

Keywords: variational inequality; subgradient extragradient method; strong convergence; inertial method
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1. Introduction

Let H be a real Hilbert space, C be a nonempty, closed and convex subset of H, and $F : H \to H$ be a mapping. The variational inequality problem (VIP in short) is to find a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of the VIP (1.1) by $\mathcal{S}$. VIP is an important branch of nonlinear analysis, closely related to many topics and has a large application in mechanics, optimization, traffic network problems, equilibrium problems, and so on; see [1–3].

The projection method is one of the effective methods for solving the VIP (1.1). The simplest projection method is the following single projection method:

$$x_1 \in C, x_{n+1} = P_C(x_n - \lambda_n F(x_n)), \quad n \in \mathbb{N}, \tag{1.2}$$

where $\mathbb{N}$ denotes the set of positive integers, $P_C : H \to C$ is the metric projection, $F : H \to H$ is an η-strongly monotone and L-Lipschitz continuous mapping, and $\lambda_n \in \left(0, \frac{2\eta}{L^2}\right)$. The sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of VIP (1.1).

However, the strong monotonicity imposed on F in (1.2) is a relatively strict condition, which is generally difficult to satisfy. To weaken the restriction of strong monotonicity on F, Korpelevich [4] introduced the following famous extragradient method (EGM in short): $x_1 \in C$, and

$$\begin{cases} y_n = P_C(x_n - \lambda_n F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda_n F(y_n)),\ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $F : H \to H$ is a monotone and L-Lipschitz continuous, and $\lambda_n \in \left(0, \frac{1}{L}\right)$. The author proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a solution of the problem (1.1). In recent two decades, many modified EGM have been proposed; see, e.g., [5–11].

Recently, Noinakorn et al. [12] introduced a new EGM with inertial technique [9,10,13] for solving VIP (1.1) as follows: $x_0, x_1 \in H$, and

$$\begin{cases} t_n = (1 - \psi_n)(x_n + \alpha_n(x_n - x_{n-1})), \\ y_n = P_C(t_n - \tau F(t_n)), \\ x_{n+1} = P_C(t_n - \tau F(y_n)),\ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $F : H \to H$ is a pseudomonotone and L-Lipschitz continuous mapping, $\{\psi_n\} \subset (0, 1)$, $0 < \tau < \frac{1}{L}$, $\alpha > 0$, and $\{\alpha_n\} \subset [0, \hat{\alpha}_n]$ with

$$\hat{\alpha}_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \dfrac{\alpha}{2}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \dfrac{\alpha}{2}, & \text{otherwise.} \end{cases}$$

The authors proved that, under some mixed conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the solution of the VIP (1.1).

Tan et al. [14] proposed the following inertial subgradient EGM solving VIP (1.1): $x_0, x_1 \in H$, and

$$\begin{cases} t_n = (1 - \psi_n)(x_n + \alpha_n(x_n - x_{n-1})), \\ y_n = P_C(t_n - \lambda_n F(t_n)), \\ T_n = \{x \in H | \langle t_n - \lambda_n F(t_n) - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(t_n - \gamma\lambda_n\rho_n F(y_n)),\ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $F : H \to H$ is a pseudomonotone and L-Lipschitz continuous mapping, $\lambda_1 > 1$, $\mu \in (0, 1)$, $\sigma > 1$, $\alpha > 0$, $\gamma \in (0, \frac{2}{\sigma})$, $\{\psi_n\} \subset (0, 1)$,

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|x_n - x_{n-1}\|}, \alpha\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise,} \end{cases}$$

and

$$\rho_n = (1 - \mu)\frac{\|t_n - y_n\|^2}{\|g_n\|^2},\ \text{with } g_n = t_n - y_n - \lambda_n(F(t_n) - F(y_n)).$$

At each step, the new stepsize λ_{n+1} is generated by

$$\lambda_{n+1} = \min\left\{\lambda_n + p_n, \frac{q_n\|t_n - y_n\|}{\|F(t_n) - F(y_n)\|}\right\},$$

where $\{p_n\} \subset [0, \infty)$ with $\sum_{n=1}^{\infty} p_n < \infty$ and $\{q_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} q_n = 1$. The authors proved that $\{x_n\}$ generated by (1.5) converges strongly to a solution of VIP (1.1).

In the inertial projection methods mentioned above, only one inertial step is included. Some authors presented the new projection methods with double inertial steps for solving VIP (1.1). For example, Yao et al. [15] proposed a subgradient exgragradient method with double inertial steps for solving a pseudomonotone VIP. The authors proved the strong and weak convergence and obtained the linear convergence rate of their method under some suitable conditions. Thong et al. [16] investigated a single projection with double inertial extrapolation steps for solving a pseudomonotone VIP. The weak convergence and linear convergence rate of their method are obtained under some suitable conditions. Li and Wang [17] introduced a subgradient extragradient method with double inertial steps for solving a quasi-monotone VIP and proved the weak convergence of the proposed method under some suitable conditions. On the recent double inertial methods for solving VIP, the reader may refer to [18, 19].

In this paper, inspired by the results [12, 14–19], we introduce a new double inertial subgradient extragradient method to solve the VIP (1.1). In our method, we use a new manner to compute t_n which includes t_n in (1.4) and (1.5) as the special case. In [15–19], the mapping F is required to be pseudomontone or quasi-monotone, while in our method the mapping F needs not to satisfy any assumption of monotonicity. Two different self-adaptive step sizes are used to deal with the Lipschitz constant of the mapping F. The strong convergence of the proposed method is proved under some new conditions. Finally, some numerical examples are given to illustrate the convergence of our method and compare with some related methods in literature. The numerical results show that our method has certain advantages over the related methods.

2. Preliminaries

In this section, we give some definitions and lemmas which will be used in the next section. In the next definitions and lemmas, we assume H be a real Hilbert space and C be a nonempty closed convex subset of H.

Definition 2.1. Let $x, y \in H$. A mapping $F : H \to H$ is said to be

(i) *monotone* if $\langle F(x) - F(y), x - y\rangle \geq 0$;

(ii) *pseudomonotone* if

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0;$$

(iii) *quasimonotone* if

$$\langle F(x), y - x\rangle > 0 \Rightarrow \langle F(y), y - x\rangle \geq 0;$$

(iv) *paramonotone on C with respect to the subset $B \subset C$*, if F is pseudomonotone on C and

$$y \in C, x \in B, \langle F(y), y - x\rangle = 0 \Rightarrow y \in B.$$

It is obvious that (i)⇒(ii)⇒(iii) and (ii) ⇒ (iv).

Definition 2.2. For any $x \in H$, there exists a unique $z \in C$ such that $z = \operatorname{argmin}_{y \in C}\|x - y\|$. The element z is denoted by $P_C(x)$, called the metric projection of $x \in H$ onto C. That is,

$$P_C(x) = \operatorname{argmin}_{y \in C}\|y - x\|.$$

For any given $\bar{x}, v \in H$ with $v \neq 0$, let $T = \{x \in H : \langle v, x - \bar{x}\rangle \leq 0\}$. Then for all $y \in H$, the projection $P_T(y)$ is defined by

$$P_T(y) = y - \max\left\{0, \frac{\langle v, y - \bar{x}\rangle}{\|v\|^2}\right\} v. \tag{2.1}$$

By (2.1), we see that the projection of any point onto a half-space can be computed explicitly; see [20, Lemma 1.2] for the details.

Two important properties of P_C are given in the following lemma. The other properties of P_C can be found in [21, Chapter 4].

Lemma 2.1. [21, Chapter 4] *Let $P_C : H \to C$ be the metric projection. Then, for all $x \in H$, the following hold:*

(i) $z = P_C(x)$ *if and only if* $\langle x - z, z - y\rangle \geq 0$ *for all* $y \in C$.

(ii) $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|P_C(x) - x\|^2$ *for all* $y \in C$.

Lemma 2.2. *For any $x, y \in H$, it holds that*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3. [22] *Assume that the sequences $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset (0, 1)$ and $\{c_n\} \subset (0, \infty)$ satisfy*

$$a_{n+1} \leq (1 - b_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n = \infty$ and $\limsup_{n\to\infty} \frac{c_n}{b_n} \leq 0$ hold, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [23] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

3. Main result

In this section, let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : H \to H$ be a mapping. Consider the following Minty VIP [24]:

$$\text{find } x^* \in C \text{ such that } \langle F(y), y - x^*\rangle \geq 0, \ \forall y \in C.$$

Denote the set of solution of Minty VIP above by Ω. Clearly, Ω is closed and convex. Since C is convex, then $\Omega \subset \mathcal{S}$ when F is continuous on C. Moreover, it is easy to show that if F is pseudomonotone on C, then $\mathcal{S} = \Omega$; see [25].

In this section, we always assume that the following conditions hold:

(A1) $\Omega \neq \emptyset$.

(A2) If $y \in C$ and $z \in \Omega$ satisfy $\langle F(y), y - z\rangle = 0$, then $y \in \Omega$.

(A3) F is L-Lipschitz continuous and sequentially weakly continuous on H.

Remark 3.1. (A1) and (A2) do not imply that F is paramonotone on C with respect to Ω because F is not assumed to be pseudomonotone on C.

In the following example, F satisfies (A1)–(A3) but does not satisfy any monotone property on H.

Example 3.1. Let $H = \mathbb{R}^2$, $C = [0, 1] \times [0, 1]$ and $F(x) = -(u + x)$ for all $x \in H$, where $u = (1, 1)^T$. It is easy to see that $\mathcal{S} = \Omega = \{u\}$ and F is Lipschitz continuous with the constant $L = 1$. For every $y \in C$, $\langle F(y), y - u\rangle = \langle u + y, u - y\rangle = 0$ implies that $y = u$. Hence (A1)–(A3) hold. For $x = (0.8177, 0.5999)^T$ and $y = (0.5629, 0.8247)^T$, we have $\langle F(x), y - x\rangle = 0.10349$ and $\langle F(y), y - x\rangle = -0.01197$, and hence F is not quasimonotone on H.

Now we introduce the following double inertial subgradient extragradient method (DISEM) for solving the VIP (1.1).

Algorithm 3.1 (DISEM)

Initialization. Choose the initial points $x_{-1}, x_0, x_1 \in H$, the constants $\gamma \in (0, 1)$, $\theta_1 > 0, \theta_2 > 0$, and $\lambda_1 > 0$, the sequences $\{\alpha_n\}_{n=1}^{\infty} \subset [\alpha, 1]$ with $\alpha \in (0, 1)$, $\{\epsilon_{1,n}\}_{n=1}^{\infty} \subset (0, \infty), \{\epsilon_{2,n}\}_{n=1}^{\infty} \subset (0, \infty)$ and $\{\psi_n\}_{n=1}^{\infty} \subset (0, 1)$ satisfying

$$\lim_{n\to\infty} \psi_n = 0,\ \sum_{n=1}^{\infty} \psi_n = \infty,\ \lim_{n\to\infty} \frac{\epsilon_{i,n}}{\psi_n} = 0, i = 1, 2.$$

Set $n = 1$.

Step 1. Given x_{n-2}, x_{n-1}, x_n, compute $t_n = \psi_n(1 - \alpha_n)x_n + (1 - \psi_n)w_n$, where $w_n = x_n + \theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2})$ with

$$\theta_{1,n} = \begin{cases} \min\left\{\theta_1, \dfrac{\epsilon_{1,n}}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta_1, & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$\theta_{2,n} = \begin{cases} \min\left\{\theta_2, \dfrac{\epsilon_{2,n}}{\|x_{n-1} - x_{n-2}\|}\right\}, & \text{if } x_{n-1} \neq x_{n-2}, \\ \theta_2, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. Compute $y_n = P_C(t_n - \lambda_n F(t_n))$. If $y_n = t_n$, stop and $t_n \in \Omega$; otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = P_{T_n}(t_n - \lambda_n F(y_n))$, where

$$T_n = \{z \in H : \langle t_n - \lambda_n F(t_n) - y_n, z - y_n\rangle \leq 0\}.$$

Step 4. Update the new step size λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \|F(t_n) - F(y_n)\| = 0, \\ \min\left\{\lambda_n, \dfrac{\gamma\|t_n - y_n\|}{\|F(t_n) - F(y_n)\|}\right\}, & \text{otherwise} \end{cases} \tag{3.3}$$

or

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \lambda_n \|F(t_n) - F(y_n)\| \le \gamma \|t_n - y_n\|, \\ \gamma \lambda_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Set $n = n + 1$ and go to **Step 1**.

Remark 3.2. If $\alpha_n \equiv 1$ in Algorithm 3.1, then $t_n = (1 - \psi_n) w_n$ is defined as a similar manner with (1.4) and (1.5). In addition, for each $n \in \mathbb{N}$, by Lemma 2.1 (i) it is easy to show that $C \subset T_n$ for each $n \in \mathbb{N}$. Moreover, since T_n is a half-space, x_{n+1} can be explicitly computed by (2.1) and so only one projection for y_n is computed at each iteration.

Remark 3.3. By (3.1), (3.2) and the hypothesis on $\{\epsilon_{i,n}\}(i = 1, 2)$ and $\{\psi_n\}$, we have

$$\lim_{n\to\infty} \frac{\theta_{i,n} \|x_{n-i+1} - x_{n-i}\|}{\psi_n} \le \lim_{n\to\infty} \frac{\epsilon_{i,n}}{\psi_n} = 0,\ i = 1, 2. \tag{3.5}$$

Since $\lim_{n\to\infty} \psi_n = 0$, $\alpha_n > \alpha$, by (3.5) we see that for each $z \in \Omega$, there exists $M_z > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\{ \|z\| + \frac{1 - \psi_n}{\alpha_n} \left(\frac{\theta_{1,n}}{\psi_n} \|x_n - x_{n-1}\| + \frac{\theta_{2,n}}{\psi_n} \|x_{n-1} - x_{n-2}\| \right) \right\} \le M_z. \tag{3.6}$$

Note that (3.3) and (3.4) are the different manners of computing λ_{n+1}. The following lemma shows that $\{\lambda_n\}$ generated by (3.3) or (3.4) has the limit which is a crucial result for proving the convergence of Algorithm 3.1.

Lemma 3.1. *The sequence $\{\lambda_n\}$ has the limit $\lambda > 0$.*

Proof. Obviously, the sequence $\{\lambda_n\}$ defined by (3.3) or (3.4) is nonnegative and nonincreasing, which implies that the limit of $\{\lambda_n\}$ exists. Set $\lim_{n\to\infty} \lambda_n = \lambda$. Now we show the desired result by the following cases:

(i) $\{\lambda_n\}$ is defined as in (3.3). If $\|F(t_n) - F(y_n)\| \neq 0$, by the L-Lipschitz continuity of F, we have

$$\frac{\gamma \|t_n - y_n\|}{\|F(t_n) - F(y_n)\|} \ge \frac{\gamma \|t_n - y_n\|}{L \|t_n - y_n\|} = \frac{\gamma}{L},$$

which together with the definition of λ_{n+1} implies that $\lambda_{n+1} \ge \min\left\{\lambda_n, \frac{\gamma}{L}\right\}$. If $\|F(t_n) - F(y_n)\| = 0$, then $\lambda_{n+1} = \lambda_n \ge \min\left\{\lambda_n, \frac{\gamma}{L}\right\}$. So by a mathematical induction we get $\lambda_n \ge \min\left\{\lambda_1, \frac{\gamma}{L}\right\}$ for all $n \in \mathbb{N}$. Therefore, we have $\lambda \ge \min\left\{\lambda_1, \frac{\gamma}{L}\right\}$.

(ii) $\{\lambda_n\}$ is defined as in (3.4). Assume that $\lambda = 0$. Then there must exist a subsequence λ_{n_k} of $\{\lambda_n\}$ such that

$$\lambda_{n_k} \|F(t_{n_k}) - F(y_{n_k})\| > \gamma \|t_{n_k} - y_{n_k}\|,\ \forall k \in \mathbb{N}.$$

Since F is L-Lipschitz continuous, we have

$$\lambda_{n_k} L \|t_{n_k} - y_{n_k}\| > \gamma \|t_{n_k} - y_{n_k}\|$$

and hence $\lambda_{n_k} > \frac{\gamma}{L}$ for all $k \in \mathbb{N}$. Letting $k \to \infty$, from $\lim_{k\to\infty} \lambda_{n_k} = 0$ we get $\frac{\gamma}{L} \le 0$. It is a contradiction. Therefore, by the two cases above we can conclude $\lambda > 0$. The proof is complete. □

Remark 3.4. By the proof process of Lemma 3.1, we see that there exists $n_0 \in \mathbb{N}$ such that λ_{n+1} defined by (3.3) or (3.4) satisfies that

$$\lambda_{n+1} \leq \frac{\gamma \|t_n - y_n\|}{\|F(t_n) - F(y_n)\|}, \ \forall n \geq n_0.$$

Lemma 3.2. *There exists $n_0 \in \mathbb{N}$ such that for each $z \in \Omega$ and $n \geq n_0$,*

$$\|x_{n+1} - z\|^2 \leq \|t_n - z\|^2 - \left(1 - \frac{\gamma\lambda_n}{\lambda_{n+1}}\right)(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2). \tag{3.7}$$

Proof. For each $z \in \Omega$ and $n \in \mathbb{N}$, by Remark 3.2 we have $z \in C \subset T_n$ for each $n \in \mathbb{N}$. From Lemma 2.1 (i) and the definition of x_{n+1} it follows that

$$\langle z - x_{n+1}, t_n - \lambda_n F(y_n) - x_{n+1} \rangle \leq 0.$$

That is

$$\langle x_{n+1} - t_n, x_{n+1} - z \rangle \leq \lambda_n \langle F(y_n), z - x_{n+1} \rangle. \tag{3.8}$$

Since $x_{n+1} \in T_n$, we have

$$\langle y_n - t_n, y_n - x_{n+1} \rangle \leq \lambda_n \langle F(t_n), x_{n+1} - y_n \rangle. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|y_n - z\|^2 - \|x_{n+1} - y_n\|^2 + 2\langle x_{n+1} - y_n, x_{n+1} - z \rangle \\
&= \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle y_n - t_n, y_n - z \rangle \\
&\quad + 2\langle x_{n+1} - y_n, x_{n+1} - z \rangle \\
&= \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle y_n - t_n, y_n - x_{n+1} \rangle \\
&\quad + 2\langle x_{n+1} - t_n, x_{n+1} - z \rangle.
\end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.9) into (3.10) we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n \langle F(y_n), z - x_{n+1} \rangle \\
&\quad + 2\lambda_n \langle F(t_n), x_{n+1} - y_n \rangle \\
&= \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n \langle F(y_n), z - y_n \rangle \\
&\quad + 2\lambda_n \langle F(t_n) - F(y_n), x_{n+1} - y_n \rangle.
\end{aligned} \tag{3.11}$$

By Remark 3.4, there exists $n_0 \in \mathbb{N}$ such that

$$\lambda_{n+1} \leq \frac{\gamma \|t_n - y_n\|}{\|F(t_n) - F(y_n)\|}, \ \forall n \geq n_0.$$

Hence by (3.11) we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n \langle F(y_n), z - y_n \rangle \\
&\quad + 2\lambda_n \|F(t_n) - F(y_n)\| \|x_{n+1} - y_n\| \\
&\leq \|t_n - z\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n \langle F(y_n), z - y_n \rangle \\
&\quad + \frac{2\gamma\lambda_n}{\lambda_{n+1}} \|t_n - y_n\| \|x_{n+1} - y_n\| \\
&\leq \|t_n - z\|^2 - \left(1 - \frac{\gamma\lambda_n}{\lambda_{n+1}}\right)(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2) \\
&\quad + 2\lambda_n \langle F(y_n), z - y_n \rangle, \ \forall n \geq n_0.
\end{aligned} \tag{3.12}$$

Since $y_n \in C$ and $z \in \Omega$, we have $\langle F(y_n), y_n - z\rangle \geq 0$. Substituting this result into (3.12) we can get (3.7). The proof is complete. □

Lemma 3.3. *The sequences* $\{x_n\}$ *and* $\{t_n\}$ *are bounded.*

Proof. By Lemma 3.1 we have

$$\lim_{n\to\infty}\left(1 - \frac{\gamma\lambda_n}{\lambda_{n+1}}\right) = 1 - \gamma < 1.$$

Hence there exist $\gamma' \in (\gamma, 1)$ and $n_1 \in \mathbb{N}$ such that

$$1 - \frac{\gamma\lambda_n}{\lambda_{n+1}} > 1 - \gamma', \ \forall n \geq n_1. \tag{3.13}$$

Set $n_2 = \max\{n_0, n_1\}$. For each $z \in \Omega$, by (3.7) and (3.13) we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|t_n - z\|^2 - (1-\gamma')\left(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2\right) \\ &\leq \|t_n - z\|^2, \ n \geq n_2. \end{aligned} \tag{3.14}$$

From (3.7) and the definition of t_n it holds

$$\begin{aligned} &\|t_n - z\| \\ =& \|\psi_n(1-\alpha_n)x_n + (1-\psi_n)w_n - z\| \\ =& \|\psi_n((1-\alpha_n)x_n - z) + (1-\psi_n)(w_n - z)\| \\ \leq& \|\psi_n((1-\alpha_n)x_n - z)\| + (1-\psi_n)\|w_n - z\| \\ =& \|\psi_n((1-\alpha_n)x_n - z)\| + (1-\psi_n)\|x_n + \theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2}) - z\| \\ \leq& \psi_n\left[(1-\alpha_n)\|x_n - z\| + \alpha_n\|z\|\right] + (1-\psi_n)[\|x_n - z\| + \theta_{1,n}\|x_n - x_{n-1}\| \\ &+ \theta_{2,n}\|x_{n-1} - x_{n-2}\|] \\ =& (1-\psi_n\alpha_n)\|x_n - z\| + \psi_n\alpha_n\|z\| + (1-\psi_n)[\theta_{1,n}\|x_n - x_{n-1}\| + \theta_{2,n}\|x_{n-1} - x_{n-2}\|] \\ =& (1-\psi_n\alpha_n)\|x_n - z\| + \psi_n\alpha_n\left[\|z\| + \frac{1-\psi_n}{\alpha_n}\left(\frac{\theta_{1,n}}{\psi_n}\|x_n - x_{n-1}\| + \frac{\theta_{2,n}}{\psi_n}\|x_{n-1} - x_{n-2}\|\right)\right] \\ \leq& (1-\psi_n\alpha_n)\|x_n - z\| + \psi_n\alpha_n M_z, \ \forall n \geq n_0. \end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.14) we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \|t_n - z\| \\ &\leq (1-\psi_n\alpha_n)\|x_n - z\| + \psi_n\alpha_n M_z \\ &\leq \max\{\|x_n - z\|, M_z\} \\ &\leq \ldots \leq \max\{\|x_{n_0} - z\|, M_z\}, \quad \forall n \geq n_2. \end{aligned} \tag{3.16}$$

It follows that $\{x_n\}$ is bounded and so is $\{t_n\}$ by (3.16). The proof is complete. □

Let $x^* = P_\Omega(0)$. We have $x^* \in \mathcal{S}$ because of $\Omega \subset \mathcal{S}$. In the position we give the main result on Algorithm 3.1 as follows.

Theorem 3.1. *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to the element* x^*.

Proof. For each $n \in \mathbb{N}$, by the definition of t_n and Lemma 2.2 we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|\psi_n(1-\alpha_n)x_n + (1-\psi_n)w_n - x^*\|^2 \\
&= \|\psi_n((1-\alpha_n)x_n - x^*) + (1-\psi_n)(w_n - x^*)\|^2 \\
&\le (1-\psi_n)^2\|w_n - x^*\|^2 + 2\psi_n\langle(1-\alpha_n)x_n - x^*, t_n - x^*\rangle \\
&= (1-\psi_n)^2\|x_n + \theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2}) - x^*\|^2 \\
&\quad + 2\psi_n[(1-\alpha_n)\langle x_n - x^*, t_n - x^*\rangle + \alpha_n\langle -x^*, t_n - x^*\rangle].
\end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned}
&\|x_n + \theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2}) - x^*\|^2 \\
&= \|x_n - x^*\|^2 + \theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\theta_{1,n}\langle x_n - x_{n-1}, x_n - x^*\rangle + 2\theta_{2,n}\langle x_{n-1} - x_{n-2}, x_n - x^*\rangle \\
&\quad + 2\theta_{1,n}\theta_{2,n}\langle x_n - x_{n-1}, x_{n-1} - x_{n-2}\rangle \\
&\le \|x_n - x^*\|^2 + \theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| \\
&\quad + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|.
\end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17) we get

$$\begin{aligned}
\|t_n - x^*\|^2 &\le (1-\psi_n)^2[\|x_n - x^*\|^2 + \theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| \\
&\quad + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] + 2\psi_n(1-\alpha_n)\|x_n - x^*\|\|t_n - x^*\| \\
&\quad + 2\psi_n\alpha_n\langle -x^*, t_n - x^*\rangle \\
&\le (1-\psi_n)^2[\|x_n - x^*\|^2 + \theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| \\
&\quad + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] + \psi_n(1-\alpha_n)(\|x_n - x^*\|^2 + \|t_n - x^*\|^2) \\
&\quad + 2\psi_n\alpha_n\langle -x^*, t_n - x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|t_n - x^*\|^2 \leq & \frac{1-\psi_n+\psi_n^2-\psi_n\alpha_n}{1-\psi_n+\psi_n\alpha_n}\|x_n - x^*\|^2 + \frac{(1-\psi_n)^2}{1-\psi_n+\psi_n\alpha_n}[\theta_{1,n}^2\|x_n - x_{n-1}\|^2 \\
& + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\
& + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\
& + \frac{2\psi_n\alpha_n}{1-\psi_n+\psi_n\alpha_n}\langle -x^*, t_n - x^*\rangle \\
= & \left(1 - \frac{2\psi_n\alpha_n}{1-\psi_n+\psi_n\alpha_n}\right)\|x_n - x^*\|^2 + \frac{\psi_n^2}{1-\psi_n+\psi_n\alpha_n}\|x_n - x^*\|^2 \\
& + \frac{(1-\psi_n)^2}{1-\psi_n+\psi_n\alpha_n}[\theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
& + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| \\
& + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] + \frac{2\psi_n\alpha_n}{1-\psi_n+\psi_n\alpha_n}\langle -x^*, t_n - x^*\rangle \\
= & (1-\gamma_n)\|x_n - x^*\|^2 + \varphi_n,
\end{aligned} \tag{3.19}$$

where $\gamma_n = \frac{2\psi_n\alpha_n}{1-\psi_n+\psi_n\alpha_n}$ and

$$\begin{aligned}
\varphi_n = & \frac{\gamma_n\psi_n}{2\alpha_n}\|x_n - x^*\|^2 + \frac{\gamma_n(1-\psi_n)^2}{2\alpha_n\psi_n}[\theta_{1,n}^2\|x_n - x_{n-1}\|^2 + \theta_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
& + 2\theta_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + 2\theta_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| \\
& + 2\theta_{1,n}\theta_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] + \gamma_n\langle -x^*, t_n - x_n\rangle + \gamma_n\langle -x^*, x_n - x^*\rangle.
\end{aligned}$$

Substituting (3.19) into (3.7) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 & \leq (1-\gamma_n)\|x_n - x^*\|^2 + \varphi_n - (1-\gamma')(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2) \\
& \leq (1-\gamma_n)\|x_n - x^*\|^2 + \varphi_n, \ \forall n \geq n_2.
\end{aligned} \tag{3.20}$$

Next we prove the strong convergence of $\{\|x_n - x^*\|^2\}$ to zero by considering the following two cases.

Case 1. Suppose there exists $N \in \mathbb{N}$ with $N > n_2$ such that $\{\|x_n - x^*\|^2\}$ is monotonically nonincreasing for $n \geq N$. Then $\{\|x_n - x^*\|^2\}$ is convergent and hence

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0. \tag{3.21}$$

Since $\lim_{n\to\infty}\psi_n = 0$, by the boundedness of $\{x_n\}$ and (3.5) we get

$$\begin{aligned}
\|t_n - x_n\| &= \|\psi_n(1-\alpha_n)x_n + (1-\psi_n)w_n - x_n\| \\
&\le \psi_n(1-\alpha_n)\|x_n\| + \|(1-\psi_n)w_n - x_n\| \\
&= \psi_n(1-\alpha_n)\|x_n\| + \|(1-\psi_n)[x_n + \theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2})] \\
&\quad - x_n\| \\
&= \psi_n(1-\alpha_n)\|x_n\| + \|(1-\psi_n)[\theta_{1,n}(x_n - x_{n-1}) + \theta_{2,n}(x_{n-1} - x_{n-2})] \\
&\quad - \psi_n x_n\| \\
&\le \psi_n(1-\alpha_n)\|x_n\| + (1-\psi_n)(\theta_{1,n}\|x_n - x_{n-1}\| + \theta_{2,n}\|x_{n-1} - x_{n-2}\|) \\
&\quad + \psi_n\|x_n\| \\
&= \psi_n(2-\alpha_n)\|x_n\| + (1-\psi_n)\psi_n\left(\frac{\theta_{1,n}\|x_n - x_{n-1}\|}{\psi_n} + \frac{\theta_{2,n}\|x_{n-1} - x_{n-2}\|}{\psi_n}\right) \\
&\to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.22}$$

On the other hand, by (3.7) and (3.16) with $z = x^*$ we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le [(1-\psi_n\alpha_n)\|x_n - x^*\| + \psi_n\alpha_n M_{x^*}]^2 - (1-\gamma')(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2) \\
&\le (1-\psi_n\alpha_n)\|x_n - x^*\|^2 + \psi_n\alpha_n M_{x^*}^2 - (1-\gamma')(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2) \\
&\le \|x_n - x^*\|^2 + \psi_n\alpha_n M_{x^*}^2 - (1-\gamma')(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2)
\end{aligned}$$

and hence

$$(1-\gamma')(\|x_{n+1} - y_n\|^2 + \|y_n - t_n\|^2) \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \psi_n\alpha_n M_{x^*}^2 \tag{3.23}$$

for all $n \ge N$. Since $\lim_{n\to\infty}\psi_n = 0$, letting $n\to\infty$ in (3.23), by (3.21) we get

$$\lim_{n\to\infty}\|y_n - x_{n+1}\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty}\|t_n - y_n\|^2 = 0. \tag{3.24}$$

From Lemma 3.2 it follows that $\{y_n\}$ is bounded.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$, where $\rightharpoonup$ denotes the weak convergence. From (3.22) and (3.24) we see that $t_{n_k} \rightharpoonup \hat{x}$, $y_{n_k} \rightharpoonup \hat{x}$ and $\hat{x} \in C$. Without loss generality we may assume that

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle -x^*, x_{n_k} - x^*\rangle = \langle -x^*, \hat{x} - x^*\rangle. \tag{3.25}$$

From (3.24) we have

$$\|t_n - x_{n+1}\| \le \|t_n - y_n\| + \|y_n - x_{n+1}\| \to 0, \text{ as } n\to\infty. \tag{3.26}$$

Let $\bar{x} \in \Omega$ be a fixed point. By the definition of y_{n_k} and Lemma 2.1 (i) we have

$$\langle y_{n_k} - (t_{n_k} - \lambda_{n_k}F(t_{n_k}))\rangle \ge \langle \bar{x} - y_{n_k}\rangle. \tag{3.27}$$

Letting $k \to \infty$ in (3.27), by Lemma 3.1, (A3) and (3.24) we get

$$\langle F(\hat{x}), \bar{x} - \hat{x} \rangle \geq 0.$$

In addition, since $\hat{x} \in C$ and $\bar{x} \in \Omega$, we have

$$\langle F(\hat{x}), \bar{x} - \hat{x} \rangle \leq 0.$$

Hence $\langle F(\hat{x}), \bar{x} - \hat{x} \rangle = 0$. From (A2) it follows that $\hat{x} \in \Omega$. This result with (3.25) and Lemma 2.1 (i) leads to that

$$\limsup_{n\to\infty} \langle -x^*, x_n - x^* \rangle = \langle -x^*, \hat{x} - x^* \rangle \leq 0. \tag{3.28}$$

Since $\lim_{n\to\infty} \psi_n = 0$, there exists $n_1 \in \mathbb{N}$ such that $\psi_n < \frac{1}{2}$ for all $n \geq n_1$, which implies that

$$\gamma_n = \frac{2\psi_n \alpha_n}{1 - \psi_n + \psi_n \alpha_n} \in (0, 1), \ \forall n \geq n_1.$$

In addition, since $\alpha\psi_n \leq \gamma_n \leq 4\psi_n$ for all $n \in \mathbb{N}$, by the hypothesis on $\{\psi_n\}$ we get

$$\lim_{n\to\infty} \gamma_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n = \infty. \tag{3.29}$$

By the definitions of γ_n and $\{\varphi_n\}$ we have

$$\begin{aligned}
\frac{\varphi_n}{\gamma_n} = & \frac{(1-\psi_n)^2}{2\alpha_n} \Big[\psi_n \frac{\theta_{1,n}^2}{\psi_n^2} \|x_n - x_{n-1}\|^2 + \psi_n \frac{\theta_{2,n}^2}{\psi_n^2} \|x_{n-1} - x_{n-2}\|^2 \\
& + \frac{2\theta_{1,n}}{\psi_n} \|x_n - x_{n-1}\| \|x_n - x^*\| + \frac{2\theta_{2,n}}{\psi_n} \|x_{n-1} - x_{n-2}\| \|x_n - x^*\| \\
& + \psi_n \frac{2\theta_{1,n}\theta_{2,n}}{\psi_n^2} \|x_n - x_{n-1}\| \|x_{n-1} - x_{n-2}\| \Big] + \langle -x^*, t_n - x_n \rangle + \langle -x^*, x_n - x^* \rangle \\
& + \frac{\psi_n}{2\alpha_n} \|x_n - x^*\|^2.
\end{aligned} \tag{3.30}$$

Letting $n \to \infty$ in (3.30), since $\{x_n\}$ is bounded, $\alpha_n > \alpha > 0$, $\psi_n \to 0$, by (3.5), (3.22) and (3.28) we get

$$\limsup_{n\to\infty} \frac{\varphi_n}{\gamma_n} \leq 0. \tag{3.31}$$

Applying Lemma 2.3 to (3.21) and using (3.29) and (3.31), we obtain $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Case 2. Suppose there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| \leq \|x_{n_i+1} - x^*\|, \forall i \in \mathbb{N}.$$

From Lemma 2.4 it follows that there exists a sequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$ such that

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\| \text{ and } \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \forall k \in \mathbb{N}. \tag{3.32}$$

Replacing n in (3.20) with m_k and using (3.32) we obtain

$$\begin{aligned}\|x_{m_k+1} - x^*\|^2 &\leq (1-\gamma_{m_k})\|x_{m_k} - x^*\|^2 + \varphi_{m_k}\\ &\leq (1-\gamma_{m_k})\|x_{m_k+1} - x^*\|^2 + \varphi_{m_k}\end{aligned}$$

and hence

$$\|x_{m_k+1} - x^*\|^2 \leq \frac{\varphi_{m_k}}{\gamma_{m_k}},\ \forall k \in \mathbb{N} \text{ with } m_k \geq n_0. \tag{3.33}$$

Note that by a similar process of showing (3.32) as in Case 1 we can get

$$\limsup_{k\to\infty} \frac{\varphi_{n_k}}{\gamma_{n_k}} \leq 0,$$

which together with (3.33) leads to that

$$\limsup_{k\to\infty} \|x_{m_k+1} - x^*\|^2 = 0. \tag{3.34}$$

Combining (3.32) with (3.34) we have

$$\limsup_{k\to\infty} \|x_k - x^*\|^2 \leq \limsup_{k\to\infty} \|x_{m_k+1} - x^*\|^2 = 0.$$

Hence $\limsup_{k\to\infty} \|x_k - x^*\|^2 = 0$. The proof is complete. □

Remark 3.5. If replacing the conditions (A1) and (A2) with the following

(A1') $\mathcal{S} \neq \emptyset$ and (A2') F is pseudomonotone on C with respect to $\mathcal{S}$, and replacing Ω with $\mathcal{S}$ in the proof lines of Lemma 3.2, Lemma 3.3 and Theorem 3.1, we still can obtain the same results. In fact, the conditions (A1') and (A2') are often used in the related literature.

4. Numerical examples

In this section, we use some numerical examples to illustrate the convergence of Algorithm 3.1 and compare the numerical results with Algorithm 3.1 of Noinakorn et al. [12] (denoted by Algorithm EXN), Algorithm 3.1 of Tan et al. [14] (denoted by Algorithm EXT). The codes of the three algorithms for the following numerical examples are written by Matlab R2016b running on a MacBook air Desktop with Core(TM) i5-4260U CPU @1.40GHz 2.00GHz, RAM 4GB.

In the following Examples 4.1–4.3, we use the following parameters and control sequences for Algorithm 3.1, Algorithm EXN and Algorithm EXT:

- Algorithm 3.1: $\epsilon_{1,n} = \epsilon_{2,n} = \frac{100}{(1+n)^2}$, $\{\alpha_n\}$, γ, λ_1, θ_1, θ_2 are as follows:

$$\begin{aligned}&\text{Case 1. } \psi_n = \frac{90}{100+n}, \alpha_n = 0.7 + 0.09^n, \gamma = 0.2, \lambda_1 = 0.3, \theta_1 = 0.3, \theta_2 = 0.5;\\ &\text{Case 2. } \psi_n = \frac{80}{100+n}, \alpha_n = 0.5 + 0.09^n, \gamma = 0.8, \lambda_1 = 0.8, \theta_1 = 0.6, \theta_2 = 0.1;\\ &\text{Case 3. } \psi_n = \frac{70}{100+n}, \alpha_n = 0.7 + \frac{1}{n}, \gamma = 0.5, \lambda_1 = 0.5, \theta_1 = 0.5, \theta_2 = 0.5,\\ &\text{Case 4. } \psi_n = \frac{60}{100+n}, \alpha_n = 0.5 + \frac{1}{n}, \gamma = 0.2, \lambda_1 = 0.3, \theta_1 = 0.3, \theta_2 = 0.5;\\ &\text{Case 5. } \psi_n = \frac{50}{100+n}, \alpha_n = 0.5, \gamma = 0.5, \lambda_1 = 0.5, \theta_1 = 0.5, \theta_2 = 0.5.\end{aligned}$$

• Algorithm EXT: $\lambda_1 = 0.5, \mu = 0.4, \gamma = 1.5, \alpha = 0.4, \psi_n = \frac{1}{n+1}, \xi_n = \frac{100}{(n+1)^2}, p_n = \frac{1}{(n+1)^{1.1}}$, and $q_n = \frac{1+n}{n}$.
• Algorithm EXN: $\tau = \frac{0.7}{L}, \alpha = 0.6, \epsilon_n = \frac{1}{(n+1)^2}, \psi_n = \frac{1}{n+2}$, where L is the Lipschitz constant of the mapping F.

Remark 4.1. The parameters and control sequences above for Algorithm EXN and Algorithm EXT were used in [12] and [14]. For the better comparison we continue to use the parameters and control sequences for Algorithm EXN and Algorithm EXT in the following numerical examples.

In the first example, since F is not pseudomonotone on C, Algorithm EXN, Algorithm EXT and many other related algorithms in the literature can not be applied to the example.

Example 4.1. Let $H = \mathbb{R}^m$, $C = [0,\pi]\times[0,\pi]\times[0,1]\times\cdots\times[0,1]$, $F : H \to H$ be a mapping defined by

$$F(x) = (x_2 + \cos(x_2), x_1 + \sin(x_1), x_3, \cdots, x_m),\ \forall x = (x_1, \cdots, x_m) \in H.$$

It follows that $\mathcal{S} = \Omega = \{0\}$ and so (A1) holds. We show that (A2) and (A3) hold. For $x = (x_1, \cdots, x_m)^T \in C$, since $\sum_{i=3}^m x_i^2 \geq 0$, $x_2 + \cos(x_2) > 0$, and $x_1 + \sin(x_1) \geq 0$, it is easy to see that $\langle F(x), x\rangle = \sum_{i=3}^m x_i^2 + x_1(x_2 + \cos(x_2)) + x_2(x_1 + \sin(x_1)) = 0$ is if and only if $x = 0$. So (A2) holds. For $x, y \in H$, we have

$$\begin{aligned}\|F(x) - F(y)\|^2 &= (x_2 - y_2 + \cos(x_2) - \cos(y_2))^2 + (x_1 - y_1 + \sin(x_1) - \sin(x_1))^2 + \sum_{i=3}^m (x_i - y_i)^2 \\ &\leq 4\sum_{i=1}^m (x_i - y_i)^2 = 4\|x - y\|^2.\end{aligned}$$

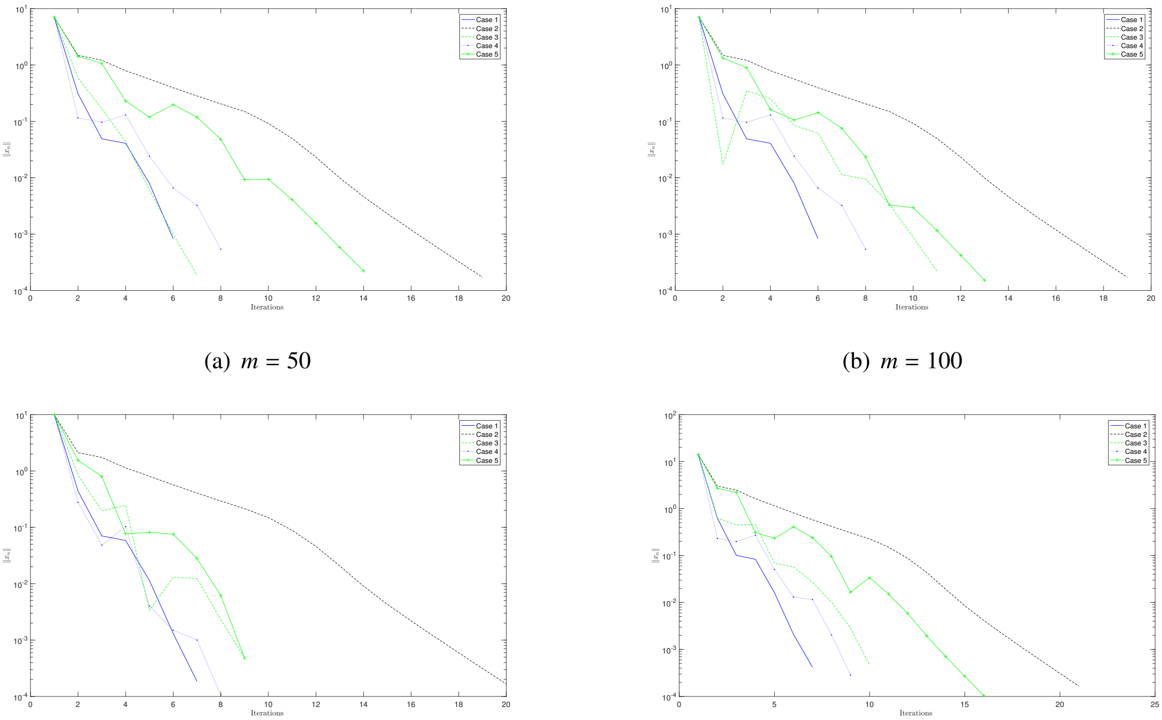
It follows that F is 2-Lipschitz continuous on H and so (A3) holds.

For $x = (2.8677, 0.9806, 0.9005, 0, \cdots, 0)^T$ and $y = (1.0688, 1.9091, 0.8044, 0, \cdots, 0)^T$, we have

$$\langle F(x), y - x\rangle > 0 \quad \text{and} \quad \langle F(y), y - x\rangle < 0.$$

Hence, F is not quasimonotone on C.

We take the initial point $x_{-1} = x_0 = x_1 = (1, 1, \cdots, 1)^T \in C$ and use $\|x_n\| \leq 10^{-4}$ as the stopping criterion of Algorithm 3.1. In the process of performing Algorithm 3.1, the step size λ_n is computed by (3.3). The computed results for Algorithm 3.1 with the different dimension m are shown in Figure 1. From the curves in Figure 1 we see that $x_n \to 0$.



(a) $m = 50$ (b) $m = 100$

(c) $m = 200$ (d) $m = 500$

Figure 1. Numerical results of Algorithm 3.1 for Example 4.1.

For comparing the computation results with other algorithms, the mappings F in the following examples are pseudomonotone on C and $\Omega \neq \emptyset$. By Remark 3.5 Algorithm 3.1 still can be applied to the examples.

Example 4.2. [9, 12, Example 4.1] Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a mapping defined by

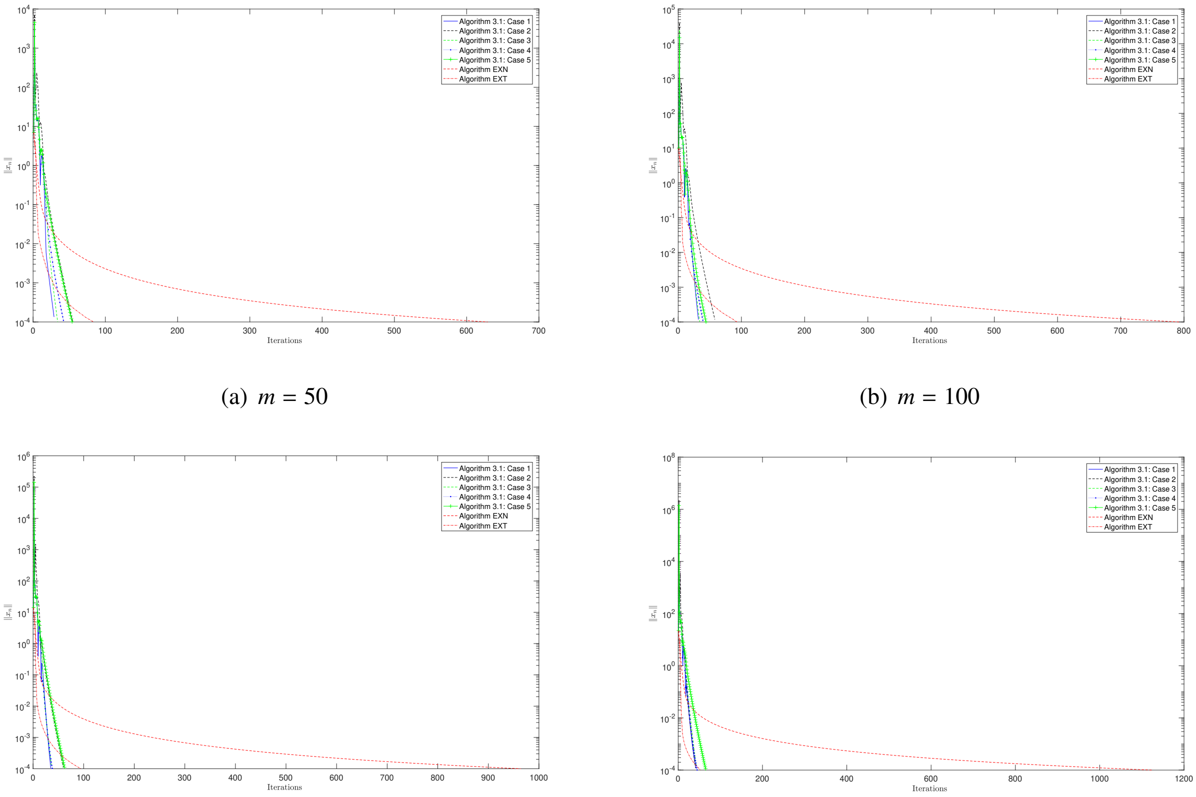
$$F = NN^T + B + D,$$

where $N = \text{rand}(m)$ is a random matrix, $B = 0.5K - 0.5K^T$ with $K = \text{rand}(m)$ is a skew-symmetric matrix, and $D = \text{diag}(\text{rand}(m, 1))$ is a diagonal matrix. The feasible set

$$C = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, i = 1, 2, \cdots, m\}.$$

It follows that F is monotone (hence it is pseudomonotone) and L-Lipschitz continuous with $L = \|F\|$. The unique solution of the VIP (1.1) with the mapping F in this example is $x^* = 0$. Since F is continuous and psuedomonotone on C, and C is convex, it has $\Omega = S$ and so (A1) holds. In addition, since the matrix $NN^T + B + D$ is positive definite [26], $\langle F(y), y\rangle = y^T(NN^T + B + D)y = 0$ implies that $y = 0$. So (A2) holds.

We choose the initial points $x_0, x_{-1}, x_1 = (1, \cdots, 1)^T$ for Algorithm 3.1 and $x_0 = x_1 = (1, \cdots, 1)^T$ for Algorithm EXN and Algorithm EXT, and use $\|x_n\| < 10^{-4}$ as the common stop criterion for all the algorithms. In the process of performing Algorithm 3.1, the step size λ_n is computed by (3.4). Figure 2 illustrates the curves and Table 1 gives the CPU time in seconds of the numerical results for these algorithms with the different dimension.



(a) $m = 50$ (b) $m = 100$

(c) $m = 200$ (d) $m = 500$

Figure 2. Numerical results of all the algorithms for Example 4.2.

Table 1. CPU time of three algorithms for Example 4.2.

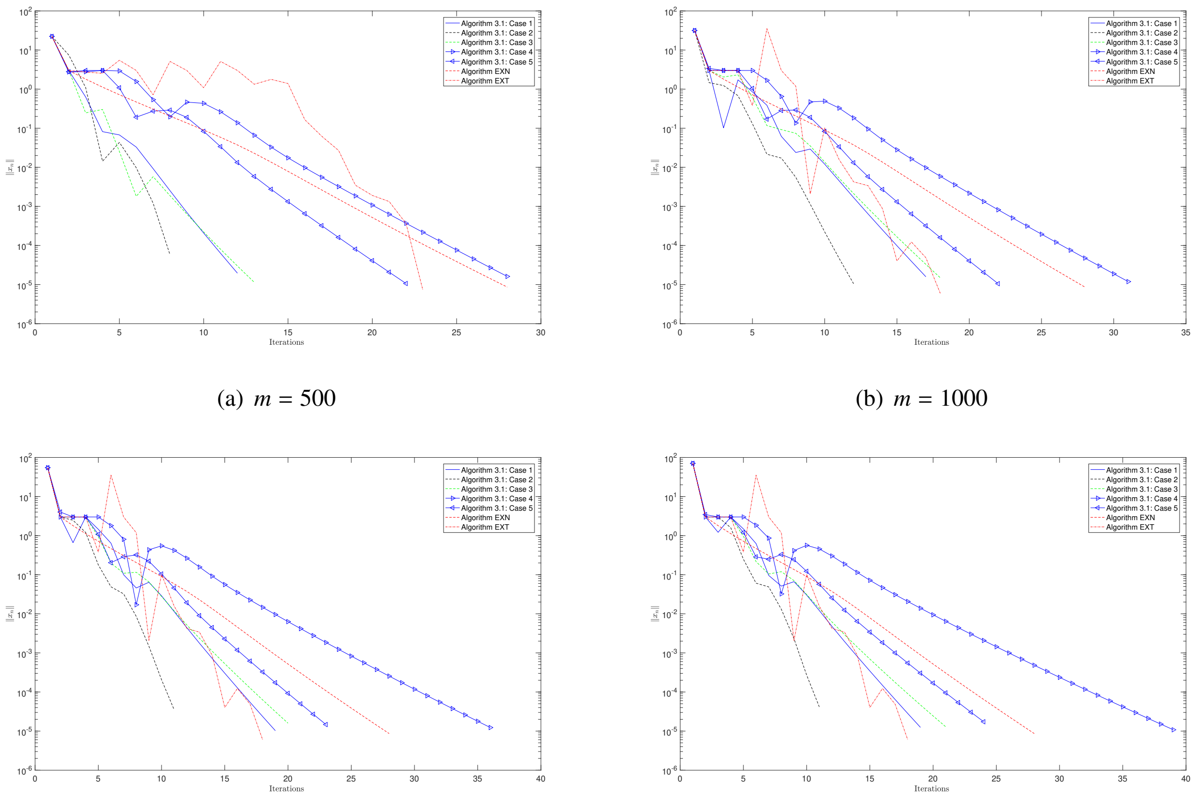
	Algorithm 3.1					Algorithm EXN	Algorithm EXT
	Case 1	Case 2	Case 3	Case 4	Case 5		
$m = 50$	0.02153	0.02071	0.01973	0.01627	0.01771	0.01526	0.02115
$m = 100$	0.02013	0.02114	0.02257	0.02331	0.03007	0.04877	0.01776
$m = 200$	0.03685	0.03234	0.03367	0.03213	0.03281	0.08586	0.02367
$m = 500$	0.21339	0.21312	0.23498	0.15074	0.20881	1.77414	0.27757

Example 4.3. [12, Example 4.3] Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$F(x) = (5 - \|x\|)x, \ \forall x \in \mathbb{R}^m.$$

It follows that F is L-Lipschitz continuous with $L = 11$ and pseudomonotone but not monotone on $\mathbb{R}^m$ (see for more details [27]). The feasible set is $C = \{x \in \mathbb{R}^m : \|x\| \leq 3\}$. The unique solution of the VIP (1.1) with the mapping F in this example is $x^* = 0$. It is obvious that the conditions (A1) and (A3) hold. For $x \in C$, since $\|x\| \leq 3$, $\langle F(x), x\rangle = (5 - \|x\|)\|x\|^2 = 0$ is if and only if $x = 0$ and so (A2) holds.

We take the initial points $x_{-1} = x_0 = x_1 = (1, 1, \cdots, 1)^T$ for our Algorithm 3.1 and $x_0 = x_1 = (1, 1, \cdots, 1)^T$ for Algorithm EXN and Algorithm EXT, and use $\|x_n\| < 10^{-5}$ as the common stop criterion for all the algorithms. In the process of performing Algorithm 3.1, the step size λ_n is computed by (3.4). Figure 3 illustrates the curves and Table 2 gives the CPU time in seconds of the numerical results for these algorithms with the different dimension m.



(a) $m = 500$ (b) $m = 1000$

(c) $m = 3000$ (d) $m = 5000$

Figure 3. Numerical results of all the algorithms for Example 4.3.

Table 2. CPU time of three algorithms for Example 4.3.

	Algorithm 3.1					Algorithm EXN	Algorithm EXT
	Case 1	Case 2	Case 3	Case 4	Case 5		
$m = 500$	0.00233	0.00147	0.00973	0.00211	0.00188	0.00249	0.00547
$m = 1000$	0.01803	0.02136	0.01901	0.02341	0.02055	0.01568	0.01612
$m = 3000$	0.01698	0.01766	0.01518	0.01809	0.01663	0.01044	0.01719
$m = 5000$	0.01104	0.01113	0.01108	0.01129	0.01167	0.01276	0.01397

For the numerical results of the three algorithms for Examples 4.2 and 4.3, we summarized as follows. For Example 4.2, Algorithm 3.1 needs the less CPU time than another two algorithms and for Example 4.3, CPU time has no obvious difference for the three algorithms. From the curves in Figure 2 we see that Algorithm 3.1 has the less numbers of iterations than the another two algorithms when the algorithms stop. From the curves in Figure 3 we see that the numbers of iterations of Algorithm 3.1 are different when the algorithm stops, which implies that the numbers of iterations are sensitive to the setting of the control parameters and sequences (especially the setting of $\{\psi_n\}$) for Algorithm 3.1. For Example 4.3, Algorithm EXN needs the most numbers of iterations and Algorithm 3.1 needs the less or more numbers of iterations than Algorithm EXT for the different setting of control parameters and sequences. Overall, from the numerical results for Examples 4.2 and 4.3, we see that our Algorithm 3.1 has certain competitiveness than the other two algorithms.

5. Conclusions

In this paper, we constructed a new double inertial subgradient extragradient algorithm for solving a non-monotone variational inequality problem in a Hilbert space. Without prior knowledge of the Lipschitz constant of the involved mapping, we use a dynamic manner to update the step-size in our algorithm. Some new conditions are used to guarantee the strong convergence of the proposed method. Some artificially constructed numerical examples and an applied problem of image recovery are solved by our algorithm and some other related algorithms. The results show the effectiveness and competitiveness of our algorithm with other compared algorithms.

Author contributions

Ziqi Zhu: Methodology, Computing numerical examples, Writing-original draft; Kaiye Zheng: Formal analysis, Writing-original draft; Shenghua Wang: Design of algorithm, Proof of conclusions. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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