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*Research article*

# Global weighted regularity for the 3D axisymmetric non-resistive MHD system

**Wenjuan Liu**[1] **and Zhouyu Li**[2,*]

[1] School of Mathematics, Northwest University, Xi'an 710069, China
[2] School of Sciences, Xi'an University of Technology, Xi'an 710054, China

* **Correspondence:** Email: zylimath@163.com.

**Abstract:** We consider the regularity criteria of axisymmetric solutions to the non-resistive MHD system with non-zero swirl in $\mathbb{R}^3$. By applying a new anisotropic Hardy-Sobolev inequality in mixed Lorentz spaces, we show that strong solutions to this system can be smoothly extended beyond the possible blow-up time $T$ if the horizontal angular component of the velocity belongs to anisotropic Lorentz spaces.

**Keywords:** non-resistive MHD system; anisotropic Lorentz spaces; regularity criteria
**Mathematics Subject Classification:** 35B65, 35Q35, 76D03

## 1. Introduction

Generally speaking, the three-dimensional incompressible non-resistive MHD system in Euclidean coordinates reads

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P - \Delta u = b \cdot \nabla b, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (u, b)|_{t=0} = (u_0, b_0), \end{cases} \tag{1.1}$$

where the unknowns $u = \left(u^1, u^2, u^3\right)$, $b = \left(b^1, b^2, b^3\right)$, and $P$ represent the velocity of the fluid, the magnetic field, and the scalar pressure function, respectively. Physically, (1.1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water. It is frequently applied in astrophysics, geophysics, cosmology, and so forth. One may check the references [7, 11, 25] for more applications and numerical studies.

Be aware that system (1.1) reduces to the classical Navier-Stokes equations when $b$ is identically zero. The global regularity of the 3D Navier-Stokes equations with large initial data remains open and it is generally viewed as one of the most challenging open problems in fluid mechanics. As a result, various efforts are made to study the solutions by using axisymmetric methods.

In this paper, we assume that the solution $(u, b)$ of system (1.1) has the following axisymmetric form:

$$\begin{cases} u(t, x) = u^r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u^z(t, r, z)e_z, \\ b(t, x) = b^r(t, r, z)e_r + b^\theta(t, r, z)e_\theta + b^z(t, r, z)e_z. \end{cases}$$

Here,

$$r = \sqrt{x_1^2 + x_2^2}, \qquad e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \qquad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \qquad e_z = (0, 0, 1).$$

In the above, $u^\theta$ is usually called the swirl component. We say $u$ is without swirl if $u^\theta = 0$.

In recent years, a great deal of mathematical effort has been dedicated to the study of the 3D axisymmetric Navier-Stokes equations. For the case of $u^\theta = 0$, Abidi [1] and Ladyzhenskaya [15] independently proved the existence, uniqueness, and regularities of generalized solutions. The first author of this paper in [19] obtained the global well-posedness of the inhomogeneous axisymmetric Navier-Stokes equations. For the case of $u^\theta \neq 0$, the authors need to impose some smallness conditions on the initial data. For more references, we recommend [4, 28] and references therein.

Many fruitful studies on the well-posedness problem of the MHD system (1.1) have been achieved in recent years, see [5, 8, 9, 13] and references therein. Now we recall some results on the axisymmetric MHD equations. Lei [17] considered a family of special axisymmetric initial data with $u_0^\theta = b_0^r = b_0^z = 0$ and showed the global well-posedness of system (1.1) without any smallness assumptions. Further improvement was made by Ai and Li [2], who weakened the initial regularity. When the angular velocity is not trivial, Liu [21] obtained the global well-posedness of system (1.1) provided that $\|ru_0^\theta\|_{L^\infty}$ and $\|\frac{b_0^\theta}{r}\|_{L^{\frac{3}{2}}}$ are small enough. Later on, Zhang and Rao [27] improved this result by removing the smallness of $\|\frac{b_0^\theta}{r}\|_{L^{\frac{3}{2}}}$.

Researchers are interested in the classical problem of finding regularity criteria of the axisymmetric MHD system. In [23], Li and Liu established the following regularity criteria for the 3D axisymmetric non-resistive MHD system in Lorentz spaces

$$\|\frac{u^\theta}{r^s}\|_{L^q(0,T^*;L^{p,\infty}(\mathbb{R}^3))} < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1 + s, \quad \frac{3}{1+s} < p \leq \infty.$$

Later, by using some inequalities in anisotropic Lorentz spaces and the generalized Hardy-Sobolev inequality, this result was extended to anisotropic Lorentz spaces in [16]. More precisely, he proved that if the initial data $(u_0, b_0) \in H^2\left(\mathbb{R}^3\right)$, $b_0^r = b_0^z = 0$, and the horizontal swirl component of velocity satisfies

$$\begin{aligned} \frac{u^\theta}{r^s} \in & L^q\left(0, T; L_{x_1}^{p_1,\infty}(\mathbb{R})L_{x_2}^{p_2,\infty}(\mathbb{R})L_{x_3}^{p_3,\infty}(\mathbb{R})\right) \quad with \quad \frac{2}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + s, \\ & -\frac{1}{2} < s \leq 0, \quad \frac{3}{1+s} < p_i \leq \infty, \quad \frac{2}{1+s} \leq q < \infty, \end{aligned} \tag{1.2}$$

or $\frac{u^\theta}{r^s} \in L^\infty(0, T; L_{x_1}^{p_1,\infty}(\mathbb{R})L_{x_2}^{p_2,\infty}(\mathbb{R})L_{x_3}^{p_3,\infty}(\mathbb{R}))$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + s$ and the norm of $\|\frac{u^\theta}{r^s}\|_{L^\infty\left(0,T;L_{x_1}^{p_1,\infty}(\mathbb{R})L_{x_2}^{p_2,\infty}(\mathbb{R})L_{x_3}^{p_3,\infty}(\mathbb{R})\right)}$ is sufficiently small, then the solution $(u, b)$ can be smoothly extended beyond $T$. For more regularity criteria on the axisymmetric MHD system, see [12, 18, 20] and references therein.

We can rewrite system (1.1) as

$$\begin{cases}\partial_t u^r + (u^r\partial_r + u^z\partial_z)\, u^r - \frac{(u^\theta)^2}{r} + \partial_r P = (\Delta - \frac{1}{r^2})u^r + (b^r\partial_r + b^z\partial_z)\, b^r - \frac{(b^\theta)^2}{r},\\ \partial_t u^\theta + (u^r\partial_r + u^z\partial_z)\, u^\theta + \frac{u^r u^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right) u^\theta + (b^r\partial_r + b^z\partial_z)\, b^\theta + \frac{b^r b^\theta}{r},\\ \partial_t u^z + (u^r\partial_r + u^z\partial_z)\, u^z + \partial_z P = \Delta u^z + (b^r\partial_r + b^z\partial_z)\, b^z,\\ \partial_t b^r + (u^r\partial_r + u^z\partial_z)\, b^r = (b^r\partial_r + b^z\partial_z)\, u^r,\\ \partial_t b^\theta + (u^r\partial_r + u^z\partial_z)\, b^\theta + \frac{u^\theta b^r}{r} = (b^r\partial_r + b^z\partial_z)\, u^\theta + \frac{u^r b^\theta}{r},\\ \partial_t b^z + (u^r\partial_r + u^z\partial_z)\, b^z = (b^r\partial_r + b^z\partial_z)\, u^z,\\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \partial_r b^r + \frac{b^r}{r} + \partial_z b^z = 0,\end{cases} \tag{1.3}$$

where the operator $\Delta \overset{def}{=} \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

In this paper, we consider the following initial data:

$$u_0 = u_0^r e_r + u_0^\theta e_\theta + u_0^z e_z, \quad b_0 = b_0^\theta e_\theta.$$

Thus, by using the uniqueness of local solutions to system (1.1), we conclude that $b^r = b^z = 0$ for all later times. Then, system (1.3) is equivalent to

$$\begin{cases}\partial_t u^r + (u^r\partial_r + u^z\partial_z)\, u^r - \frac{(u^\theta)^2}{r} + \partial_r P = (\Delta - \frac{1}{r^2})u^r - \frac{(b^\theta)^2}{r},\\ \partial_t u^\theta + (u^r\partial_r + u^z\partial_z)\, u^\theta + \frac{u^r u^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right) u^\theta,\\ \partial_t u^z + (u^r\partial_r + u^z\partial_z)\, u^z + \partial_z P = \Delta u^z,\\ \partial_t b^\theta + (u^r\partial_r + u^z\partial_z)\, b^\theta = \frac{u^r b^\theta}{r},\\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0.\end{cases} \tag{1.4}$$

We can also write the vorticity field $w$ in cylindrical coordinates

$$w = \nabla \times u = w^r\,(t, r, z)\, e_r + w^\theta\,(t, r, z)\, e_\theta + w^z\,(t, r, z)\, e_z,$$

where

$$w^r = -\partial_z u^\theta, \quad w^\theta = \partial_z u^r - \partial_r u^z, \quad w^z = \frac{\partial_r\left(r u^\theta\right)}{r}.$$

According to system (1.4), the quantity $\left(w^r, w^\theta, w^z\right)$ verifies

$$\begin{cases}\partial_t w^r + (u^r\partial_r + u^z\partial_z)\, w^r = (\Delta - \frac{1}{r^2})w^r + (w^r\partial_r + w^z\partial_z)\, u^r,\\ \partial_t w^\theta + (u^r\partial_r + u^z\partial_z)\, w^\theta = \left(\Delta - \frac{1}{r^2}\right) w^\theta + \frac{u^r}{r} w^\theta + \frac{1}{r}\partial_z (u^\theta)^2 - \frac{1}{r}\partial_z (b^\theta)^2,\\ \partial_t w^z + (u^r\partial_r + u^z\partial_z)\, w^z = \Delta u^z + (w^r\partial_r + w^z\partial_z)\, u^z.\end{cases} \tag{1.5}$$

We notice that condition (1.2) is concerned with the case $-\frac{1}{2} < s \le 0$. Thus, a natural and interesting problem is whether or not the range of indicator $s$ in condition (1.2) can be extended. The goal of this paper is to give a positive answer. Inspired by [16, 26], we obtain the regularity criteria of system (1.4) in anisotropic Lorentz spaces with $0 \le s < \infty$. Let us state our main result.

**Theorem 1.1.** *Let* $(u, b)$ *be an axially symmetric solution to the MHD system* (1.1) *associated with the initial data* $(u_0, b_0) \in H^m\left(\mathbb{R}^3\right)$, $m \geq 3$, *and* $b_0^r = b_0^z = 0$. *If the horizontal swirl component of velocity satisfies*

$$\begin{gathered}\frac{u^\theta}{r^s} \in L^q\left(0, T; L_{x_1}^{p_1,\infty}(\mathbb{R}) L_{x_2}^{p_2,\infty}(\mathbb{R}) L_{x_3}^{p_3,\infty}(\mathbb{R})\right) \quad with \quad \frac{2}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + s, \\ 0 \leq s < \infty, \quad \frac{3}{1+s} < p_i \leq \infty, \quad \frac{2}{1+s} \leq q < \infty,\end{gathered} \tag{1.6}$$

*or*

$$\|\frac{u^\theta}{r^s}\|_{L^\infty\left(0,T; L_{x_1}^{p_1,\infty}(\mathbb{R}) L_{x_2}^{p_2,\infty}(\mathbb{R}) L_{x_3}^{p_3,\infty}(\mathbb{R})\right)} < \epsilon \quad with \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + s, \tag{1.7}$$

*where* $\epsilon = \epsilon\left(s, r{u_0}^\theta\right) << 1$, *then (u,b) can be smoothly extended beyond* $T$.

**Remark 1.1.** *In [26], the authors established several new anisotropic Hardy-Sobolev inequalities in mixed Lebesgue spaces and mixed Lorentz spaces. They also derived regularity criteria of the 3D axisymmetric Navier-Stokes system. We extend the related regularity criteria to the MHD system. In addition, compared to the results in [23], thanks to the new anisotropic Hardy-Sobolev inequality, we generalize the result to the anisotropic Lorentz space.*

**Remark 1.2.** *We extend the results in [16] to the case of* $0 \leq s < \infty$.

The remaining of this paper is organized as follows: In Section 2, we provide the definition of anisotropic Lorentz spaces and gather some elementary inequalities. The proof of Theorem 1.1 is given in Section 3.

**Notations.** We shall always denote $\int_{\mathbb{R}^3} \cdot dx = 2\pi \int_0^\infty \int_{\mathbb{R}} \cdot r dr dz$ and the letter $C$ as a generic constant which may vary from line to line. The Fourier transform $\hat{f}$ of a Schwartz function $f$ on $\mathbb{R}^n$ is defined as $\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$. Furthermore, for $s \geq 0$, we define $\Lambda^s f$ by $\widehat{\Lambda^s f}(\xi) = \left(\sum_{i=1}^n |\xi_i|^2\right)^{\frac{s}{2}} \hat{f}(\xi)$, where the notation $\Lambda$ stands for the square root of the negative Laplacian $(-\Delta)^{\frac{1}{2}}$. Similarly, we denote $\widehat{\Lambda_{x_i}^s f}(\xi) = |\xi_i|^s \hat{f}(\xi)$ and $\widehat{\Lambda_{x_1,x_2,\cdots x_k}^s f}(\xi) = \left(\sum_{i=1}^k |\xi_i|^2\right)^{\frac{s}{2}} \hat{f}(\xi)$.

## 2. Preliminaries

First, we recall the definition of Lorentz spaces, see [24] for details. Given $1 \leq p < \infty, 1 \leq q \leq \infty$, a measurable function $f$ then belongs to the Lorentz spaces $L^{p,q}\left(\mathbb{R}^3\right)$ if $\|f\|_{L^{p,q}\left(\mathbb{R}^3\right)} < \infty$, where

$$\|f\|_{L^{p,q}\left(\mathbb{R}^3\right)} := \begin{cases} \left(\int_0^\infty t^{q-1} |x \in \mathbb{R}^3 : |f(x) > t|^{\frac{q}{p}}\, dt\right)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup\limits_{t>0} t \left|\left\{x \in \mathbb{R}^3 : |f(x)| > t\right\}\right|^{\frac{1}{p}}, & \text{if } q = \infty. \end{cases}$$

The anisotropic Lorentz space $L^{\vec{p},\vec{q}}\left(\mathbb{R}^3\right)$ was first introduced in [3, 10, 14], and its norm is determined by

$$\|f\|_{L^{\vec{p},\vec{q}}\left(\mathbb{R}^3\right)} := \|f\|_{L_1^{p_1,q_1} L_2^{p_2,q_2} L_3^{p_3,q_3}\left(\mathbb{R}^3\right)} = \left\|\left\|\|f\|_{L_{x_1}^{p_1,q_1}(\mathbb{R})}\right\|_{L_{x_2}^{p_2,q_2}(\mathbb{R})}\right\|_{L_{x_3}^{p_3,q_3}(\mathbb{R})}.$$

For the convenience of the reader, we present some technical lemmas which will be useful later.

**Lemma 2.1** (Hölder's inequality [10, 14]). *Let $f \in L^{\vec{r_1},\vec{s_1}}\left(\mathbb{R}^3\right)$ and $g \in L^{\vec{r_2},\vec{s_2}}\left(\mathbb{R}^3\right)$. Then, there exists a constant $C > 0$ such that*

$$\|fg\|_{L^{\vec{r},\vec{s}}(\mathbb{R}^3)} \leq C\|f\|_{L^{\vec{r_1},\vec{s_1}}(\mathbb{R}^3)}\|g\|_{L^{\vec{r_2},\vec{s_2}}(\mathbb{R}^3)},$$

*where* $\frac{1}{\vec{r}} = \frac{1}{\vec{r_1}} + \frac{1}{\vec{r_2}}$, $\frac{1}{\vec{s}} = \frac{1}{\vec{s_1}} + \frac{1}{\vec{s_2}}$, $0 < \vec{r_1}, \vec{r_2}, \vec{s_1}, \vec{s_2} \leq \infty$.

**Lemma 2.2** (Sobolev inequality [10, 14]). *Assume that $1 \leq \vec{l} \leq \infty$. It then holds that*

$$\|f\|_{L^{\vec{p},\vec{l}}(\mathbb{R}^3)} \leq C\|\Lambda^s f\|_{L^{\vec{r},\vec{l}}(\mathbb{R}^3)},$$

*with $1 < r_i < p_i < \infty$ and*

$$\sum_{i=1}^{3}\left(\frac{1}{r_i} - \frac{1}{p_i}\right) = s.$$

The subsequent lemmas are crucial in substantiating our findings.

**Lemma 2.3** ( [26]). *Suppose that $\mathbb{R}^n = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \cdots \mathbb{R}^{k_i} \times \mathbb{R}^{n-\sum_{j=1}^{i} k_j}$ and $n \geq \sum_{j=1}^{i} k_j$. Let $r_1 = \sqrt{x_1^2 + \cdots x_{k_1}^2}$, $r_2 = \sqrt{x_{k_1+1}^2 + \cdots x_{k_1+k_2}^2}$, ..., $r_i = \sqrt{x_{\sum_{j=1}^{i-1} k_j+1}^2 + \cdots x_{\sum_{j=1}^{i} k_j}^2}$, $0 < p, q \leq \infty$, $1 < \left(\vec{p_j}\right)_l \leq \infty$, $1 \leq \left(\vec{q_j}\right)_l \leq \infty$ and $0 < \alpha_j < \frac{k_j}{\left(\vec{p_j}\right)_l}$, $1 \leq j \leq i$, $1 \leq l \leq k_j$. Then, for all $f \in C_0^\infty(\mathbb{R}^n)$, we have that it holds that*

$$\left\|\frac{f(x_1, x_2, \cdots x_n)}{\prod_{j=1}^{i} |r_j|^{\alpha_j}}\right\|_{L^{\vec{p_1},\vec{q_1}}(\mathbb{R}^{k_1})\cdots L^{\vec{p_i},\vec{q_i}}(\mathbb{R}^{k_i})L^{p,q}\left(\mathbb{R}^{n-\sum_{j=1}^{i} k_j}\right)} \leq C\left\|\Lambda_{x_1,\cdots x_{\sum_{j=1}^{i} k_j}}^{\sum_{j=1}^{i} \alpha_j} f\right\|_{L^{\vec{p_1},\vec{q_1}}(\mathbb{R}^{k_1})\cdots L^{\vec{p_i},\vec{q_i}}(\mathbb{R}^{k_i})L^{p,q}\left(\mathbb{R}^{n-\sum_{j=1}^{i} k_j}\right)}.$$

**Lemma 2.4** (Gagliardo-Nirenberg inequality [6]). *Let $0 \leq \sigma < s < \infty$ and $1 \leq q, r \leq \infty$. Then we have*

$$\|\Lambda^\sigma u\|_{L^p(\mathbb{R}^3)} \leq C\|u\|_{L^q(\mathbb{R}^3)}^{\theta}\|\Lambda^s u\|_{L^r(\mathbb{R}^3)}^{1-\theta},$$

*where $\frac{3}{p} - \sigma = \theta\frac{3}{q} + (1-\theta)\left(\frac{3}{r} - s\right)$ and $0 \leq \theta \leq 1 - \frac{\sigma}{s}\left(\theta \neq 0 \ if \ s - \sigma \geq \frac{3}{r}\right)$.*

**Lemma 2.5** ( [22]). *Assume that $u$ is a smooth axisymmetric vector field and $w = \nabla \times u$. Then it holds that*

$$\frac{u^r}{r} = \Delta^{-1}\partial_z\left(\frac{w^\theta}{r}\right) - 2\frac{\partial_r}{r}\Delta^{-2}\partial_z\left(\frac{w^\theta}{r}\right).$$

*In addition, for $1 < p < \infty$, it is valid that*

$$\left\|\nabla\frac{u^r}{r}\right\|_{L^p(\mathbb{R}^n)} \leq C\left\|\frac{w^\theta}{r}\right\|_{L^p(\mathbb{R}^n)},$$

*and*

$$\left\|\nabla^2\frac{u^r}{r}\right\|_{L^p(\mathbb{R}^n)} \leq C\left\|\partial_z\frac{w^\theta}{r}\right\|_{L^p(\mathbb{R}^n)}.$$

## 3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To begin, motivated by Li and Liu [23], the following lemma can be obtained.

**Lemma 3.1** (Continuation criterion [23])**.** *Assume that* $(u_0, b_0) \in H^m\left(\mathbb{R}^3\right)$ *with* $m \geq 3$ *and* $b_0^r = b_0^z = 0$. *Let* $(u, b)$ *be an axially symmetric local solution of system* (1.1). *If*

$$\sup_{t\in[0,T)} \|\frac{w^\theta}{r}(\cdot, t)\|_{L^2\left(\mathbb{R}^3\right)} < +\infty,$$

*then the solution* $(u, b)$ *can be smoothly extended beyond* $T$.

Now, we introduce the following new variables:

$$\Gamma := ru^\theta, \quad \Pi := \frac{b^\theta}{r}.$$

By taking advantage of system (1.4), we obtain

$$\begin{cases} \partial_t\Gamma + (u^r\partial_r + u^z\partial_z)\,\Gamma = \left(\Delta - \frac{2}{r}\partial_r\right)\Gamma, \\ \partial_t\Pi + (u^r\partial_r + u^z\partial_z)\,\Pi = 0. \end{cases} \tag{3.1}$$

The following proposition states fundamental estimates of system (1.1) which do not need the axisymmetric assumption.

**Proposition 3.1.** *Let* $(u, b)$ *be a smooth solution of system* (1.1) *with* $(u_0, b_0) \in H^m$ $(m \geq 3)$. *Then, we have for any* $t \in \mathbb{R}^+$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\nabla u\,(\tau)\,\|_{L^2}^2\,d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2, \tag{3.2}$$

$$\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad \forall p \in [2, +\infty], \tag{3.3}$$

*and*

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma_0\|_{L^p}, \quad \forall p \in [2, +\infty]. \tag{3.4}$$

*Proof.* Taking the inner product of $(1.1)_1$ and $(1.1)_2$ with $u$ and $b$, respectively, integrating by parts, and summing the results together, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 \leq 0.$$

Applying Gronwall's inequality leads to the desired result (3.2). The estimates for $\Gamma$ and $\Pi$ are classical for the heat equation when $p < \infty$, and follow from the maximum principle when $p = \infty$. We omit the details here, see [22]. □

Now we are in a position to derive the estimates of $(\Omega, J)$. We deduce from system (1.5) that the pair $(\Omega, J) \overset{def}{=} \left(\frac{w^\theta}{r}, \frac{w^r}{r}\right)$ satisfies

$$\begin{cases} \partial_t\Omega + (u^r\partial_r + u^z\partial_z)\,\Omega = \left(\Delta + \frac{2}{r}\partial_r\right)\Omega - \partial_z\Pi^2 - 2\frac{u^\theta}{r}J, \\ \partial_tJ + (u^r\partial_r + u^z\partial_z)\,J = \left(\Delta + \frac{2}{r}\partial_r\right)J + (w^r\partial_r + w^z\partial_z)\,\frac{u^r}{r}. \end{cases} \tag{3.5}$$

**Proposition 3.2.** *Under the assumptions of Theorem 1.1, the following estimate of* $(\Omega, J)$ *holds:*

$$\sup_{0\le t\le T}\left(\|\Omega(t)\|_{L^2}^2+\|J(t)\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\Omega(t)\|_{L^2}^2+\|\nabla J(t)\|_{L^2}^2\right)\,dt<\infty.$$

*Proof.* Multiplying $(3.5)_1$ and $(3.5)_2$ by $\Omega$ and $J$, respectively, integrating over $\mathbb{R}^3$, and using the divergence-free condition, we observe

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Omega(t)\|_{L^2}^2+\|J(t)\|_{L^2}^2\right)+\|\nabla\Omega(t)\|_{L^2}^2+\|\nabla J(t)\|_{L^2}^2\\
&=\int(w^r\partial_r+w^z\partial_z)\frac{u^r}{r}J\,dx-\int\partial_z\Pi^2\Omega\,dx-2\int\frac{u^\theta}{r}J\Omega\,dx\\
&=-2\pi\int_{-\infty}^{+\infty}\int_0^\infty\partial_z u^\theta\partial_r\frac{u^r}{r}J\,rdrdz+2\pi\int_{-\infty}^{+\infty}\int_0^\infty\frac{\partial_r(ru^\theta)}{r}\partial_z\frac{u^r}{r}J\,rdrdz\\
&\qquad-\int\partial_z\Pi^2\Omega\,dx-2\int\frac{u^\theta}{r}J\Omega\,dx\\
&=\int u^\theta\partial_r\frac{u^r}{r}\partial_zJ\,dx-\int u^\theta\partial_z\frac{u^r}{r}\partial_rJ\,dx-\int\partial_z\Pi^2\Omega\,dx-2\int\frac{u^\theta}{r}J\Omega\,dx\\
&\le\int|u^\theta||\nabla\frac{u^r}{r}||\nabla J|\,dx-\int\partial_z\Pi^2\Omega\,dx-2\int\frac{u^\theta}{r}J\Omega\,dx\\
&:=I_1+I_2+I_3.
\end{aligned}\tag{3.6}$$

For $I_2$, integration by parts, Young's inequality, and (3.3) yield

$$|I_2|=|\int\Pi^2\partial_z\Omega\,dx|\le C\|\Pi\|_{L^\infty}\|\Pi\|_{L^2}\|\partial_z\Omega\|_{L^2}\le C\|\Pi_0\|_{L^\infty}^2\|\Pi_0\|_{L^2}^2+\frac{1}{4}\|\nabla\Omega\|_{L^2}^2.$$

For any $\frac{3}{1+s}<p_i\le\infty$, by using Lemmas 2.1 and 2.2, and (3.4) we achieve

$$\begin{aligned}
|I_1|&=\int|ru^\theta|^{\frac{s}{s+1}}|\frac{u^\theta}{r^s}|^{\frac{1}{1+s}}|\nabla\frac{u^r}{r}||\nabla J|\,dx\\
&\le\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{1}{1+s}}\|ru^\theta\|_{L^\infty_{x_1}L^\infty_{x_2}L^\infty_{x_3}(\mathbb{R}^3)}^{\frac{s}{s+1}}\|\nabla\frac{u^r}{r}\|_{L_{x_1}^{\frac{2p_1(1+s)}{p_1(1+s)-2},2}L_{x_2}^{\frac{2p_2(1+s)}{p_2(1+s)-2},2}L_{x_3}^{\frac{2p_3(1+s)}{p_3(1+s)-2},2}(\mathbb{R}^3)}\|\nabla J\|_{L^2}\\
&\le\frac{1}{8}\|\nabla J\|_{L^2}^2+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{2}{1+s}}\|\Gamma_0\|_{L^\infty_{x_1}L^\infty_{x_2}L^\infty_{x_3}(\mathbb{R}^3)}^{\frac{2s}{s+1}}\|\nabla\frac{u^r}{r}\|^2_{L_{x_1}^{\frac{2p_1(1+s)}{p_1(1+s)-2},2}L_{x_2}^{\frac{2p_2(1+s)}{p_2(1+s)-2},2}L_{x_3}^{\frac{2p_3(1+s)}{p_3(1+s)-2},2}(\mathbb{R}^3)}\\
&\le\frac{1}{8}\|\nabla J\|_{L^2}^2+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{2}{1+s}}\|\Lambda^{\Sigma_{i=1}^3\frac{1}{p_i(1+s)}}\nabla\frac{u^r}{r}\|_{L^2}^2.
\end{aligned}$$

$(i)$ **under the assumption that** (1.6) **holds.**

We get from Lemmas 2.4 and 2.5, and Young's inequality that

$$\begin{aligned}
|I_1|&\le\frac{1}{8}\|\nabla J\|_{L^2}^2+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{2}{1+s}}\|\nabla\frac{u^r}{r}\|_{L^2}^{2-\Sigma_{i=1}^3\frac{2}{p_i(1+s)}}\|\Lambda\nabla\frac{u^r}{r}\|_{L^2}^{\Sigma_{i=1}^3\frac{2}{p_i(1+s)}}\\
&\le\frac{1}{8}\|\nabla J\|_{L^2}^2+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{2}{1+s}}\|\Omega\|_{L^2}^{2-\Sigma_{i=1}^3\frac{2}{p_i(1+s)}}\|\nabla\Omega\|_{L^2}^{\Sigma_{i=1}^3\frac{2}{p_i(1+s)}}\\
&\le\frac{1}{8}\left(\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\right)+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}^{\frac{2}{1+s-\Sigma_{i=1}^3\frac{1}{p_i}}}\|\Omega\|_{L^2}^2.
\end{aligned}$$

The term $I_3$ can be bounded by

$$|I_3| \leq \int \frac{|u^\theta|}{r}|\Omega|^2\,dx + \int \frac{|u^\theta|}{r}|J|^2\,dx := I_{31} + I_{32}. \tag{3.7}$$

We shall estimate $I_{31}$ and $I_{32}$ in the following two cases:
**Case 1.** $0 \leq s < 1$

Lemma 2.1 yields that

$$\begin{aligned}|I_{31}| = |\int \frac{|u^\theta|}{r^s}\frac{|\Omega|^2}{r^{1-s}}\,dx| &\leq C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\frac{\Omega^2}{r^{1-s}}\|_{L^{\frac{p_1}{p_1-1},1}_{x_1}L^{\frac{p_2}{p_2-1},1}_{x_2}L^{\frac{p_3}{p_3-1},1}_{x_3}(\mathbb{R}^3)}\\ &\leq C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\frac{\Omega}{r^{\frac{1-s}{2}}}\|^2_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)}.\end{aligned}$$

Due to $0 < \frac{1-s}{2} < \frac{p_1-1}{p_1}, \frac{p_2-1}{p_2}$, we invoke Lemma 2.3 with $k_1 = 2$, $k_2 = 1$, $i = 1$ to get

$$\|\frac{\Omega}{r^{\frac{1-s}{2}}}\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)} \leq C\|\Lambda^{\frac{1-s}{2}}_{x_1,x_2}\Omega\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)}.$$

Noting that $0 \leq \sum_{i=1}^3 \frac{1}{2p_i} + \frac{1-s}{2} < 1$ when $\frac{3}{1+s} < p_i$, Lemmas 2.2 and 2.4 allow us to conclude that

$$\|\Lambda^{\frac{1-s}{2}}_{x_1,x_2}\Omega\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)} \leq C\|\Lambda^{\sum_{i=1}^3\frac{1}{2p_i}+\frac{1-s}{2}}\Omega\|_{L^2} \leq C\|\Omega\|_{L^2}^{\frac{1+s-\sum_{i=1}^3\frac{1}{p_i}}{2}}\|\nabla\Omega\|_{L^2}^{\frac{1-s+\sum_{i=1}^3\frac{1}{p_i}}{2}},$$

which implies

$$|I_{31}| \leq C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\Omega\|_{L^2}^{1+s-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla\Omega\|_{L^2}^{1-s+\sum_{i=1}^3\frac{1}{p_i}}. \tag{3.8}$$

Similarly, we have

$$|I_{32}| \leq C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|J\|_{L^2}^{1+s-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla J\|_{L^2}^{1-s+\sum_{i=1}^3\frac{1}{p_i}}. \tag{3.9}$$

From (3.8), (3.9), and Young's inequality, we obtain

$$\begin{aligned}|I_3| &\leq C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\Omega\|_{L^2}^{1+s-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla\Omega\|_{L^2}^{1-s+\sum_{i=1}^3\frac{1}{p_i}}\\ &\quad+\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|J\|_{L^2}^{1+s-\sum_{i=1}^3\frac{1}{p_i}}\|\nabla J\|_{L^2}^{1-s+\sum_{i=1}^3\frac{1}{p_i}}\\ &\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s-\sum_{i=1}^3\frac{1}{p_i}}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\Omega\|_{L^2}^2+\|J\|_{L^2}^2\right)+\frac{1}{8}\left(\|\nabla\Omega\|_{L^2}^2+\|\nabla J\|_{L^2}^2\right).\end{aligned}$$

Substituting the above estimates into (3.6), we know that, for $0 \leq s < 1$,

$$\begin{aligned}&\frac{d}{dt}\left(\|\Omega(t)\|_{L^2}^2+\|J(t)\|_{L^2}^2\right)+\|\nabla\Omega(t)\|_{L^2}^2+\|\nabla J(t)\|_{L^2}^2\\ &\quad\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s-\sum_{i=1}^3\frac{1}{p_i}}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\Omega\|_{L^2}^2+\|J\|_{L^2}^2\right).\end{aligned}\tag{3.10}$$

**Case 2.** $s \geq 1$

By applying Lemmas 2.1, 2.2, 2.4, and (3.4) we get

$$\begin{aligned}
|I_{31}| &= \int |\frac{u^\theta}{r^s}|^{\frac{2}{1+s}} |ru^\theta|^{\frac{s-1}{s+1}} |\Omega|^2 \, dx \\
&\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|ru^\theta\|^{\frac{s-1}{s+1}}_{L^\infty_{x_1} L^\infty_{x_2} L^\infty_{x_3}(\mathbb{R}^3)} \|\Omega^2\|_{L^{\frac{p_1(1+s)}{p_1(1+s)-2},1}_{x_1} L^{\frac{p_2(1+s)}{p_2(1+s)-2},1}_{x_2} L^{\frac{p_3(1+s)}{p_3(1+s)-2},1}_{x_3}(\mathbb{R}^3)} \\
&\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|\Gamma_0\|^{\frac{s-1}{s+1}}_{L^\infty_{x_1} L^\infty_{x_2} L^\infty_{x_3}(\mathbb{R}^3)} \|\Omega\|^2_{L^{\frac{2p_1(1+s)}{p_1(1+s)-2},2}_{x_1} L^{\frac{2p_2(1+s)}{p_2(1+s)-2},2}_{x_2} L^{\frac{2p_3(1+s)}{p_3(1+s)-2},2}_{x_3}(\mathbb{R}^3)} \\
&\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|\Omega\|^{2-\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \|\nabla\Omega\|^{\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2}.
\end{aligned}$$

Along the same line as the proof of $I_{31}$, we infer that

$$|I_{32}| \leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|J\|^{2-\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \|\nabla J\|^{\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2}.$$

Collecting all estimates above, we conclude from Young's inequality that

$$\begin{aligned}
|I_3| &\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|\Omega\|^{2-\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \|\nabla\Omega\|^{\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \\
&\quad + \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|J\|^{2-\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \|\nabla J\|^{\sum_{i=1}^3 \frac{2}{p_i(1+s)}}_{L^2} \\
&\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s-\sum_{i=1}^3 \frac{1}{p_i}}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \left(\|\Omega\|^2_{L^2} + \|J\|^2_{L^2}\right) + \frac{1}{8}\left(\|\nabla\Omega\|^2_{L^2} + \|\nabla J\|^2_{L^2}\right).
\end{aligned}$$

Putting all the estimates above into (3.6) yields for $s \geq 1$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Omega(t)\|^2_{L^2} + \|J(t)\|^2_{L^2}\right) + \|\nabla\Omega(t)\|^2_{L^2} + \|\nabla J(t)\|^2_{L^2} \\
&\quad \leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s-\sum_{i=1}^3 \frac{1}{p_i}}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \left(\|\Omega\|^2_{L^2} + \|J\|^2_{L^2}\right).
\end{aligned} \tag{3.11}$$

We obtain from (3.10) and (3.11) that, for $0 \leq s < \infty$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Omega(t)\|^2_{L^2} + \|J(t)\|^2_{L^2}\right) + \|\nabla\Omega(t)\|^2_{L^2} + \|\nabla J(t)\|^2_{L^2} \\
&\quad \leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s-\sum_{i=1}^3 \frac{1}{p_i}}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \left(\|\Omega\|^2_{L^2} + \|J\|^2_{L^2}\right),
\end{aligned} \tag{3.12}$$

which along with Gronwall's inequality leads to the desired result.

($ii$) **under the assumption that** (1.7) **holds.**

By virtue of Lemma 2.5 and Young's inequality, we obtain

$$\begin{aligned}
|I_1| &\leq \frac{1}{8}\|\nabla J\|^2_{L^2} + C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|\Lambda\nabla\frac{u^r}{r}\|^2_{L^2} \\
&\leq C \|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1} L^{p_2,\infty}_{x_2} L^{p_3,\infty}_{x_3}(\mathbb{R}^3)} \|\nabla\Omega\|^2_{L^2} + \frac{1}{8}\|\nabla J\|^2_{L^2}.
\end{aligned}$$

For $I_3$, similar to that in (3.7), Young's inequality yields that

$$|I_3| \le \int \frac{|u^\theta|}{r}|\Omega|^2\,dx + \int \frac{|u^\theta|}{r}|J|^2\,dx := I'_{31} + I'_{32}.$$

We will estimate $I'_{31}$ and $I'_{32}$ in the following two cases:
**Case** $1'$**.** $0 \le s < 1$

Using Lemma 2.1, one finds

$$\begin{aligned}|I'_{31}| &\le C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\frac{\Omega^2}{r^{1-s}}\|_{L^{\frac{p_1}{p_1-1},1}_{x_1}L^{\frac{p_2}{p_2-1},1}_{x_2}L^{\frac{p_3}{p_3-1},1}_{x_3}(\mathbb{R}^3)}\\ &\le C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\frac{\Omega}{r^{\frac{1-s}{2}}}\|^2_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)}.\end{aligned}$$

Since $0 < \frac{1-s}{2} < \frac{p_1-1}{p_1}, \frac{p_2-1}{p_2}$, we can apply Lemma 2.3 with $k_1 = 2$, $k_2 = 1$, $i = 1$ to get

$$\|\frac{\Omega}{r^{\frac{1-s}{2}}}\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)} \le C\|\Lambda^{\frac{1-s}{2}}_{x_1,x_2}\Omega\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)}.$$

From Lemma 2.2, we infer that

$$\|\Lambda^{\frac{1-s}{2}}_{x_1,x_2}\Omega\|_{L^{\frac{2p_1}{p_1-1},2}_{x_1}L^{\frac{2p_2}{p_2-1},2}_{x_2}L^{\frac{2p_3}{p_3-1},2}_{x_3}(\mathbb{R}^3)} \le C\|\nabla\Omega\|_{L^2},$$

which implies

$$|I'_{31}| \le C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\nabla\Omega\|^2_{L^2}. \tag{3.13}$$

Analogously to the treatments of (3.13), we get

$$|I'_{32}| \le C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\nabla J\|^2_{L^2}. \tag{3.14}$$

(3.13) and (3.14) lead us to conclude that

$$|I_3| \le C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2} + \|\nabla J\|^2_{L^2}\right).$$

Substituting the above estimates into (3.6), we get that, for $0 \le s < 1$,

$$\begin{aligned}&\frac{d}{dt}\left(\|\Omega(t)\|^2_{L^2} + \|J(t)\|^2_{L^2}\right) + \|\nabla\Omega(t)\|^2_{L^2} + \|\nabla J(t)\|^2_{L^2}\\ \le& C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\nabla\Omega\|^2_{L^2} + C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2} + \|\nabla J\|^2_{L^2}\right).\end{aligned} \tag{3.15}$$

**Case** $2'$**.** $s \ge 1$

Applying Lemmas 2.1 and 2.2, and (3.4), we infer that

$$\begin{aligned}|I_{31}'| &\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|ru^\theta\|^{\frac{s-1}{s+1}}_{L^{\infty}_{x_1}L^{\infty}_{x_2}L^{\infty}_{x_3}(\mathbb{R}^3)}\|\Omega\|^2_{L^{\frac{2p_1(1+s)}{p_1(1+s)-2},2}_{x_1}L^{\frac{2p_2(1+s)}{p_2(1+s)-2},2}_{x_2}L^{\frac{2p_3(1+s)}{p_3(1+s)-2},2}_{x_3}(\mathbb{R}^3)}\\ &\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\nabla\Omega\|^2_{L^2}.\end{aligned}$$

Similarly, we obtain

$$|I_{32}'| \leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\|\nabla J\|^2_{L^2}.$$

Thus, we can see that

$$|I_3| \leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2}+\|\nabla J\|^2_{L^2}\right).$$

Plugging the above estimates into (3.6), we know that, for $s \geq 1$,

$$\begin{aligned}&\frac{d}{dt}\left(\|\Omega(t)\|^2_{L^2}+\|J(t)\|^2_{L^2}\right)+\|\nabla\Omega(t)\|^2_{L^2}+\|\nabla J(t)\|^2_{L^2}\\ &\quad\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2}+\|\nabla J\|^2_{L^2}\right).\end{aligned} \tag{3.16}$$

We infer from (3.15) and (3.16) that, for $0 \leq s < \infty$,

$$\begin{aligned}&\frac{d}{dt}\left(\|\Omega(t)\|^2_{L^2}+\|J(t)\|^2_{L^2}\right)+\|\nabla\Omega(t)\|^2_{L^2}+\|\nabla J(t)\|^2_{L^2}\\ &\leq C\|\frac{u^\theta}{r^s}\|^{\frac{2}{1+s}}_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2}+\|\nabla J\|^2_{L^2}\right)+C\|\frac{u^\theta}{r^s}\|_{L^{p_1,\infty}_{x_1}L^{p_2,\infty}_{x_2}L^{p_3,\infty}_{x_3}(\mathbb{R}^3)}\left(\|\nabla\Omega\|^2_{L^2}+\|\nabla J\|^2_{L^2}\right).\end{aligned} \tag{3.17}$$

We choose

$$\epsilon = (4C)^{-max\left\{1,\frac{1+s}{2}\right\}}, \tag{3.18}$$

where $C$ is a sufficiently large constant and $C = C(s, ru_0^\theta)$. Together with Gronwall's inequality, we obtain

$$\sup_{0\leq t\leq T}\left(\|\Omega(t)\|^2_{L^2}+\|J(t)\|^2_{L^2}\right)+\int_0^T\left(\|\nabla\Omega(t)\|^2_{L^2}+\|\nabla J(t)\|^2_{L^2}\right)\,dt \leq C.$$

Thus, we complete the proof of Proposition 3.2. □

Now we are in a position to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* With the help of Lemma 3.1 and Proposition 3.2, we naturally infer that the solution $(u, b)$ can be smoothly extended beyond $T$. □

## Author contributions

Wenjuan Liu and Zhouyu Li: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article.

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