Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 20919–20938.
DOI:10.3934/math.20241018
Received: 14 March 2024
Revised: 27 May 2024
Accepted: 30 May 2024
Published: 28 June 2024

Research article

The worst-case scenario: robust portfolio optimization with discrete distributions and transaction costs

Ebenezer Fiifi Emire Atta Mills[1,2,*]

[1] School of Mathematical Sciences, Wenzhou-Kean University, Wenzhou, China
[2] Academy of Research for Sustainability (AIRs), Wenzhou-Kean University, Wenzhou, China

* **Correspondence:** Email: eattamil@kean.edu.

Abstract: This research introduces min-max portfolio optimization models that incorporating transaction costs and focus on robust Entropic value-at-risk. This study offers a unified approach to handl the distribution of random parameters that affect the reward and risk aspects. Utilizing the duality theorem, the study transforms the optimization models into manageable forms, thereby accommodating the underlying random variables' discrete box and ellipsoidal distributions. The impact of transaction costs on optimal portfolio selection is examined through numerical examples under a robust return-risk framework. The results underscore the importance of the proposed model in safeguarding capital and reducing exposure to extreme risks, thus outperforming other strategies documented in the literature. This demonstrates the model's effectiveness in balancing maximizing returns and minimizing potential losses, making it a valuable tool for investors that seek to navigate uncertain financial markets.

Keywords: portfolio optimization; entropic value-at-risk; transaction costs; uncertainty; robust return-risk
Mathematics Subject Classification: 91B05, 91G10

1. Introduction

The nature of financial markets is fundamentally characterized by uncertainty, as the behavior of asset returns demonstrates stochastic tendencies that challenge conventional portfolio optimization strategies [12, 20, 40]. In this context, the importance of risk management is of the utmost significance, and the necessity for robust optimization models that can withstand various market scenarios is apparent [18, 19, 26].

Historically, the establishment of the modern portfolio theory can be traced to Harry Markowitz in his seminal work on the mean-variance model. This innovative approach introduced the concept of diversification, thereby highlighting the significance of considering both return and risk in portfolio selection. The contributions of [25] revolutionized the field, leading to significant advancements in the discipline and paving the way for numerous later models and theories [22]. Nevertheless, like any pioneering study, the mean-variance model had intrinsic limitations [6, 33]. Researchers recognized the necessity for more precise risk measures, which prompted JP Morgan to introduce Value-at-Risk (VaR) in the early 1990s. VaR provided a more tangible comprehension of risk by emphasizing potential losses in a portfolio within specific periods. However, it had several drawbacks, especially when applied to distributions that were not normal [3, 4]. As a result, Conditional Value-at-Risk (CVaR) emerged as a risk measure that addressed the limitations of VaR and offered a more coherent approach [4, 17]. CVaR considers both the likelihood and size of a loss. Additionally, it accommodates distributions that are not symmetrical, while still maintaining the desired coherence properties [30].

The research conducted by [31] and [1] provided a comprehensive analyses of the development of these risk measures and the inherent challenges they provide. The study by [39] emphasized that many existing models, as seen [41], used unrealistic assumptions, such as assuming complete knowledge of the distribution of random parameters. This is an important insight since there are a lot of uncertainties in the real world of finance, and making uninformed assumptions can lead to misguided outcomes.

Against this backdrop, this research introduces a significant paradigm shift: adopting the Entropic Value-at-Risk (EVaR) as the primary risk measure. The coherence of CVaR as a risk measure has been demonstrated in previous studies [4]. However, it is worth noting that CVaR mostly relies on the extreme values of the distribution. Furthermore, empirical evidence has demonstrated that it lacks smoothness. In a notable study, [1] showcased the potential of EVaR in computationally addressing challenging stochastic optimization problems that would otherwise be intractable using CVaR. [5] demonstrated that EVAR is a computationally efficient and coherent risk measure that can quantify risk—it meets the axioms of monotonicity, sub-additivity, positive homogeneity, and translation invariance as introduced by [4]. The portfolio optimization strategy that utilized EVaR was examined by [38]. Then, the results were juxtaposed with alternative downside risk measures, namely VaR and CVaR. Results from their research showed that EVaR exhibited the most effective risk resolution. Recently, [15] introduced EVaR for uncertain random variables of the portfolio selection problem, established a mean-EVaR model, and demonstrated its superior diversification over the mean-variance model through numerical examples and indices.

EVaR has emerged as a significant risk measure, capturing the uncertainty in asset returns with a probabilistic approach [1]. However, in real-world scenarios, the exact probability distribution of returns might be uncertain [32]. This leads to the introduction of the worst-case EVaR (WEVaR), which considers the most adverse distribution within a set of plausible distributions. Such a measure ensures that the portfolio is optimized, keeping the worst possible outcome in mind, thus providing a more conservative and safer approach to investments. This approach was pursued by [34] albeit with the use of CVaR. [34], examined the worst-case CVaR (WCVaR) based portfolio optimization under uncertain distributions, simplifying them to linear programming for asset allocation in power markets. [21] applied WCVaR with parallelepiped uncertainty to enhance portfolio robustness and mitigate risk. Furthermore, there has been an increase in the exploration of theories and methodologies that address incomplete information across various fields, as illustrated in [7, 8, 24, 37]. [23] presented a distributionally robust optimization model using kernel density estimation and mean EVaR, offering tractable solutions and showing promising empirical results for portfolio and project management. Taking inspiration from the aforementioned research works, this study aims to present robust portfolio optimization models that incorporate WEVaR, ensuring that the portfolios are optimized for expected returns and resilient to extreme market downturns. This study explores the mathematical intricacies of EVaR and its worst-case counterpart, providing a comprehensive framework for their integration into the portfolio optimization models. Furthermore, this study examines discrete distributions, particularly the box and ellipsoidal distributions, to understand their implications in the robust optimization context.

However, the introduction of EVaR is just one facet of this research's novelty. Beyond selecting assets based on expected returns and risks, practitioners must grapple with the on-ground realities of the financial world, chief among them being transaction costs. The findings of [9] showed that unexpected returns to transaction costs contributed to about 40% of losses in the financial market. Recognizing this critical gap, this research integrates transaction costs into the portfolio optimization model. In light of the vital role that transaction costs play in portfolio optimization, the discussion extends beyond the foundational perspectives provided by [29] and [42], who highlighted the impact of taxes, liquidity costs, and brokerage fees on portfolio decisions [36]. Given the dynamic nature of portfolio management, this paper integrates the concept of transaction costs influencing the strategic decisions of portfolio adjustments. This approach acknowledges the findings of [2], who demonstrated how transaction costs can affected prices and liquidity over more extended periods and aligns with the analyses by [10], who explored the implications of transaction costs on risk assessment and portfolio diversification strategies. By adopting a WEVaR framework, this study explicitly accommodates transaction costs, thereby enhancing the model's practical relevance in handling the complexities of real-world portfolio construction.

[34] and [35] investigated portfolio optimization models that assessed return and risk under partially known information about the random variables. Their methodology employed min-max optimization to consider the WCVaR, with separate worst-case distributions assumed to evaluate returns and risks. This approach might be considered overly cautious, leading to different distributions for the random variable in the return and risk computations. Ideally, the distribution should remain consistent across the return and risk evaluations.

Drawing on the robust optimization frameworks highlighted in [34] and [35], this research advances these models by ensuring a consistent distribution of random parameters across return and risk dimensions. This uniformity is significant. Without it, the models may inadvertently harbor biases or inconsistencies, potentially resulting in sub-optimal portfolio selections. In contrast to [34, 35] robust return-risk portfolio strategies, this study accounts for uniform uncertainty distribution in both return and risk assessments, thus providing a more integrated and unified approach.

This research pursues a comprehensive approach to portfolio optimization by integrating the worst-case EVaR framework with transaction costs, uniquely ensuring a consistent distribution across the risk and return dimensions. The study merges these critical aspects, thus presenting a robust optimization model that effectively balances rigorous risk assessment with practical challenges in financial markets. This approach significantly advances robust portfolio management, thereby enhancing portfolio resilience and adaptability under uncertain market conditions.

This study is structured as follows: Section 2 presents the robust portfolio optimization models, including the concept of WEVaR. In Section 3, the paper discusses how the worst-case portfolio strategies can be reformulated as tractable problems when the distribution information of the random returns is incomplete. In Section 4, transaction costs are incorporated into the proposed models. Section 5 presents a practical application. The final portion offers the concluding remarks.

2. Robust portfolio optimization models with WEVaR

This section presents a coherent risk measure, distribution, and the proposed robust risk-return optimization models.

2.1. Entropic Value-at-Risk (EVaR) and its technique

In the portfolio, the decision variables $x \in \mathcal{X}$ represent the weights of the risky assets, with N assets under consideration. The random returns of these assets are denoted by $r \in \mathbb{R}^m$, where the returns' probability distribution is described by the density function $p(\cdot)$. The set of admissible portfolios, $\mathcal{X}$, consists of all portfolios x that satisfy certain constraints, which is defined as follows:

$$\mathcal{X} = x \in \mathbb{R}^N : \iota' x = 1, \iota = [1, 1, \ldots, 1]' \text{ and } x \geq 0, Mx \leq b, \tag{2.1}$$

where ι is a vector of all ones. This constraint ensures that the sum of the weights in the portfolio equals 1, thus reflecting a fully invested portfolio. The conditions $x \geq 0$, and $Mx \leq b$ further impose no short selling and comply with the additional constraints represented by the matrix M and vector b, which could specify the maximum allocations or any other type of constraints such as transaction costs.

The EVaR value at α significance level for the loss random variable with x and r is denoted by $\text{EVaR}_p(x)$. $\text{EVaR}_p(x)$ is defined as in the sample version illustrated by [13] as follows:

$$\text{EVaR}_p(x) = \min_{z>0} \left\{ z ln\left(\frac{1}{\alpha} \Sigma_{i=1}^{m} p_i \text{exp}\left(\frac{x_i}{z} \right) \right) \right\}, \tag{2.2}$$

where $\alpha \in (0, 1)$, z is the threshold level, and $p = (p_1, p_2, \ldots, p_m)$ is the probability of m random returns.

2.2. Worst-case EVaR (WEVaR)

EVaR can be defined under a known probability distribution of r; however, in reality, the random vector r's distribution may only be known to belong to a set $\mathcal{P}$ such that $p(\cdot) \in \mathcal{P}$. Following [34]'s approach to WCVaR and considering this study's adaptation of EVaR, the WEVaR is defined as follows:

Definition 1. *The WEVaR for a defined $x \in \mathcal{X}$ associated with $\mathcal{P}$ is expressed as follows:*

$$\text{WEVaR}(x) = \sup_{p(\cdot) \in \mathcal{P}} \text{EVaR}_p(x). \tag{2.3}$$

Given that the set $\mathcal{P}$ over which the supremum is taken is compact (closed and bounded), it follows from the Extreme Value Theorem that the supremum is attained. Therefore, one can replace sup with max.

Definition 2. *The WEVaR for a defined $x \in \mathcal{X}$ associated with $\mathcal{P}$ is expressed as follows:*

$$\text{WEVaR}(x) = \max_{p(\cdot) \in \mathcal{P}} \text{EVaR}_p(x). \tag{2.4}$$

Since many distributions used in modeling financial data can be approximated by discrete samples [16], this study assumes that random returns follow a discrete distribution. Under an incomplete known discrete distribution, the sample space of r is given as $\mathcal{P}_d = \{r_{[1]}, r_{[2]}, \ldots, r_{[m]}\}$ with the probability of occurrence, $\Pr\{r_{[k]}\} = p_k$, such that $\sum_{k=1}^{m} p_k = 1$, $p_k \geq 0$ $(k = 1, 2, \ldots, m)$.

One useful distribution is the ellipsoidal distribution. The ellipsoidal distribution is chosen for modeling financial returns because it can effectively capture correlations and variations in financial data. It balances robustness and computational efficiency, which are essential for portfolio optimization in volatile markets [11]. Thus, the probability distribution for the uncertainty set of r is defined as follows:

$$p \in \mathcal{P}^{\mathcal{E}} = \{p : p = p^0 + S\zeta, \|\zeta\| \leq 1, p^0 + S\zeta \geq 0, \iota' S\zeta = 0\}, \tag{2.5}$$

where ζ is a vector within the unit ball such that $\|\zeta\| = \sqrt{\zeta'\zeta}$, p^0 is a known distribution, and the center of the ellipsoid, $S \in \mathbb{R}^{m\times N}$ is a scaling matrix of the ellipsoid. The restrictions, $p^0 + S\zeta \geq 0$ and $\iota' S\zeta = 0$, guarantee that p satisfies the conditions of a probability distribution. See [34] for more details. Another widely used distribution is that of the box distribution. The probability of random returns characterized by the polytopic or box uncertainty set are defined as follows:

$$p \in \mathcal{P}^{\mathcal{B}} = \{p : p = p^0 + \zeta, \iota'\zeta = 0, \zeta_l \leq \zeta \leq \zeta_u\}, \tag{2.6}$$

where ζ_u and ζ_l are given constant vectors. See [35] for more details.

Under the defined discrete distributions, WEVaR can be computed by the following:

$$\text{WEVaR}_p^d(x) = \max_{p \in \mathcal{P}_d} \text{EVaR}_p(x). \tag{2.7}$$

2.3. Portfolio strategies with WEVaR

Let $R = \{r_{[1]}, r_{[2]}, \ldots, r_{[m]}\}$ be the profit matrix. Then, the expected profit can be expressed as follows:

$$\mathbb{E}_p(x) = x'\mathbb{E}_p[r] = x'Rp. \tag{2.8}$$

Similar to the definition of the worst-case EVaR in Eq (2.4), one can establish the worst-case returns as follows:

$$\text{W}\mathbb{E}(x) = \inf_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p(x). \tag{2.9}$$

Given that the set $\mathcal{P}$ is compact, one can replace inf with min. Therefore,

$$\text{W}\mathbb{E}(x) = \min_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p(x). \tag{2.10}$$

Three profit-risk optimization models can be proposed based on the worst-case profit and worst-case risk measures, WEVaR, that is, the maximal the robust profit with robust EVaR constraint, the minimal robust EVaR with the robust return constraint, and the robust return and robust EVaR model using a utility function. Given V^* and V_*, which denote the required risk and the required return levels, respectively, as well as $\lambda > 0$, which is the risk aversion parameter, the three worst-case profit-risk optimization models can be described as follows:

Model 1:

$$\begin{aligned} \max_{x \in X} \quad & \min_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p(x) \\ \text{subject to} \quad & \max_{p(\cdot) \in \mathcal{P}} \text{EVaR}_p(x) \leq V^*. \end{aligned} \tag{2.11}$$

This model aims to maximize the worst-case expected return of the portfolio X. The agent seeks to ensure that the expected return is as high as possible under the least favorable probability distribution within the set $\mathcal{P}$. This approach is particularly conservative, and focuses on the robustness against the worst outcomes in terms of returns. The constraint specifies that the worst-case scenario of EVaR should not exceed a predefined threshold V_*. This limits the acceptable level of risk, ensuring that even in the most adverse conditions, the risk taken by the portfolio does not surpass a set limit.

Model 2:

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \max_{p(\cdot) \in \mathcal{P}} \mathrm{EVaR}_p(x) \\ \text{subject to} \quad & \min_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p(x) \geq V_*. \end{aligned} \tag{2.12}$$

In contrast to Model 1, this model focuses on minimizing the maximum EVaR across all possible distributions in $\mathcal{P}$. The goal is to ensure that the highest possible risk is kept as low as possible, effectively managing the most extreme risk scenarios. Here, the agent ensures that , under the most favorable probability distribution, the expected return remains above a certain level V_*. This condition guarantees that the portfolio does not sacrifice a return below a minimum acceptable threshold while controlling for maximum risk.

Model 3:

$$\max_{x \in \mathcal{X}} \quad \left[\min_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p(x) - \lambda \max_{p(\cdot) \in \mathcal{P}} \mathrm{EVaR}_p(x) \right]. \tag{2.13}$$

Model 3 aims to find an optimal balance between maximizing the expected returns and minimizing the risk. The objective function integrates the minimum expected return and the maximum EVaR, which is weighted by a factor λ. This parameter adjusts the relative importance of risk aversion compared to the pursuit of the returns, allowing the agent to tailor the balance based on their risk appetite and the market conditions.

Therefore, the strategies simultaneously mitigate against the worst-case in reward and risk measures. This assumption can be considered excessively conservative as it necessitates the satisfaction of each constraint for all potential realizations of uncertain parameters, particularly under the worst-case scenarios. A practical and feasible alternative assumes the same worst-case distribution for risk and reward. Considering the assumption of the same distribution, one can rewrite Model 2 (i.e., Eq (2.12)) as follows:

$$\min_{x \in \mathcal{X}} \quad \max_{p(\cdot) \in \mathcal{P}} \{ \mathrm{EVaR}_p(x) : \mathbb{E}_p(x) \geq V_* \}. \tag{2.14}$$

Model 2 is selected for this study, since the other models can be similarly pursued. Eq (2.14) has a larger robust feasible set than Eq (2.12). This allows one to attain an improved optimal value while simultaneously meet all potential realizations of the constraints.

3. Robust portfolio strategies with WEVaR under discrete distributions

The section aims to broaden the findings of [39] to encompass the WEVaR scenario. The assumption of box and ellipsoidal discrete distributions will be investigated.

The robust optimization problem, Eq (2.14), can be rewritten in the following form:

$$\min_{x \in \mathcal{X}} \quad \max_{p(\cdot) \in \mathcal{P}} \left\{ \min_{z>0} z \ln \left(\frac{1}{\alpha} \sum_{i=1}^{m} p_i \exp\left(\frac{x_i}{z} \right) \right) : \mathbb{E}_p(x) \geq V_* \right\}. \tag{3.1}$$

Motivated by Lemma 1 in [41] with regards to CVaR, from the convexity property, WEVaR can be expressed as follows:

$$\max_{p(\cdot)\in\mathcal{P}} \min_{z>0} z \ln\left(\frac{1}{\alpha}\sum_{i=1}^{m} p_i \exp\left(\frac{x_i}{z}\right)\right) = \min_{z>0} \max_{p(\cdot)\in\mathcal{P}} z \ln\left(\frac{1}{\alpha}\sum_{i=1}^{m} p_i \exp\left(\frac{x_i}{z}\right)\right). \tag{3.2}$$

To prove the convexity property and justify the interchangeability of the minimization and maximization operators, let's first establish that the function $f(p, z) = z \ln\left(\frac{1}{\alpha}\sum_{i=1}^{m} p_i \exp\left(\frac{x_i}{z}\right)\right)$ is convex in p. This follows because the exponential function $\exp\left(\frac{x_i}{z}\right)$ is convex in x_i, and a weighted sum of convex functions where weights $p_i \geq 0$ and $\sum_{i=1}^{m} p_i = 1$ remains convex. Next, consider the function's concavity in z when p is fixed. The function $-\exp\left(\frac{x_i}{z}\right)$ is concave since the exponential of a linear function is convex, and the negative of a convex function is concave. Furthermore, the logarithmic function $\ln(\cdot)$, being concave, when applied to a sum of exponentials (a log-sum-exp function), exhibits concavity in z under certain conditions. With these properties established, the minimax theorem applies, stating that if $f(p, z)$ is convex in p and concave in z, and both $\mathcal{P}$ and the range of z are convex sets, then $\max_{p\in\mathcal{P}} \min_{z>0} f(p, z) = \min_{z>0} \max_{p\in\mathcal{P}} f(p, z)$. This theorem supports the interchange of the max and min operators under the convex-concave conditions provided, which is crucial to validate the form of Eq (3.2).

In the next subsection, this paper presents a new version of Model 2 that considers the discrete distributions $\mathcal{P}^{\mathcal{B}}$ and $\mathcal{P}^{\mathcal{E}}$.

3.1. Model 2 reformulated for box distribution

From the convexity property and auxiliary vector u, WEVaR under $\mathcal{P}^{\mathcal{B}}$ can be derived as follows :

$$\begin{aligned} &\max_{p\in\mathcal{P}^{\mathcal{B}}} \min_{z>0} \left\{ z \ln\left(\frac{1}{\alpha}\sum_{i=1}^{m} p_i \exp\left(\frac{x_i}{z}\right)\right)\right\} = \\ &\min_{z,u} \max_{p\in\mathcal{P}^{\mathcal{B}}} \left\{ z \ln\left(\frac{1}{\alpha} u' p\right) : u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, i = 1, 2, \ldots, m\right\}. \end{aligned} \tag{3.3}$$

Therefore, optimization problem, Eq (3.1), can be rewritten as follows with $(x, z, u) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^m$:

$$\begin{aligned} \min_{x,z,u} \quad & \max_{p\in\mathcal{P}^{\mathcal{B}}} \left\{ z \ln\left(\frac{1}{\alpha} u' p\right) : x' R p \geq V_* \right\} \\ \text{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, i = 1, 2, \ldots, m, \\ & x \in \mathcal{X}. \end{aligned} \tag{3.4}$$

Let's focus on the inner problem of Eq (3.4). Given the box uncertainty set, the following can be derived:

$$\begin{aligned} \max_{\zeta} \quad & z \ln\left(\frac{1}{\alpha} u' (p^0 + \zeta)\right) \\ \text{subject to} \quad & x' R (p^0 + \zeta) \geq V_*, \\ & \iota' \zeta = 0, \\ & \zeta_l \leq \zeta \leq \zeta_u. \end{aligned} \tag{3.5}$$

The first constraint ensures that the expected profit is not less than V_*. From the method of a Lagrangian duality, the Lagrangian for the primal problem, Eq (3.5), is as follows:

$$L(\zeta, \Lambda, \Gamma, \eta, \kappa) = z\ln\left(\frac{1}{\alpha}u'(p^0+\zeta)\right) - \Lambda(x'R(p^0+\zeta) - V_*) + \Gamma\iota'\zeta + \eta'(\zeta - \zeta_l) + \kappa'(\zeta_u - \zeta).$$

Here, Λ corresponds to the constraint $x'R(p^0+\zeta) \geq V_*$, Γ ensures the sum constraint $\iota'\zeta = 0$, η and κ are vectors of Lagrange multipliers for the box constraints $\zeta_l \leq \zeta \leq \zeta_u$.

To find the optimal ζ, differentiate L with respect to ζ and set the derivative to zero. For simplicity, consider the derivative of the logarithmic term separately: $\frac{\partial}{\partial\zeta}\left(z\ln\left(\frac{1}{\alpha}u'(p^0+\zeta)\right)\right) = z\frac{1}{\frac{1}{\alpha}u'(p^0+\zeta)}\alpha u = \frac{zu}{u'(p^0+\zeta)}$. Adding the derivatives of the constraint terms, the full derivative is $\frac{\partial L}{\partial\zeta} = \frac{zu}{u'(p^0+\zeta)} - \Lambda Rx' + \Gamma\iota + \eta - \kappa$. Setting $\frac{\partial L}{\partial\zeta} = 0$ and solving for ζ: $\frac{zu}{u'(p^0+\zeta)} - \Lambda Rx' + \Gamma\iota + \eta - \kappa = 0$. Rearranging for ζ, one can obtain the following:

$$\zeta = -\frac{\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'}.$$

Substituting this expression for ζ into the Lagrangian, the dual function is as follows :

$$\begin{aligned} g(\Lambda, \Gamma, \eta, \kappa) = & z\ln\left(\frac{u'\left(p^0 - \frac{\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'}\right)}{\alpha}\right) \\ & - \frac{\iota'\Gamma(\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z)}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'} \\ & - \Lambda\left(-V_* + Rx'\left(p^0 - \frac{\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'}\right)\right) \\ & + \eta'\left(-\frac{\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'} - \zeta_l\right) \\ & + \kappa'\left(\frac{\eta'p^0 - \kappa'p^0 + \iota'\Gamma p^0 - \Lambda p^0 Rx' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda Rx'} + \zeta_u\right). \end{aligned} \tag{3.6}$$

Then, the dual problem for Eq (3.5) is as follows:

$$\begin{aligned} \min_{\Lambda,\Gamma,\eta,\kappa} \quad & g(\Lambda, \Gamma, \eta, \kappa) \\ \text{subject to} \quad & \Lambda \geq 0, \\ & \eta \geq 0, \\ & \kappa \geq 0. \end{aligned} \tag{3.7}$$

The dual problem is solvable and has an optimal value equal to the optimal value of Eq (3.5), as it is evident that the primal problem is bounded above and that a strictly solvable feasible point exists.

Substituting Eq (3.7) into Eq (3.4), the following proposition is derived.

Proposition 1. *Under the box discrete distribution with $\zeta_u > 0$ and $\zeta_l < 0$, if p^0 satisfies $R \cdot p^0 > \iota \cdot V_*$, then the robust portfolio strategy, Eq (3.1), can be reformulated as follows:*

$$
\begin{aligned}
\min_{x,z,u,\Lambda,\Gamma,\eta,\kappa} \quad & z \ln\left(\frac{u'\left(p^0 - \frac{\eta' p^0 - \kappa' p^0 + \iota'\Gamma p^0 - \Lambda p^0 R x' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda R x'}\right)}{\alpha}\right) \\
& - \frac{\iota'\Gamma(\eta' p^0 - \kappa' p^0 + \iota'\Gamma p^0 - \Lambda p^0 R x' + z)}{\eta' - \kappa' + \iota'\Gamma - \Lambda R x'} \\
& - \Lambda\left(- V_* + Rx'\left(p^0 - \frac{\eta' p^0 - \kappa' p^0 + \iota'\Gamma p^0 - \Lambda p^0 R x' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda R x'}\right)\right) \\
& + \eta'\left(- \frac{\eta' p^0 - \kappa' p^0 + \iota'\Gamma p^0 - \Lambda p^0 R x' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda R x'} - \zeta_l\right) \\
& + \kappa'\left(\frac{\eta' p^0 - \kappa' p^0 + \iota'\Gamma p^0 - \Lambda p^0 R x' + z}{\eta' - \kappa' + \iota'\Gamma - \Lambda R x'} + \zeta_u\right) \\
\textit{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, \quad i = 1, 2, \ldots, m, \\
& \Lambda \geq 0, \\
& \eta \geq 0, \\
& \kappa \geq 0. \\
& x \in \mathcal{X}.
\end{aligned} \tag{3.8}
$$

Proposition 1 presents a reformulation of the robust optimization problem under the assumption of a box distribution for asset returns. It translates the problem of managing the return and risk under uncertainty into a more tractable form, thereby leveraging on the properties of the function $f(p, z)$. The problem initially formulated in Eq (3.1) is transformed into a dual formulation in Eq (3.4), thus allowing for linearization of the constraints and optimization over the auxiliary variables $(x, z, u, \Lambda, \Gamma, \eta, \kappa)$ which represent different aspects of the optimization problem such as scaling and the Lagrange multipliers for various constraints.

Therefore, the proposed robust return-EVaR model under the box distribution is reformulated by Proposition 1 as a tractable optimization problem that may be solved using various techniques. Although an optimal solution exists, its uniqueness isn't guaranteed due to potential multiple local optima. This can be tackled with global optimization and interior point methods to ensure a robust applicability.

3.2. Model 2 reformulated for ellipsoidal distribution

Similar to the first of the previous section, the robust portfolio strategy can be written in the following form:

$$
\begin{aligned}
\min_{x,z,u} \quad & \max_{p \in \mathcal{P}^{\mathcal{E}}} \left\{ z \ln\left(\frac{1}{\alpha} u' p\right) : x' R p \geq V_* \right\} \\
\text{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, \quad i = 1, 2, \ldots, m,
\end{aligned}
$$

$$x \in \mathcal{X}. \tag{3.9}$$

Considering the inner problem of Eq (3.9) under $\mathcal{P}^{\mathcal{E}}$, then:

$$\begin{aligned}
\max_{\zeta} \quad & z \ln\left(\frac{1}{\alpha} u'(p^0 + S\zeta)\right) \\
\text{subject to} \quad & x'R(p^0 + S\zeta) \geq V_*, \\
& p^0 + S\zeta \geq 0, \\
& \iota' S\zeta = 0, \\
& \left\|\zeta\right\| \leq 1.
\end{aligned} \tag{3.10}$$

The following assumptions are made to make the problem tractable:

1) S is positive definite; and
2) ζ is not zero, ensuring $\left\|\zeta\right\| \neq 0$.

From the method of Lagrangian duality, the Lagrangian for the primal problem (3.10) is defined as:

$$L(\zeta, v, \tau, \sigma, \theta) = z \ln\left(\frac{1}{\alpha} u'(p^0 + S\zeta)\right) - v(x'R(p^0 + S\zeta) - V_*) - \tau'(p^0 + S\zeta) + \sigma'\iota' S\zeta - \theta(\left\|\zeta\right\| - 1).$$

Here, $v \geq 0$ is a non-negative dual variable associated with the return constraint $x'R(p^0 + S\zeta) \geq V_*$, penalizing violations. It scales the penalty imposed when this constraint is violated. $\tau \geq 0$ is a non-negative dual variable to ensure the non-negativity of the probabilities $p^0 + S\zeta \geq 0$. σ is a dual variable for the constraint $\iota' S\zeta = 0$, which ensures that the total probability remains constant. $\theta \geq 0$ is a non-negative dual variable which enforces the norm constraint $\left\|\zeta\right\| \leq 1$, and penalizes deviations beyond the unit ball. θ ensures that the adjustments ζ remain within the unit ball, thus maintaining the definition of the ellipsoid.

From the equation obtained from differentiating the Lagrangian function with respect to ζ:

$$\frac{Sz \cdot u'}{u'(p^0 + S\zeta)} - vx'RS - \tau' S + \sigma'\iota' S - \theta \frac{\zeta}{\left\|\zeta\right\|} = 0. \tag{3.11}$$

Using the assumptions, one can express ζ as follows:

$$\zeta = S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right). \tag{3.12}$$

Substituting this into the Lagrangian, the dual function is expressed as follows:

$$\begin{aligned}
g(v, \tau, \sigma, \theta) = z \ln & \left(\frac{1}{\alpha} u'(p^0 + S(S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right)))\right) \\
& - v(x'R(p^0 + S(S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right))) - V_*) \\
& - \tau'(p^0 + S(S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right))) \\
& + \sigma'\iota' S(S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right)) \\
& - \theta(\left\|S^{-1}\left(vx'RS + \tau' S - \sigma'\iota' S\right)\right\| - 1).
\end{aligned}$$

Hence, the dual problem for Eq (3.10) is as follows:

$$\begin{aligned} \min_{v,\tau,\sigma,\theta} \quad & g(v,\tau,\sigma,\theta) \\ \text{subject to} \quad & v \geq 0, \\ & \tau \geq 0. \end{aligned} \tag{3.13}$$

By substituting Eq (3.13) into Eq (3.9), the following proposition is derived.

Proposition 2. *Under the ellipsoidal discrete distribution, if the distribution* p^0 *satisfies* $R \cdot p^0 > \iota \cdot V_*$*, the robust optimization problem, Eq (3.1), can be reformulated as follows:*

$$\begin{aligned} \min_{x,z,u,v,\tau,\sigma,\theta} \quad & g(v,\tau,\sigma,\theta) \\ \text{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, \quad i = 1,2,\ldots,m, \\ & v \geq 0, \\ & \tau \geq 0, \\ & x \in \mathcal{X}. \end{aligned} \tag{3.14}$$

The objective function $g(v,\tau,\sigma,\theta)$ in Proposition 2 is designed to minimize potential losses under the worst-case scenarios dictated by the ellipsoidal distribution model, thus rendering the problem tractable. This function incorporates the logarithm of transformed probabilities, adapted through ellipsoidal distribution adjustments, making it well-suited for robust optimization techniques. The constraints include auxiliary variables $u_i \geq \exp\left(\frac{x_i}{z}\right)$, which ensure that control variables appropriately scale with the following risk adjustments; $u_i \geq 0$, which guarantees non-negativity; and $x \in \mathcal{X}$, which ensures that decision variables remain within the feasible set of portfolio choices. This structured framework, premised on the assumption that p^0 satisfies $R \cdot p^0 > \iota \cdot V_*$, enables portfolio managers to systematically manage risk in environments marked by uncertainty and complex dependencies among asset returns.

Therefore, this paper obtains a tractable optimization model from the robust return-EVaR model to solve the asset allocation problem.

4. Incorporating transaction costs

Transaction costs arise during portfolio revision because a portfolio of assets, rather than cash, is often employed as the starting point for investment decisions. This necessitates liquidating certain assets to fund new investments, thus incurring transaction costs.

To explore this concept in greater detail, consider an investment portfolio that includes investments in either part or all of N risky assets alongside a risk-free asset. Inspired by the approach of [14], this paper integrates transaction costs into the proposed frameworks, as outlined in Eq (3.8) and Eq (3.14). This integration is achieved by adjusting the decision variables $x \in \mathcal{X}$ to reflect the transaction costs, similar to the modifications in their mean-variance and mean-CVaR frameworks.

Eq (3.8) can be rewritten as follows:

$$
\begin{aligned}
\min_{x,z,u,\Lambda,\Gamma,\eta,\kappa} \quad & z\ln\left(\frac{u'\left(p^0-\frac{\eta' p^0-\kappa' p^0+\iota'\Gamma p^0-\Lambda p^0 R x'+z}{\eta'-\kappa'+\iota'\Gamma-\Lambda R x'}\right)}{\alpha}\right) \\
& -\frac{\iota'\Gamma(\eta' p^0-\kappa' p^0+\iota'\Gamma p^0-\Lambda p^0 R x'+z)}{\eta'-\kappa'+\iota'\Gamma-\Lambda R x'} \\
& -\Lambda\left(-V_*+Rx'\left(p^0-\frac{\eta' p^0-\kappa' p^0+\iota'\Gamma p^0-\Lambda p^0 R x'+z}{\eta'-\kappa'+\iota'\Gamma-\Lambda R x'}\right)\right) \\
& +\eta'\left(-\frac{\eta' p^0-\kappa' p^0+\iota'\Gamma p^0-\Lambda p^0 R x'+z}{\eta'-\kappa'+\iota'\Gamma-\Lambda R x'}-\zeta_l\right) \\
& +\kappa'\left(\frac{\eta' p^0-\kappa' p^0+\iota'\Gamma p^0-\Lambda p^0 R x'+z}{\eta'-\kappa'+\iota'\Gamma-\Lambda R x'}+\zeta_u\right) \\
\text{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, \quad i=1,2,\ldots,m, \\
& \Lambda \geq 0, \quad \eta \geq 0, \quad \kappa \geq 0, \\
& x = x^0 + x^b - x^s, \\
& y + \iota' x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s \leq 1, \\
& x^b \geq 0, \quad x^s \geq 0,
\end{aligned}
\tag{4.1}
$$

where x^0 is the proportion of funds initially assigned to risky assets, x^b is the proportion of funds used to buy risky assets, and $x^s (x^s \leq x^0)$ is the proportion of funds obtained by selling shares of risky assets. Therefore, x becomes the proportion of funds in a risky asset after rebalancing the initial portfolio, and y is the proportion of funds invested in the risk-free asset. The costs of buying and selling a risky asset are denoted as c^b and c^s for the purchase and sale transactions, respectively. The complementary constraint, $x^b \cdot x^s = 0$, was not considered because [27, 28] proved that the complementary constraint could be removed in the presence of a risk-free asset.

Similarly, one can rewrite Eq (3.14) as follows:

$$
\begin{aligned}
\min_{x,z,u,v,\tau,\sigma,\theta} \quad & g(v,\tau,\sigma,\theta) \\
\text{subject to} \quad & u_i \geq \exp\left(\frac{x_i}{z}\right), u_i \geq 0, \quad i=1,2,\ldots,m, \\
& v \geq 0, \quad \tau \geq 0, \\
& x = x^0 + x^b - x^s, \\
& y + \iota' x + \sum_{j=1}^{n} c_j^b x_j^b + \sum_{j=1}^{n} c_j^s x_j^s \leq 1, \\
& x^b \geq 0, \quad x^s \geq 0.
\end{aligned}
\tag{4.2}
$$

5. Practical application

In this section, the applicability of the robust WEVaR-return optimization model with transaction cost under the box discrete distribution $\mathcal{P}^{\mathcal{B}}$ is demonstrated. This research uses the dataset utilized by [41]. Two sub-indices of the Hong Kong Stock Exchange's Hang Seng Index have been chosen to construct a portfolio. These sub-indices are the Hang Seng Property Index (HSNP) and the Hang Seng Commercial/Industrial Index. A total of 1800 samples of daily returns and variation for the two assets were gathered during two periods, with 900 samples collected per period. Period 1 dated 27/7/1990 to 06/01/1994, and period 2 dated 07/01/1994 to 19/06/1997. The mean returns and variances of the two assets in line with the stated periods are displayed in Table 1.

Table 1. Mean returns and variances of two indices in different time frames.

Mean (10^{-3})			Variance (10^{-3})	
Period	HSNP	HSNC	HSNP	HSNC
Period 1	1.5859	1.1383	0.2417	0.2062
Period 2	0.2902	0.2625	0.2740	0.2110

This study normalizes the initial portfolio. Therefore, $x_j^0 = 0.5$ for $j = 1, 2$. For feasibility purposes, the following parameters are employed: $R = \begin{bmatrix} 1.5859 & 0.2902 \\ 1.1383 & 0.2625 \end{bmatrix}$, $p^0 = (0.5, 0.5)'$, $\alpha = 0.95$, and $V_* = 0.0007$. It is easy to verify that $R \cdot p^0 > \iota \cdot V_*$ satisfies the problem in Proposition 1 and thus Eq (4.1).

First, this paper shows the effect of the transaction costs on the optimal portfolio to be rebalanced. To demonstrate the impact, consider two scenarios with different transaction costs, $c_j^b = c_j^s = 0$, and $c_j^b = c_j^s = 0.01$. Figure 1 shows that the transaction costs lower the efficient frontier. Then, the EVaR-return optimization model with the transaction cost and the proposed WEVaR-return optimization model with the transaction cost are investigated.

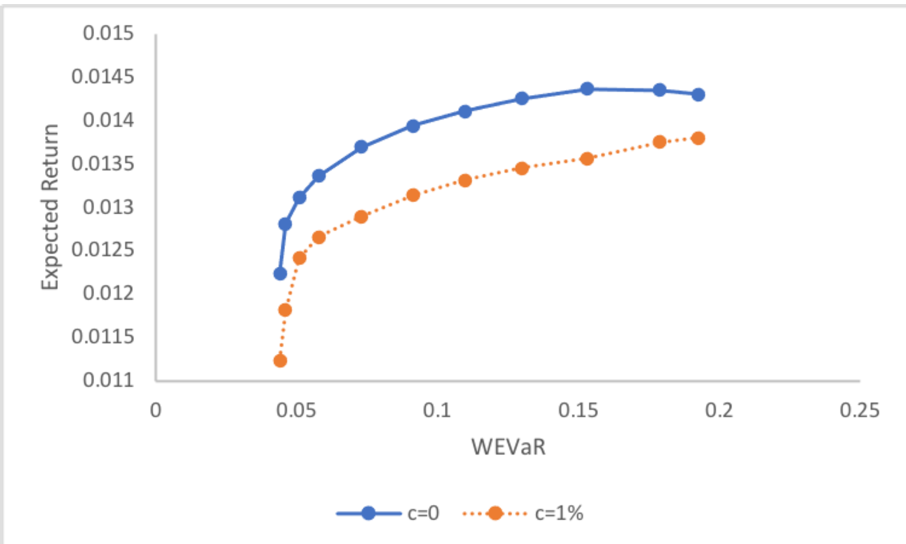

Figure 1. The efficient frontier of robust WEVaR-return optimization model under the box discrete distribution with transaction costs, $c_j^b = c_j^s = 0$ and 0.01.

Considering the same dataset, the task is to maximize the reward with the EVaR constraint and the transaction costs. Figures 2 and 3 display the outcomes under two transaction costs. Both figures show that with or without the transaction costs and the same risk value, the reward under EVaR is greater than WEVaR. This is consistent with how WEVaR is defined.

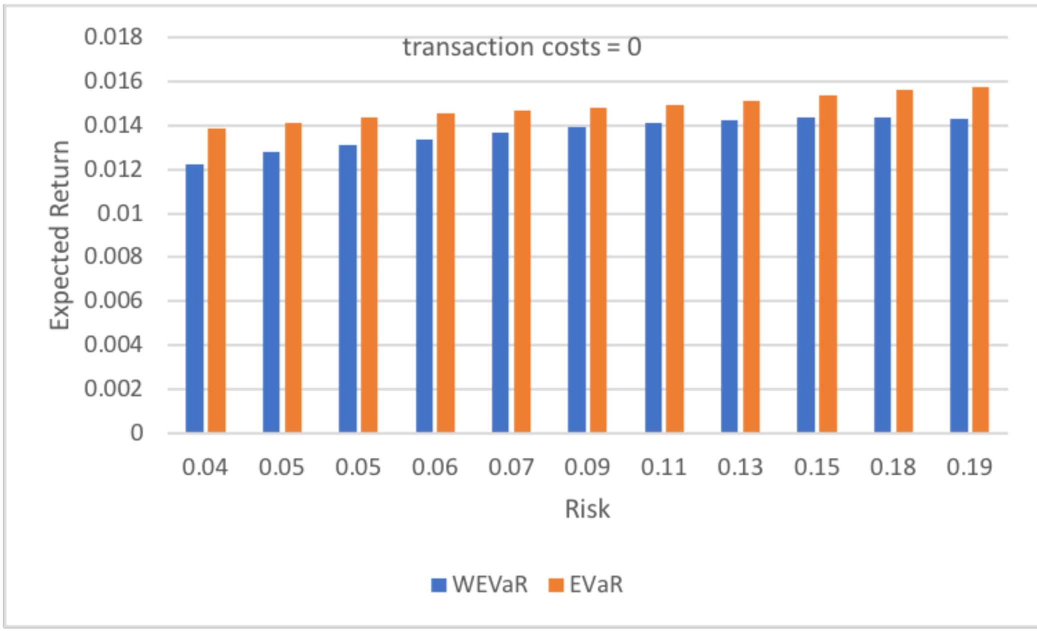

Figure 2. Comparison between EVaR and WEVaR consideration with $c_j^s = c_j^b = 0$.

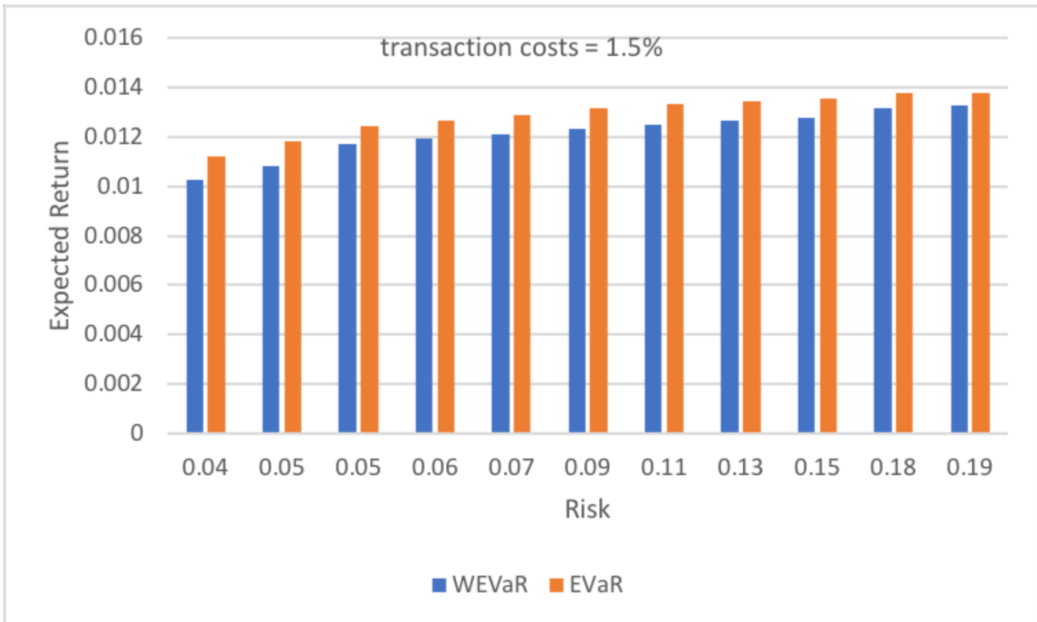

Figure 3. Comparison between EVaR and WEVaR consideration with $c_j^s = c_j^b = 0.015$.

5.1. Comparative analysis of portfolio strategies

This section evaluates the performance of various portfolio strategies using simulated data via a Monte Carlo simulation to determine which strategy offers the best risk-adjusted returns, capital preservation, and tail risk mitigation. Then, this study compares the proposed WEVaR portfolio with the following strategies: mean-EVaR, Mean-Variance (MV), and Mean-Variance Conditional Value at Risk (MV-CVaR).

From Table 2, the proposed WEVaR is the most conservative with the highest bond allocation (40%). The MV Portfolio is aggressive, with a 50% allocation to Equities. The mean-EVaR Portfolio balances Equities (35%) and Commodities (35%), while the MV-CVaR Portfolio leans towards Equities (40%). In the comparative analysis of the portfolio strategies, the measure of asset allocation variability reveals significant insights into the diversification practices. Specifically, Equities exhibit the greatest variation across portfolios with a range of 0.25, and a standard deviation of 0.09, indicating a broad spectrum of equity exposures reflective of varying risk tolerances and market outlooks. However, Bonds and Commodities show tighter ranges of 0.15 and 0.10, respectively, with correspondingly lower standard deviations (0.056 for Bonds and 0.05 for Commodities), suggesting a more uniform approach to these asset classes across the strategies. This variability analysis underlines how different portfolios adjust their asset allocations to either embrace or mitigate risk, with the MV portfolio appearing particularly aggressive due to its higher equity allocation, thus providing a quantitative foundation to the assertion that MV and MV-CVaR strategies emphasize diversification more markedly.

Table 2. Portfolio weights for different strategies.

Asset class	Proposed WEVaR Portfolio	mean-EVaR Portfolio	MV Portfolio	MV-CVaR Portfolio
Equities	0.25	0.35	0.50	0.40
Bonds	0.40	0.30	0.25	0.35
Commodities	0.35	0.35	0.25	0.25

Table 3 provides a comprehensive overview of the key risk metrics for the four portfolio strategies. These measurements provide crucial insights into the risk profiles and performances of the strategies. As described, the WEVaR portfolio is a risk-averse option that demonstrates the lowest standard deviation (0.12) and WEVaR value (0.05). This indicates its capacity to preserve capital and limit extreme losses. In contrast, the mean-EVaR demonstrates a comparatively higher WEVaR value of 0.07, indicating an increased level of risk exposure. The MV-CVaR strategy demonstrates the most favorable performance in CVaR with a value of 0.09, highlighting its efficacy in mitigating extreme losses. Regarding the risk-adjusted returns, MV-CVaR stands out with the highest Sharpe ratio of 1.25. This was achieved with a mean return of 0.1725 and a risk-free rate of 1%. WEVaR follows closely behind, which has a Sharpe ratio of 1.15, a mean return of 0.148, and a risk-free rate of 1%. These results indicate that both strategies have strong risk-adjusted performances. These measures enable investors to make well-informed decisions, tailoring their portfolios based on risk preferences and financial goals.

Table 3. Risk metrics for different portfolios.

Portfolio	proposed WEVaR	mean-EVaR	MV	MV-CVaR
Standard Deviation	0.12	0.14	0.15	0.13
proposed-WEVaR	0.05	0.07	0.08	0.06
EVaR	0.06	0.08	0.09	0.07
CVaR	0.08	0.10	0.11	0.09

In this comprehensive portfolio analysis presented in Table 4, the proposed WEVaR and MV-CVaR portfolios have exhibited significant advantages. These portfolios can better preserve capital during difficult market conditions, such as the COVID-19 pandemic, and the 2008 financial crisis, outperforming the mean-EVaR and MV portfolios.

Table 4. Scenario analysis–portfolio performance.

Scenario	proposed WEVaR return (%)	mean- EVaR return (%)	MV return (%)	MV-CVaR return (%)
Normal market	8.5	9.2	9.7	9.0
Stress (2008 Crisis)	-15.3	-18.5	-21.8	-17.2
Stress (COVID-19)	-9.7	-11.2	-13.5	-10.5

In summary, when considering risk-adjusted returns measured by the Sharpe ratio, the MV-CVaR portfolio emerges as the top performer, closely followed by WEVaR, suggesting that MV-CVaR provides the most favorable risk-adjusted returns. Furthermore, the WEVaR and MV-CVaR portfolios mitigate tail risks, as evidenced by their lower WEVaR and CVaR values. While all the portfolios maintain diversified asset allocations, the MV and MV-CVaR portfolios notably emphasize the importance of diversification. This analysis underscores the appeal of the WEVaR and MV-CVaR portfolios for investors that seek capital preservation and risk management in their investment strategies. Based on the evaluated criteria, the portfolio strategies can be ranked as follows:

- proposed WEVaR (for capital preservation and tail risk mitigation);
- MV-CVaR (for risk-adjusted returns and diversification benefits);
- mean-EVaR; and
- MV.

It's important to note that choosing the best strategy depends on the individual risk tolerance, the investment objectives, and the market conditions. The WEVaR portfolio stands out for its resilience in preserving capital and mitigating extreme risks.

6. Conclusions

This research examined robust portfolio optimization, and particularly emphasized the WEVaR-return optimization model under discrete distributions. The complexities of financial optimization problems, especially when considering worst-case scenarios, necessitated a fresh approach, and this work aimed to address these challenges head-on.

The primary contributions of this study are as follows: This paper provided a comprehensive framework for the WEVaR-return optimization model that considered both box and ellipsoidal discrete distributions. This framework offers a nuanced understanding of the underlying risk structures in portfolio management. In contrast to the existing robust return-risk portfolio strategies, this study considered the same uncertainty distribution in both the return and the risk. Moreover, this paper successfully reformulated the robust optimization problems through meticulous mathematical derivations to make them more tractable. A method to address the inherent non-convexities in the optimization process was presented by leveraging the convexity properties and utilizing the duality theorem. Furthermore, the integration of the transaction costs into the optimization models was pursued. This essential incorporation ensured that the suggested frameworks retain their pertinence and applicability since they consider the real-world costs that investors encounter while adjusting their portfolios.

Demonstrating the practicality of the model, this study utilized real-world data obtained from the Hang Seng Index. Through simulations, the proposed WEVaR portfolio exhibited an exceptional resilience in preserving capital and mitigating extreme risks. This translates into concrete benefits for investors, as they can rely on a robust methodology to navigate volatile market conditions and achieve their financial objectives with a greater confidence.

In essence, the practical implications of this study lie in empowering investors and portfolio managers with a practical toolset to optimize portfolios, manage risks, and ultimately enhance their investment outcomes in dynamic financial environments.

Possible directions for future investigation could employ the use of machine learning methods to enhance the models. Such advancements could lead to precision of the optimization process outcomes.

Author contributions

The author has read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The author declares the use of Grammarly in the creation of this article.

Acknowledgments

The author acknowledges support from Wenzhou-Kean Internal Research Support Program, China–Grant/Award Number: IRSPG202204, and Wenzhou-Kean University Academy of Interdisciplinary Research for Sustainability, China.

Conflict of interest

The author declares there is no conflict of interest.

References

1. A. Ahmadi-Javid, Entropic value-at-risk: A new coherent risk measure, *J. Optim. Theory Appl.*, **155** (2012), 1105–1123. https://dx.doi.org/10.1007/s10957-011-9968-2

2. Y. Amihud, H. Mendelson, Liquidity and stock returns, *Financ. Anal. J.*, **42** (1986), 43–48. http://dx.doi.org/10.2469/faj.v42.n3.43

3. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Thinking coherently, *Risk*, **10** (1997), 68–71.

4. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk, *Math. Financ.*, **9** (1999), 203–228. http://dx.doi.org/10.1111/1467-9965.00068

5. A. Axelrod, L. Carlone, G. Chowdhary, S. Karaman, Data-driven prediction of EVAR with confidence in time-varying datasets, *2016 IEEE 55th Conference on Decision and Control (CDC)*, Las Vegas, NV, USA, 2016, 5833–5838. http://dx.doi.org/10.1109/CDC.2016.7799166

6. D. P. Baron, On the utility theoretic foundations of mean-variance analysis, *J. Financ.*, **32** (1977), 1683–1697. http://dx.doi.org/10.1111/j.1540-6261.1977.tb03363.x

7. A. Ben-Tal, D. Bertsimas, D. B. Brown, A soft robust model for optimization under ambiguity, *Oper. Res.*, **58** (2010), 1220–1234. http://dx.doi.org/10.1287/opre.1100.0821

8. D. Bertsimas, D. B. Brown, Constructing uncertainty sets for robust linear optimization, *Oper. Res.*, **57** (2009), 1483–1495. http://dx.doi.org/10.1287/opre.1080.0646

9. M. Borkovec, I. Domowitz, B. Kiernan, V. Serbin, Portfolio optimization and the cost of trading, *J. Invest.*, **19** (2010), 63–76. http://dx.doi.org/10.3905/joi.2010.19.2.063

10. J. P. Bouchaud, J. Bonart, J. Donier, M. Gould, *Trades, quotes and prices: financial markets under the microscope*, Cambridge: Cambridge University Press, 2018. http://dx.doi.org/https://doi.org/10.1017/9781316659335

11. S. P. Boyd, L. Vandenberghe, *Convex optimization*, Cambridge: Cambridge University Press, 2004.

12. S. Caçador, J. M. Dias, P. Godinho, Portfolio selection under uncertainty: a new methodology for computing relative-robust solutions, *Int. T. Oper. Res.*, **28** (2021), 1296–1329. http://dx.doi.org/10.1111/itor.12674

13. D. Cajas, Entropic portfolio optimization: a disciplined convex programming framework, *SSRN Electronic Journal*, **2021** (2021), 3792520.

14. A. H. Chen, F. J. Fabozzi, D. S. Huang, Portfolio revision under mean-variance and mean-cvar with transaction costs, *Rev. Quant. Finan. Acc.*, **39** (2012), 509–526. http://dx.doi.org/10.1007/s11156-012-0292-1

15. S. Chennaf, J. B. Amor, Entropic value at risk to find the optimal uncertain random portfolio, *Soft Comput.*, **27** (2023), 15185–15197. http://dx.doi.org/10.1007/s00500-023-08547-5

16. J. Dufitinema, S. Pynnönen, T. Sottinen, Maximum likelihood estimators from discrete data modeled by mixed fractional brownian motion with application to the nordic stock markets, *Commun. Stat. Simul. C.*, **51** (2022), 5264–5287. http://dx.doi.org/10.1080/03610918.2020.1764581

17. P. Embrechts, S. I. Resnick, G. Samorodnitsky, Extreme value theory as a risk management tool, *N. Am. Actuar. J.*, **3** (1999), 30–41. http://dx.doi.org/10.1080/10920277.1999.10595797

18. D. Goldfarb, G. Iyengar, Robust portfolio selection problems, *Math. Oper. Res.*, **28** (2003), 1–38. http://dx.doi.org/10.1287/moor.28.1.1.14260

19. F. Hooshmand, Z. Anoushirvani, S. A. MirHassani, Model and efficient algorithm for the portfolio selection problem with real-world constraints under value-at-risk measure, *Int. T. Oper. Res.*, **30** (2023), 2665–2690. http://dx.doi.org/10.1111/itor.13239

20. R. P. Huang, Z. S. Xu, S. J. Qu, X. G. Yang, M. Goh, Robust portfolio selection with distributional uncertainty and integer constraints, *J. Oper. Res. Soc. China*, **11** (2023), 1–27. http://dx.doi.org/10.1007/s40305-023-00466-4

21. G. Kara, A. Özmen, G.-W. Weber, Stability advances in robust portfolio optimization under parallelepiped uncertainty, *Cent. Eur. J. Oper. Res.*, **27** (2019), 241–261. http://dx.doi.org/10.1007/s10100-017-0508-5

22. J. Kriens, J. T. van Lieshout, Notes on the Markowitz portfolio selection method, *Stat. Neerl.*, **42** (1988), 181–191. http://dx.doi.org/10.1111/j.1467-9574.1988.tb01232.x

23. W. Liu, L. Yang, B. Yu, Distributionally robust optimization based on Kernel density estimation and mean-entropic value-at-risk, *INFORMS Journal on Optimization*, **5** (2022), 68–91. http://dx.doi.org/10.1287/ijoo.2022.0076

24. G. M. Luo, Mixed complementarity problems for robust optimization equilibrium in bimatrix game, *Appl. Math.*, **57** (2012), 503–520. http://dx.doi.org/10.1007/s10492-012-0029-4

25. H. Markowitz, Portfolio selection, *J. Financ.*, **7** (1952), 77–91. http://dx.doi.org/10.1111/j.1540-6261.1952.tb01525.x

26. E. J. Menvouta, S. Serneels, T. Verdonck, Portfolio optimization using cellwise robust association measures and clustering methods with application to highly volatile markets, *Journal of Finance and Data Science*, **9** (2023), 100097. http://dx.doi.org/10.1016/j.jfds.2023.100097

27. J. E. Mitchell, S. Braun, Rebalancing an investment portfolio in the presence of convex transaction costs, submitted for publication.

28. J. E. Mitchell, S. Braun, Rebalancing an investment portfolio in the presence of convex transaction costs, including market impact costs, *Optim. Method. Softw.*, **28** (2013), 523–542. http://dx.doi.org/10.1080/10556788.2012.717940

29. K. Muthuraman, S. Kumar, Multidimensional portfolio optimization with proportional transaction costs, *Math. Financ.*, **16** (2006), 301–335. http://dx.doi.org/10.1111/j.1467-9965.2006.00273.x

30. G. C. Pflug, Some remarks on the value-at-risk and the conditional value-at-risk, In: *Probabilistic constrained optimization*, Boston: Springer, 2000, 272–281. http://dx.doi.org/10.1007/978-1-4757-3150-7_15

31. R. T. Rockafellar, S. Uryasev, Optimization of conditional value-at-risk, *J. Risk*, **3** (2000), 21–41.

32. R. Sehgal, A. Mehra, Robust reward–risk ratio portfolio optimization, *Int. T. Oper. Res.*, **28** (2021), 2169–2190. http://dx.doi.org/10.1111/itor.12652

33. R. L. Sun, T. F. Ma, S. Z. Liu, Portfolio selection based on semivariance and distance correlation under minimum variance framework, *Stat. Neerl.*, **73** (2019), 373–394. http://dx.doi.org/10.1111/stan.12174

34. X. J. Tong, F. L. Wu, L. Q. Qi, Worst-case cvar based portfolio optimization models with applications to scenario planning, *Optim. Method. Softw.*, **24** (2009), 933–958. http://dx.doi.org/10.1080/10556780902865942

35. X. J. Tong, F. L. Wu, Robust reward–risk ratio optimization with application in allocation of generation asset, *Optimization*, **63** (2014), 1761–1779. http://dx.doi.org/10.1080/02331934.2012.672419

36. X. T. Wang, Z. Li, L. Zhuang, Risk preference, option pricing and portfolio hedging with proportional transaction costs, *Chaos Soliton. Fract.*, **95** (2017), 111–130. https://dx.doi.org/10.1016/j.chaos.2016.12.010

37. L. J. Xu, Y. J. Zhou, New robust reward-risk ratio models with CVaR and standard deviation, *J. Math.*, **2022** (2022), 8304411. http://dx.doi.org/10.1155/2022/8304411

38. C. L. Zheng, Y. Chen, Portfolio selection based on relative entropy coherent risk measure, *Systems Engineering-Theory & Practice*, **34** (2014), 648–655.

39. Y. J. Zhou, L. Yang, L. J. Xu, B. Yu, Inseparable robust reward–risk optimization models with distribution uncertainty, *Japan J. Indust. Appl. Math.*, **33** (2016), 767–780. http://dx.doi.org/10.1007/s13160-016-0230-z

40. S. S. Zhu, D. Li, S. Y. Wang, Robust portfolio selection under downside risk measures, *Quant. Financ.*, **9** (2009), 869–885. http://dx.doi.org/10.1080/14697680902852746

41. S. S. Zhu, M. Fukushima, Worst-case conditional value-at-risk with application to robust portfolio management, *Oper. Res.*, **57** (2009), 1155–1168. http://dx.doi.org/10.1287/opre.1080.0684

42. J. X. Zhu, Optimal financing and dividend distribution with transaction costs in the case of restricted dividend rates, *ASTIN Bull.*, **47** (2017), 239–268. http://dx.doi.org/10.1017/asb.2016.29

AIMS Press