AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 20939–20955.
DOI: 10.3934/math.20241019
Received: 16 April 2024
Revised: 01 June 2024
Accepted: 24 June 2024
Published: 28 June 2024

Research article

The best approximation problems between the least-squares solution manifolds of two matrix equations

Yinlan Chen* **and Yawen Lan**

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

* **Correspondence:** Email: cylfzg@hbnu.edu.cn.

Abstract: In this paper, we will deal with the following two classes of best approximation problems about the linear manifolds: **Problem 1.** Given matrices A_1, B_1, C_1, and $D_1 \in \mathbb{R}^{m\times n}$, find $d(L_1, L_2) = \min_{X\in L_1, Y\in L_2} \|X - Y\|$, and find $\hat{X} \in L_1, \hat{Y} \in L_2$ such that $\|\hat{X} - \hat{Y}\| = d(L_1, L_2)$, where $L_1 = \{X \in \mathbb{SR}^{n\times n} \mid \|A_1X - B_1\| = \min\}$ and $L_2 = \{Y \in \mathbb{SR}^{n\times n} \mid \|C_1Y - D_1\| = \min\}$. **Problem 2.** Given matrices $A_2, B_2, E_2, F_2 \in \mathbb{R}^{m\times n}$ and $C_2, D_2, G_2, H_2 \in \mathbb{R}^{n\times p}$, find $d(L_3, L_4) = \min_{X\in L_3, Y\in L_4} \|X - Y\|$, and find $\tilde{X} \in L_3, \tilde{Y} \in L_4$ such that $\|\tilde{X} - \tilde{Y}\| = d(L_3, L_4)$, where $L_3 = \left\{X \in \mathbb{R}^{n\times n} \,\middle|\, \|A_2X - B_2\|^2 + \|XC_2 - D_2\|^2 = \min\right\}$ and $L_4 = \left\{Y \in \mathbb{R}^{n\times n} \,\middle|\, \|E_2Y - F_2\|^2 + \|YG_2 - H_2\|^2 = \min\right\}$. We obtain explicit formulas for $d(L_1, L_2)$ and $d(L_3, L_4)$, and all the matrices in question by using the singular value decompositions and the canonical correlation decompositions of matrices.

Keywords: linear manifold; best approximation; singular value decomposition; canonical correlation decomposition
Mathematics Subject Classification: 15A24, 15A60

1. Introduction

Matrix equationsplay important roles in structural vibration systems [1, 2], automatic control [3], and other fields [4, 5]. Observe that the linear matrix equation

$$AX = B \tag{1.1}$$

and the linear matrix equations

$$AX = B, \; XC = D \tag{1.2}$$

have been extensively studied, and some profound results were established. The various special solutions of Eq (1.1) have been studied in [6–9]. The various special solutions of Eq (1.2) have been considered in [10–14]. Don [15] studied the general symmetric solution to Eq (1.1) by applying a formula for the partitioned minimum-norm reflexive generalized inverse. Dai [16] derived the symmetric solution of Eq (1.1) by utilizing the singular value decomposition (SVD). Sun [17] provided the least-squares symmetric solution for Eq (1.1) by using the SVD. Rao and Mitra [18] studied the common solutionof Eq (1.2). Yuan [19] considered the least-squares solution and the least-squares symmetric solution with the minimum-normof Eq (1.2) by applying the SVD. Obviously, the least-squares solution of the matrix equations form a linear manifold. It is known thatthe manifold distance is widely used in many aspects, such asin pattern recognition [20], image recognition [21–23], and Riemannian manifolds [24–27]. There are some results about the distance between linear manifolds. For example, Kass [28] obtained the expression of the distance from a point y to an affine subspace (linear manifold) $L = \{x|Ax = b\}$. Dupré and Kass [29] and Yuan [30] further proposed the specific formulas for the distance between two affine subspaces. Grover [31] derived an expression for the distance of a matrix A from any unital C^*-subalgebra of $\mathbb{C}^{n\times n}$ by the SVD. In addition, Du and Deng [32] established a new characterization of gaps between two subspaces of a Hilbert space. Baksalary and Trenkler [33] investigated the angles and distances between two given subspaces of $\mathbb{C}^n$. Scheffer and Vahrenhold [34, 35] proposed an algorithm to approximate the geodesic distances and approximate weighted geodesic distances on a 2-manifold in $\mathbb{R}^3$, respectively. Inspired by the works of Kass [28] and Yuan [30], in this paper we will discuss the best approximation problem between two least-squares symmetric solution manifolds of Eq (1.1) and the best approximation problem between two least-squares solution manifolds of Eq (1.2). Namely, we consider the following two problems:

Problem 1. Given A_1, B_1, C_1, and $D_1 \in \mathbb{R}^{m\times n}$, find $d(L_1, L_2) = \min_{X\in L_1, Y\in L_2} \|X - Y\|$, and find $\hat{X} \in L_1, \hat{Y} \in L_2$ such that $\|\hat{X} - \hat{Y}\| = d(L_1, L_2)$, where

$$L_1 = \{X \in \mathbb{SR}^{n\times n} \mid \|A_1X - B_1\| = \min\},$$
$$L_2 = \{Y \in \mathbb{SR}^{n\times n} \mid \|C_1Y - D_1\| = \min\},$$

$\mathbb{SR}^{n\times n}$ stands for the set of all $n \times n$ symmetric matrices, and $\|\cdot\|$ is the Frobenius norm.

Problem 2. Given matrices $A_2, B_2, E_2, F_2 \in \mathbb{R}^{m\times n}$ and $C_2, D_2, G_2, H_2 \in \mathbb{R}^{n\times p}$, find $d(L_3, L_4) = \min_{X\in L_3, Y\in L_4} \|X - Y\|$, and find $\tilde{X} \in L_3, \tilde{Y} \in L_4$ such that $\|\tilde{X} - \tilde{Y}\| = d(L_3, L_4)$, where

$$L_3 = \left\{X \in \mathbb{R}^{n\times n} \,\middle|\, \|A_2X - B_2\|^2 + \|XC_2 - D_2\|^2 = \min\right\},$$
$$L_4 = \left\{Y \in \mathbb{R}^{n\times n} \,\middle|\, \|E_2Y - F_2\|^2 + \|YG_2 - H_2\|^2 = \min\right\}.$$

The paper is organized as follows. In Section 2, we introduce some important lemmas. In Sections 3 and 4, we derive the expressions of the matrices $\hat{X}, \hat{Y}, \tilde{X}, \tilde{Y}$ and present the explicit expressions for $d(L_1, L_2)$ and $d(L_3, L_4)$ of Problems 1 and 2 by utilizing the singular value decomposition and the canonical correlation decomposition (CCD). Finally, in Section 5, we provide a simple recipe for numerical computation to solve Problem 2.

2. Preliminaries

In order to solve Problems 1 and 2, we need the following lemmas.

Lemma 2.1. *[19] Given A_1, B_1, C_1, and $D_1 \in \mathbb{R}^{m\times n}$, let the SVDs of the matrices A_1 and C_1 be*

$$A_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^{\top}, \quad C_1 = P_1 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} Q_1^{\top},$$

where $\Sigma_1 = \mathrm{diag}(\gamma_1, \ldots, \gamma_{s_1}) > 0$ *(that is,* $\Sigma_1 > 0$ *means that* Σ_1 *is a symmetric positive definite matrix),* $s_1 = \mathrm{rank}(A_1)$, $\Sigma_2 = \mathrm{diag}(\theta_1, \ldots, \theta_{t_1}) > 0$, $t_1 = \mathrm{rank}(C_1)$*, and* $U_1 = [U_{11}, U_{12}]$, $V_1 = [V_{11}, V_{12}]$, $P_1 = [P_{11}, P_{12}]$, $Q_1 = [Q_{11}, Q_{12}]$ *are orthogonal matrices with* $U_{11} \in \mathbb{R}^{m\times s_1}$, $V_{11} \in \mathbb{R}^{n\times s_1}$, $P_{11} \in \mathbb{R}^{m\times t_1}$, $Q_{11} \in \mathbb{R}^{n\times t_1}$. *Let*

$$U_1^\top B_1 V_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{13} & B_{14} \end{bmatrix}, \quad P_1^\top D_1 Q_1 = \begin{bmatrix} D_{11} & D_{12} \\ D_{13} & D_{14} \end{bmatrix}.$$

Then, the solution sets L_1 *and* L_2 *can be expressed as*

$$\begin{aligned} L_1 &= \{X \in \mathbb{SR}^{n\times n} |\ X = X_1 + V_{12} X_{14} V_{12}^\top\}, \\ L_2 &= \{Y \in \mathbb{SR}^{n\times n} |\ Y = Y_1 + Q_{12} Y_{14} Q_{12}^\top\}, \end{aligned} \tag{2.1}$$

where

$$X_1 = V_1 \begin{bmatrix} W_1 * (\Sigma_1 B_{11} + B_{11}^\top \Sigma_1) & \Sigma_1^{-1} B_{12} \\ B_{12}^\top \Sigma_1^{-1} & 0 \end{bmatrix} V_1^\top, \tag{2.2}$$

$$Y_1 = Q_1 \begin{bmatrix} W_2 * (\Sigma_2 D_{11} + D_{11}^\top \Sigma_2) & \Sigma_2^{-1} D_{12} \\ D_{12}^\top \Sigma_2^{-1} & 0 \end{bmatrix} Q_1^\top, \tag{2.3}$$

and $W_1 * (\Sigma_1 B_{11} + B_{11}^\top \Sigma_1)$ *represents the Hadamard product of* W_1 *and* $\Sigma_1 B_{11} + B_{11}^\top \Sigma_1$, $W_1 = [w_{ij}^{(1)}]_{s_1\times s_1}$ *with* $w_{ij}^{(1)} = \frac{1}{\gamma_i^2+\gamma_j^2}$ $(i, j = 1, \ldots, s_1)$, $W_2 = [w_{ij}^{(2)}]_{t_1\times t_1}$ *with* $w_{ij}^{(2)} = \frac{1}{\theta_i^2+\theta_j^2}$ $(i, j = 1, \ldots, t_1)$*, and* $X_{14} \in \mathbb{SR}^{(n-s_1)\times(n-s_1)}$, $Y_{14} \in \mathbb{SR}^{(n-t_1)\times(n-t_1)}$ *are arbitrary symmetric matrices.*

Lemma 2.2. *[19] Suppose that* $A_2, B_2, E_2, F_2 \in \mathbb{R}^{m\times n}$ *and* $C_2, D_2, G_2, H_2 \in \mathbb{R}^{n\times p}$. *Assume that the SVDs of the matrices* A_2, C_2, E_2*, and* G_2 *are*

$$\begin{aligned} A_2 &= U_2 \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} V_2^\top, \quad C_2 = P_2 \begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0 \end{bmatrix} Q_2^\top, \\ E_2 &= U_3 \begin{bmatrix} \Sigma_5 & 0 \\ 0 & 0 \end{bmatrix} V_3^\top, \quad G_2 = P_3 \begin{bmatrix} \Sigma_6 & 0 \\ 0 & 0 \end{bmatrix} Q_3^\top, \end{aligned} \tag{2.4}$$

where $\Sigma_3 = \mathrm{diag}(\lambda_1, \ldots, \lambda_{s_2}) > 0$, $s_2 = \mathrm{rank}(A_2)$, $\Sigma_4 = \mathrm{diag}(\rho_1, \ldots, \rho_{t_2}) > 0$, $t_2 = \mathrm{rank}(C_2)$, $\Sigma_5 = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{s_3}) > 0$, $s_3 = \mathrm{rank}(E_2)$, $\Sigma_6 = \mathrm{diag}(\eta_1, \ldots, \eta_{t_3}) > 0$, $t_3 = \mathrm{rank}(G_2)$*, and* $U_2 = [U_{21}, U_{22}]$, $V_2 = [V_{21}, V_{22}]$, $P_2 = [P_{21}, P_{22}]$, $Q_2 = [Q_{21}, Q_{22}]$, $U_3 = [U_{31}, U_{32}]$, $V_3 = [V_{31}, V_{32}]$, $P_3 = [P_{31}, P_{32}]$, $Q_3 = [Q_{31}, Q_{32}]$ *are orthogonal matrices with* $U_{21} \in \mathbb{R}^{m\times s_2}$, $V_{21} \in \mathbb{R}^{n\times s_2}$, $P_{21} \in \mathbb{R}^{n\times t_2}$, $Q_{21} \in \mathbb{R}^{p\times t_2}$, $U_{31} \in \mathbb{R}^{m\times s_3}$, $V_{31} \in \mathbb{R}^{n\times s_3}$, $P_{31} \in \mathbb{R}^{n\times t_3}$*, and* $Q_{31} \in \mathbb{R}^{p\times t_3}$. *Let*

$$\begin{aligned} U_2^\top B_2 P_2 &= \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}, \quad V_2^\top D_2 Q_2 = \begin{bmatrix} D_{21} & D_{22} \\ D_{23} & D_{24} \end{bmatrix}, \\ U_3^\top F_2 P_3 &= \begin{bmatrix} K_{21} & K_{22} \\ K_{23} & K_{24} \end{bmatrix}, \quad V_3^\top H_2 Q_3 = \begin{bmatrix} H_{21} & H_{22} \\ H_{23} & H_{24} \end{bmatrix}. \end{aligned}$$

Then, the solution sets L_3 *and* L_4 *can be expressed as*

$$\begin{aligned} L_3 &= \{X \in \mathbb{R}^{n\times n} |\ X = X_2 + V_{22} X_{24} P_{22}^\top\}, \\ L_4 &= \{Y \in \mathbb{R}^{n\times n} |\ Y = Y_2 + V_{32} Y_{24} P_{32}^\top\}, \end{aligned} \tag{2.5}$$

where

$$X_2 = V_2 \begin{bmatrix} W_3 * (\Sigma_3 B_{21} + D_{21}\Sigma_4) & \Sigma_3^{-1} B_{22} \\ D_{23}\Sigma_4^{-1} & 0 \end{bmatrix} P_2^\top, \tag{2.6}$$

$$Y_2 = V_3 \begin{bmatrix} W_4 * (\Sigma_5 K_{21} + H_{21}\Sigma_6) & \Sigma_5^{-1} K_{22} \\ H_{23}\Sigma_6^{-1} & 0 \end{bmatrix} P_3^\top, \tag{2.7}$$

and $W_3 = [w_{ij}^{(3)}]_{s_2\times t_2}$ *with* $w_{ij}^{(3)} = \frac{1}{\lambda_i^2+\rho_j^2}$ $(i = 1, \ldots, s_2;\ j = 1, \ldots, t_2)$*,* $W_4 = [w_{ij}^{(4)}]_{s_3\times t_3}$ *with* $w_{ij}^{(4)} = \frac{1}{\epsilon_i^2+\eta_j^2}$ $(i = 1, \ldots, s_3;\ j = 1, \ldots, t_3)$*, and* $X_{24} \in \mathbb{R}^{(n-s_2)\times(n-t_2)}$*,* $Y_{24} \in \mathbb{R}^{(n-s_3)\times(n-t_3)}$ *are arbitrary matrices.*

Lemma 2.3. *Let* $F_{26},\ F_{56} \in \mathbb{R}^{h_1\times k_1}$*, and* $C_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_{h_1}) > 0,\ S_A = \mathrm{diag}\,(\beta_1, \ldots, \beta_{h_1}) > 0$ *satisfy* $\alpha_i^2 + \beta_i^2 = 1$ $(i = 1, \ldots, h_1)$*. Then,*

$$\Phi(T_{23}) = \|C_A T_{23} + F_{26}\|^2 + \|S_A T_{23} + F_{56}\|^2 = \min$$

if and only if

$$T_{23} = -(C_A F_{26} + S_A F_{56}). \tag{2.8}$$

Proof. Let $F_{26} = [f_{ij}], F_{56} = [g_{ij}] \in \mathbb{R}^{h_1\times k_1}$, and $T_{23} = [t_{ij}] \in \mathbb{R}^{h_1\times k_1}$. Then,

$$\Phi(T_{23}) = \sum_{i=1}^{h_1}\sum_{j=1}^{k_1}\left((\alpha_i t_{ij} + f_{ij})^2 + (\beta_i t_{ij} + g_{ij})^2\right).$$

Now, we minimize the quantities

$$\varphi_1 = (\alpha_i t_{ij} + f_{ij})^2 + (\beta_i t_{ij} + g_{ij})^2\ (i = 1, \ldots, h_1;\ j = 1, \ldots, k_1).$$

It is easy to obtain the minimizers

$$t_{ij} = -\frac{\alpha_i f_{ij} + \beta_i g_{ij}}{\alpha_i^2 + \beta_i^2} = -(\alpha_i f_{ij} + \beta_i g_{ij})\ (i = 1, \ldots, h_1;\ j = 1, \ldots, k_1). \tag{2.9}$$

(2.8) follows from (2.9) straightforwardly. □

Lemma 2.4. *Suppose that the matrices* $F_{12},\ F_{15} \in \mathbb{R}^{r_1\times h_1}$*. Let* $C_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_{h_1}) > 0,\ S_A = \mathrm{diag}(\beta_1, \ldots, \beta_{h_1}) > 0$ *satisfy* $\alpha_i^2 + \beta_i^2 = 1$ $(i = 1, \ldots, h_1)$*. Then,*

$$\Phi(T_{12},\ J_{12}) = \|T_{12}C_A - J_{12} + F_{12}\|^2 + \|T_{12}S_A + F_{15}\|^2 = \min \tag{2.10}$$

if and only if

$$T_{12} = -F_{15}S_A^{-1},\ J_{12} = F_{12} - F_{15}S_A^{-1}C_A. \tag{2.11}$$

Proof. Let $F_{12} = [f_{ij}],\ F_{15} = [k_{ij}],\ T_{12} = [t_{ij}],\ J_{12} = [q_{ij}] \in \mathbb{R}^{r_1\times h_1}$. Then, the minimization problem of (2.10) is equivalent to

$$\Phi(T_{12},\ J_{12}) = \sum_{i=1}^{r_1}\sum_{j=1}^{h_1}\left((t_{ij}\alpha_j - q_{ij} + f_{ij})^2 + (t_{ij}\beta_j + k_{ij})^2\right). \tag{2.12}$$

Clearly, the function of concerning variables t_{ij} and q_{ij} in (2.12) is

$$\varphi_2 = (t_{ij}\alpha_j - q_{ij} + f_{ij})^2 + (t_{ij}\beta_j + k_{ij})^2 \ (i = 1, \ldots, r_1;\ j = 1, \ldots, h_1).$$

It is easy to verify that the function φ_2 attains the smallest value at

$$\frac{\partial \varphi_2}{\partial t_{ij}} = 0,\ \ \frac{\partial \varphi_2}{\partial q_{ij}} = 0 \ (i = 1, \ldots, r_1;\ j = 1, \ldots, h_1),$$

which yields

$$t_{ij} = -k_{ij}\beta_j^{-1},\ \ q_{ij} = f_{ij} - k_{ij}\beta_j^{-1}\alpha_j \ (i = 1, \ldots, r_1;\ j = 1, \ldots, h_1). \tag{2.13}$$

By rewriting (2.13) in matrix forms, we immediately obtain (2.11). □

Lemma 2.5. *[36] Assume that* $C_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_{h_1}) > 0,\ S_A = \mathrm{diag}(\beta_1, \ldots, \beta_{h_1}) > 0$ *with* $\alpha_i^2 + \beta_i^2 = 1\ (i = 1, \ldots, h_1)$*, and* $F_{22}, F_{25}, F_{55} \in \mathbb{R}^{h_1\times h_1}$. *Then, the problem*

$$\Phi(T_{22}, J_{22}) = \|C_A T_{22} C_A - J_{22} + F_{22}\|^2 + 2\|C_A T_{22} S_A + F_{25}\|^2 + \|S_A T_{22} S_A + F_{55}\|^2 = \min$$

has unique symmetric solutions $T_{22}, J_{22} \in \mathbb{R}^{h_1\times h_1}$ *with the forms*

$$\begin{aligned} T_{22} &= W_5 * (C_A F_{25} S_A + S_A F_{25}^{\top} C_A + S_A F_{55} S_A), \\ J_{22} &= F_{22} + W_5 * (C_A^2 F_{25} S_A C_A + C_A S_A F_{25}^{\top} C_A^2 + C_A S_A F_{55} S_A C_A), \end{aligned} \tag{2.14}$$

where $W_5 = [w_{ij}^{(5)}]_{h_1\times h_1},\ w_{ij}^{(5)} = \frac{1}{\alpha_i^2\alpha_j^2 - 1}\ (i, j = 1, \ldots, h_1)$.

Lemma 2.6. *[36] Let* $M_{22},\ M_{25},\ M_{52},\ M_{55} \in \mathbb{R}^{h_2\times h_3}$*, and let* $C_3 = \mathrm{diag}(\kappa_1, \ldots, \kappa_{h_2}) > 0,\ S_3 = \mathrm{diag}(\sigma_1, \ldots, \sigma_{h_2}) > 0$ *satisfy* $\kappa_i^2 + \sigma_i^2 = 1\ (i = 1, \ldots, h_2),\ C_4 = \mathrm{diag}(\delta_1, \ldots, \delta_{h_3}) > 0$*, and* $S_4 = \mathrm{diag}(\zeta_1, \ldots, \zeta_{h_3}) > 0$ *satisfy* $\delta_i^2 + \zeta_i^2 = 1\ (i = 1, \ldots, h_3)$. *Then,*

$$\begin{aligned} \Phi(R_{22}, Z_{22}) =& \|C_3 R_{22} C_4 - Z_{22} + M_{22}\|^2 + \|S_3 R_{22} C_4 + M_{52}\|^2 \\ &+ \|C_3 R_{22} S_4 + M_{25}\|^2 + \|S_3 R_{22} S_4 + M_{55}\|^2 = \min \end{aligned}$$

if and only if

$$\begin{aligned} R_{22} &= W_6 * (C_3 M_{25} S_4 + S_3 M_{52} C_4 + S_3 M_{55} S_4), \\ Z_{22} &= M_{22} + W_6 * (C_3^2 M_{25} S_4 C_4 + C_3 S_3 M_{52} C_4^2 + C_3 S_3 M_{55} S_4 C_4), \end{aligned} \tag{2.15}$$

where $W_6 = [w_{ij}^{(6)}]_{h_2\times h_3}$ *with* $w_{ij}^{(6)} = \frac{1}{\kappa_i^2\delta_j^2 - 1}\ (i = 1, \ldots, h_2; j = 1, \ldots, h_3)$.

3. The solution of Problem 1

Let $V_{12} \in \mathbb{R}^{n\times(n-s_1)}$ and $Q_{12} \in \mathbb{R}^{n\times(n-t_1)}$ be column orthogonal matrices, and assume that $n - s_1 = \mathrm{rank}(V_{12}) \geq \mathrm{rank}(Q_{12}) = n - t_1$. Let the CCD [37] of the matrix pair $[V_{12}, Q_{12}]$ be

$$V_{12} = F\Sigma_{A_1} N_1^{\top},\ \ Q_{12} = F\Sigma_{C_1} N_2^{\top}, \tag{3.1}$$

in which $F \in \mathbb{R}^{n\times n}$, $N_1 \in \mathbb{R}^{(n-s_1)\times(n-s_1)}$, and $N_2 \in \mathbb{R}^{(n-t_1)\times(n-t_1)}$ are orthogonal matrices, and

$$\Sigma_{A_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & C_A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_A & 0 \\ 0 & 0 & I \end{bmatrix} \begin{matrix} r_1 \\ h_1 \\ f_1 \\ t_1-h_1-k_1 \\ h_1 \\ k_1 \end{matrix}, \quad \Sigma_{C_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 \\ h_1 \\ f_1 \\ t_1-h_1-k_1 \\ h_1 \\ k_1 \end{matrix},$$
$$\qquad r_1 \quad h_1 \quad k_1 \qquad\qquad\qquad r_1 \quad h_1 \quad f_1$$

$\mathrm{rank}(V_{12}) = r_1+h_1+k_1$, $h_1 = \mathrm{rank}([V_{12}, Q_{12}])+\mathrm{rank}(Q_{12}^\top V_{12})-\mathrm{rank}(V_{12})-\mathrm{rank}(Q_{12})$, $r_1 = \mathrm{rank}(V_{12})+\mathrm{rank}(Q_{12})-\mathrm{rank}([V_{12}, Q_{12}])$, $k_1 = \mathrm{rank}(V_{12})-\mathrm{rank}(Q_{12}^\top V_{12})$, $f_1 = n-t_1-r_1-h_1$, and

$$C_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_{h_1}), \quad S_A = \mathrm{diag}(\beta_1, \ldots, \beta_{h_1})$$

with

$$1 > \alpha_1 \geq \cdots \geq \alpha_{h_1} > 0, \quad 0 < \beta_1 \leq \cdots \leq \beta_{h_1} < 1, \quad \alpha_i^2+\beta_i^2 = 1 \ (i = 1, \ldots, h_1).$$

It follows from (2.1) and (3.1) that for any $X \in L_1$ and $Y \in L_2$, we have

$$\begin{aligned} \|X-Y\| &= \|F\Sigma_{A_1}N_1^\top X_{14}N_1\Sigma_{A_1}^\top F^\top - F\Sigma_{C_1}N_2^\top Y_{14}N_2\Sigma_{C_1}^\top F^\top + (X_1-Y_1)\| \\ &= \|\Sigma_{A_1}N_1^\top X_{14}N_1\Sigma_{A_1}^\top - \Sigma_{C_1}N_2^\top Y_{14}N_2\Sigma_{C_1}^\top + F^\top(X_1-Y_1)F\|. \end{aligned}$$

Write

$$F^\top(X_1-Y_1)F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{12}^\top & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{13}^\top & F_{23}^\top & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{14}^\top & F_{24}^\top & F_{34}^\top & F_{44} & F_{45} & F_{46} \\ F_{15}^\top & F_{25}^\top & F_{35}^\top & F_{45}^\top & F_{55} & F_{56} \\ F_{16}^\top & F_{26}^\top & F_{36}^\top & F_{46}^\top & F_{56}^\top & F_{66} \end{bmatrix}, \tag{3.2}$$

$$N_1^\top X_{14}N_1 = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^\top & T_{22} & T_{23} \\ T_{13}^\top & T_{23}^\top & T_{33} \end{bmatrix}, \quad N_2^\top Y_{14}N_2 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12}^\top & J_{22} & J_{23} \\ J_{13}^\top & J_{23}^\top & J_{33} \end{bmatrix}. \tag{3.3}$$

Then,

$$\begin{aligned} d(L_1, L_2) &= \min_{X\in L_1, Y\in L_2} \|X-Y\| \\ &= \left\| \begin{bmatrix} T_{11}-J_{11}+F_{11} & T_{12}C_A-J_{12}+F_{12} & F_{13}-J_{13} & F_{14} & T_{12}S_A+F_{15} & T_{13}+F_{16} \\ C_AT_{12}^\top-J_{12}^\top+F_{12}^\top & C_AT_{22}C_A-J_{22}+F_{22} & F_{23}-J_{23} & F_{24} & C_AT_{22}S_A+F_{25} & C_AT_{23}+F_{26} \\ F_{13}^\top-J_{13}^\top & F_{23}^\top-J_{23}^\top & F_{33}-J_{33} & F_{34} & F_{35} & F_{36} \\ F_{14}^\top & F_{24}^\top & F_{34}^\top & F_{44} & F_{45} & F_{46} \\ S_AT_{12}^\top+F_{15}^\top & S_AT_{22}C_A+F_{25}^\top & F_{35}^\top & F_{45}^\top & S_AT_{22}S_A+F_{55} & S_AT_{23}+F_{56} \\ T_{13}^\top+F_{16}^\top & T_{23}^\top C_A+F_{26}^\top & F_{36}^\top & F_{46}^\top & T_{23}^\top S_A+F_{56}^\top & T_{33}+F_{66} \end{bmatrix} \right\| \\ &= \min. \end{aligned} \tag{3.4}$$

It follows from (3.2)–(3.4) that $d(L_1, L_2) = \min_{X\in L_1, Y\in L_2} \|X - Y\|$ if and only if

$$\|T_{11} - J_{11} + F_{11}\| = \min, \tag{3.5}$$
$$\|T_{13} + F_{16}\| = \min,\ \|T_{33} + F_{66}\| = \min, \tag{3.6}$$
$$\|F_{13} - J_{13}\| = \min,\ \|F_{23} - J_{23}\| = \min,\ \|F_{33} - J_{33}\| = \min, \tag{3.7}$$
$$\|C_A T_{23} + F_{26}\|^2 + \|S_A T_{23} + F_{56}\|^2 = \min, \tag{3.8}$$
$$\|T_{12}C_A - J_{12} + F_{12}\|^2 + \|T_{12}S_A + F_{15}\|^2 = \min, \tag{3.9}$$
$$\|C_A T_{22} C_A - J_{22} + F_{22}\|^2 + 2\|C_A T_{22} S_A + F_{25}\|^2 + \|S_A T_{22} S_A + F_{55}\|^2 = \min\,. \tag{3.10}$$

From (3.5), we obtain

$$T_{11} = J_{11} - F_{11}, \tag{3.11}$$

where $J_{11} \in \mathbb{SR}^{r_1\times r_1}$ is an arbitrary matrix. By (3.6) and (3.7), we can find that

$$T_{13} = -F_{16},\ \ T_{33} = -F_{66},\ \ J_{13} = F_{13},\ \ J_{23} = F_{23},\ \ J_{33} = F_{33}. \tag{3.12}$$

By applying Lemma 2.3, from (3.8) we have

$$T_{23} = -(C_A F_{26} + S_A F_{56}). \tag{3.13}$$

Solving the minimization problem (3.9) by using Lemma 2.4, we obtain

$$T_{12} = -F_{15}S_A^{-1},\ \ J_{12} = F_{12} - F_{15}S_A^{-1}C_A. \tag{3.14}$$

Applying Lemma 2.5, we can obtain (2.14) from (3.10). Inserting (2.14) and (3.11)–(3.14) into (3.3) and (3.4), we obtain

$$X_{14} = N_1 \begin{bmatrix} J_{11} - F_{11} & -F_{15}S_A^{-1} & -F_{16} \\ -S_A^{-1}F_{15}^\top & T_{22} & -(C_A F_{26} + S_A F_{56}) \\ -F_{16}^\top & -(F_{26}^\top C_A + F_{56}^\top S_A) & -F_{66} \end{bmatrix} N_1^\top,$$

$$Y_{14} = N_2 \begin{bmatrix} J_{11} & F_{12} - F_{15}S_A^{-1}C_A & F_{13} \\ F_{12}^\top - C_A S_A^{-1} F_{15}^\top & J_{22} & F_{23} \\ F_{13}^\top & F_{23}^\top & F_{33} \end{bmatrix} N_2^\top,$$

$$d(L_1, L_2) = \left\| \begin{bmatrix} 0 & 0 & 0 & F_{14} & 0 & 0 \\ 0 & 0 & 0 & F_{24} & C_A T_{22} S_A + F_{25} & S_A^2 F_{26} - C_A S_A F_{56} \\ 0 & 0 & 0 & F_{34} & F_{35} & F_{36} \\ F_{14}^\top & F_{24}^\top & F_{34}^\top & F_{44} & F_{45} & F_{46} \\ 0 & S_A T_{22} C_A + F_{25}^\top & F_{35}^\top & F_{45}^\top & S_A T_{22} S_A + F_{55} & C_A^2 F_{56} - S_A C_A F_{26} \\ 0 & F_{26}^\top S_A^2 - F_{56}^\top S_A C_A & F_{36}^\top & F_{46}^\top & F_{56}^\top C_A^2 - F_{26}^\top C_A S_A & 0 \end{bmatrix} \right\|. \tag{3.15}$$

The relation of (3.15) can be equivalently written as

$$\begin{aligned} d(L_1, L_2) =& \Big(2\sum_{i=1}^{3} \|F_{i4}\|^2 + 2\sum_{j=5}^{6} \|F_{3j}\|^2 + 2\sum_{k=5}^{6} \|F_{4k}\|^2 + \|F_{44}\|^2 + \|S_A T_{22} S_A + F_{55}\|^2 \\ &+ 2\|C_A T_{22} S_A + F_{25}\|^2 + 2\|S_A^2 F_{26} - C_A S_A F_{56}\|^2 + 2\|C_A^2 F_{56} - S_A C_A F_{26}\|^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.16}$$

As a summary of the above discussion, we have proved the following result.

Theorem 3.1. *Given the matrices A_1, B_1, C_1, and $D_1 \in \mathbb{R}^{m\times n}$, the explicit expression for $d(L_1, L_2)$ of Problem 1 can be expressed as* (3.16), *and the matrices $\hat{X}$ and $\hat{Y}$ are given by*

$$\hat{X} = X_1 + V_{12}\hat{X}_{14}V_{12}^{\top},\ \ \hat{Y} = Y_1 + Q_{12}\hat{Y}_{14}Q_{12}^{\top}, \tag{3.17}$$

where

$$\hat{X}_{14} = N_1 \begin{bmatrix} J_{11} - F_{11} & -F_{15}S_A^{-1} & -F_{16} \\ -S_A^{-1}F_{15}^{\top} & T_{22} & -(C_A F_{26} + S_A F_{56}) \\ -F_{16}^{\top} & -(F_{26}^{\top}C_A + F_{56}^{\top}S_A) & -F_{66} \end{bmatrix} N_1^{\top},$$

$$\hat{Y}_{14} = N_2 \begin{bmatrix} J_{11} & F_{12} - F_{15}S_A^{-1}C_A & F_{13} \\ F_{12}^{\top} - C_A S_A^{-1}F_{15}^{\top} & J_{22} & F_{23} \\ F_{13}^{\top} & F_{23}^{\top} & F_{33} \end{bmatrix} N_2^{\top},$$

$J_{11} \in \mathbb{SR}^{r_1\times r_1}$ *is an arbitrary matrix, and X_1, Y_1, T_{22}, J_{22} are given by* (2.2), (2.3), *and* (2.14), *respectively.*

4. The solution of Problem 2

Suppose that $V_{22} \in \mathbb{R}^{n\times(n-s_2)}$, $V_{32} \in \mathbb{R}^{n\times(n-s_3)}$, $P_{22} \in \mathbb{R}^{n\times(n-t_2)}$, and $P_{32} \in \mathbb{R}^{n\times(n-t_3)}$ are column orthogonal matrices, and assume that $n - s_2 = \mathrm{rank}(V_{22}) \geq \mathrm{rank}(V_{32}) = n - s_3$ and $n - t_2 = \mathrm{rank}(P_{22}) \geq \mathrm{rank}(P_{32}) = n - t_3$. Let the CCDs of the matrix pairs $[V_{22}, V_{32}]$ and $[P_{22}, P_{32}]$ be

$$\begin{aligned} V_{22} &= M\Sigma_{A_2}N_3^{\top},\ \ V_{32} = M\Sigma_{E_2}N_4^{\top}, \\ P_{22} &= W\Sigma_{C_2}N_5^{\top},\ \ P_{32} = W\Sigma_{G_2}N_6^{\top}, \end{aligned} \tag{4.1}$$

in which $M \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $N_3 \in \mathbb{R}^{(n-s_2)\times(n-s_2)}$, $N_4 \in \mathbb{R}^{(n-s_3)\times(n-s_3)}$, $N_5 \in \mathbb{R}^{(n-t_2)\times(n-t_2)}$, and $N_6 \in \mathbb{R}^{(n-t_3)\times(n-t_3)}$ are all orthogonal matrices, and

$$\Sigma_{A_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_3 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{matrix} r_2 \\ h_2 \\ f_2 \\ s_3 - h_2 - k_2 \\ h_2 \\ k_2 \end{matrix}, \quad \Sigma_{E_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 \\ h_2 \\ f_2 \\ s_3 - h_2 - k_2 \\ h_2 \\ k_2 \end{matrix},$$
$$\qquad\quad \begin{matrix} r_2 & h_2 & k_2 \end{matrix} \qquad\qquad\qquad\qquad \begin{matrix} r_2 & h_2 & f_2 \end{matrix}$$

$\mathrm{rank}(V_{22}) = r_2 + h_2 + k_2$, $h_2 = \mathrm{rank}([V_{22}, V_{32}]) + \mathrm{rank}(V_{32}^{\top}V_{22}) - \mathrm{rank}(V_{22}) - \mathrm{rank}(V_{32})$, $r_2 = \mathrm{rank}(V_{22}) + \mathrm{rank}(V_{32}) - \mathrm{rank}([V_{22}, V_{32}])$, $k_2 = \mathrm{rank}(V_{22}) - \mathrm{rank}(V_{32}^{\top}V_{22})$, $f_2 = n - s_3 - r_2 - h_2$, and

$$C_3 = \mathrm{diag}(\kappa_1, \ldots, \kappa_{h_2}),\ \ S_3 = \mathrm{diag}(\sigma_1, \ldots, \sigma_{h_2})$$

with

$$1 > \kappa_1 \geq \cdots \geq \kappa_{h_2} > 0,\ \ 0 < \sigma_1 \leq \cdots \leq \sigma_{h_2} < 1,\ \ \kappa_i^2 + \sigma_i^2 = 1\ (i = 1, \ldots, h_2).$$

$$\Sigma_{C_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & C_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_4 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{matrix} r_3 \\ h_3 \\ f_3 \\ t_3 - h_3 - k_3 \\ h_3 \\ k_3 \end{matrix}, \quad \Sigma_{G_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 \\ h_3 \\ f_3 \\ t_3 - h_3 - k_3 \\ h_3 \\ k_3 \end{matrix},$$
$$\begin{matrix} r_3 & h_3 & k_3 \end{matrix} \qquad\qquad \begin{matrix} r_3 & h_3 & f_3 \end{matrix}$$

$\mathrm{rank}(P_{22}) = r_3 + h_3 + k_3$, $h_3 = \mathrm{rank}([P_{22}, P_{32}]) + \mathrm{rank}(P_{32}^\top P_{22}) - \mathrm{rank}(P_{22}) - \mathrm{rank}(P_{32})$, $r_3 = \mathrm{rank}(P_{22}) + \mathrm{rank}(P_{32}) - \mathrm{rank}([P_{22}, P_{32}])$, $k_3 = \mathrm{rank}(P_{22}) - \mathrm{rank}(P_{32}^\top P_{22})$, $f_3 = n - t_3 - r_3 - h_3$, and

$$C_4 = \mathrm{diag}(\delta_1, \ldots, \delta_{h_3}), \quad S_4 = \mathrm{diag}(\zeta_1, \ldots, \zeta_{h_3})$$

with

$$1 > \delta_1 \geq \cdots \geq \delta_{h_3} > 0, \quad 0 < \zeta_1 \leq \cdots \leq \zeta_{h_3} < 1, \quad \delta_i^2 + \zeta_i^2 = 1 \ (i = 1, \ldots, h_3).$$

It follows from (2.5) and (4.1) that, for any $X \in L_3$ and $Y \in L_4$, we have

$$\begin{aligned} \|X - Y\| &= \|M\Sigma_{A_2} N_3^\top X_{24} N_5 \Sigma_{C_2}^\top W^\top - M\Sigma_{E_2} N_4^\top Y_{24} N_6 \Sigma_{G_2}^\top W^\top + (X_2 - Y_2)\| \\ &= \|\Sigma_{A_2} N_3^\top X_{24} N_5 \Sigma_{C_2}^\top - \Sigma_{E_2} N_4^\top Y_{24} N_6 \Sigma_{G_2}^\top + M^\top (X_2 - Y_2) W\|. \end{aligned}$$

If we set

$$M^\top (X_2 - Y_2) W = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} \\ M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} \end{bmatrix}, \tag{4.2}$$

$$N_3^\top X_{24} N_5 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}, \quad N_4^\top Y_{24} N_6 = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}, \tag{4.3}$$

then

$$\begin{aligned} d(L_3, L_4) &= \min_{X \in L_3, Y \in L_4} \|X - Y\| \\ &= \left\| \begin{bmatrix} R_{11} - Z_{11} + M_{11} & R_{12}C_4 - Z_{12} + M_{12} & M_{13} - Z_{13} & M_{14} & R_{12}S_4 + M_{15} & R_{13} + M_{16} \\ C_3 R_{21} - Z_{21} + M_{21} & C_3 R_{22} C_4 - Z_{22} + M_{22} & M_{23} - Z_{23} & M_{24} & C_3 R_{22} S_4 + M_{25} & C_3 R_{23} + M_{26} \\ M_{31} - Z_{31} & M_{32} - Z_{32} & M_{33} - Z_{33} & M_{34} & M_{35} & M_{36} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\ S_3 R_{21} + M_{51} & S_3 R_{22} C_4 + M_{52} & M_{53} & M_{54} & S_3 R_{22} S_4 + M_{55} & S_3 R_{23} + M_{56} \\ R_{31} + M_{61} & R_{32} C_4 + M_{62} & M_{63} & M_{64} & R_{32} S_4 + M_{65} & R_{33} + M_{66} \end{bmatrix} \right\| \\ &= \min. \end{aligned} \tag{4.4}$$

It follows from (4.2)–(4.4) that $d(L_3, L_4) = \min_{X \in L_3, Y \in L_4} \|X - Y\|$ if and only if

$$\|R_{11} - Z_{11} + M_{11}\| = \min, \tag{4.5}$$

$$\|M_{13} - Z_{13}\| = \min,\ \|M_{23} - Z_{23}\| = \min,\ \|M_{33} - Z_{33}\| = \min,\ \|M_{31} - Z_{31}\| = \min, \tag{4.6}$$

$$\|M_{32} - Z_{32}\| = \min,\ \|R_{13} + M_{16}\| = \min,\ \|R_{31} + M_{61}\| = \min,\ \|R_{33} + M_{66}\| = \min, \tag{4.7}$$

$$\|C_3R_{23} + M_{26}\|^2 + \|S_3R_{23} + M_{56}\|^2 = \min, \tag{4.8}$$

$$\|R_{32}C_4 + M_{62}\|^2 + \|R_{32}S_4 + M_{65}\|^2 = \min, \tag{4.9}$$

$$\|R_{12}C_4 - Z_{12} + M_{12}\|^2 + \|R_{12}S_4 + M_{15}\|^2 = \min, \tag{4.10}$$

$$\|C_3R_{21} - Z_{21} + M_{21}\|^2 + \|S_3R_{21} + M_{51}\|^2 = \min, \tag{4.11}$$

$$\begin{aligned}&\|C_3R_{22}C_4 - Z_{22} + M_{22}\|^2 + \|S_3R_{22}C_4 + M_{52}\|^2\\ &+\|C_3R_{22}S_4 + M_{25}\|^2 + \|S_3R_{22}S_4 + M_{55}\|^2 = \min.\end{aligned} \tag{4.12}$$

From (4.5), we obtain

$$R_{11} = Z_{11} - M_{11}, \tag{4.13}$$

where $Z_{11} \in \mathbb{R}^{r_2\times r_3}$ is an arbitrary matrix. By (4.6) and (4.7), we get

$$\begin{aligned}&Z_{13} = M_{13},\ \ Z_{23} = M_{23},\ \ Z_{33} = M_{33},\ \ Z_{31} = M_{31},\\ &Z_{32} = M_{32},\ \ R_{13} = -M_{16},\ \ R_{31} = -M_{61},\ \ R_{33} = -M_{66}.\end{aligned} \tag{4.14}$$

It follows from (4.8), (4.9), and Lemma 2.3 that

$$R_{23} = -(C_3M_{26} + S_3M_{56}),\ \ R_{32} = -(M_{62}C_4 + M_{65}S_4). \tag{4.15}$$

By using Lemma 2.4, solving minimization problems (4.10) and (4.11) yields

$$\begin{aligned}&R_{12} = -M_{15}S_4^{-1},\ \ Z_{12} = M_{12} - M_{15}S_4^{-1}C_4,\\ &R_{21} = -S_3^{-1}M_{51},\ \ Z_{21} = M_{21} - C_3S_3^{-1}M_{51}.\end{aligned} \tag{4.16}$$

The relation of (2.15) follows from Lemma 2.6 and (4.12). Substituting (2.15) and (4.13)–(4.16) into (4.3) and (4.4) leads to

$$X_{24} = N_3\begin{bmatrix} Z_{11} - M_{11} & -M_{15}S_4^{-1} & -M_{16}\\ -S_3^{-1}M_{51} & R_{22} & -(C_3M_{26} + S_3M_{56})\\ -M_{61} & -(M_{62}C_4 + M_{65}S_4) & -M_{66}\end{bmatrix}N_5^{\top},$$

$$Y_{24} = N_4\begin{bmatrix} Z_{11} & M_{12} - M_{15}S_4^{-1}C_4 & M_{13}\\ M_{21} - C_3S_3^{-1}M_{51} & Z_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{bmatrix}N_6^{\top},$$

$$d(L_3, L_4) = \left\|\begin{bmatrix} 0 & 0 & 0 & M_{14} & 0 & 0\\ 0 & 0 & 0 & M_{24} & C_3R_{22}S_4 + M_{25} & S_3^2M_{26} - C_3S_3M_{56}\\ 0 & 0 & 0 & M_{34} & M_{35} & M_{36}\\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46}\\ 0 & S_3R_{22}C_4 + M_{52} & M_{53} & M_{54} & S_3R_{22}S_4 + M_{55} & C_3^2M_{56} - S_3C_3M_{26}\\ 0 & M_{62}S_4^2 - M_{65}S_4C_4 & M_{63} & M_{64} & M_{65}C_4^2 - M_{62}C_4S_4 & 0\end{bmatrix}\right\|. \tag{4.17}$$

The relation of (4.17) can be equivalently written as

$$\begin{aligned}
&d(L_3, L_4)\\
=&\Big(\sum_{i=1}^{6}\|M_{i4}\|^2+\sum_{j=1}^{6}\|M_{4j}\|^2-\|M_{44}\|^2+\sum_{k=5}^{6}\|M_{3k}\|^2+\sum_{n=5}^{6}\|M_{n3}\|^2\\
&+\|S_3^2M_{26}-C_3S_3M_{56}\|^2+\|M_{62}S_4^2-M_{65}S_4C_4\|^2+\|C_3^2M_{56}-S_3C_3M_{26}\|^2\\
&+\|M_{65}C_4^2-M_{62}C_4S_4\|^2+\|C_3R_{22}S_4+M_{25}\|^2+\|S_3R_{22}C_4+M_{52}\|^2+\|S_3R_{22}S_4+M_{55}\|^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4.18}$$

By now, we have proved the following theorem.

Theorem 4.1. *Given $A_2, B_2, E_2, F_2 \in \mathbb{R}^{m\times n}$ and $C_2, D_2, G_2, H_2 \in \mathbb{R}^{n\times p}$, $d(L_3, L_4)$ can be expressed by* (4.18) *and the matrices $\tilde{X}$ and $\tilde{Y}$ are given by*

$$\tilde{X} = X_2 + V_{22}\tilde{X}_{24}P_{22}^{\top}, \quad \tilde{Y} = Y_2 + V_{32}\tilde{Y}_{24}P_{32}^{\top}, \tag{4.19}$$

where

$$\tilde{X}_{24} = N_3\begin{bmatrix} Z_{11}-M_{11} & -M_{15}S_4^{-1} & -M_{16}\\ -S_3^{-1}M_{51} & R_{22} & -(C_3M_{26}+S_3M_{56})\\ -M_{61} & -(M_{62}C_4+M_{65}S_4) & -M_{66}\end{bmatrix}N_5^{\top}, \tag{4.20}$$

$$\tilde{Y}_{24} = N_4\begin{bmatrix} Z_{11} & M_{12}-M_{15}S_4^{-1}C_4 & M_{13}\\ M_{21}-C_3S_3^{-1}M_{51} & Z_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{bmatrix}N_6^{\top}, \tag{4.21}$$

$Z_{11} \in \mathbb{R}^{r_2\times r_3}$ is an arbitrary matrix, and X_2, Y_2, R_{22}, Z_{22} are given by (2.6), (2.7), *and* (2.15)*, respectively.*

5. Numerical algorithm and numerical example

According to Theorem 4.1, we have the following algorithm to solve Problem 2.

Algorithm 1.

1) Input matrices $A_2, B_2, C_2, D_2, E_2, F_2, G_2$, and H_2.

2) Compute the SVDs of the matrices A_2, C_2, E_2, and G_2 according to (2.4).

3) Compute the CCDs of the matrix pairs $[V_{22}, V_{23}]$ and $[P_{22}, P_{23}]$ by (4.1).

4) Calculate the matrices $M_{ij}, i, j = 1, \ldots, 6$ following (4.2).

5) Randomly choose the matrix Z_{11}.

6) Calculate the matrices $R_{ij},\ i, j = 1, 2, 3;\ Z_{mn},\ m, n = 2, 3;\ Z_{12}$ and Z_{13} by (2.15) and (4.13)–(4.16).

7) Compute $d(L_3, L_4), \tilde{X}_{24}$, and $\tilde{Y}_{24}$ by (4.18), (4.20), and (4.21), respectively.

8) Compute matrices $\tilde{X}$ and $\tilde{Y}$ by (4.19).

Example 5.1. Let $m = 9,\ n = 8,\ p = 7$, and the matrices $A_2, B_2, C_2, D_2, E_2, F_2, G_2$ and H_2 be given by

$$A_2 = \begin{bmatrix} 0.0108 & -0.0027 & -0.0064 & -0.0067 & -0.0012 & -0.0016 & -0.0021 & 0.0079 \\ -0.0081 & 0.0307 & 0.0208 & 0.0652 & -0.0081 & -0.2083 & 0.1079 & 0.0530 \\ -0.0005 & 0.0786 & -0.2736 & 0.1436 & 0.0529 & 0.1079 & -0.1713 & -0.0905 \\ -0.0004 & -0.0655 & 0.1064 & 0.0393 & -0.0948 & -0.2018 & 0.2075 & 0.0974 \\ -0.0023 & -0.0688 & 0.1772 & -0.0304 & -0.0782 & -0.1758 & 0.1999 & 0.0966 \\ 0.0049 & -0.0563 & 0.1564 & -0.2363 & 0.0333 & 0.2868 & -0.1181 & -0.0524 \\ -0.0044 & 0.1064 & -0.2607 & 0.0795 & 0.1052 & 0.1970 & -0.2556 & -0.1294 \\ -0.0042 & -0.0849 & 0.2096 & -0.2453 & 0.0060 & 0.2716 & -0.0695 & -0.0384 \\ 0.0135 & 0.0302 & -0.0541 & 0.0603 & 0.0025 & -0.1140 & 0.0312 & 0.0295 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0532 & 0.5681 & 0.0157 & 0.0004 & 0.0160 & 0.1990 & 0.7426 & 0.4204 \\ 0.0129 & 0.7590 & 0.1846 & 0.7179 & 0.8347 & 0.3237 & 0.0864 & 0.0545 \\ 0.9298 & 0.2514 & 0.0249 & 0.4608 & 0.0692 & 0.9583 & 0.3043 & 0.9984 \\ 0.4425 & 0.4036 & 0.7946 & 0.7032 & 0.1901 & 0.1580 & 0.4579 & 0.1898 \\ 0.7017 & 0.0773 & 0.3014 & 0.7888 & 0.0771 & 0.7160 & 0.5443 & 0.2690 \\ 0.9423 & 0.2019 & 0.4865 & 0.0087 & 0.7508 & 0.2349 & 0.4249 & 0.1090 \\ 0.0165 & 0.8895 & 0.4931 & 0.2505 & 0.1460 & 0.8112 & 0.2247 & 0.9505 \\ 0.2601 & 0.6580 & 0.0603 & 0.0932 & 0.3902 & 0.7843 & 0.3564 & 0.6575 \\ 0.4770 & 0.6348 & 0.7141 & 0.7846 & 0.4591 & 0.7429 & 0.0878 & 0.8769 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -0.0000 & -0.0805 & 0.0352 & -0.1027 & -0.0855 & 0.1902 & 0.0583 \\ -0.1759 & 0.1462 & 0.3348 & -0.0477 & -0.0999 & -0.2107 & -0.0952 \\ 0.0123 & 0.1782 & 0.2681 & -0.0759 & -0.3082 & -0.1380 & -0.0847 \\ 0.0499 & -0.1696 & -0.3077 & 0.0798 & 0.2626 & 0.1524 & 0.0874 \\ 0.4614 & 0.1417 & -0.2276 & 0.0424 & -0.3935 & 0.0007 & -0.0244 \\ 0.2170 & -0.0272 & -0.3974 & 0.1836 & 0.1722 & -0.0373 & 0.0165 \\ -0.2272 & -0.1740 & -0.0035 & -0.0161 & 0.3051 & 0.1199 & 0.0681 \\ -0.3750 & -0.0817 & 0.2230 & -0.0368 & 0.2828 & -0.0384 & 0.0020 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3101 & 0.6758 & 0.6771 & 0.0803 & 0.8809 & 0.9752 & 0.5119 \\ 0.8763 & 0.2769 & 0.7085 & 0.8187 & 0.4956 & 0.4534 & 0.6392 \\ 0.0841 & 0.4658 & 0.0967 & 0.1608 & 0.1558 & 0.4297 & 0.1369 \\ 0.4781 & 0.6627 & 0.3859 & 0.7805 & 0.6804 & 0.4992 & 0.2636 \\ 0.1173 & 0.6215 & 0.0542 & 0.9701 & 0.6157 & 0.0577 & 0.7184 \\ 0.7110 & 0.0966 & 0.7896 & 0.3762 & 0.6068 & 0.4427 & 0.5008 \\ 0.5470 & 0.7411 & 0.1857 & 0.8037 & 0.7711 & 0.2363 & 0.1614 \\ 0.9489 & 0.8545 & 0.3430 & 0.1492 & 0.2734 & 0.1988 & 0.3628 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -0.0000 & -0.1050 & 0.0648 & 0.5265 & -0.2047 & -0.1511 & -0.0259 & -0.1679 \\ -0.0201 & -0.0467 & -0.0623 & 0.1041 & -0.0245 & -0.0919 & 0.0901 & 0.0313 \\ 0.0180 & -0.0012 & -0.0350 & -0.1734 & 0.0834 & 0.0341 & 0.0225 & 0.0621 \\ -0.0013 & 0.1083 & 0.0180 & -0.0455 & 0.0105 & -0.0459 & 0.0235 & -0.0169 \\ -0.0356 & 0.0959 & -0.0512 & -0.3490 & 0.1167 & 0.1187 & 0.0211 & 0.1290 \\ -0.0301 & 0.0022 & 0.0372 & 0.0451 & -0.0492 & 0.0759 & -0.0707 & -0.0240 \\ 0.0215 & -0.0004 & 0.0052 & -0.0316 & 0.0224 & -0.0046 & -0.0021 & -0.0049 \\ 0.0380 & -0.0180 & -0.0262 & -0.1668 & 0.0901 & 0.0279 & 0.0138 & 0.0480 \\ 0.0237 & 0.0367 & 0.0571 & -0.0051 & -0.0045 & 0.0202 & -0.0572 & -0.0543 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.9122 & 0.6673 & 0.7554 & 0.3068 & 0.6356 & 0.0003 & 0.2779 & 0.4272 \\ 0.0289 & 0.3513 & 0.2998 & 0.4126 & 0.3406 & 0.6659 & 0.9696 & 0.1547 \\ 0.4573 & 0.2242 & 0.4836 & 0.1212 & 0.2281 & 0.9499 & 0.2658 & 0.0178 \\ 0.4250 & 0.1710 & 0.3992 & 0.7180 & 0.3658 & 0.6640 & 0.4039 & 0.5602 \\ 0.4875 & 0.0040 & 0.3787 & 0.5228 & 0.0370 & 0.0245 & 0.2682 & 0.3987 \\ 0.9081 & 0.4887 & 0.5079 & 0.3866 & 0.1795 & 0.0502 & 0.7804 & 0.1791 \\ 0.9027 & 0.4806 & 0.7677 & 0.5034 & 0.1183 & 0.4226 & 0.7104 & 0.7852 \\ 0.6292 & 0.2774 & 0.3868 & 0.0710 & 0.6928 & 0.1443 & 0.7576 & 0.3096 \\ 0.8653 & 0.7670 & 0.6426 & 0.5960 & 0.7242 & 0.3325 & 0.2346 & 0.5268 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -0.0000 & 0.0001 & -0.0389 & 0.0260 & 0.0293 & 0.0064 & -0.1110 \\ 0.1108 & -0.0970 & 0.1646 & -0.1058 & 0.0241 & 0.0816 & -0.1040 \\ -0.0429 & 0.0256 & -0.0328 & 0.0263 & -0.0180 & -0.0330 & 0.0953 \\ -0.0444 & 0.1511 & -0.1401 & 0.0763 & -0.0706 & -0.0955 & 0.0666 \\ -0.0482 & 0.1376 & -0.1571 & 0.0864 & -0.0471 & -0.0817 & 0.0087 \\ -0.0612 & -0.2212 & 0.1033 & -0.0319 & 0.1272 & 0.1054 & 0.0301 \\ -0.0633 & -0.1386 & 0.0463 & 0.0026 & 0.0871 & 0.0510 & 0.0343 \\ 0.1740 & 0.1614 & 0.0636 & -0.0941 & -0.1495 & -0.0349 & -0.0310 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.1207 & 0.4066 & 0.1842 & 0.1781 & 0.5861 & 0.3966 & 0.0846 \\ 0.7193 & 0.8691 & 0.2463 & 0.8898 & 0.7267 & 0.7302 & 0.7662 \\ 0.2726 & 0.8748 & 0.8731 & 0.3545 & 0.3923 & 0.6070 & 0.7799 \\ 0.6791 & 0.3647 & 0.4329 & 0.9014 & 0.8862 & 0.7190 & 0.7917 \\ 0.6768 & 0.5848 & 0.2961 & 0.5733 & 0.9603 & 0.2510 & 0.8145 \\ 0.2611 & 0.8055 & 0.4341 & 0.4515 & 0.5780 & 0.0533 & 0.5921 \\ 0.4829 & 0.7472 & 0.5875 & 0.4787 & 0.3247 & 0.7591 & 0.4995 \\ 0.4825 & 0.2034 & 0.7149 & 0.7367 & 0.0422 & 0.7859 & 0.5803 \end{bmatrix}.$$

By using Algorithm 1, the matrices $\tilde{X}$ and $\tilde{Y}$ can be calculated as

$$\tilde{X} = \begin{bmatrix} 10.3125 & 1.6856 & 11.5768 & 2.9404 & 5.8608 & 10.0334 & 5.6616 & 10.2796 \\ 3.1558 & 0.9953 & 3.4535 & 2.8974 & 2.4965 & 3.0032 & -0.6049 & 4.8444 \\ -1.4733 & -0.3261 & -1.7530 & -0.0089 & 0.4039 & -2.1998 & 0.0685 & -1.0581 \\ -3.5213 & -2.8300 & -1.5642 & -1.5348 & -2.5738 & -2.0156 & -1.0689 & -2.5506 \\ -10.9338 & -1.0780 & -6.9596 & -0.2293 & -2.0804 & -8.1569 & -0.4377 & -6.7851 \\ -0.6373 & -2.9051 & 1.4927 & -1.7029 & -2.0441 & 0.8884 & -0.4478 & -0.4140 \\ -2.6335 & -2.8486 & -0.2289 & -0.3784 & -3.8493 & -0.8503 & -2.6912 & -2.7217 \\ -1.3059 & -0.8998 & 3.7576 & 0.6759 & 2.5416 & 0.4244 & 3.9367 & 0.5737 \end{bmatrix},$$

$$\tilde{Y} = \begin{bmatrix} 11.4671 & 0.9183 & 10.6494 & 0.8231 & 8.1120 & 12.1095 & 2.8832 & 9.8136 \\ 2.2225 & 0.8241 & 3.2230 & 4.2523 & 0.3029 & 3.5958 & 1.9810 & 3.4372 \\ -5.1358 & -0.7684 & -0.8481 & -0.6715 & -0.8387 & -0.0539 & -1.1688 & 0.8117 \\ -3.9963 & 0.2281 & 0.0877 & -0.4146 & -2.2147 & -3.4810 & 1.3341 & -1.6580 \\ -11.6730 & 1.0370 & -5.1128 & -1.6334 & -4.8345 & -10.7077 & 0.3348 & -7.9691 \\ -0.6932 & -2.1950 & 1.5918 & -1.3304 & -4.3691 & -3.4050 & -1.2438 & -1.3081 \\ -3.1939 & -3.0659 & -0.6151 & -0.2020 & -3.1712 & 1.4496 & -2.2899 & 1.2293 \\ -1.4378 & -1.0018 & 1.9367 & -0.5103 & 1.7462 & 3.3388 & 3.4293 & 4.0501 \end{bmatrix},$$

and the distance between L_3 and L_4 is $d(L_3, L_4) = 13.7969$, which implies that there is no common element between linear manifolds L_3 and L_4.

6. Conclusions

In this paper, by utilizing the singular value decompositions and the canonical correlation decompositions of matrices, we have achieved the explicit representation for the optimal approximation distance $d(L_1, L_2)$ of linear manifolds L_1 and L_2 and the matrices $\hat{X} \in L_1, \hat{Y} \in L_2$ satisfying $\|\hat{X} - \hat{Y}\| = d(L_1, L_2)$ in Problem 1 (see Theorem 3.1), and the explicit representation for the optimal approximation distance $d(L_3, L_4)$ of linear manifolds L_3 and L_4 and the matrices $\tilde{X} \in L_3, \tilde{Y} \in L_4$ satisfying $\|\tilde{X} - \tilde{Y}\| = d(L_3, L_4)$ in Problem 2 (see Theorem 4.1). Also, we have provided a simple recipe for constructing the optimal approximation solution of Problem 2, which can serve as the basis for numerical computation.The approach is demonstrated by a simple numerical example and reasonable results are produced.

Author contributions

Yinlan Chen: Conceptualization, Methodology, Project administration, Supervision, Writing-review & editing; Yawen Lan: Investigation, Software, Validation, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. E. Mottershead, Y. M. Ram, Inverse eigenvalue problems in vibration absorption: passive modification and active control, *Mech. Syst. Signal Process.*, **20** (2006), 5–44. https://doi.org/10.1016/j.ymssp.2005.05.006
2. B. Dong, M. M. Lin, M. T. Chu, Parameter reconstruction of vibration systems from partial eigeninformation, *J. Sound Vibration*, **327** (2009), 391–401. https://doi.org/10.1016/j.jsv.2009.06.026
3. S. A. Avdonin, M. I. Belishev, S. A. Ivano, Boundary control and a matrix inverse problem for the equation $u_{tt} - u_{xx} + V(x)u = 0$, *Math. USSR Sb.*, **72** (1992), 287–310. https://doi.org/10.1070/SM1992v072n02ABEH002141
4. Y. X. Yuan, A symmetric inverse eigenvalue problem in structural dynamic model updating, *Appl. Math. Comput.*, **213** (2009), 516–521. https://doi.org/10.1016/j.amc.2009.03.045
5. Y. X. Yuan, H. Dai, An inverse problem for undamped gyroscopic systems, *J. Comput. Appl. Math.*, **236** (2012), 2574–2581. https://doi.org/10.1016/j.cam.2011.12.015
6. L. Wu, The re-positive definite solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, **174** (1992), 145–151. https://doi.org/10.1016/0024-3795(92)90048-F
7. K. W. E. Chu, Symmetric solutions of linear matrix equations by matrix decompositions, *Linear Algebra Appl.*, **119** (1989), 35–50. https://doi.org/10.1016/0024-3795(89)90067-0
8. L. J. Zhao, X. Y. Hu, L. Zhang, Least squares solutions to $AX = B$ for bisymmetric matrices under a central principal submatrix constraint and the optimal approximation, *Linear Algebra Appl.*, **428** (2008), 871–880. https://doi.org/10.1016/j.laa.2007.08.019
9. Q. W. Wang, Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations, *Comput. Math. Appl.*, **49** (2005), 641–650. https://doi.org/10.1016/j.camwa.2005.01.014
10. S. K. Mitra, The matrix equations $AX = C$, $XB = D$, *Linear Algebra Appl.*, **59** (1984), 171–181.
11. S. K. Mitra, A pair of simultaneous linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ and a matrix programming problem, *Linear Algebra Appl.*, **131** (1990), 107–123. https://doi.org/10.1016/0024-3795(90)90377-O
12. A. Dajić, J. J. Koliha, Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators, *J. Math. Anal. Appl.*, **333** (2007), 567–576. https://doi.org/10.1016/j.jmaa.2006.11.016
13. Y. Y. Qiu, A. D. Wang, Least squares solutions to the equations $AX = B$, $XC = D$ with some constraints, *Appl. Math. Comput.*, **204** (2008), 872–880. https://doi.org/10.1016/j.amc.2008.07.035
14. Y. H. Liu, Some properties of submatrices in a solution to the matrix equations $AX = C$, $XB = D$, *J. Appl. Math. Comput.*, **31** (2009), 71–80. https://doi.org/10.1007/s12190-008-0192-7

15. F. J. H. Don, On the symmetric solutions of a linear matrix equation, *Linear Algebra Appl.*, **93** (1987), 1–7. https://doi.org/10.1016/S0024-3795(87)90308-9
16. D. Hua, On the symmetric solutions of linear matrix equations, *Linear Algebra Appl.*, **131** (1990), 1–7. https://doi.org/10.1016/0024-3795(90)90370-r
17. J. G. Sun, Two kinds of inverse eigenvalue problems for real symmetric matrices (Chinese), *Math. Numer. Sinica*, **3** (1988), 282–290.
18. C. R. Rao, S. K. Mitra, *Generalized inverse of matrices and its applications*, John Wiley & Sons, 1971.
19. Y. X. Yuan, Least-squares solutions to the matrix equations $AX = B$ and $XC = D$, *Appl. Math. Comput.*, **216** (2010), 3120–3125. https://doi.org/10.1016/j.amc.2010.04.002
20. R. Hettiarachchi, J. F. Peters, Multi-manifold LLE learning in pattern recognition, *Pattern Recogn.*, **48** (2015), 2947–2960. https://doi.org/10.1016/j.patcog.2015.04.003
21. R. Souvenir, R. Pless, Image distance functions for manifold learning, *Image Vision Comput.*, **25** (2007), 365–373. https://doi.org/10.1016/j.imavis.2006.01.016
22. J. X. Du, M. W. Shao, C. M. Zhai, J. Wang, Y. Y. Tang, C. L. P. Chen, Recognition of leaf image set based on manifold-manifold distance, *Neurocomputing*, **188** (2016), 131–188. https://doi.org/10.1016/j.neucom.2014.10.113
23. L. K. Huang, J. W. Lu, Y. P. Tan, Multi-manifold metric learning for face recognition based on image sets, *J. Vis. Commun. Image Represent.*, **25** (2014), 1774–1783. https://doi.org/10.1016/j.jvcir.2014.08.006
24. C. Y. Chen, J. P. Zhang, R. Fleischer, Distance approximating dimension reduction of Riemannian manifolds, *IEEE Trans. Syst. Man Cybernet. Part B*, **40** (2010), 208–217. https://doi.org/10.1109/TSMCB.2009.2025028
25. H. R. Chen, Y. F. Sun, J. B. Gao, Y. L. Hu, B. C. Yin, Solving partial least squares regression via manifold optimization approaches, *IEEE Trans. Neural Netw. Learn. Syst.*, **30** (2019), 588–600. https://doi.org/10.1109/TNNLS.2018.2844866
26. M. Shahbazi, A. Shirali, H. Aghajan, H. Nili, Using distance on the Riemannian manifold to compare representations in brain and in models, *NeuroImage*, **239** (2021), 118271. https://doi.org/10.1016/j.neuroimage.2021.118271
27. K. Sharma, R. Rameshan, Distance based kernels for video tensors on product of Riemannian matrix manifolds, *J. Vis. Commun. Image Represent.*, **75** (2021), 103045. https://doi.org/10.1016/j.jvcir.2021.103045
28. S. Kass, Spaces of closest fit, *Linear Algebra Appl.*, **117** (1989), 93–97. https://doi.org/10.1016/0024-3795(89)90550-8
29. A. M. Dupré, S. Kass, Distance and parallelism between flats in $\mathbb{R}^n$, *Linear Algebra Appl.*, **171** (1992), 99–107. https://doi.org/10.1016/0024-3795(92)90252-6
30. Y. X. Yuan, On the approximation between affine subspaces (Chinese), *J. Nanjing Univ. Math. Biq.*, **17** (2000), 244–249.
31. P. Grover, Orthogonality to matrix subspaces, and a distance formula, *Linear Algebra Appl.*, **445** (2014), 280–288. https://doi.org/10.1016/j.laa.2013.11.040

32. H. K. Du, C. Y. Deng, A new characterization of gaps between two subspaces, *Proc. Amer. Math. Soc.*, **133** (2005), 3065–3070.
33. O. M. Baksalary, G. Trenkler, On angles and distances between subspaces, *Linear Algebra Appl.*, **431** (2009), 2243–2260. https://doi.org/10.1016/j.laa.2009.07.021
34. C. Scheffer, J. Vahrenhold, Approximating geodesic distances on 2-manifolds in $\mathbb{R}^3$, *Comput. Geom.*, **47** (2014), 125–140. https://doi.org/10.1016/j.comgeo.2012.05.001
35. C. Scheffer, J. Vahrenhold, Approximating geodesic distances on 2-manifolds in $\mathbb{R}^3$: the weighted case, *Comput. Geom.*, **47** (2014), 789–808. https://doi.org/10.1016/j.comgeo.2014.04.003
36. G. P. Xu, M. S. Wei, D. S. Zheng, On solutions of matrix equation $AXB + CYD = F$, *Linear Algebra Appl.*, **279** (1998), 93–109. https://doi.org/10.1016/S0024-3795(97)10099-4
37. G. H. Golub, H. Y. Zha, Perturbation analysis of the canonical correlations of matrix pairs, *Linear Algebra Appl.*, **210** (1994), 3–28. https://doi.org/10.1016/0024-3795(94)90463-4