Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 20826–20842.
DOI: 10.3934/math.20241013
Received: 05 May 2024
Revised: 14 June 2024
Accepted: 18 June 2024
Published: 27 June 2024

Research article

Inversion formulas for space-fractional Bessel heat diffusion through Tikhonov regularization

Fethi Bouzeffour*

Department of Mathematics, College of Sciences, King Saud University, P. O Box 2455 Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: fbouzaffour@ksu.edu.sa.

Abstract: This article explores the generalized Gauss-Weierstrass transform associated with the space-fractional Bessel diffusion equation. Explicit inversion formulae for this transform are developed using best approximation methods and reproducing kernel theory. To address the inherent ill-posedness of this transform, Tikhonov regularization is implemented. Furthermore, the convergence rate of the regularized solutions is rigorously established.

Keywords: Weierstrass transform; best approximation; Hankel transform; reproducing kernel; Tikhonov regularization
Mathematics Subject Classification: 44A15, 46E22, 35K05

1. Introduction

In [1], S. Omri and L. T. Rachdi define the Gauss-Weierstrass transform $\mathscr{W}_{\nu,t}$ associated with the Hankel transform as follows:

$$\mathscr{W}_{\nu,t}(f)(x) = \frac{1}{2^{2\nu+1}\Gamma(\mu+1)} \int_0^\infty \frac{e^{-\frac{x^2+y^2}{4t}}}{t^{\nu+1}} \mathscr{J}_\nu(i\frac{xy}{2t}) f(y) y^{2\nu+1}\, dy, \tag{1.1}$$

where $\mathscr{J}_\nu(\cdot)$ is the normalized Bessel function defined in (2.2). This integral transform, which generalizes the usual Weierstrass transform [2–4], is used to solve the heat equation problem:

$$\begin{cases} \partial_t u(x,t) = \mathscr{B}_\nu(u)(x,t), \\ u(x,0) = f(x), \end{cases}$$

where the Bessel differential operator is given by

$$\mathscr{B}_\nu := \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}, \quad \nu \geq -\frac{1}{2}. \tag{1.2}$$

The authors in [1] established practical, real inversion formulas for Hankel-type heat diffusion, building on the ideas of Saitoh, Matsuura, Fujiwara, and Yamada [2–6], and utilizing the theory of reproducing kernels [2–4].

Recently, many researchers have adapted and applied this same method to various types of Gauss-Weierstrass integral transforms associated with several kinds of differential and difference-differential operators. For instance, Soltani pioneered the exploration of L^p-Fourier multipliers for the Dunkl operator on the real line [7], extremal functions on Sobolev-Dunkl spaces [8], multiplier operators and extremal functions related to the dual Dunkl-Sonine operator [9], and extremal functions on Sturm-Liouville hypergroups [10]. More recently, the same authors examined Dunkl-Weinstein multiplier operators [11]. Additional research was conducted by Dziri and Kroumi [12], as well as by Ghobber and Mejjaoli [13]. For further work related to existing results on inverse problems, some important findings can be found in [14–16].

In this work, we consider the space-fractional diffusion equation associated with the Bessel operator, which is given by

$$\begin{cases} \partial_t u(x,t) + (-\mathscr{B}_\nu)^{\alpha/2} u(x,t) = 0, & x \geq 0,\ t > 0, \\ u(x,0) = \phi(x), & t > 0, \end{cases} \tag{1.3}$$

where the parameters ν and α are restricted by the condition $\nu \geq -\frac{1}{2}$, $1 \leq \alpha \leq 2$, and the space-fractional Bessel operator $(-\mathscr{B}_\nu)^{\alpha/2}$, which is defined pointwise by the principal value integral [17],

$$(-\mathscr{B}_\nu)^{\alpha/2}\phi(x) = c_{\alpha,\nu} \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{\phi(x) - \tau_\nu^\xi \phi(\xi)}{\xi^{\alpha+1}}\, d\xi, \tag{1.4}$$

here, the normalization constant $c_{\nu,\alpha}$ is given by

$$c_{\nu,\alpha} = \frac{2^{\alpha+\nu}\Gamma\left(\nu + \frac{\alpha}{2} + 1\right)}{\Gamma(\nu+1)|\Gamma\left(-\frac{\alpha}{2}\right)|}.$$

In this work, we introduce the generalized Gauss-Weierstrass transform associated with the Bessel operator by

$$(\mathscr{W}_{\alpha,\nu,t}\phi)(y) = \int_0^\infty \mathscr{S}^{\alpha,\nu}(x,y,\,t)\phi(x)\sigma_\nu(dx),$$

where $\mathscr{S}^{\alpha,\nu}(x,y,\,t)$ is the fractional heat kernel, which will be defined later. For $\alpha = 2$, this integral transform simplifies to the Gauss-Weierstrass transform defined in (2.3). Thus, it can be considered a one-parameter extension of the transform (2.3). The principal motivation for considering the generalized Gauss-Weierstrass integral transform is that for $\nu = \frac{n}{2} - 1$, it reduces to the ordinary Gauss-Weierstrass transform for radial functions on the Euclidean space $\mathbf{R}^n$. Since the Bessel operator coincides with the radial part of the Laplace operator $\Delta = \sum_{i=1}^m \partial_i^2$, this transform provides a significant extension. For more details, the reader is referred to the paper [18]. By using the properties of the Fourier-Bessel transform $\mathscr{F}_\nu$ and its connection with the $*$-convolution product (see Section 2), we first show that the transform $\mathscr{W}_{\alpha,\nu,t}$ is a one-to-one bounded linear operator from a Sobolev space $\mathcal{H}_\nu^s$ into $L_\nu^2(0,\infty)$. By the same argument as the standard Gauss-Weierstrass transform, we can assume that the operator $\mathscr{W}_{\alpha,\nu,t}^{-1}$ is unbounded or that its range is not closed, which causes the ill-posed problem in solving the operator equation

$$\mathscr{W}_{\alpha,\nu,t}\phi = \psi.$$

Then, for stable reconstruction of ϕ, some regularization techniques are necessary. The Tikhonov regularization techniques are widely applicable (e.g., Bakushinsky and Goncharsky [19], Baumeister [20], Tikhonov and Arsenin [21], Tikhonov et al. [22]). In our case, the Tikhonov regularization can be stated as follows: For given data $\psi \in L^2_\nu(0,\infty)$, we search for a minimizer of a functional given by

$$J_\gamma(\phi) = \frac{1}{2}\|\mathscr{W}_{\alpha,\nu,t}\phi - \psi\|^2_{L^2_\nu(0,\infty)} + \frac{\gamma}{2}\|\phi\|^2_{\mathcal{H}^s_\nu}, \quad \phi \in \mathcal{H}^s_\nu,$$

with a parameter $\gamma > 0$, which is called a regularizing parameter. We show that the above variational problem has a unique solution denoted by $\mathscr{R}_{\gamma,\psi}$ and called the regularized solution; it is also referred to as the extremal solution by Soltani [7]. The following theorem is the main result of the paper, which provides a real inversion of the generalized Gauss-Weierstrass transform.

Theorem 1.1. *Let $s > \nu + 1$. For every $\phi \in \mathcal{H}^s_\nu$ and $\psi = \mathscr{W}_{\alpha,\nu,t}(\phi)$, we have:*

$$\lim_{\gamma \to 0^+} \|\mathscr{R}_{\gamma,\psi} - \phi\|_{\mathcal{H}^{(s)}_\nu} = 0.$$

Moreover, the set $\{\mathscr{R}_{\gamma,\psi}\}_{\gamma>0}$ converges uniformly to ϕ as $\gamma \to 0^+$.

Our paper is organized as follows:

- Section 2 serves as an introductory section that provides an overview of fundamental concepts. Topics covered include the Fourier-Bessel transform, generalized translation, generalized convolution, fractional Bessel operator, and the space-fractional Bessel diffusion equation, setting the stage for understanding subsequent content.
- Section 3 is devoted to introducing the generalized Gauss-Weierstrass transform and establishing its principal properties.
- Section 4 states the main results of the paper and provides their proofs.

2. Preliminaries

Before revealing our main results, it is essential to establish the groundwork by introducing key notations and collecting pertinent facts about the Bessel operator. This section serves as a primer, elucidating the significance of the Fourier-Bessel transform and the Delsarte translation, which will be pivotal for the subsequent analysis.

2.1. Fourier-Bessel transform

The normalized Bessel function is defined as follows:

$$\mathscr{J}_\nu(x) := \Gamma(\nu+1)\,(2/x)^\nu\, J_\nu(x), \quad \nu > -1, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function [23] and $J_\nu(\cdot)$ is the Bessel function of the first kind, see [23, (10.16.9)]. Then

$$\mathscr{J}_\nu(x) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{(\nu+1)_k\, k!}. \tag{2.2}$$

The normalized Bessel function arises as the unique solution to the eigenvalue problem related to the Bessel equation. More precisely, the functions defined as $x \to \mathscr{J}_\nu(\lambda x)$ stand as the unique solution to the eigenvalue problem [23, (10.13.5)]

$$\begin{cases} \mathscr{B}_\nu \phi(x) = -\lambda^2 \phi(x), \\ \phi(0) = 1, \quad \phi'(0) = 0. \end{cases}$$

The function $\mathscr{J}_\nu(\cdot)$ is an entire analytic function with even symmetry. Notably, there are straightforward special cases that hold:

$$\mathscr{J}_{-1/2}(x) = \cos x, \quad \mathscr{J}_{1/2}(x) = \frac{\sin x}{x}.$$

We introduce the following notation:

- $L^p_\nu(0,\infty)$ $(1 \leq p)$ represents the Lebesgue space associated with the measure

$$\sigma_\nu(dx) = \frac{x^{2\nu+1}}{2^\nu \Gamma(\nu+1)}\, dx. \tag{2.3}$$

  The norm $\|\phi\|_{L^p_\nu(0,\infty)}$ is the conventional norm given by

$$\|\phi\|_{L^p_\nu(0,\infty)} = \Big(\int_0^\infty |\phi(x)|^p \,\sigma_\nu(dx) \Big)^{1/p}.$$

- $S_*(\mathbb{R})$ signifies the space of even functions on $\mathbb{R}$ that are infinitely differentiable and decrease rapidly, along with all their derivatives.

For $\nu \geq -1/2$, the Fourier-Bessel transform $\mathscr{F}_\nu \phi$ of $\phi \in L^1_\nu(0,\infty)$ is defined as:

$$\mathscr{F}_\nu \phi(x) := \int_0^\infty \phi(t) \mathscr{J}_\nu(tx) \sigma_\nu(dx), \quad \nu \geq -1/2. \tag{2.4}$$

This integral transform can be extended to establish an isometry of $L^2_\nu(0,\infty)$. For any function ϕ belonging to $L^1_\nu(0,\infty) \cap L^2_\nu(0,\infty)$, the following relationships hold [24, Prop. 5.III.2]

$$\int_0^\infty |\phi(x)|^2 \,\sigma_\nu(dx) = \int_0^\infty |\mathscr{F}_\nu \phi(t)|^2 \,\sigma_\nu(dt). \tag{2.5}$$

Furthermore, its inverse is expressed as:

$$\phi(x) = \int_0^\infty \mathscr{F}_\nu \phi(t)\, \mathscr{J}_\nu(tx)\, \sigma_\nu(dt). \tag{2.6}$$

Moving forward, our focus shifts to the exploration of the generalized translation operator linked to the Bessel operator. This operator is symbolized as τ^x_ν and operates on functions belonging to $L^1_\nu(0,\infty)$ according to the following expression [25, §3.4.1]:

$$\tau^x_\nu \phi(y) = \begin{cases} \int_0^\pi \phi(\sqrt{x^2+y^2+2xy\cos\theta}) \sin^{2\nu}\theta \, d\theta, & \text{if } \nu > -1/2, \\ \\ \frac{1}{2}(\phi(x+y) + \phi(x-y)), & \text{if } \nu = -1/2. \end{cases} \tag{2.7}$$

With the help of this translation operator, one defines the convolution of $\phi \in L^1_\nu(0,\infty)$ and $\psi \in L^p_\nu(0,\infty)$ for $p \in [1,\ \infty)$ as the element $f *_\nu g$ of $L^p_\nu(0,\ \infty)$ given by

$$(\phi *_\nu \psi)(x) := \int_0^\infty (\tau^x_\nu \phi)(y)\,\psi(y)\sigma_\nu(dy), \quad \nu \geq -1/2. \tag{2.8}$$

The following properties are obvious.

- $\mathscr{F}_\nu(\tau^x_\nu \phi)(t) = \mathscr{J}_\nu(xt)\mathscr{F}_\nu \phi(t)$,
- $\mathscr{F}_\nu(\phi *_\nu \psi)(x) = \mathscr{F}_\nu \phi(x) \mathscr{F}_\nu \psi(x)$.

2.2. *Space-fractional diffusion equation*

In this section, we consider the space-fractional Bessel diffusion equation [17, 18]

$$\begin{cases} \partial_t u(x,t) + (-\mathscr{B}_\nu)^{\alpha/2} u(x,t) = 0, & x \geq 0,\ t > 0, \\ u(x,0) = \phi(x), & u(\infty,t) = 0,\ t > 0, \end{cases} \tag{2.9}$$

where the parameters ν and α are restricted by the condition $\nu \geq -\frac{1}{2}$, $1 \leq \alpha \leq 2$, and the space-fractional Bessel operator $(-\mathscr{B}_\nu)^{\alpha/2}$, which is defined pointwise by the principal value integral [17],

$$(-\mathscr{B}_\nu)^{\alpha/2}\phi(x) = c_{\alpha,\nu} \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{\phi(x) - \tau^\xi_\nu \phi(\xi)}{\xi^{\alpha+1}}\, d\xi, \tag{2.10}$$

here, the normalization constant $c_{\nu,\alpha}$ is given by

$$c_{\nu,\alpha} = \frac{2^{\alpha+\nu}\Gamma\left(\nu + \frac{\alpha}{2} + 1\right)}{\Gamma(\nu+1)|\Gamma\left(-\frac{\alpha}{2}\right)|}.$$

Moreover, the Fourier-Bessel transform of the fractional Bessel operator is given by [17]

$$\mathscr{F}_\nu\left((-\mathscr{B}_\nu)^{\alpha/2}\phi\right)(\xi) = \xi^\alpha \mathscr{F}_\nu \phi(\xi). \tag{2.11}$$

Let us denote the Fourier-Bessel transform of a function $u(x,t)$ with respect to x as $\hat{u}(\xi,t)$, where $\xi \geq 0$. Applying the Fourier-Bessel transform to both sides of the equation in (2.9), we obtain:

$$\begin{cases} \partial_t \widehat{u}(\xi,t) = -\xi^\alpha \widehat{u}(\xi,t), \\ \widehat{u}(\xi,t) = \widehat{\phi}(\xi). \end{cases}$$

Then

$$\widehat{u}(\xi,t) = \widehat{\phi}(\xi) e^{-\xi^\alpha t}.$$

Therefore,

$$u(x,t) = (\mathscr{G}^{\alpha,\nu}_t * \phi)(x),$$

where

$$\mathscr{G}^{\alpha,\nu}_t(x) = \mathscr{G}^{\alpha,\nu}(x,t) = \int_0^\infty e^{-\xi^\alpha t} \mathscr{J}_\nu(\xi x)\,\sigma_\nu(d\xi). \tag{2.12}$$

Using the following scaling rules for the Fourier-Bessel transform:

$$\int_0^\infty f(ax)\mathscr{J}_\nu(\lambda x)\sigma_\nu(dx) = \frac{1}{a^{2\nu+2}}\int_0^\infty f(x)\mathscr{J}_\nu(\lambda x/a)\sigma_\nu(dx),\ \ a > 0,$$

we obtain the following scaling property of the kernel $\mathscr{G}^{\alpha,\nu}(x,t)$

$$\mathscr{G}^{\alpha,\nu}(x,t) = t^{-2(\nu+1)/\alpha}\mathscr{G}^{\alpha,\nu}(xt^{-1/\alpha},1), \quad t > 0, \quad x \geq 0.$$

Consequently by introducing the similarity variable x/t^α, we can write

$$\mathscr{G}^{\alpha,\nu}(x,t) = t^{-2(\nu+1)/\alpha}\mathscr{K}_{\alpha,\nu}(xt^{-1/\alpha}),$$

where

$$\mathscr{K}_{\alpha,\nu}(x) = \int_0^\infty e^{-\xi^\alpha}\mathscr{J}_\nu(\xi x)\sigma_\nu(d\xi). \tag{2.13}$$

Particular cases of the density $\mathscr{K}^{\alpha,\nu}$ are the following [26]:

- The density $\mathscr{K}_{2,\nu}(x)$, where $\nu \geq -\frac{1}{2}$, corresponds to the Gaussian density kernel:

$$\mathscr{K}_{2,\nu}(x) = \frac{e^{-\frac{x^2}{4}}}{2^{\nu+1}}. \tag{2.14}$$

- The density $\mathscr{K}_{1,\nu}$, where $\nu \geq -\frac{1}{2}$, corresponds to the Poisson density:

$$\mathscr{K}_{1,\nu}(x) = \frac{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}}\frac{1}{(1+x^2)^{\nu+\frac{3}{2}}}. \tag{2.15}$$

More generally, for $1 < \alpha < 2$, we have [17, Proposition 4.1]:

$$\mathscr{K}_{\alpha,\nu}(x) = \frac{1}{\alpha 2^\nu}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{\Gamma(\frac{2}{\alpha}(n+\nu+1))}{\Gamma(\nu+1+n)}\left(\frac{x^2}{4}\right)^n. \tag{2.16}$$

Definition 2.1. *For $\nu \geq -\frac{1}{2}$ and $0 < \alpha < 2$, the generalized heat kernel is defined as:*

$$\mathscr{G}^{\alpha,\nu}(x,y,t) = \int_0^\infty e^{-t\xi^\alpha}\mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)\sigma_\nu(d\xi), \quad x,y \in [0,\infty). \tag{2.17}$$

Lemma 2.1. *The heat kernel $\mathscr{G}^{\alpha,\nu}(x,y,t)$ possesses the following properties:*

i) $\mathscr{G}^{\alpha,\nu}(x,y,t) = \tau_\nu^y\mathscr{G}^{\alpha,\nu}(x,t)$.
ii) $\mathscr{G}^{\alpha,\nu}(x,y,t) = \mathscr{G}^{\alpha,\nu}(y,x,t)$.
iii) $\mathscr{G}^{\alpha,\nu}(x,y,t) > 0$.
iv) $\|\mathscr{G}^{\alpha,\nu}(x,y,t)\|_{L^1_\nu(0,\infty)} = 1$.
v) $\mathscr{G}^{\alpha,\nu}(x,y,t) = t^{-2(\nu+1)/\alpha}\mathscr{G}^{\alpha,\nu}(xt^{-1/\alpha},yt^{-1/\alpha},1)$.
vi) $\mathscr{F}_\nu(\mathscr{G}^{\alpha,\nu}(\cdot,y,t))(\xi) = e^{-t\xi^\alpha}\mathscr{J}_\nu(y\xi)$.

Proof. The proofs for properties *i*), *ii*), *v*), and *vi*) are straightforward. Property *iii*) follows from [18, Theorem 7] and the positivity of the generalized translation operator τ_ν^y. The proof for property *iv*) is derived by setting $\xi = 0$ in the formula from property *vi*. □

3. The generalized Weierstrass transform

Definition 3.1. *The generalized Weierstrass transform associated with the fractional Bessel operator, denoted as $\mathscr{W}_{\alpha,\nu,t}$, is defined on $L^2(d\sigma_\nu)$ by the following expression:*

$$(\mathscr{W}_{\alpha,\nu,t}\phi)(y) = (\mathscr{S}_t^{\alpha,\nu} *_\nu \phi)(y) = \int_0^\infty \mathscr{S}^{\alpha,\nu}(x, y,\ t)\phi(x)\sigma_\nu(dx),$$

where $\mathscr{S}^{s,\nu}(x, y,\ t)$ is the generalized heat defined in (2.17)*.*

Let $s > \nu + 1$. We define the space $\mathcal{H}_\nu^{(s)}$ as follows [1]:

$$\mathcal{H}_\nu^{(s)} := \{\phi \in L_\nu^2(0, \infty) : (1 + \xi^2)^{s/2}\mathscr{F}_\nu(\phi)(\xi) \in L_\nu^2(0, \infty)\}. \tag{3.1}$$

This space is equipped with an inner product defined by:

$$\langle\phi, \psi\rangle_{\mathcal{H}_\nu^{(s)}} = \int_0^\infty (1 + \xi^2)^s \mathscr{F}_\nu(\phi)(\xi)\mathscr{F}_\nu(\psi)(\xi)\, \sigma_\nu(d\xi),$$

and a norm:

$$\|\phi\|_{\mathcal{H}_\nu^{(s)}} = \sqrt{\langle\phi, \phi\rangle_{\mathcal{H}_\nu^{(s)}}}.$$

The space $\mathcal{H}_\nu^{(s)}$ features a reproducing kernel, which is defined by:

$$\mathscr{K}_s(x, y) = \int_0^\infty \frac{\mathscr{J}_\nu(\xi x)\mathscr{J}_\nu(\xi y)}{(1 + \xi^2)^s}\, \sigma_\nu(d\xi), \quad \text{for } (x, y) \in [0, \infty) \times [0, \infty). \tag{3.2}$$

Additionally, this space satisfies the following inclusions:

$$\mathcal{H}_\nu^{(s)} \subset L_\nu^2(0, \infty), \quad \mathscr{F}_\nu(\mathcal{H}_\nu^{(s)}) \subset L_\nu^1(0, \infty) \cap L_\nu^2(0, \infty).$$

For further details concerning the space $\mathcal{H}_\nu^{(s)}$, readers are referred to the paper [1].

Theorem 3.1. *i) Let $\phi \in C_0(\mathbb{R}) \cap L_\nu^2(0, \infty)$. For $t > 0$ and $x \in [0, \infty)$, the function $\mathscr{W}_{\alpha,\mu,t}\phi(x)$ solves the following heat equation.*

$$\partial_t u(x, t) = -(-\Delta_\nu)^{\gamma/2} u(x, t),$$

with the initial condition

$$\lim_{t\to 0^+} \mathscr{W}_{\alpha,\nu,t}\phi = \phi \quad in \quad L_\nu^2(0, \infty).$$

ii) The integral transform $\mathscr{W}_{\alpha,\nu,t}$, for $t > 0$, is a one-to-one bounded linear operator from $\mathcal{H}_\nu^{(s)}$ into $L_\nu^2(0, \infty)$, and we have:

$$\|\mathscr{W}_{\alpha,\nu,t}\phi\|_{L_\nu^2(0,\infty)} \le \|\phi\|_{\mathcal{H}_\nu^{(s)}}, \quad \phi \in \mathcal{H}_\nu^{(s)}.$$

Proof. The claim *i*) follows from [17, Theorem 4.5]. From Lemma 2.1, for all $\phi \in L_\nu^2(0, \infty)$, we have:

$$\begin{aligned}\|\mathscr{W}_{\alpha,\nu,t}\phi\|_{L_\nu^2(0,\infty)} &= \|(\mathscr{S}_t^{\alpha,\nu} *_\nu \phi)\|_{L_\nu^2(0,\infty)} \\ &\le \|\mathscr{S}_t^{\alpha,\nu}\|_{L_\nu^1(0,\infty)}\|\phi\|_{L_\nu^2(0,\infty)} \\ &= \|\phi\|_{L_\nu^2(0,\infty)} = \|\mathscr{F}_\nu\phi\|_{L_\nu^2(0,\infty)} \le \|\phi\|_{\mathcal{H}_\nu^{(s)}}.\end{aligned}$$

This inequality shows that the transform $\mathscr{W}_{\alpha,\nu,t}$ is indeed bounded. To complete the proof of *ii*), it remains to show that this transform is one-to-one. Let $\phi \in \mathcal{H}_\nu^{(s)}$ such that $\mathscr{W}_{\alpha,\nu,t}\phi = 0$. Then

$$\mathscr{F}_\nu(\mathscr{W}_{\alpha,\nu,t}\phi)(\xi) = e^{-t\xi^\alpha}\mathscr{F}_\nu\phi(\xi) = 0,$$

from the injectivity of the Fourier-Bessel transform, we get $\phi = 0$. This shows that $\mathscr{W}_{\alpha,\nu,t}$ is one-to-one. □

We denote by $\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}$, the space $\mathcal{H}_\nu^{(s)}$ equipped with the inner product

$$\langle\phi|\psi\rangle_{\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}} = \gamma\langle\phi|\psi\rangle_{\mathcal{H}_\nu^{(s)}} + \langle\mathscr{W}_{\alpha,\nu,t}\phi|\mathscr{W}_{\alpha,\nu,t}\psi\rangle_{L_\nu^2(0,\infty)},$$

and the norm

$$\|\phi\|_{\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}} = \left(\gamma\|\phi\|^2_{\mathcal{H}_\nu^{(s)}} + \|\mathscr{W}_{s,t}^\nu\phi\|^2_{L_\nu^2(0,\infty)}\right)^{1/2}.$$

Then, we have the following main result:

Theorem 3.2. *Let ξ, $t > 0$ and $s > \nu + 1$. Then the Hilbert space $\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}$ admits the following reproducing kernel:*

$$\mathscr{K}_{\nu,s,\alpha,\gamma}(x,y) = \int_0^\infty \frac{\mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\sigma_\nu(d\xi),$$

that is

(i) For all $x \in [0,\infty)$, the function $y \mapsto \mathscr{K}_{\nu,s,\alpha,\gamma}(x,y)$ belongs to $\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}$.
(ii) For all $\phi \in \mathcal{H}_{\nu,\alpha,\gamma}^{(s)}$ and any $y \in [0,\infty)$.

$$\langle\phi, \mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot,y)\rangle_{\mathcal{H}_{\nu,\alpha,\gamma}^{(s)}} = \phi(y).$$

Proof. For all $x \in [0,\infty)$, consider the function

$$\xi \mapsto \frac{\mathscr{J}_\nu(\xi x)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}},$$

which belongs to both $L_\nu^1(0,\infty)$ and $L_\nu^2(0,\infty)$. Then, by the Plancherel theorem for the Fourier-Bessel transform, the function

$$\mathscr{K}_{\nu,s,\alpha,\gamma}(x,y) = \mathscr{F}_\nu\left(\frac{\mathscr{J}_\nu(\xi x)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\right)(y), \tag{3.3}$$

is well-defined. Following this,

$$\xi \mapsto (1+\xi^2)^{s/2}\mathscr{F}_\nu(\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot,y))(\xi),$$

is a member of $L_\nu^2(0,\infty)$. This demonstrates that for all $y \geq 0$, the function $\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot,y)$ belongs to $\mathcal{H}_\nu^{(s)}$. This establishes part (i) of the theorem.

Let $\phi \in \mathcal{H}_\nu^{(s)}$ and $y \in [0,\infty)$. By Eq (3.3), we have

$$\langle\phi, \mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot,y)\rangle_{\mathcal{H}_\nu^{(s)}} = \int_0^\infty \frac{(1+\xi^2)^s\mathscr{J}_\nu(\xi y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathscr{F}_\nu\phi(\xi)\,\sigma_\nu(d\xi). \tag{3.4}$$

From the relation

$$\mathscr{W}_{\alpha,\nu,t}\phi = \mathscr{F}_\nu^{-1}\left(e^{-t\xi^\alpha}\mathscr{F}_\nu(\phi)\right),$$

the action of $\mathscr{W}_{\alpha,\nu,t}$ on the kernel $\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y)$ is then:

$$\begin{aligned}\mathscr{W}_{\alpha,\nu,t}(\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y)) &= \mathscr{S}^\nu_{s,t} * \mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y)\\ &= \mathscr{F}_\nu^{-1}\left(\mathscr{F}_\nu(\mathscr{S}^\nu_{s,t}) \cdot \mathscr{F}_\nu(\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y))\right)\\ &= \mathscr{F}_\nu^{-1}\left(\frac{e^{-t\xi^\alpha}\mathscr{J}_\nu(\xi y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\right).\end{aligned}$$

Therefore

$$\langle \mathscr{W}_{\alpha,\nu,t}\phi, \mathscr{W}_{\alpha,\nu,t}\mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y)\rangle_{L^2_\nu(0,\infty)} = \int_0^\infty \frac{e^{-2t\xi^\alpha}\mathscr{J}_\nu(\xi y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\,\sigma_\nu(d\xi). \tag{3.5}$$

Combining Eqs (3.4) and (3.5), we get

$$\begin{aligned}\langle \phi, \mathscr{K}_{\nu,s,\alpha,\gamma}(\cdot, y)\rangle_{\mathcal{H}^{(s)}_{\nu,\alpha,\gamma}} &= \gamma\int_0^\infty \frac{(1+\xi^2)^s\mathscr{J}_\nu(\xi y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathscr{F}_\nu\phi(\xi)\,\sigma_\nu(d\xi)\\ &\quad+ \int_0^\infty \frac{e^{-2t\xi^\alpha}\mathscr{J}_\nu(\xi y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathscr{F}_\nu\phi(\xi)\,\sigma_\nu(d\xi)\\ &= \int_0^\infty \mathscr{J}_\nu(\xi y)\mathscr{F}_\nu\phi(\xi)\,\sigma_\nu(d\xi)\\ &= \phi(y).\end{aligned}$$

This confirms the reproducing property (ii). □

4. Tikhonov regularized method

We now consider the variational functional associated with the generalized Weierstrass integral transform $\mathscr{W}_{\alpha,\nu,t}$, defined as

$$J_\gamma(\phi) = \frac{1}{2}\|\mathscr{W}_{\alpha,\nu,t}\phi - \psi\|^2_{L^2_\nu(0,\infty)} + \frac{\gamma}{2}\|\phi\|^2_{\mathcal{H}^{(s)}_\nu}, \quad \phi \in \mathcal{H}^{(s)}_\nu. \tag{4.1}$$

For $\gamma > 0$, the functional J_γ is strictly convex and $J_\gamma(\phi) \geq \frac{\gamma}{2}\|\phi\|_\nu$. Hence, J_γ has a unique minimizer, which can be characterized by the first-order condition

$$\langle J'_\gamma(\phi), \varphi\rangle = 0, \text{ for all } \varphi \in \mathcal{H}^{(s)}_\nu, \tag{4.2}$$

where $J'_\gamma(\phi)$ is the Fréchet differential of J_γ.

We denote by $\mathscr{R}_{\gamma,\psi}$ the regularized solution of the Eq (4.2), that is,

$$\mathscr{R}_{\gamma,\psi} = \min_{\phi\in\mathcal{H}^s_\nu} J_\gamma(\phi). \tag{4.3}$$

The following theorem is our second main result.

Theorem 4.1. *For $\nu \geq -1/2$, $\gamma > 0$, and $\psi \in L^2_\nu(0, \infty)$. Then there is a unique function $\mathscr{R}_{\gamma,\psi} \in \mathcal{H}^{(s)}_\nu$, where the infimum of the functional J_γ, defined by*

$$J_\gamma(\phi) = \frac{1}{2}\|\mathscr{W}_{\alpha,\nu,t}\phi - \psi\|^2_{L^2_\nu(0,\infty)} + \frac{\gamma}{2}\|\phi\|^2_{\mathcal{H}^{(s)}_\nu}, \quad \phi \in \mathcal{H}^s_\nu, \tag{4.4}$$

is attained. Furthermore, the regularized function $\mathscr{R}_{\gamma,\psi}$ is given by

$$\mathscr{R}_{\gamma,\psi}(x) = \int_0^\infty \mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\psi(y)\sigma_\nu(dy), \tag{4.5}$$

where

$$\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y) = \int_0^\infty \frac{e^{-t\xi^\alpha}\mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\sigma_\nu(d\xi).$$

Proof. Observe that

$$\begin{aligned} J_\gamma(\phi + \varepsilon\Delta\phi) - J_\gamma(\phi) =& \frac{1}{2}\Big\{\|\mathscr{W}_{\alpha,\nu,t}(\phi + \varepsilon\Delta\phi) - \psi\|^2_{L^2_\nu(0,\infty)} - \|\mathscr{W}_{\alpha,\nu,t}\phi - \psi\|^2_{L^2_\nu(0,\infty)}\Big\} \\ &+ \frac{\gamma}{2}\Big\{\|\phi + \varepsilon\Delta\phi\|^2_{\mathcal{H}^{(s)}_\nu} - \|\phi\|^2_{\mathcal{H}^{(s)}_\nu}\Big\}, \end{aligned}$$

where $\Delta\phi$ denotes the increment.

Since,

$$(\mathscr{F}_\nu\mathscr{W}_{\alpha,\nu,t}\phi)(\xi) = e^{-t\xi^\alpha}(\mathscr{F}_\nu\phi)(\xi). \tag{4.6}$$

Taking into account Eq (4.6) and using the Plancheral formula for the Fourier-Bessel transform to get

$$\begin{aligned} J_\gamma(\phi + \varepsilon\Delta\phi) - J_\gamma(\phi) =& \frac{1}{2}\Big\{\|e^{-t\xi^\alpha}\mathscr{F}_\nu(\phi + \varepsilon\Delta\phi)(\xi) - \mathscr{F}_\nu\psi(\xi)\|^2_{L^2_\nu(0,\infty)} \\ &- \|e^{-t\xi^\alpha}\mathscr{F}_\nu\phi(\xi) - \mathcal{F}_\nu\psi(\xi)\|^2_{L^2_\nu(0,\infty)}\Big\} \\ &+ \frac{\gamma}{2}\Big\{\|(\phi + \varepsilon\Delta\phi\|^2_{\mathcal{H}^{s,\gamma}_{\nu,\alpha}} - \|\phi\|^2_{\mathcal{H}^{(s)}_\nu}\Big\} \\ &= \varepsilon\Big\{\operatorname{Re}\langle e^{-t\xi^\alpha}\mathscr{F}_\nu\phi(\xi) - \mathscr{F}_\nu\psi(\xi),\ e^{-t\xi^\alpha}\mathscr{F}_\nu\Delta\phi(\xi)\rangle_{L^2_\nu(0,\infty)} \\ &+ \gamma\operatorname{Re}\langle(1+\xi^2)^s\mathscr{F}_\nu\phi(\xi), \mathscr{F}_\nu\Delta\phi\rangle_{L^2_\nu(0,\infty)}\Big\} + \frac{\varepsilon^2}{2}\Big\{\|e^{-t\xi^\alpha}\mathscr{F}_\nu\Delta\phi(\xi)\|^2_{L^2_\nu(0,\infty)} \\ &+ \gamma\|(1+\xi^2)^s\mathscr{F}_\nu\Delta\phi(\xi)\|^2_{\mathcal{H}^{(s)}_\nu}\Big\}. \end{aligned}$$

Hence, the Fréchet differential of J_γ can be written as

$$\langle J'_\gamma(\phi),\ \Delta\phi\rangle_{L^2_\nu(0,\infty)} = \langle e^{-2t\xi^\alpha}\mathscr{F}_\nu\phi(\xi) - e^{-t\xi^\alpha}\mathscr{F}_\nu\psi(\xi) + \gamma(1+\xi^2)^s\mathscr{F}_\nu\phi,\ \mathscr{F}_\nu\Delta\phi\rangle_{L^2_\nu(0,\infty)}.$$

By the Perseval formula for the Fourier-Bessel transform, it follows that the regularized solution $\mathcal{R}_{\gamma,\psi}(\xi)$ is given by

$$e^{-2t\xi^\alpha}\mathscr{F}_\nu\mathcal{R}_{\gamma,\psi}(\xi) - e^{-t\xi^\alpha}\mathscr{F}_\nu\psi(\xi) + \gamma(1+\xi^2)^s\mathscr{F}_\nu\mathcal{R}_{\gamma,\psi}(\xi) = 0.$$

Therefore

$$\mathscr{F}_\nu\mathcal{R}_{\gamma,\psi}(\xi) = \frac{e^{-t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathscr{F}_\nu\psi(\xi).$$

It is easy to see that

$$\xi \to \frac{e^{-t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} \mathscr{F}_\nu \psi(\xi) \in L^1_\nu(0,\infty) \cap L^2_\nu(0,\infty).$$

By the inversion formula for the Fourier-Bessel transform, we have

$$\mathscr{R}_{\gamma,\psi}(x) = \mathcal{F}_\nu^{-1}\Big(\frac{e^{-t\xi^\alpha}\mathcal{F}_\nu\psi(\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\Big)(x).$$

We have

$$\begin{aligned}
\mathscr{R}_{\gamma,\psi}(x) &= \mathcal{F}_\nu^{-1}\Big(\frac{e^{-t\xi^\alpha}\mathcal{F}_\nu\psi(\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\Big)(x) \\
&= \int_0^\infty \frac{e^{-t\xi^\alpha}\mathcal{F}_\nu\psi(\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} \mathscr{J}_\nu(x\xi)\sigma_\nu(d\xi) \\
&= \int_0^\infty \int_0^\infty \frac{e^{-t\xi^\alpha}\psi(y)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} \mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)\sigma_\nu(dy)\sigma_\nu(d\xi) \\
&= \int_0^\infty \Big(\int_0^\infty \frac{e^{-t\xi^\alpha}\mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\sigma_\nu(d\xi)\Big)\psi(y)\sigma_\nu(dy) \\
&= \int_0^\infty \mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\psi(y)\sigma_\nu(dy),
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y) &= \int_0^\infty \frac{e^{-t\xi^\alpha}\mathscr{J}_\nu(x\xi)\mathscr{J}_\nu(y\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\sigma_\nu(d\xi) \\
&= \mathscr{F}_\nu\Big(\frac{e^{-t\xi^\alpha}\mathscr{J}_\nu(x\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\Big)(y).
\end{aligned}$$

□

In the following theorem, we will provide an error estimate for the inversion formula.

Theorem 4.2. *Let $s > \nu + 1$. For all $\psi_1, \psi_2 \in L^2_\nu(0,\infty)$, the following inequality holds:*

$$\|\mathscr{R}_{\gamma,\psi_1} - \mathscr{R}_{\gamma,\psi_2}\|_{\mathcal{H}^{(s)}_\nu} \le \frac{1}{4\gamma^{1/2}}\|\psi_1 - \psi_2\|_{L^2_\nu(0,\infty)}.$$

Proof. Consider any $\psi_1, \psi_2 \in L^2_\nu(0,\infty)$. The squared norm of the difference between the operators $\mathscr{R}_{\gamma,\psi_1}$ and $\mathscr{R}_{\gamma,\psi_2}$ in the space $\mathcal{H}^{(s)}_\nu$ are given by:

$$\|\mathscr{R}_{\gamma,\psi_1} - \mathscr{R}_{\gamma,\psi_2}\|^2_{\mathcal{H}^{(s)}_\nu} = \int_0^\infty (1+\xi^2)^s |\mathscr{F}_\nu(\mathscr{R}_{\gamma,\psi_1})(\xi) - \mathscr{F}_\nu(\mathscr{R}_{\gamma,\psi_2})(\xi)|^2 \sigma_\nu(d\xi).$$

Using the formula for the Fourier-Bessel transform of the extremal function $\mathscr{R}_{\gamma,\psi_i}$:

$$\mathscr{F}_\nu(\mathscr{R}_{\gamma,\psi_i})(\xi) = \frac{e^{-t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathcal{F}_\nu(\psi_i)(\xi) \quad \text{for } i = 1, 2,$$

we can express the integral as:

$$\|\mathscr{R}_{\gamma,\psi_1} - \mathscr{R}_{\gamma,\psi_2}\|^2_{\mathcal{H}_\nu^{(s)}} = \int_0^\infty \frac{(1+\xi^2)^s e^{-2t\xi^\alpha}}{\left(\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}\right)^2} |\mathscr{F}_\nu(\psi_1)(\xi) - \mathscr{F}_\nu(\psi_2)(\xi)|^2 \sigma_\nu(d\xi).$$

By using the inequality

$$\frac{(1+\xi^2)^s e^{-2t\xi^\alpha}}{\left(\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}\right)^2} \leq \frac{1}{4\gamma},$$

we can further estimate:

$$\begin{aligned}\|\mathscr{R}_{\gamma,\psi_1} - \mathscr{R}_{\gamma,\psi_2}\|^2_{\mathcal{H}_\nu^{(s)}} &\leq \frac{1}{4\gamma} \int_0^\infty |\mathscr{F}_\nu(\psi_1)(\xi) - \mathscr{F}_\nu(\psi_2)(\xi)|^2 \sigma_\nu(d\xi)\\ &= \frac{1}{4\gamma} \|\psi_1 - \psi_2\|^2_{L^2_\nu(0,\infty)}.\end{aligned}$$

This completes the proof. □

Proposition 4.1. *Let $s > \nu + 1$, $\gamma > 0$, and $\psi \in L^2_\nu(0,\infty)$. We have the following estimate:*

$$\int_0^\infty |\mathscr{R}_{\gamma,\psi}(\xi)|^2 \sigma_\nu(d\xi) \leq \frac{a_{\nu,\alpha}}{\gamma} \int_0^\infty e^{\xi^\alpha} |\psi(\xi)|^2 \sigma_\nu(d\xi).$$

where

$$a_{\nu,\alpha} = \frac{\Gamma\left(\frac{2(\nu+1)}{\alpha}\right)\Gamma(s-\nu-1)}{\alpha 2^{2\nu+3}\Gamma(s)\Gamma(\nu+1)}.$$

Proof. From (4.9) and applying the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}|\mathscr{R}_{\gamma,\psi}(\xi)|^2 &\leq \left(\int_0^\infty |\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\psi(y)| \sigma_\nu(dy)\right)^2\\ &\leq \int_0^\infty e^{-y^\alpha/2} \sigma_\nu(dy) \int_0^\infty e^{y^\alpha} |\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)|^2 |\psi(y)|^2 \sigma_\nu(dy).\end{aligned}$$

Integrating over $[0,\infty)$ with respect to the measure $\sigma_\nu(dx)$, we obtain:

$$\begin{aligned}\|\mathscr{R}_{\gamma,\psi}(\xi)\|^2_{L^2_\nu(0,\infty)} &\leq \left(\int_0^\infty |\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\psi(y)| \sigma_\nu(dy)\right)^2\\ &\leq \int_0^\infty e^{-y^\alpha/2} \sigma_\nu(dy) \int_0^\infty e^{y^\alpha} \|\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\|^2_{L^2_\nu(0,\infty)} |\psi(y)|^2 \sigma_\nu(dy).\end{aligned}$$

However,

$$\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y) = \mathscr{F}_\nu\left(\frac{e^{-t\xi^\alpha} \mathscr{J}_\nu(x\xi)}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\right)(y),$$

it follows that

$$\|\mathscr{N}_{\nu,s,\alpha,\gamma}(x,y)\|^2_{L^2_\nu(0,\infty)} = \int_0^\infty \frac{e^{-2t\xi^\alpha} |\mathscr{J}_\nu(x\xi)|^2}{(\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha})^2} \sigma_\nu(d\xi)$$

$$\leq \frac{1}{4\gamma}\int_0^\infty \frac{1}{(1+\xi^2)^s}\sigma_\nu(d\xi).$$

Therefore,

$$\|\mathscr{R}_{\gamma,\psi}(\xi)\|^2_{L^2_\nu(0,\infty)} \leq \int_0^\infty e^{-y^\alpha/2}\sigma_\nu(dy)\frac{1}{4\gamma}\int_0^\infty \frac{1}{(1+\xi^2)^s}\sigma_\nu(d\xi)\int_0^\infty e^{y^\alpha}|\psi(y)|^2\sigma_\nu(dy). \tag{4.7}$$

We complete the proof by using the relation (4.7) and the fact that:

$$\int_0^\infty e^{-y^\alpha}\sigma_\nu(dy) = \frac{\Gamma\left(\frac{2(\nu+1)}{\alpha}\right)}{\alpha 2^\nu \Gamma(\nu+1)},$$

and

$$\int_0^\infty \frac{\sigma_\nu(d\xi)}{(1+\xi^2)^s} = \frac{\Gamma(s-\nu-1)}{2^{\nu+1}\Gamma(s)}.$$

□

Theorem 4.3. *Let $s > \nu + 1$. For every $\phi \in \mathcal{H}^{(s)}_\nu$ and $\psi = \mathscr{W}_{\alpha,\nu,t}(\phi)$, we have:*

$$\lim_{\gamma\to 0^+} \|\mathscr{R}_{\gamma,\psi} - \phi\|_{\mathcal{H}^{(s)}_\nu} = 0. \tag{4.8}$$

Moreover, the set $\{\mathscr{R}_{\gamma,\psi}\}_{\gamma>0}$ converges uniformly to ϕ as $\gamma \to 0^+$.

Proof. Let $\phi \in \mathcal{H}^{(s)}_\nu$ and $\psi = \mathscr{W}_{\alpha,\nu,t}(\phi)$. Utilizing the formula given in Eq (4.6), the Fourier-Bessel transform of the extremal function $\mathscr{R}_{\gamma,\psi}$ takes the form:

$$\begin{aligned}\mathscr{F}_\nu(\mathscr{R}_{\gamma,\psi})(\xi) &= \frac{e^{-t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathcal{F}_\nu(\psi)(\xi)\\ &= \frac{e^{-2t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}}\mathcal{F}_\nu(\phi)(\xi).\end{aligned}$$

We can express the norm of the difference between $\mathscr{R}_{\gamma,\psi}$ and ϕ in $\mathcal{H}^{(s)}_\nu$ as:

$$\begin{aligned}\|\mathscr{R}_{\gamma,\psi} - \phi\|^2_{\mathcal{H}^{(s)}_\nu} &= \int_0^\infty (1+\xi^2)^s \left|\frac{e^{-2t\xi^\alpha}}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} - 1\right|^2 |\mathcal{F}_\nu(\phi)(\xi)|^2\sigma_\nu(d\xi)\\ &= \int_0^\infty \frac{\gamma^2(1+\xi^2)^{3s}}{\left(\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}\right)^2}|\mathcal{F}_\nu(\phi)(\xi)|^2\sigma_\nu(d\xi).\end{aligned}$$

Using the dominated convergence theorem and observing that

$$\frac{\gamma^2(1+\xi^2)^{3s}}{\left(\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}\right)^2}|\mathcal{F}_\nu(\phi)(\xi)|^2 \leq (1+\xi^2)^s|\mathcal{F}_\nu(\phi)(\xi)|^2,$$

and given that $\phi \in \mathcal{H}^{(s)}_\nu$, we deduce that

$$\lim_{\gamma\to 0^+} \|\mathscr{R}_{\gamma,\psi} - \phi\|_{\mathcal{H}^{(s)}_\nu} = 0.$$

The function $\mathscr{F}_\nu(\phi)$ belongs to both $L^1_\nu(0,\infty)$ and $L^2_\nu(0,\infty)$. Applying the inversion formula for the Fourier-Bessel transform, we compute the deviation of the function ϕ under the operator $\mathscr{R}_{\gamma,\psi}$ as follows:

$$(\mathscr{R}_{\gamma,\psi} - \phi)(\xi) = -\int_0^\infty \frac{\gamma(1+\xi^2)^s}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} \mathcal{F}_\nu(\phi)(\xi)\sigma_\nu(\xi)d\xi.$$

Thus, for all $\xi \in [0,+\infty[$, the magnitude of the deviation is bounded by:

$$\left|(\mathscr{R}_{\gamma,\psi} - \phi)(\xi)\right| \leq \int_0^\infty \frac{\gamma(1+\xi^2)^s}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} |\mathcal{F}_\nu(\phi)(\xi)|\sigma_\nu(\xi)d\xi.$$

Employing the dominated convergence theorem and noting that:

$$\frac{\gamma(1+\xi^2)^s}{\gamma(1+\xi^2)^s + e^{-2t\xi^\alpha}} |\mathcal{F}_\nu(\phi)(\xi)| \leq |\mathcal{F}_\nu(\phi)(\xi)|,$$

we deduce that the supremum over all $\xi \geq 0$ of the deviation approaches zero as γ tends towards zero

$$\sup_{\xi\in[0,+\infty[} \left|(\mathscr{R}_{\gamma,\psi} - \phi)(\xi)\right| \to 0, \text{ as } \gamma \to 0.$$

This completes the proof of convergence in the $\mathcal{H}^{(s)}_\nu$ norm and uniform convergence as $\gamma \to 0^+$. □

Example 4.1. *As an illustrative example, consider the fractional heat equation on* $(0,\infty)\times(0,\infty)$

$$\partial_t u(x,t) = -(-\partial_{xx})^{1/2} u(x,t),$$

with the initial condition

$$\lim_{t\searrow 0} \|u(\cdot,t) - \phi\|_{2,\nu} = 0.$$

To apply the Tikhonov regularization method to this fractional heat equation, we consider the integral operator $\mathscr{W}_{1,-1/2,t} \colon \mathcal{H}^{(1)}_{-1/2} \to L^2(0,\infty)$ *defined by*

$$(\mathscr{W}_{1,-1/2,t}\phi)(y) = \frac{1}{\sqrt{8\pi}} \int_0^\infty \frac{\phi(x+y)+\phi(x-y)}{x^2+t^2}\, x\, dx,$$

where the Sobolev space $\mathcal{H}^{(1)}_{-1/2}$ *is realized by the reproducing kernel Hilbert space* $\mathscr{K}_1(x,y)$ *given by*

$$\mathscr{K}_1(x,y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\cos(xu)\cos(yu)}{u^2+1} du = \sqrt{\frac{\pi}{32}} \left(\exp(-|x-y|) + \exp(-x-y)\right),$$

where $x, y \geq 0$.

We now consider the following best approximation problem, that is, the Tikhonov functional. For any $\psi \in L^2(0,\infty)$ *and* $\gamma > 0$*, we aim to solve*

$$\inf_{\phi\in\mathcal{H}^{(1)}_{-1/2}} \left\{ \frac{\gamma}{2}\|\phi\|^2_{\mathcal{H}^{(1)}_{-1/2}} + \frac{1}{2}\|\mathscr{W}_{1,-1/2,t}\phi - \psi\|^2_{L^2(0,\infty)} \right\}.$$

Then for the RKHS $\mathcal{H}^{(s)}_{-1/2,1,\gamma}$ *consisting of all the members of* $\mathcal{H}^{(1)}_{-1/2}$ *with the norm*

$$\|\phi\|_{\mathcal{H}^{(1)}_{-1/2}} = \sqrt{\gamma\|\phi\|^2_{\mathcal{H}^{(1)}_{-1/2}} + \|\mathscr{W}_{1,-1/2,t}\phi\|^2_{L^2(0,\infty)}},$$

the reproducing kernel $\mathscr{N}_{-1/2,1,1,\gamma}(x, y)$ *can be calculated directly using the Fourier integrals as follows:*

$$\mathscr{N}_{-1/2,1,1,\gamma}(x, y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\exp(-tu)\cos(xu)\cos(yu)}{\gamma(1 + u^2) + \exp(-2tu)}\, du.$$

Hence, the unique member of $\mathcal{H}^{(1)}_{-1/2}$ *with the minimum* $\mathcal{H}^{(1)}_{-1/2}$ *norm—the function* $\mathscr{R}_{\gamma,\psi}$ *which attains the infimum—is given by*

$$\mathscr{R}_{\gamma,\psi}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathscr{N}_{-1/2,1,1,\gamma}(x, y)\psi(y)\, dy. \tag{4.9}$$

For $\phi \in \mathcal{H}^{(1)}_{-1/2}$ *and for* $\psi = \mathscr{W}_{1,-1/2,t}\phi$*, we have the formula*

$$\lim_{\gamma\to 0} \mathscr{R}_{\gamma,\psi}(x) = \phi(x),$$

uniformly on $(0, \infty)$.

5. Conclusions

In this paper, we have explored the generalized Gauss-Weierstrass integral transform associated with the Bessel operator, emphasizing its application to space-fractional diffusion equations. This extension is significant because the Bessel operator coincides with the radial part of the Laplace operator, thereby broadening the scope of the classical transform.

We utilized the properties of the Fourier-Bessel transform and its connection with the $*$-convolution product to demonstrate that the transform $\mathscr{W}_{\alpha,\nu,t}$ is a one-to-one bounded linear operator from the Sobolev space $\mathcal{H}^s_\nu$ into $L^2_\nu(0, \infty)$. Given the ill-posed nature of the inverse problem, we applied Tikhonov regularization techniques to achieve stable reconstruction of functions. Our main theorem established the convergence of the regularized solution $\mathscr{R}_{\gamma,\psi}$ to the original function ϕ as the regularization parameter γ approaches zero.

The implications of our findings extend beyond theoretical interest, offering potential applications in solving inverse problems associated with fractional diffusion equations. Future research could further investigate the numerical implementation of the regularized inversion process and explore other types of fractional differential operators within this framework.

Overall, our work provides a robust foundation for the generalized Gauss-Weierstrass transform's utility in fractional calculus and opens avenues for further investigation into its applications in various mathematical contexts.

Use of AI tools declaration

The author declares he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to extend their sincere appreciation to the Researchers Supporting Project number (RSPD2024R974), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. S. Omri, L. T. Rachdi, Weierstrass transform associated with the Hankel operator, *Bull. Math. Anal. Appl.*, **1** (2009), 1–16.
2. S. Saitoh, Best approximation, Tikhonov regularization and reproducing kernels, *Kodai Math. J.*, **28** (2005), 359–367. https://doi.org/10.2996/kmj/1123767016
3. S. Saitoh, Theory of reproducing kernels: Applications to approximate solutions of bounded linear operator equations on Hilbert spaces, *Amer. Math. Soc. Transl. Series 2*, **230** (2010).
4. S. Saitoh, Y. Sawano, Theory of reproducing kernels and applications, *Devel. Math.*, **44** (2016). https://doi.org/10.1007/978-981-10-0530-5
5. T. Matsuura, S. Saitoh, Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley-Wiener spaces, *Appl. Anal.*, **85** (2006), 901–915. https://doi.org/10.1080/00036810600643662
6. T. Matsuura, S. Saitoh, M. Yamada, Representations of inverse functions by the integral transform with the sign kernel, *Frac. Calc. Appl. Anal.*, **2** (2007), 161–168.
7. F. Soltani, L^p-Fourier multipliers for the Dunkl operator on the real line, *J. Funct. Anal.*, **209** (2004), 16–35.
8. F. Soltani, Extremal functions on Sobolev-Dunkl spaces, *Integral Transf. Spec. Funct.*, **24** (2013), 582–595. https://doi.org/10.1080/10652469.2012.725167
9. F. Soltani, Multiplier operators and extremal functions related to the dual Dunkl-Sonine operator, *Acta Math. Sci.*, **33** (2013), 430–442. https://doi.org/10.1016/S0252-9602(13)60010-7
10. F. Soltani, Extremal functions on Sturm-Liouville hypergroups, *Complex Anal. Oper. Theory*, **8** (2014), 311–325. https://doi.org/10.1007/s11785-013-0303-9
11. F. Soltani, I. Maktouf, Dunkl-Weinstein multiplier operators and applications to reproducing kernel theory, *Mediterr. J. Math.*, **21** (2024). https://doi.org/10.1007/s00009-024-02623-2
12. M. Dziri, A. Kroumi, Tikhonov regularisation for the Weierstrass transform associated with the Kontorovich-Lebedev transform, *Complex Vari. Ellip. Equat.*, **68** (2022), 1119–1131. https://doi.org/10.1080/17476933.2022.2038584
13. S. Ghobber, H. Mejjaoli, Reproducing kernel theory associated with the generalized stockwell transform and applications, *Complex Anal. Oper. Theory*, **17** (2023). https://doi.org/10.1007/s11785-023-01407-y

14. H. Liu, M. Xie, B. Pan, N. Li, J. Zhang, M. Lu, et al., In-Situ intercalated pyrolytic Graphene/Serpentine Hybrid as an efficient lubricant additive in paraffin oil, *Colloids Surf. A Phys. Eng. Aspects*, **652** (2022), 129929. https://doi.org/10.1016/j.colsurfa.2022.129929
15. M. Ding, H. Liu, G. H. Zheng, On inverse problems for several coupled PDE systems arising in mathematical biology, *J. Math. Bio.*, **87** (2023). https://doi.org/10.1007/s00285-023-02021-4
16. W. Yin, B. Zhang, P. Meng, L. Zhou, D. Qi, A neural network method for inversion of turbulence strength, *J. Nonlinear Math. Phys.*, **31** (2024), 22. https://doi.org/10.1007/s44198-024-00186-0
17. F. Bouzeffour, M. Garayev, On the fractional Bessel operator, *Integr. Trans. Special Funct.*, 2001, 1–7. https://doi.org/10.1080/10652469.2021.1925268
18. F. Bouzeffour, Continuation of radial positive definite functions and their characterization, *Fractal Fract.*, **7** (2023). https://doi.org/10.3390/fractalfract7080623
19. A. Bakushinsky, A. Goncharsky, *Ill-Posed problems: Theory and applications*, Springer Dordrecht, (1994). https://doi.org/10.1007/978-94-011-1026-6
20. J. Baumeister, *Stable solution of inverse problems*, Teubner Verlag Wiesbaden, 1987. https://doi.org/10.1007/978-3-322-83967-1
21. A. N. Tikhonov, V. Y. Arsenin, *Solutions of Ill-Posed Problems*, Washington: Winston & Sons, 1977.
22. A. N. Tikhonov, A. S. Leonov, A. G. Yagola, *Nonlinear Ill-Posed Problems*, London: Chapman and Hall, 1998.
23. E. T. Whittaker, G. N. Watson, *A course of modern analysis*, Cambridge: Cambridge University Press, 1952.
24. K. Trimeche, *Generalized harmonic analysis and wavelet packets*, CRC Press, 2001. https://doi.org/10.1201/9781482283174
25. W. R. Bloom , H. Heyer, Harmonic analysis of probability measures on hypergroups, *De Gruyter Stu. Math.*, **20** (1995). https://doi.org/10.1017/S0013091500023130
26. E. M. Stein, W. Guido, *Introduction to fourier analysis on euclidean spaces (PMS-32)*, Princeton University Press, 1971. Available from: `http://www.jstor.org/stable/j.ctt1bpm9w6.12`