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Research article

Stability analysis of a class of nonlinear magnetic diffusion equations and its fully implicit scheme

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Abstract: We studied a class of nonlinear magnetic diffusion problems with step-function resistivity $\eta(e)$ in electromagnetically driven high-energy-density physics experiments. The stability of the nonlinear magnetic diffusion equation and its fully implicit scheme, based on the step-function resistivity approximation model $\eta_\delta(e)$ with smoothing, were studied. A rigorous theoretical analysis was established for the approximate model of one-dimensional continuous equations using Gronwall's theorem. Following this, the stability of the fully implicit scheme was proved using bootstrapping and other methods. The correctness of the theoretical proof was verified through one-dimensional numerical experiments.

Keywords: nonlinear magnetic diffusion equation; step-function resistivity; stability; implicit finite volume method
Mathematics Subject Classification: 35L03, 35L65, 65M08

1. Introduction

This article investigates the stability of the nonlinear magnetic diffusion equation and its fully implicit discrete scheme for the following equation system in [1]:

$$\begin{cases} \dfrac{\partial}{\partial t}B(x,t) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta(e)}{\mu_0}\dfrac{\partial}{\partial x}B(x,t)\right), \\ \\ \dfrac{\partial}{\partial t}(e(x,t) + \dfrac{1}{2\mu_0}B^2(x,t)) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta(e)\,B(x,t)}{\mu_0^2}\dfrac{\partial}{\partial x}B(x,t)\right), \end{cases} \tag{1.1}$$

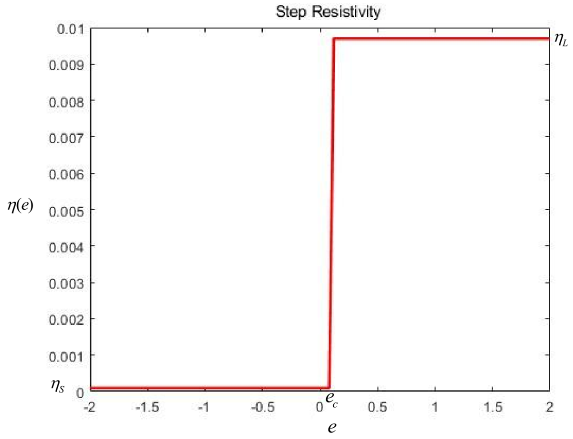
where B is the magnetic field, e is the internal energy density (that is, internal energy per volume), μ_0 is the vacuum permeability constant ($\mu_0 = 4\pi \times 10^{-7}$N/A^2), and $\eta(e)$ is the resistivity in the material. The relationship between $\eta(e)$ and e results in nonlinearity of the diffusion term $\frac{\partial}{\partial x}(\frac{\eta(e)}{\mu_0}\frac{\partial}{\partial x}B(x,t))$ in (1.1).

The resistivity $\eta(e)$ in the equation system (1.1) is a step-function, as shown in Figure 1a:

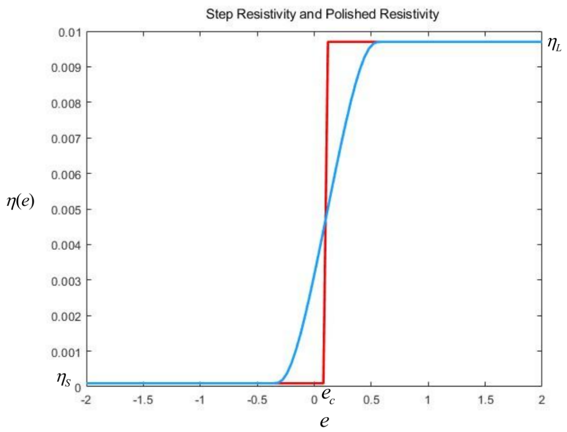
$$\eta(e) = \eta(x,t) = \begin{cases} \eta_S = 9.7 \times 10^{-5}, & e \in [0, e_c], \\ \eta_L = 9.7 \times 10^{-3}, & e \in (e_c, +\infty), \end{cases} \tag{1.2}$$

$e_c = 0.11084958$, representing the critical value of internal energy density.

In electromagnetic loading experiments [2], when the magnetic field outside the metal wall is relatively small (below 10 T), the driving current is also very small, the heating in the metal is weak, the temperature rise is slow, and the change in metal resistivity is not significant. At this point, the diffusion of the magnetic field exhibits behavioral characteristics similar to common diffusion phenomena such as thermal diffusion and concentration diffusion. When the magnetic field outside the metal wall reaches a strong magnetic field level of 100 T, the diffusion of the magnetic field in the metal will exhibit a nonlinear magnetic diffusion wave phenomenon. Compared to ordinary magnetic diffusion in metals, nonlinear magnetic diffusion waves have higher penetration rates and velocities, which can cause rapid magnetic flux leakage and device load erosion in high-energy-density physical experiments. Although nonlinear magnetic diffusion waves in metals with strong magnetic fields were proposed as early as 1970, it was not until after 2000 that phenomena related to nonlinear magnetic diffusion waves gradually attracted people's attention with the widespread development of electromagnetic driven high-energy-density physics experiments. The fundamental reason for the formation of nonlinear magnetic diffusion waves is that during the process of metal temperature rise caused by magnetic diffusion, the metal resistivity also changes accordingly [3]. Before the metal forms a highly conductive plasma, the overall resistivity shows an upward trend. After metal gasification, as the temperature increases, the degree of metal vapor ionization increases, and the resistivity gradually decreases. In [4, 5], authors such as B. Xiao assume that the electrical resistivity of metals undergoes a sudden change of several orders of magnitude after reaching a critical temperature, while the electrical resistivity before and after the sudden change is independent of temperature. They consider an approximate theoretical analytical solution for one-dimensional steep-gradient surface magnetic diffusion waves under the step-function resistivity model. In [1], C. H. Yan et al. designed an explicit finite volume discretization scheme for one-dimensional magnetic field diffusion problems based on the step resistivity model. By relaxing the time step, the formulas for excessive magnetic flux transport and total internal energy transport were truncated when solving strong magnetic diffusion problems. On the basis of using the truncated magnetic flux transport capacity and total internal energy transport capacity, the program can allow for larger time steps without causing oscillation dispersion. In addition, there are also some studies on magnetic diffusion problems, such as [6–8].



(a) step-function resistivity.

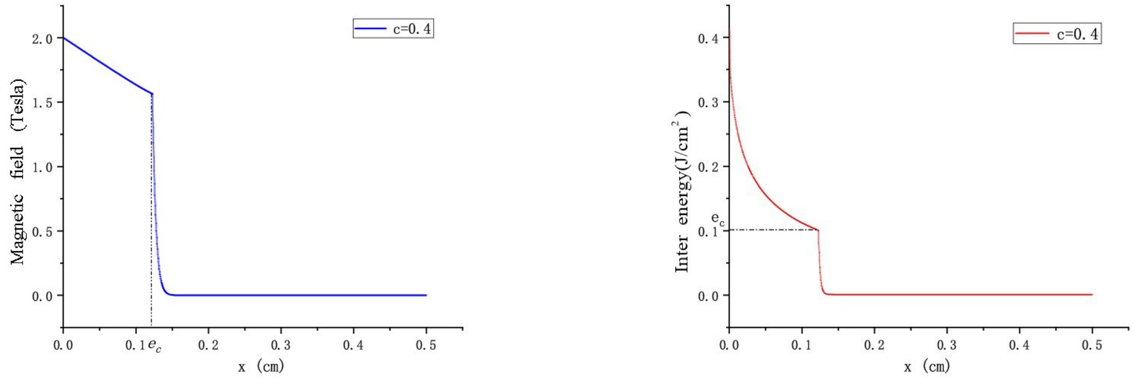


(b) resistivity of smoothed step-function.

Figure 1. Comparison between step-function and smoothed step-function resistivity.

The stability of solutions is an important issue in the study of differential equations. Stability generally refers to the behavior of the solution remaining unchanged or tending to a certain equilibrium state when there is a small disturbance in the initial or boundary conditions of the equation. In [9], Y. L. Zhou et al. studied a class of parallel nature difference schemes for the initial boundary value problem of quasi-linear parabolic systems, and proved the unconditional stability of the constructed parallel nature difference scheme solutions under the discrete $W_2^{(2,1)}$ norm. In [10], author G. W. Yuan proved the uniqueness and stability of the obtained difference solution under the general non-uniform grid difference scheme. In [11, 12], based on the non-uniform grid difference scheme, the authors constructed and developed an implicit discrete scheme that maintains the conservation of the implicit scheme while maintaining the required accuracy and unconditional stability for parallel computing through various methods such as estimation correction, to meet the needs of large-scale numerical solutions to radiation fluid dynamics problems.

The magnetic diffusion problem studied in this article is also based on the step-function resistivity model. We first reproduced the results of equation system (1.1) in [1] (under explicit finite volume discretization scheme):



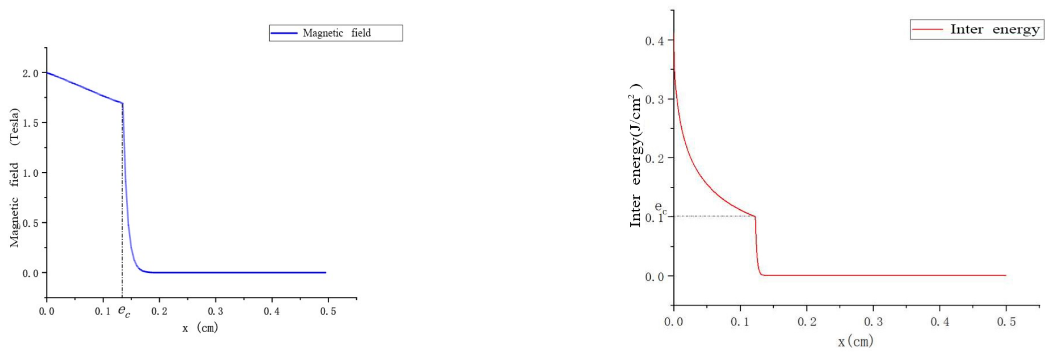
(a) Magnetic field under explicit finite volume with step-function resistivity.

(b) Internal energy density under explicit finite volume with step-function resistivity.

Figure 2. Magnetic field and internal energy density with step-function resistivity.

Note: The c in the above figure is the time step influence factor. In this paper, the solution under the explicit finite volume scheme at $c = 0.4$ is considered as the true solution of the problem.

Next, in the magnetic diffusion equation system (1.1), the smoothed step-function resistivity $\eta_\delta(e)$ is used, where δ is used to describe the distance from the smooth curve inflection point to e_c, as shown in Figure 1b. The experimental results of the implicit finite volume method are as follows:



(a) Magnetic field under implicit finite volume with smoothed step-function resistivity, $\eta_\delta(e), \delta = 0.01$.

(b) Internal energy density under implicit finite volume with smoothed step-function resistivity, $\eta_\delta(e), \delta = 0.01$.

Figure 3. Magnetic field and internal energy density with smoothed step-function resistivity.

The above experiment indicates that by replacing the step-function resistivity $\eta(e)$ in equation system (1.1) and using the smoothed step-function resistivity model $\eta_\delta(e)$, the experimental results in [1] can be well reproduced. Can the modified resistivity maintain the stability of the solution to the magnetic diffusion equation? What are the advantages of the corrected resistivity compared to the step-function resistivity? These are the starting points of this study and will be answered one by one in the following text. Below, we will first theoretically prove that the solution of the one-dimensional nonlinear magnetic diffusion equation and its fully implicit scheme under the smoothed step-function resistivity are stable with initial values. Then, the correctness and stability of the magnetic diffusion model under the smoothed step-function resistivity in the implicit finite volume discrete scheme are further verified through comparative experiments of explicit and implicit schemes [13].

2. Mathematical preliminaries

A measurable function $u[0,T] \to X$ that satisfies the following conditions:

$$(1)\ \|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p dt\right)^{1/p} < \infty, 1 \le p < \infty,$$
$$(2)\ \|u\|_{L^\infty(0,T;X)} := \mathbf{ess}sup_{0<t\le T}\|u(t)\| < \infty,$$

forms the $L^p(0,T;X)$ space.

The polishing function $J(x)$ satisfies

$$J(x) = \begin{cases} ce^{\frac{1}{x^2-1}}, |x| < 1, \\ 0, \qquad |x| \ge 1, \end{cases} \qquad c = \frac{1}{\int_{-1}^{1} e^{\frac{1}{x^2-1}} \mathrm{d}t}, \tag{2.1}$$

and the conclusions are as follows.

Lemma 1 ([14]). *For any $\epsilon > 0$, when taking $J_\epsilon(x) = \frac{1}{\epsilon} J\left(\frac{x}{\epsilon}\right)$, $J(x)$ and $J_\epsilon(x)$ satisfy the following properties:*

(1) $J(x) \in C^\infty(\mathbb{R})$, and when $|x| \ge 1, k \in \mathbb{N}$, $J^{(k)}(x) = 0$;

(2) $\int_{\mathbb{R}} J(x)\, \mathrm{d}x = \int_{\mathbb{R}} J_\epsilon(x)\, \mathrm{d}x = 1$.

Then, the step-function resistivity (1.2) can be smoothed to the following continuous differentiable function:

$$\eta_\delta(e) = \eta_\delta(x,t) = \begin{cases} \eta_\epsilon(e), & e \in [e_c - \epsilon, e_c + \epsilon], \\ \eta(e), & else, \end{cases} \tag{2.2}$$

where,

$$\eta_\epsilon(e) = \eta_\epsilon(x,t) = \int_I \eta(y) J_\epsilon(x-y)\, dy, \quad I = [e_c - \epsilon, e_c + \epsilon]. \tag{2.3}$$

According to [14], it is easy to know that the resistivity (2.3) after the effect of the polishing function (2.1) satisfies the following properties:

$$\eta_\epsilon \in C^\infty(\mathbb{R}), \quad \text{and} \quad \eta'_\epsilon(x) = \int_{\mathbb{R}} \eta(y) J'_\epsilon(x-y)\, dy, \qquad x, y \in I. \tag{2.4}$$

Thereby, $\eta_\delta(e)$ in (2.2) is continuously differentiable across all real number fields. Further, $\eta_\delta(e)$ converges to $\eta(e)$: $\eta(e)$ is integrable on $I_\epsilon = [e_c - \epsilon, e_c + \epsilon]$. By the Lemma 1, for each $x \in I_\epsilon = [e_c - \epsilon, e_c + \epsilon]$, there is $\int_{\mathbb{R}} J_\epsilon(x-y)\, \mathrm{d}y = 1$, and for any $\epsilon > 0$, it is easily available that

$$\begin{aligned} &|\eta_\delta(x) - \eta(x)| \\ =&\left|\int_R \eta(y) J_\epsilon(x-y)\, \mathrm{d}y - \int_R \eta(x) J_\epsilon(x-y)\, \mathrm{d}y\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_R |\eta(y) - \eta(x)| \, J_\epsilon(x-y)\, \mathrm{d}y \\
&\leq C \int_{e_c-\epsilon}^{e_c+\epsilon} |\eta(y) - \eta(x)| \, dy \\
&= C \int_{e_c-\epsilon}^{e_c} |\eta_S - \eta(x)| \, dy + \int_{e_c}^{e_c+\epsilon} |\eta_L - \eta(x)| \, dy \\
&\leq 2C(\eta_L - \eta_S)\epsilon,
\end{aligned} \tag{2.5}
$$

where $C = max|J_\epsilon(x-y)|$, and η_S, η_L are the minimum and maximum values of $\eta(e)$. Thus, it can be concluded that $lim_{\epsilon\to 0}\eta_\delta(x) = \eta(x)$.

Remark: This provides us with a theoretical basis for a stability proof by replacing the step-function resistivity $\eta(e)$ in equation system (1.1) with the smoothed resistivity $\eta_\delta(e)$ (continuous, differentiable).

Lemma 2. *(Young's inequality) If $p > 1, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall a, b \geq 0$, the following inequality holds:*

$$
a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q},
$$

and specifically, when $a = \sqrt{\varepsilon}u, b = \frac{v}{2\sqrt{\varepsilon}}$, Young's inequality can be expressed as

$$
u \cdot v \leq \varepsilon u^2 + \frac{1}{4\varepsilon} v^2. \tag{2.6}
$$

Lemma 3. *(Continuous Gronwall' inequality) Let $g(t)$ and $h(t)$ be non negative integrable functions, and satisfy $f'(t) \leq f(t)g(t) + h(t)$. The following inequality holds:*

$$
f(t) \leq e^{\int_0^t g(s)ds}(f(0) + \int_0^t h(s)ds).
$$

Lemma 4. *(Discrete Gronwall' inequality) [15] Let $\{f^n\}$, $\{g^n\}$, and $\{h^n\}$ be sequences of non-negative functions satisfying, $\frac{f^{n+1}-f^n}{\Delta t} \leq f^{n+1}g^{n+1} + h^{n+1}$, for $\forall \alpha > 1$, such that*

$$
\Delta t \max_{1\leq n\leq N} g^n \leq \frac{\alpha - 1}{\alpha},
$$

where $\Delta t > 0$. Then, the following inequality holds:

$$
f^n \leq Ce^{3\tau \max_{1\leq n\leq N} g^n}(f^0 + \sum_{k=0}^{n} h^k \Delta t), \tag{2.7}
$$

where C and τ are constants that depend on the initial conditions.

Lemma 5. *(Abel's identity) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real or complex functions. If $Q_n = \sum_{i=1}^{n} b_i$, the following identity holds:*

$$
S = \sum_{i=1}^{n} a_i b_i = Q_n a_n - \sum_{i=1}^{n-1} Q_i(a_{i+1} - a_i). \tag{2.8}
$$

Lemma 6. *(Embedding inequality)*

$$\|w(\cdot,s)\|_\infty^2 \lesssim \varepsilon\|w_x(\cdot,s)\|_2^2 + \frac{1}{\varepsilon}\|w(\cdot,s)\|_2^2, \tag{2.9}$$

derived from the Poincaré inequality, $w(\cdot,t) \in H_0^1(0,l)$. Then, the inequality $\|w(\cdot,t)\|_2 \lesssim \|w_x(\cdot,t)\|_2$ holds. Substituting this inequality into (2.9) *yields $\|w(\cdot,t)\|_\infty^2 \lesssim \|w_x(\cdot,t)\|_2^2$.*

Lemma 7. *(Discrete embedding inequality) (space direction)*

$$\|w_h^k\|_\infty^2 \lesssim \varepsilon\|\delta w_h^k\|_2^2 + \frac{1}{\varepsilon}\|w_h^k\|_2^2, \tag{2.10}$$

and similar to Lemma 6, it is easy to derive $\|w_h^k(\cdot,t)\|_\infty^2 \lesssim \|\delta w_h^k(\cdot,t)\|_2^2$.

Remark: The conclusions of Lemmas 6 and 7 hold only in one dimension.

Lemma 8. *(Discrete embedding inequality) (time direction)*

$$\|w_h^k\|_2^2 \leq \sum_{m=1}^{k}(\varepsilon\|\Delta_\tau w_h^m\|_2^2 + \frac{1}{2\varepsilon}\|w_h^m\|^2) + \|w_h^0\|_2^2. \tag{2.11}$$

3. Stability proof of one-dimensional magnetic diffusion equation

3.1. Homogeneous boundary conditions

This section proves that the equation is stable with initial values. Now, we will transform the non-zero boundary value problem into a non-zero initial value problem.

$$\begin{cases} \dfrac{\partial}{\partial t}B(x,t) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta_\epsilon(e)}{\mu_0}\dfrac{\partial}{\partial x}B(x,t)\right), \\ B(0,t) = 0.2, \quad B(0.5,t) = 0, t \in (0,1], \\ B(x,0) = 0, x \in (0,0.5]. \end{cases} \tag{3.1}$$

Let $B(x,t) = u(x,t) + v(x,t)$, and $v(x,t)$ satisfies the boundary conditions in (3.1), that is,

$$v(x,t)|_{x=0} = 0.2, \qquad v(x,t)|_{x=0.5} = 0.$$

Construct auxiliary functions $v(x,t) = 0.2 - \frac{2}{5}x$, $x \in [0,0.5]$, based on boundary conditions. Then, the equation satisfying the definite solution condition in (3.1) with respect to $u(x,t)$ is

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta_\epsilon(e)}{\mu_0}\dfrac{\partial}{\partial x}u(x,t)\right), \\ u(x,0) = -(0.2 - \dfrac{2}{5}x), \quad x \in (0,0.5], \\ u(x,t)|_{x=0} = 0, u(x,t)|_{x=0.5} = 0, \quad t \in (0,1]. \end{cases} \tag{3.2}$$

Remark: The equation to be proved below indicates the stability with respect to initial values, which also implies the stability of the original equation with respect to boundary values.

3.2. *The proof of stability*

Consider the one-dimensional magnetic diffusion equation with Dirichlet boundary as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}B(x,t) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta(e(B))}{\mu_0}\dfrac{\partial}{\partial x}B(x,t)\right), \\ B(0,t) = B(l,t) = 0, t \in (0,T], \\ B(x,0) = \varphi, x \in (0,l], \end{cases} \tag{3.3}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\widetilde{B}(x,t) = \dfrac{\partial}{\partial x}\left(\dfrac{\eta(e(\widetilde{B}))}{\mu_0}\dfrac{\partial}{\partial x}\widetilde{B}(x,t)\right), \\ \widetilde{B}(0,t) = \widetilde{B}(l,t) = 0, t \in (0,T], \\ \widetilde{B}(x,0) = \widetilde{\varphi}, x \in (0,l], \end{cases} \tag{3.4}$$

where, B and $\widetilde{B}$ are the magnetic field, and e and $\widetilde{e}$ are the internal energy density ($e = e(B)$, $\widetilde{e} = e(\widetilde{B})$).

The solutions of Eqs (3.3) and (3.4) belong to $L^\infty(0,T;H_0^1(0,l)) \cap L^2(0,T;H^2(0,l))$, and the following is an inequality for energy estimation:

$$\sup_{0\le t\le T} \|B_x(\cdot,t)\|_2^2 + \int_0^T \|B_{xx}(\cdot,t)\|_2^2 dt \le C\|B_x(\cdot,0)\|_2^2. \tag{3.5}$$

On the premise of not causing misunderstandings, for the convenience of labeling and calculation, in this section, we still use $\eta(e)$ to represent the step-function resistivity after polishing. Let $w(x,t) = B(x,t) - \widetilde{B}(x,t)$, and based on the differentiability of the smoothed resistivity $\eta(e)$, it can be assumed that the derivatives of η and e satisfy the following relationship:

$$|\eta_x + \eta_e| \le c_1, \quad e_B \le c_2. \tag{3.6}$$

Remark: The c_1 in (3.6) depends on the value of ϵ in (2.3).

Subtract the first equation in (3.3) from the first equation in (3.4) to obtain

$$B_t - \widetilde{B}_t = \frac{1}{\mu_0}[\eta(e(B))_x B_x - \eta(e(\widetilde{B}))_x \widetilde{B}_x + \eta(e(B))B_{xx} - \eta(e(\widetilde{B}))\widetilde{B}_{xx}]. \tag{3.7}$$

The above equation can be changed to

$$\mu_0 w_t = \eta(e(B))_x w_x + \widetilde{B}_x(\eta(e(B))_x - \eta(e(\widetilde{B}))_x) + \eta(e(B))w_{xx} + \widetilde{B}_{xx}(\eta(e(B)) - \eta(e(\widetilde{B}))). \tag{3.8}$$

According to the Lagrange mean value theorem, $\eta(e(B)) - \eta(e(\widetilde{B}))$ in (3.8) can be resolved as

$$\eta(e(B)) - \eta(e(\widetilde{B})) = \eta_e(\zeta)(e(B) - e(\widetilde{B})) = \eta_e(\zeta)e'(\xi)(B - \widetilde{B}) = \eta_e(\zeta)e'(\xi)w, \tag{3.9}$$

where, $\eta_e(\zeta)$ represents the first derivative of η with respect to e, ζ is the value between $e(B)$ and $e(\widetilde{B})$, $e'(\xi)$ represents the first derivative of e with respect to B and $\widetilde{B}$, and ξ is the value between B and $\widetilde{B}$.

Substituting (3.9) into (3.8) yields

$$\mu_0 w_t = \eta(e(B))_x w_x + \widetilde{B}_x(\eta(e(B))_x - \eta(e(\widetilde{B}))_x) + \eta(e(B))w_{xx} + \widetilde{B}_{xx}\eta_e(\zeta)e'(\xi)w. \tag{3.10}$$

Multiply w_{xx} on both sides of (3.10) and integrate on $x \in (0, l)$ to obtain

$$\mu_0 \int_0^l w_t w_{xx} dx - \int_0^l \eta(e(B)) w_{xx}^2 dx$$
$$= \int_0^l \eta(e(B))_x w_x w_{xx} dx + \int_0^l \widetilde{B}_x (\eta(e(B))_x - \eta(e(\widetilde{B}))_x) w_{xx} dx + \int_0^l \widetilde{B}_{xx} \eta_e(\zeta) e'(\xi) w w_{xx} dx. \quad (3.11)$$

The first term at the left end of the above equation can be written by the partial integration method as follows:

$$\int_0^l w_t w_{xx} dx = w_t w_x|_0^l - \int_0^l w_{tx} w_x dx = -\frac{1}{2} \int_0^l \frac{d}{dt} w_x^2 dx = -\frac{1}{2} \frac{d}{dt} ||w_x||_2^2. \quad (3.12)$$

By combining (1.2) $\eta(e) \geq \eta_S > 0$ with (3.12), the left end of (3.11) can be simplified as

$$\mu_0 \int_0^l w_t w_{xx} dx - \int_0^l \eta(e(B)) w_{xx}^2 dx \leq -\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 - \eta_S \int_0^l w_{xx}^2 dx$$
$$= -\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 - \eta_S ||w_{xx}^2||_2^2,$$

that is,

$$\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 + \eta_S ||w_{xx}^2||_2^2 \leq -(\mu_0 \int_0^l w_t w_{xx} dx - \int_0^l \eta(e(B)) w_{xx}^2 dx). \quad (3.13)$$

Take absolute values on both sides of (3.13) and obtain from (3.11)

$$\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 + \eta_S ||w_{xx}^2||_2^2$$
$$\leq |\mu_0 \int_0^l w_t w_{xx} dx - \int_0^l \eta(e(B)) w_{xx}^2 dx|$$
$$\leq \int_0^l |\eta(e(B))_x w_x w_{xx}| dx + \int_0^l |\widetilde{B}_x (\eta(e(B))_x - \eta(e(\widetilde{B}))_x) w_{xx}| dx + \int_0^l |\widetilde{B}_{xx} \eta_e(\zeta) e'(\xi) w w_{xx}| dx. \quad (3.14)$$

On the basis of the assumption (3.6), (3.14) can be resolved as

$$\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 + \eta_S ||w_{xx}||_2^2$$
$$\leq c_1 \int_0^l |w_x w_{xx}| dx + 2c_1 \int_0^l |\widetilde{B}_x w_{xx}| dx + c_1 c_2 \int_0^l |\widetilde{B}_{xx} w w_{xx}| dx, \quad (3.15)$$

and according to the Lemma 2 (Young's inequality), (3.15) can be transformed into

$$\frac{\mu_0}{2} \frac{d}{dt} ||w_x||_2^2 + \eta_S ||w_{xx}||_2^2$$
$$\leq c_1 \int_0^l \left(\varepsilon_1 |w_{xx}|^2 + \frac{1}{4\varepsilon_1} |w_x|^2 \right) dx + 2c_1 \int_0^l \left(\varepsilon_2 |w_{xx}|^2 + \frac{1}{4\varepsilon_2} |\widetilde{B}_x|^2 \right) dx$$
$$+ c_1 c_2 \int_0^l \left(\varepsilon_3 |w_{xx}|^2 + \frac{1}{4\varepsilon_3} |w|^2 |\widetilde{B}_{xx}|^2 \right) dx$$

$$\leq c_3\|w_{xx}\|_2^2 + c_4\|w_x\|_2^2 + c_5\|\widetilde{B}_x\|_2^2 dx + c_6\int_0^l |w|^2|\widetilde{B}_{xx}|^2 dx, \tag{3.16}$$

where $c_3 = c_1\varepsilon_1 + 2c_1\varepsilon_2 + c_1c_2\varepsilon_3$.

By the Lemma 6 (embedding inequality)

$$c_6\int_0^l |w|^2|\widetilde{B}_{xx}|^2 dx \leq c_6 \sup_{0\leq x\leq l}|w|^2\|\widetilde{B}_{xx}\|_2^2 \leq c_6\|w\|_\infty^2\|\widetilde{B}_{xx}\|_2^2 \leq c_7\|w_x\|_2^2\|\widetilde{B}_{xx}\|_2^2. \tag{3.17}$$

Remark: The conclusion of (3.17) only holds for one-dimensional cases.

Substitute (3.17) into (3.16), and from the energy estimation inequality (3.5), obtain

$$\begin{aligned}\frac{\mu_0}{2}\frac{d}{dt}\|w_x\|_2^2 + \eta_S\|w_{xx}\|_2^2 &\leq c_3\|w_{xx}\|_2^2 + \|w_x\|_2^2(c_4 + c_7\|\widetilde{B}_{xx}\|_2^2) + c_5\|\widetilde{B}_x\|_2^2 \\ &\leq c_3\|w_{xx}\|_2^2 + c_8\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_5\sup_{0\leq t\leq T}\|\widetilde{B}_x\|_2^2 \\ &\leq c_3\|w_{xx}\|_2^2 + c_8\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_9,\end{aligned} \tag{3.18}$$

where $c_9 = c_5C\|B_x(\cdot,0)\|_2^2$.

According to (3.18)

$$\eta_S\|w_{xx}\|_2^2 \leq c_3\|w_{xx}\|_2^2 + c_8\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_9, \tag{3.19}$$

so

$$c_3\|w_{xx}\|_2^2 \leq \frac{c_3c_8}{\eta_S - c_3}\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + \frac{c_3c_9}{\eta_S - c_3}. \tag{3.20}$$

Remark: $c_3 = c_1\varepsilon_1 + 2c_1\varepsilon_2 + c_1c_2\varepsilon_3$ can ensure that $\eta_S - c_3 > 0$.

Substitute (3.20) into the right-hand end of (3.18), and organize it to obtain

$$\frac{d}{dt}\|w_x\|_2^2 + \frac{2\eta_S}{\mu_0}\|w_{xx}\|_2^2 \leq c_{10}\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_{11}. \tag{3.21}$$

In (3.21), on the one hand

$$\frac{d}{dt}\|w_x\|_2^2 \leq c_{10}\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_{11}. \tag{3.22}$$

In (3.22), take $f'(t) = \frac{d}{dt}\|w_x\|_2^2$, $f(t) = \|w_x\|_2^2$, $g(t) = c_{10}(1 + \|\widetilde{B}_{xx}\|_2^2)$, $h(t) = c_{11}$, by using the Lemma 3 (continuous Gronwall' inequality), it can be concluded that

$$\|w_x(\cdot,t)\|_2^2 \leq e^{c_{10}\int_0^T(1+\|\widetilde{B}_{xx}\|_2^2)dt}\|w_x(\cdot,0)\|_2^2 = c_{12}\|\varphi_x - \widetilde{\varphi}_x\|_2^2 + c_{13}. \tag{3.23}$$

On the other hand, as can be seen from (3.21).

$$\frac{2\eta_S}{\mu_0}\|w_{xx}(\cdot,t)\|_2^2 \leq c_{10}\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2) + c_{11}. \tag{3.24}$$

Integrate the two sides of Eq (3.24) in the time direction on $[0,T]$ and obtain from (3.23)

$$\frac{2\eta_S}{\mu_0}\int_0^T \|w_{xx}(\cdot,t)\|_2^2 ds \leq \int_0^T c_{10}\|w_x\|_2^2(1 + \|\widetilde{B}_{xx}\|_2^2)ds + \int_0^T c_{11}dt \leq c_{14}\|\varphi_x - \widetilde{\varphi}_x\|_2^2 + c_{11}T. \tag{3.25}$$

From (3.23) and (3.25), it can be concluded that

$$\|w_x(x,t)\|_2^2 + a\int_0^t \|w_{xx}(x,t)\|_2^2 ds \le C\|\varphi_x - \widetilde{\varphi}_x\|_2^2 + c. \tag{3.26}$$

Thus, the stability of the magnetic diffusion equation is proven.

3.3. Stability of fully implicit scheme for magnetic diffusion equation

The following proves the stability of the fully implicit scheme corresponding to Eq (3.3) or (3.4):

$$\begin{cases} \mu_0 \Delta_\tau B_j^{n+1} = \eta(e(B_j^{n+1}))\delta^2 B_j^{n+1} + \delta\eta(e(B_j^{n+1}))\delta B_j^{n+1}, \\ B_0^n = B_J^n = 0, \\ B_j^0 = \varphi(x_j), \end{cases} \tag{3.27}$$

taking

$$\begin{cases} \mu_0 \Delta_\tau \widetilde{B}_j^{n+1} = \eta(e(\widetilde{B}_j^{n+1}))\delta^2 \widetilde{B}_j^{n+1} + \delta\eta(e(\widetilde{B}_j^{n+1}))\delta \widetilde{B}_j^{n+1}, \\ \widetilde{B}_0^n = \widetilde{B}_J^n = 0, \\ \widetilde{B}_j^0 = \widetilde{\varphi}(x_j). \end{cases} \tag{3.28}$$

The energy estimation in the discrete scheme is

$$\sup_{0\le n\le N} \|\delta B_j^{n+1}\|_2^2 + \sum_{n=0}^{N} \|\delta^2 B_j^{n+1}\|_2^2 dt \le C\|\delta B_j^0\|_2^2. \tag{3.29}$$

Let $w_j^{n+1} = B_j^{n+1} - \widetilde{B}_j^{n+1}$, and subtract the first equation in (3.27) from the first equation in (3.28) to obtain:

$$\mu_0 \Delta_\tau \left(B_j^{n+1} - \widetilde{B}_j^{n+1}\right) = \left(\eta(e(B_j^{n+1}))\delta^2 B_j^{n+1} - \eta(e(\widetilde{B}_j^{n+1}))\delta^2 \widetilde{B}_j^{n+1}\right)$$
$$+\delta\eta(e(B_j^{n+1}))\delta B_j^{n+1} - \delta\eta(e(\widetilde{B}_j^{n+1}))\delta \widetilde{B}_j^{n+1}.$$

After applying the Lagrange mean value theorem to the above equation, the following is obtained:

$$\begin{aligned} &\mu_0 \Delta_\tau w_j^{n+1} - \left(\eta(e(B_j^{n+1}))\delta^2 w_j^{n+1} + \delta^2 \widetilde{B}_j^{n+1} \delta\eta(\zeta_j^{n+1})\delta e(\xi_j^{n+1}) w_j^{n+1}\right) \\ =&\delta\eta(e(B_j^{n+1}))\delta w_j^{n+1} + \delta\widetilde{B}_j^{n+1}(\delta\eta(e(B_j^{n+1})) - \delta\eta(e(\widetilde{B}_j^{n+1}))), \end{aligned} \tag{3.30}$$

where $\delta\eta(\zeta_j^{n+1})$ represents the first derivative of η with respect to e, ζ_j^{n+1} is the value between $e(B_j^{n+1})$ and $e(\widetilde{B}_j^{n+1})$, $\delta e(\xi_j^{n+1})$ represents the first derivative of e with respect to B_j^{n+1} or $\widetilde{B}_j^{n+1}$, and ξ_j^{n+1} is the value between B_j^{n+1} and $\widetilde{B}_j^{n+1}$.

Multiply $\delta^2 w_j^{n+1}$ on both sides of (3.30) , and sum $j = 1, 2, \ldots, J-1$ to obtain

$$\begin{aligned} &\mu_0 \sum_{j=1}^{J-1} \delta^2 w_j^{n+1} \Delta_\tau w_j^{n+1} - \sum_{j=1}^{J-1} \eta(e(B_j^{n+1}))(\delta^2 w_j^{n+1})^2 \\ =&\sum_{j=1}^{J-1} \delta^2 \widetilde{B}_j^{n+1} \delta\eta(\zeta_j^{n+1})\delta e(\xi_j^{n+1}) w_j^{n+1} \delta^2 w_j^{n+1} + \sum_{j=1}^{J-1} \delta\eta(e(B_j^{n+1}))\delta w_j^{n+1} \delta^2 w_j^{n+1} \\ &+ \sum_{j=1}^{J-1} \delta\widetilde{B}_j^{n+1}(\delta\eta(e(B_j^{n+1})) - \delta\eta(e(\widetilde{B}_j^{n+1})))\delta^2 w_j^{n+1}. \end{aligned} \tag{3.31}$$

We can consider the first term in equation (3.31)

$$\sum_{j=1}^{J-1}\delta^2 w_j^{n+1}\Delta_\tau w_j^{n+1}.$$

Let

$$\begin{aligned} a_j &= \frac{w_j^{n+1}-w_{j-1}^{n+1}}{h},\\ Q_j &= \frac{w_j^{n+1}-w_j^{n}}{\Delta t},\\ b_j &= \frac{w_j^{n+1}-w_j^{n}}{\Delta t}-\frac{w_{j-1}^{n+1}-w_{j-1}^{n}}{\Delta t}. \end{aligned} \tag{3.32}$$

Using Lemma 5 (Abel's identity), it can be obtained that

$$\begin{aligned} \sum_{j=1}^{J-1}\delta^2 w_j^{n+1}\Delta_\tau w_j^{n+1} =& \sum_{j=1}^{J-1}\frac{w_j^{n+1}-w_j^{n}}{\Delta t}\frac{(a_{j+1}-a_j)}{h}\\ =&\frac{1}{h}\sum_{j=1}^{J-1}Q_j(a_{j+1}-a_j)\\ =&\frac{1}{h}(a_J Q_J-\sum_{j=1}^{J}a_j b_j)\\ =&-\frac{1}{h}\sum_{j=1}^{J}\left(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}\right)\left(\frac{w_j^{n+1}-w_j^{n}}{\Delta t}-\frac{w_{j-1}^{n+1}-w_{j-1}^{n}}{\Delta t}\right)\\ =&-\frac{1}{\Delta t}\sum_{i=j}^{J}\left(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}\right)\left(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}-\frac{w_j^{n}-w_{j-1}^{n}}{h}\right). \end{aligned} \tag{3.33}$$

According to the inequality

$$u(u-v)=\frac{1}{2}\Big(u^2-u^2+(u-v)^2\Big)\geq\frac{1}{2}\Big(u^2-v^2\Big). \tag{3.34}$$

Let $u=\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}$,$v=\frac{w_j^{n}-w_{j-1}^{n}}{h}$, and from (3.34), (3.33) can be changed to

$$\begin{aligned} &\frac{1}{\Delta t}\sum_{j=1}^{J}\left(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}\right)\left(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h}-\frac{w_j^{n}-w_{j-1}^{n}}{h}\right)\\ \geq&\frac{1}{2}\frac{1}{\Delta t}\sum_{j=1}^{J}\left[(\frac{w_j^{n+1}-w_{j-1}^{n+1}}{h})^2-(\frac{w_j^{n}-w_{j-1}^{n}}{h})^2\right]\\ =&\frac{1}{2\Delta t}(\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^{n}\|_2^2), \end{aligned} \tag{3.35}$$

that is

$$\mu_0 \sum_{j=1}^{J-1} \delta^2 w_j^{n+1} \Delta_\tau w_j^{n+1} \leq -\frac{\mu_0}{2\Delta t}(\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2). \tag{3.36}$$

From $\eta > \eta_S$ and substituting (3.36) into (3.31), it can be concluded that

$$\begin{aligned}
&\frac{\mu_0}{2\Delta t}\left(\left\|\delta w_h^{n+1}\right\|_2^2 - \left\|\delta w_h^n\right\|_2^2\right) + \eta_S \left\|\delta^2 w_h^{n+1}\right\|_2^2 \\
&= \sum_{j=1}^{J-1} |\delta^2 \widetilde{B}_j^{n+1} \delta\eta(\zeta_j^{n+1}) \delta e(\xi_j^{n+1}) w_j^{n+1} \delta^2 w_j^{n+1}| + \sum_{j=1}^{J-1} |\delta\eta(e(B_j^{n+1})) \delta w_j^{n+1} \delta^2 w_j^{n+1}| \\
&+ \sum_{j=1}^{J-1} |\delta \widetilde{B}_j^{n+1} (\delta\eta(e(B_j^{n+1})) - \delta\eta(e(\widetilde{B}_j^{n+1}))) \delta^2 w_j^{n+1}|.
\end{aligned} \tag{3.37}$$

By the assumption (3.6)

$$\begin{aligned}
&\frac{\mu_0}{2\Delta t}\left(\left\|\delta w_h^{n+1}\right\|_2^2 - \left\|\delta w_h^n\right\|_2^2\right) + \eta_S \left\|\delta^2 w_h^{n+1}\right\|_2^2 \\
&\leq c_1 c_2 \sum_{j=1}^{J-1} |\delta^2 \widetilde{B}_j^{n+1} w_j^{n+1} \delta^2 w_j^{n+1}| + c_1 \sum_{j=1}^{J-1} |\delta w_j^{n+1} \delta^2 w_j^{n+1}| + 2c_1 \sum_{j=1}^{J-1} |\delta \widetilde{B}_j^{n+1} \delta^2 w_j^{n+1}|.
\end{aligned} \tag{3.38}$$

Applying Lemma 2 (Young's inequality) to Eq (3.38) yields

$$\begin{aligned}
&\frac{\mu_0}{2\Delta t}\left(\left\|\delta w_h^{n+1}\right\|_2^2 - \left\|\delta w_h^n\right\|_2^2\right) + \eta_S \left\|\delta^2 w_h^{n+1}\right\|_2^2 \\
&\leq c_1 c_2 \left(\varepsilon_1 \|\delta^2 w_h^{n+1}\|_2^2 + \frac{1}{4\varepsilon_1} \|\delta^2 \widetilde{B}_h^{n+1}\|_2^2 \sup_{0 \leq h \leq J-1} |w_h^{n+1}|_2^2\right) + c_1 \left(\varepsilon_2 \|\delta^2 w_h^{n+1}\|_2^2 + \frac{1}{4\varepsilon_2} \|\delta w_h^{n+1}\|_2^2\right) \\
&+ 2c_1 \left(\varepsilon_3 \|\delta^2 w_h^{n+1}\|_2^2 + \frac{1}{4\varepsilon_3} \|\delta \widetilde{B}_h^{n+1}\|_2^2\right).
\end{aligned} \tag{3.39}$$

From the energy estimation inequality (3.29) and Lemma 7 (discrete embedding inequality, space direction), (3.39) can be expressed as

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|\delta w_h^{n+1}\right\|_2^2 - \left\|\delta w_h^n\right\|_2^2\right) + \frac{2\eta_S}{\mu_0} \left\|\delta^2 w_h^{n+1}\right\|_2^2 \\
&\leq c_4 \|\delta^2 w_h^{n+1}\|_2^2 + c_5 \|w_h^{n+1}(\cdot, t)\|_\infty^2 \|\delta^2 \widetilde{B}_h^{n+1}\|_2^2 + c_6 \|\delta w_h^{n+1}\|_2^2 + c_7 \sup_{0 \leq n \leq N} \|\delta B_h^{n+1}\|_2^2 \\
&\leq c_4 \|\delta^2 w_h^{n+1}\|_2^2 + c_5 \|w_h^{n+1}(\cdot, t)\|_\infty^2 \|\delta^2 \widetilde{B}_h^{n+1}\|_2^2 + c_6 \|\delta w_h^{n+1}\|_2^2 + c_7 \sup_{0 \leq n \leq N} \|\delta \widetilde{B}_h^{n+1}\|_2^2.
\end{aligned} \tag{3.40}$$

Let $a = \frac{2\eta_S}{\mu_0}$, and according to (3.40)

$$a \left\|\delta^2 w_h^{n+1}\right\|_2^2 \leq c_4 \|\delta^2 w_h^{n+1}\|_2^2 + c_5 \|w_h^{n+1}(\cdot, t)\|_\infty^2 \|\delta^2 \widetilde{B}_h^{n+1}\|_2^2 + c_6 \|\delta w_h^{n+1}\|_2^2 + c_7 \sup_{0 \leq n \leq N} \|\delta \widetilde{B}_h^{n+1}\|_2^2, \tag{3.41}$$

so

$$c_4 \left\|\delta^2 w_h^{n+1}\right\|_2^2 \leq \frac{c_4 c_5}{a - c_4} \|w_h^{n+1}(\cdot, t)\|_\infty^2 \|\delta^2 \widetilde{B}_h^{n+1}\|_2^2 + \frac{c_4 c_6}{a - c_4} \|\delta w_h^{n+1}\|_2^2 + \frac{c_4 c_7}{a - c_4} \sup_{0 \leq n \leq N} \|\delta \widetilde{B}_h^{n+1}\|_2^2. \tag{3.42}$$

Substituting (3.42) into the right-hand side of (3.40) yields

$$\frac{1}{\Delta t}\left(\left\|\delta w_h^{n+1}\right\|_2^2-\left\|\delta w_h^{n}\right\|_2^2\right)+a\left\|\delta^2 w_h^{n+1}\right\|_2^2\leq c_9\|\delta w_h^{n+1}\|_2^2+c_{10}\|w_h^{n+1}(\cdot,t)\|_\infty^2\|\delta^2\widetilde{B}_h^{n+1}\|_2^2+c_8. \tag{3.43}$$

Summing the two sides of (3.43) with respect to n, from inequality (3.29), it can be concluded that:

$$\begin{aligned}&\left\|\delta w_h^{n+1}\right\|_2^2+a\sum_{k=0}^{n}\left\|\delta^2 w_h^{k+1}\right\|_2^2\Delta t\\ \leq& c_{11}\sup_{1\leq k\leq n}\|w_h^{k+1}\|_\infty^2\sum_{k=0}^{n}\|\delta^2\widetilde{B}_h^{n+1}\|_2^2\Delta t+c_9\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+\|\delta w_h^0\|_2^2+c_8\\ \leq& c_{12}\sup_{1\leq k\leq n}\|w_h^{k+1}\|_\infty^2+c_9\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+\|\delta w_h^0\|_2^2+c_8.\end{aligned} \tag{3.44}$$

According to Lemma 7, the right-hand side of the (3.44) inequality can be written as

$$\begin{aligned}&\left\|\delta w_h^{n+1}\right\|_2^2+a\sum_{k=0}^{n}\left\|\delta^2 w_h^{n+1}\right\|_2^2\Delta t\\ \leq& c_{12}\sup_{1\leq k\leq n}(\varepsilon\|\delta w_h^{k+1}\|_2^2+\frac{1}{\varepsilon}\|w_h^{k+1}\|_2^2)+c_9\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+\|\delta w_h^0\|_2^2+c_8.\end{aligned} \tag{3.45}$$

From (3.45),

$$\sup_{1\leq k\leq n}\left\|\delta w_h^{k+1}\right\|_2^2\leq c_{12}\sup_{1\leq k\leq n}(\varepsilon\|\delta w_h^{k+1}\|_2^2+\frac{1}{\varepsilon}\|w_h^{k+1}\|_2^2)+c_9\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+\|\delta w_h^0\|_2^2+c_8, \tag{3.46}$$

and then,

$$\begin{aligned}&c_{12}\varepsilon\sup_{1\leq k\leq n}\left\|\delta w_h^{n+1}\right\|_2^2\\ \leq&\sup_{1\leq k\leq n}\frac{(c_{12})^2}{1-c_{12}\varepsilon}\|w_h^{k+1}\|_2^2+\frac{c_9c_{12}\varepsilon}{1-c_{12}\varepsilon}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+\frac{c_{12}\varepsilon}{1-c_{12}\varepsilon}\|\delta w_h^0\|_2^2+\frac{c_8c_{12}\varepsilon}{1-c_{12}\varepsilon}.\end{aligned} \tag{3.47}$$

Substituting (3.47) into the right-hand of (3.45) yields

$$\left\|\delta w_h^{n+1}\right\|_2^2+a\sum_{k=0}^{n}\left\|\delta^2 w_h^{n+1}\right\|_2^2\Delta t\leq c_{13}\sup_{1\leq k\leq n}\|w_h^{k+1}\|_2^2+c_{14}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+c_{15}\|\delta w_h^0\|_2^2+c_{16}. \tag{3.48}$$

On the basis of Lemma 8 (discrete embedding inequality, time direction), $\|w_h^{k+1}\|_2^2$ on the right hand of (3.48) can be expressed as

$$\|w_h^{k+1}\|_2^2\leq\sum_{m=0}^{k+1}(\varepsilon\|\Delta_\tau w_h^m\|_2^2+\frac{1}{2\varepsilon}\|w_h^m\|^2)+\|w_h^0\|_2^2. \tag{3.49}$$

Using the bootstrapping and fully utilizing the properties of the format itself, it can be concluded from (3.30) that

$$\begin{aligned}\Delta_\tau w_j^{n+1} =&\frac{1}{\mu_0}\left(\eta(e(B_j^{n+1}))\delta^2 w_j^{n+1}+\delta^2\widetilde{B}_j^{n+1}\delta\eta(\zeta_j^{n+1})\delta e(\xi_j^{n+1})w_j^{n+1}\right)\\ &+\frac{1}{\mu_0}\left(\delta\eta(e(B_j^{n+1}))\delta w_j^{n+1}+\delta\widetilde{B}_j^{n+1}(\delta\eta(e(B_j^{n+1}))-\delta\eta(e(\widetilde{B}_j^{n+1})))\right).\end{aligned}$$

Substitute the above equation into (3.49), use the energy estimation inequality (3.29) and the assumption condition (3.6), and repeat the above steps to obtain

$$\|w_h^k\|_2^2\le\sum_{m=0}^{k}\varepsilon(c_{15}\|\delta^2 w_h^m\|_2^2+c_{16}\|\delta w_h^m\|_2^2+c_{17}\|w_h^m\|_2^2)+\frac{1}{2\varepsilon}\sum_{m=0}^{k}\|w_h^m\|_2^2+\|w_h^0\|_2^2,\tag{3.50}$$

that is,

$$\|w_h^k\|_2^2\le c_{18}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2+c_{19}\sum_{m=0}^{k}\|\delta w_h^m\|_2^2+c_{20}\sum_{m=0}^{k}\|w_h^m\|_2^2+\|w_h^0\|_2^2.\tag{3.51}$$

For the $\|w_h^m\|_2^2$ in the above equation, the embedding theorem is applied to obtain: $\|w_h^m\|_2^2\le\|w_h^m\|_\infty^2\le\|\delta w_h^m\|_2^2$. Thus(3.51) can be simplified as

$$\|w_h^k\|_2^2\le c_{18}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2+c_{21}\sum_{m=0}^{k}\|\delta w_h^m\|_2^2+\|w_h^0\|_2^2.\tag{3.52}$$

Substituting (3.52) into (3.48) yields

$$\begin{aligned}&\left\|\delta w_h^{n+1}\right\|_2^2+a\sum_{k=0}^{n}\left\|\delta^2 w_h^{k+1}\right\|_2^2\Delta t\\ \le& c_{13}\sup_{1\le k\le n+1}(c_{18}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2+c_{21}\sum_{m=0}^{k}\|\delta w_h^m\|_2^2+\|w_h^0\|_2^2)+c_{14}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+c_{15}\|\delta w_h^0\|_2^2+c_{16}.\end{aligned}\tag{3.53}$$

Similarly, by

$$\begin{aligned}&a\sum_{k=0}^{n}\left\|\delta^2 w_h^{k+1}\right\|_2^2\Delta t\\ \le& c_{13}\sup_{1\le k\le n+1}(c_{18}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2+c_{21}\sum_{m=0}^{k}\|\delta w_h^m\|_2^2+\|w_h^0\|_2^2)+c_{14}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+c_{15}\|\delta w_h^0\|_2^2,\end{aligned}\tag{3.54}$$

it can be inferred that

$$\begin{aligned}&a\sup_{1\le k\le n+1}\sum_{k=0}^{n}\left\|\delta^2 w_h^{k+1}\right\|_2^2\Delta t\\ \le& c_{13}\sup_{1\le k\le n+1}(c_{18}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2+c_{21}\sum_{m=0}^{k}\|\delta w_h^m\|_2^2+\|w_h^0\|_2^2)+c_{14}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t+c_{15}\|\delta w_h^0\|_2^2.\end{aligned}\tag{3.55}$$

Thereby it can be deduced that

$$c_{13}c_{18}\sup_{1\le k\le n+1}\sum_{m=0}^{k}\|\delta^2 w_h^m\|_2^2 \le c_{22}\sum_{m=0}^{k}(\|\delta w_h^m\|_2^2+\|w_h^0\|_2^2)+c_{23}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t + c_{24}\|\delta w_h^0\|_2^2. \tag{3.56}$$

Substituting (3.56) into the right end of (3.53) yields

$$\left\|\delta w_h^{n+1}\right\|_2^2 + a\sum_{k=0}^{n}\left\|\delta^2 w_h^k\right\|_2^2\Delta t \le c_{25}\sum_{k=0}^{n}\|\delta w_h^k\|_2^2\Delta t + c_{26}(\|w_h^0\|_2^2+\|\delta w_h^0\|_2^2). \tag{3.57}$$

By Lemma 2.7 (discrete Gronwall inequality), we have

$$\frac{f^n-f^{n-1}}{\Delta t}=\frac{\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t-\sum_{k=0}^{n-1}\|\delta w_h^{k+1}\|_2^2\Delta t}{\Delta t}=\|\delta w_h^{n+1}\|_2^2, \tag{3.58}$$

and from (3.57),

$$\left\|\delta w_h^{n+1}\right\|_2^2 \le c_{25}\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t + c_{26}(\|w_h^0\|_2^2+\|\delta w_h^0\|_2^2). \tag{3.59}$$

Then, take $f^n=\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\Delta t$, $g^{n+1}=c_{25}$,$h^{n+1}=c_{26}(\|w_h^0\|_2^2+\|\delta w_h^0\|_2^2)$, and we can obtain

$$\sum_{k=0}^{n}\left\|\delta w_h^{k+1}\right\|_2^2\Delta t \le c_{27}(\|w_h^0\|_2^2+\|\delta w_h^0\|_2^2). \tag{3.60}$$

Substituting (3.60) into (3.57) yields

$$\left\|\delta w_h^{n+1}\right\|_2^2 + a\sum_{k=0}^{n}\left\|\delta^2 w_h^{k+1}\right\|_2^2\Delta t \le C(\|w_h^0\|_2^2+\|\delta w_h^0\|_2^2). \tag{3.61}$$

Thus the stability of the discrete scheme is demonstrated.

4. Numerical verification

4.1. Verification of correctness of discrete scheme

Next, we will verify the correctness of the implicit finite volume discretization scheme for the magnetic diffusion equation with constant resistivity, and consider the following magnetic diffusion equation:

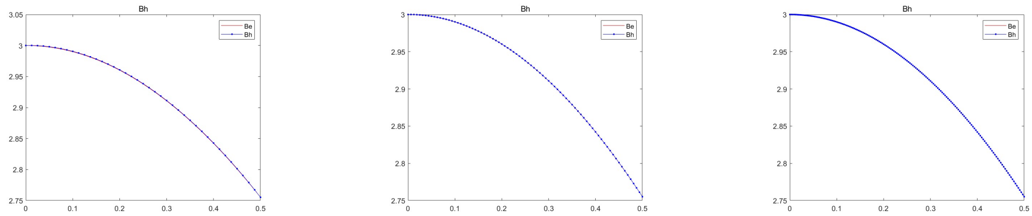
$$\frac{\partial}{\partial t}B(x,t)-\frac{\partial}{\partial x}(\frac{\eta(e)}{\mu_0}\frac{\partial}{\partial x}B(x,t))=2t+\frac{2cos(x)\eta(e)}{\mu_0}. \tag{4.1}$$

The solution interval is $(x,t)\in[0,0.5]\times[0,1]$. Take $\mu_0=4\pi$. The number of mesh segments is, respectively $N=40,80,160$, and the number of nodes is, respectively, $N_1=41,81,161$. The resistivity $\eta=9.7\times 10-3$. The length of the line segment $L=0.5$, the average length of the grid $dx=L/N$,

$T = 1$, and the time step $dt = dx * dx$. It is easy to know that the true solution to this problem is $B(x,t) = 2cos(x) + t^2$. The error used in the experiment is L_2, that is, $Error_{L_2} = \sqrt{\frac{\Sigma_{n=1}^{N_1}(B-B_{exact})^2}{N_1}}$.

Table 1. Error order test (space).

dt	N	$Error_{L^2}$	*Error ratio*
	40	1.44E-04	–
$dx * dx$	80	3.64E-05	3.97
	160	9.12E-06	3.99

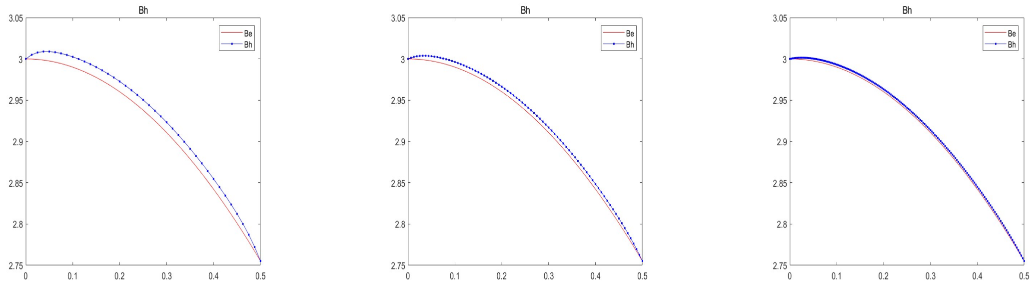


(a) N=40. **(b)** N=80. **(c)** N=160.

Figure 4. Comparison of different space steps.

Table 2. Error Order test (time).

N	dt	$Error_{L^2}$	*Error ratio*
40	0.01	9.20E-03	–
40	0.005	4.60E-03	2.00
40	0.0025	2.30E-03	2.00



(a) dt=0.01. **(b)** dt=0.005. **(c)** dt=0.0025.

Figure 5. Comparison of different time steps.

Conclusions: From the above comparative experiments, it can be seen that when the grid size increases by 2 times with a fixed time scale, the error ratio between the experimental results and the true solution is close to 4. When the grid size is fixed, and the time scale increases by 2 times, the error ratio between the experimental results and the true solution is equal to 2, which conforms to the expected experimental errors of $o(h^2)$ and $o(t)$, thus verifying the correctness of the implicit finite volume discretization scheme in this experiment.

4.2. Stability experiments with different resistivities

In this experiment, the true solution is denoted as B, and the perturbation solution is denoted as B_ε. B^{100} represents the true solution at time step $dt = 0.01$, and B_ε^{100} represents the perturbation solution at time step $dt = 0.01$. The error ratio still uses L_2.

Experiment (1): Step-function resistivity

$$\eta(e) = \eta(x,t) = \begin{cases} \eta_S = 9.7 \times 10^{-5}, & e \in [0, e_c], \\ \eta_L = 9.7 \times 10^{-3}, & e \in (e_c, +\infty). \end{cases}$$

Table 3. Error ratio of magnetic field under step resistivity.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|B^{100} - B_\varepsilon^{100}\|\|$	ratio	$\|\|B^{1000} - B_\varepsilon^{1000}\|\|$	ratio	$\|\|B^{10000} - B_\varepsilon^{10000}\|\|$	ratio
0.1	0.1779	–	0.2364	–	0.2409	–
0.01	0.0191	9.31	0.0309	7.65	0.0328	7.34
0.001	8.00E-04	23.87	0.0049	6.31	0.0032	10.25
0.0001	8.02E-05	9.98	0.0015	3.27	1.77E-04	18.10

Table 4. Error ratio of internal energy density under step resistivity.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|e^{100} - e_\varepsilon^{100}\|\|$	ratio	$\|\|e^{1000} - e_\varepsilon^{1000}\|\|$	ratio	$\|\|e^{10000} - e_\varepsilon^{10000}\|\|$	ratio
0.1	1.96E-02	–	0.0224	–	0.0231	–
0.01	6.60E-03	2.97	0.0039	5.74	0.0033	7.00
0.001	2.10E-03	3.14	8.56E-04	4.55	6.36E-04	5.19
0.0001	7.78E-06	269.95	2.69E-04	3.18	5.33E-05	11.93

Conclusions: Under the same time step dt, when there is a small disturbance in the initial value, the error ratio changes significantly, indicating that the solution of the magnetic diffusion equation under step-function resistivity cannot be stable based on the initial value.

Experiment (2): Constant resistivity $\eta(e) = 9.7e - 3$.

Table 5. Error ratio of magnetic field under constant resistivity $\eta(e) = 9.7e - 3$.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|B^{100} - B_{\varepsilon}^{100}\|\|$	ratio	$\|\|B^{1000} - B_{\varepsilon}^{1000}\|\|$	ratio	$\|\|B^{10000} - B_{\varepsilon}^{10000}\|\|$	ratio
0.1	0.0895	–	0.0895	–	0.0895	–
0.01	0.009	9.94	0.009	9.94	0.009	9.94
0.001	8.95E-04	10.05	8.95E-04	10.05	8.95E-04	10.05
0.0001	8.95E-05	10.00	8.95E-05	10.00	8.95E-05	10.00

Table 6. Error ratio of internal energy density under constant resistivity.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|e^{100} - e_{\varepsilon}^{100}\|\|$	ratio	$\|\|e^{1000} - e_{\varepsilon}^{1000}\|\|$	ratio	$\|\|e^{10000} - e_{\varepsilon}^{10000}\|\|$	ratio
0.1	0.0186	–	0.0191	–	0.0197	–
0.01	0.00185	10.05	0.002	9.55	0.002	9.85
0.001	1.82E-04	10.16	1.9703E-04	10.15	2.0298E-04	9.85
0.0001	1.82E-05	10.00	1.9708E-05	10.00	2.0303E-05	10.00

Experiment (3): Linear resistivity $\eta(e) = 9.7e - 3$.

$$\eta(e) = \frac{\eta_L - \eta_S}{2e_c} e + \eta_S , e \in [0, 2e_c],$$

where, $e_c = 0.11084958$.

Table 7. Error ratio of magnetic field under linear resistivity $\eta(e) = 9.7e - 3$.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|B^{100} - B_{\varepsilon}^{100}\|\|$	ratio	$\|\|B^{1000} - B_{\varepsilon}^{1000}\|\|$	ratio	$\|\|B^{10000} - B_{\varepsilon}^{10000}\|\|$	ratio
0.1	4.37E-04	–	9.80E-05	–	8.08E-05	–
0.01	4.05E-05	10.78	8.68E-06	11.29	7.30E-06	11.06
0.001	4.02E-06	10.09	8.57E-07	10.13	7.22E-07	10.12
0.0001	4.01E-07	10.01	8.56E-08	10.01	7.26E-08	9.94

Table 8. Error ratio of internal energy density under linear resistivity.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|e^{100} - e_{\varepsilon}^{100}\|\|$	ratio	$\|\|e^{1000} - e_{\varepsilon}^{1000}\|\|$	ratio	$\|\|e^{10000} - e_{\varepsilon}^{10000}\|\|$	ratio
0.1	0.1821	–	0.4073	–	0.4729	–
0.01	0.0202	9.01	0.0443	9.19	0.0514	9.20
0.001	0.002	10.10	0.0045	9.84	0.0052	9.88
0.0001	2.04E-04	9.79	4.47E-04	10.07	5.18E-04	10.03

Experiment (4): The step-function resistivity after polishing.

Table 9. Error ratio of magnetic field under the step-function resistivity after polishing.

	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|B^{100} - B^{100}_{\varepsilon}\|\|$	ratio	$\|\|B^{1000} - B^{1000}_{\varepsilon}\|\|$	ratio	$\|\|B^{10000} - B^{10000}_{\varepsilon}\|\|$	ratio
0.1	9.89E-02	–	9.93E-02	–	9.94E-02	–
0.01	9.90E-03	9.99	9.90E-03	10.03	9.98E-03	9.96
0.001	9.89E-04	10.01	9.98E-04	9.92	9.94E-04	10.04
0.0001	9.87E-05	10.02	9.88E-05	10.11	9.87E-05	10.07

Table 10. Error ratio of internal energy density under the step-function resistivity after polishing.

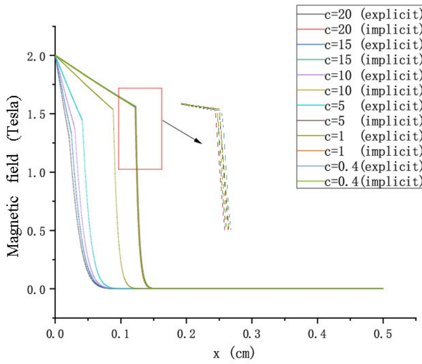
	dt=0.01		dt=0.001		dt=0.0001	
ε	$\|\|e^{100} - e^{100}_{\varepsilon}\|\|$	ratio	$\|\|e^{1000} - e^{1000}_{\varepsilon}\|\|$	ratio	$\|\|e^{10000} - e^{10000}_{\varepsilon}\|\|$	ratio
0.1	0.0012	–	0.01891	–	0.0189	–
0.01	1.19E-04	10.08	0.0019	9.95	0.00198	9.55
0.001	1.19E-05	9.98	1.8903E-04	10.05	1.9998E-04	9.90
0.0001	1.19E-06	10.03	1.8708E-05	10.10	1.9803E-05	10.10

Conclusion: From experiments (2), (3), and (4), it can be seen that under the same time step dt, when there is a small disturbance in the initial value, the error ratio does not change much. Especially, the solution of the magnetic diffusion equation under the smoothed step-function resistivity model has good stability. This is also the advantage of smoothed step-function resistivity $\eta_\delta(e)$ compared to step-function resistivity $\eta(e)$.

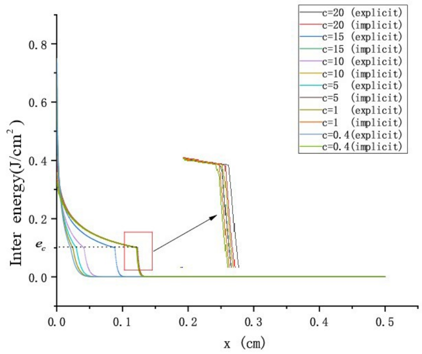
4.3. Comparison experiment of explicit and implicit schemes under step-function resistivity

In the comparison experiment between explicit and implicit schemes, we take $dt = \frac{c\mu_0 * (dx)^2}{\eta_L}$, where, $\mu_0 = 4\pi$, $dx = L/N$, $\eta_L = 100 \times 9.7 \times 10 - 3$. Therefore, c is the factor that affects dt, and the larger c is, the larger the time step dt.

Conclusions: From the comparison experiment in the figure above, it is evident that when we use the curve at $c = 0.4$ as the true solution graph, as the value of c increases (that is, as the time step increases), the explicit solution gradually diverges from the true solution. In contrast, the implicit solutions remain nearly identical to the true solution, with differences only noticeable upon close inspection. This observation further demonstrates the strong stability and weak time step constraints of the fully implicit method. These characteristics particularly underscore the superiority of the fully implicit method, especially when dealing with models exhibiting strong nonlinearity.



(a) Comparison of magnetic field.



(b) Comparison of internal energy density.

Figure 6. Comparison of magnetic field and internal energy density.

Author contributions

Gao Chang: Conceptualization, Writing–original draft, Data curation, Software, Investigation; Chunsheng Feng: Conceptualization, Software, Writing–review and editing, Methodology; Jianmeng He: Conceptualization, Writing–review and editing, Validation, Investigation; Shi Shu: Conceptualization, Supervision, Formal analysis, Methodology, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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