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Research article

Mean chain transitivity and almost mean shadowing property of iterated function systems

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Abstract: In this paper, we introduce the notions of mean chain transitivity, mean chain mixing, totally mean chain transitivity, and almost mean shadowing property to iterated function systems (*IFS*). We study the interrelations of these notions. We prove that an iterated function system is chain transitive if one of the constituent maps is surjective, and it has almost mean shadowing property.

Keywords: transitivity; shadowing; pseudo-orbits; chains; iterated function systems
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1. Introduction

A classical dynamical system consists of a phase space M together with a unique function g, where, by iterating this function, we obtain the orbits of points. However, we can find many systems with some finite maps rather than a single map that acts on the phase space. Indeed, we can find many natural processes involved with two or more interactions whose evolutions evolve with discrete time [1, 2]. Therefore, there is a need to extend the study of dynamical systems by considering more than one mapping. Mathematicians have studied such systems either as non-autonomous systems or as iterated function systems (*IFS*). Therefore, these systems originate from a common study, specifically, the study of classical dynamical systems. Hence, important concepts in dynamics, including transitivity and shadowing [3], could be extended to *IFS*s.

In a dynamical system, generally, the future state follows from the initial state. Therefore, it is often deterministic. However, they often appear chaotic, i.e., minor changes in the initial state bring dramatically different long-term behavior. Both topological transitivity and shadowing are dynamical properties that are closely related to the chaoticity of dynamical systems. Usually, in chaos, topological transitivity is a part of its definition, or it is implied by it (at least in some spaces), or it implies chaos. Indeed, it is a part of the definition in Devaney's chaos [4], while in Li–Yorke chaos [5], if a function is topologically transitive (*TT*), then it is chaotic, but the converse is not valid. Moreover, a *TT* map g has points that eventually move under iteration from one arbitrary small neighborhood to any other. As a result, one cannot break the corresponding system into a pair of invariant subsystems under g. Recently, mathematicians have studied this property intensively since it is a global characteristic in the dynamical systems theory.

Topological transitivity was introduced to dynamical systems by Birkhoff [6] in the 1920s. The term 'topologically transitive (*TT*)' is not a unified one. Instead, some authors use 'regionally transitive' [7,8], 'nomadic' [9], 'topologically ergodic' [10], cf. [8], and 'topologically indecomposable (or irreducible)' [11]. A mapping g of a dynamical system (M, g) is *TT* if, for any pair of non-empty open sets $W_1, W_2 \subset M$, there is some $k > 0$ such that $g^k(W_1) \cap W_2 \neq \emptyset$. The notion of topological transitivity was introduced to *IFS*s by Bahabadi in [3]. Devi and Mangang [12] have also discussed this notion in *IFS*s by giving several examples, and they also extend the notions of equicontinuity, sensitivity, and distality to *IFS*s. Moreover, Mangang [13] has studied the notions of mean equicontinuity, mean sensitivity, and mean distality of the product dynamical systems.

A natural generalization of topological transitivity is chain transitivity. It connects any two points of the phase space by a chain with any desired error bound. It is an essential notion of a dynamical system. For example, if a dynamical system is chain transitive, then several shadowing properties, including thick shadowing and shadowing, are equivalent [14]. In a dynamical system, there might be a circumstance where for any error bound γ, we could not find a γ-chain but it may be simpler to obtain an η-average (or η-mean) chain with any average (or mean) error bound γ. It leads to the introduction of average (or mean) chain transitivity to dynamical systems.

In dynamical systems theory, our main goal is to study the nature of all its orbits. Likewise, in *IFS*s, we study the orbit behaviors of the system. Yet, in particular cases, it is unlikely to compute the accurate initial value of a point, which gives rise to the approximate values of the orbits. Thus, we obtain pseudo-orbits of the system. The notion of shadowing puts these pseudo-orbits close to the actual orbits of the system. It was introduced independently by Anosov [15] and Bowen [16] in the 1970s. Shadowing plays an essential part in developing the qualitative theory of dynamical systems. In systems with shadowing property, any pseudo-orbit is followed uniformly by a true orbit over an arbitrarily long duration of time. Usually, it is crucial in systems with chaos, where even an arbitrarily small error in the initial position leads to a large divergence of orbits. Moreover, the shadowing lemma in [16] roughly states that shadowing is a common phenomenon in chaotic dynamical systems. In recent years, shadowing has developed intensively and has become a notion of great interest. Many researchers have introduced different aspects of shadowing in dynamical systems, including average shadowing [17], h-shadowing [18], ergodic shadowing [19], thick shadowing [20], and $\underline{d}$-shadowing [20]. Consequently, these aspects of shadowing have also been extended to *IFS*s; for references, one can see, [3, 21–23].

Ruchi Das and Mukta Garg introduced the notions of average (or mean) chain properties and the almost average (or mean) shadowing property to dynamical systems in [24]. Unlike the classical shadowing property, the notion of the almost mean shadowing property deals with pseudo-orbits with very small mean errors. In [25], the authors have also investigated the chaotic behavior of maps with almost average (or mean) shadowing property.

Motivated by this, in this work we wish to study the concepts of mean chain properties and the almost mean shadowing property in *IFS*s. In Section 2, we give some preliminary discussions on dynamical systems and *IFS*s. In Section 3, we introduce the notions of mean chain transitive (*MCT*), mean chain mixing (*MCM*), and totally mean chain transitive (*TMCT*) to *IFS*s and study the relations among them. We also give an example of an *IFS* that is not chain transitive (*CT*) but *MCM* (Example 3.5). In Section 4, we introduce the notion of almost mean shadowing property (*AMSP*) to *IFS*s and study some of its basic properties. We also study the relation between *CT* and *AMSP* in *IFS*s. In particular, in Theorem 4.7, we find that an *IFS* is *CT* if one of the constituent maps is surjective, and it has *AMSP*.

2. Preliminaries

We consider (M, g) to be a dynamical system where, (M, d) is a compact metric space and f is a self-continuous map on M. Put $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}\}$. Then the set $O(s, g) = \{g^n(s) : n \in \mathbb{Z}_+\}$ is said to be the orbit of $s \in M$ under (M, g).

Let (M, g) be a dynamical system. Let $\gamma, \eta > 0$, then

i) A finite sequence $\{s_0, s_1, \ldots, s_n\}$ in M is an η-chain if $d(g(s_i), s_{i+1}) < \eta$, $\forall\ 0 \leq i \leq n-1$. When i is not bounded above, it is called an η-pseudo-orbit.

ii) An η-pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-shadowed by $s \in M$ if $d(g^i(s), s_i) < \gamma$, $\forall\ i \geq 0$.

A dynamical system (M, g) is said

a) To have shadowing property (*SP*) if $\forall\ \gamma > 0$, $\exists\ \eta > 0$ such that every η-pseudo-orbit is γ-shadowed by some point in M.

b) To be chain transitive (*CT*) if $\forall\ \eta > 0$, and for any pair of points $s, t \in M$, $\exists$ an η-chain joining s and t.

c) To be chain mixing (*CM*) if $\forall\ \eta > 0$, and for any pair of points $s, t \in M$, $\exists\ N > 0$, such that $\forall\ n \geq N$, $\exists$ an η-chain joining s and t of length n.

Hutchinson introduced *IFS*s in [26] and were popularized by Barnsley [27]. Moreover, Barnsley and Demko [28] first named the word *IFS*, and it has garnered much attention since then. Let Λ be a non-empty finite set; an *IFS* $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ is a family of continuous mappings $g_\alpha : M \to M$, where $\alpha \in \Lambda$, and (M, d) is a compact metric space. Put $\Lambda^{\mathbb{Z}_+} = \{\langle\alpha_i\rangle : \alpha_i \in \Lambda\ \forall i \in \mathbb{Z}_+\}$. We use the short notation

$$\mathfrak{F}_{\sigma_i} = g_{\alpha_{i-1}} \circ g_{\alpha_{i-2}} \circ \cdots \circ g_{\alpha_1} \circ g_{\alpha_0}.$$

Let $\sigma = \langle\alpha_i\rangle$ be a typical member of $\Lambda^{\mathbb{Z}_+}$. An infinite sequence $\{s_i\}_{i\in\mathbb{Z}_+}$ in M is an orbit of $\mathfrak{F}$ if $\exists$ $\sigma \in \Lambda^{\mathbb{Z}_+}$, such that $s_i = \mathfrak{F}_{\sigma_i}(s_0)$, where $\mathfrak{F}_{\sigma_i}(s_0) = g_{\alpha_{i-1}} \circ g_{\alpha_{i-2}} \circ \cdots \circ g_{\alpha_1} \circ g_{\alpha_0}(s_0)$ and $\mathfrak{F}_{\sigma_0}(s_0) = s_0$. So, for any $\sigma \in \Lambda^{\mathbb{Z}_+}$, we define $O_\sigma(s) = \{\mathfrak{F}_{\sigma_i}(s) : i \in \mathbb{Z}_+\}$ as an orbit of $s \in M$ related to σ. For an *IFS* $\mathfrak{F}$ and for a fixed integer $n > 0$, we define $\Lambda^n = \{(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) : \alpha_i \in \Lambda,\ 0 \leq i \leq n-1\}$; $f_\mu = g_{\alpha_{n-1}} \circ \cdots \circ g_{\alpha_1} \circ g_{\alpha_0}$; and $\mathfrak{F}^n = \{f_\mu \mid \mu \in \Lambda^n\}$.

Bahabadi [3] extended the notions of *SP*, average (or mean) *SP* (*MSP*), *TT*, *CT* and *CM* to *IFS*s. An *IFS* $\mathfrak{F}$ is *TT* if for any pair of non-empty open sets $W_1, W_2 \subset M$, $\exists\ \sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathfrak{F}_{\sigma_k}(W_1) \cap W_2 \neq \emptyset$ for some $k \geq 0$.

Let $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ be an *IFS* and let $\gamma, \eta > 0$, then

i) A finite sequence $\{s_0, s_1, \ldots, s_n\}$ in M is an η-chain if $\exists$ $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ such that $d(g_{\alpha_i}(s_i), s_{i+1}) < \eta$, $\forall$ $0 \le i \le n-1$. When i is not bounded above, it is called an η-pseudo-orbit.

ii) An η-pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-shadowed by $s \in M$ if $\exists$ $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $d(\mathfrak{F}_{\sigma_i}(s), s_i) < \gamma$, $\forall$ $i \ge 0$.

iii) $\{s_i\}_{i\in\mathbb{Z}_+}$ is an η-mean-pseudo-orbit if $\exists$ $N > 0$, and $\exists$ $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that for any $n \ge N$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \eta.$$

iv) An η-mean-pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-mean shadowed by $s \in M$ if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\mathfrak{F}_{\sigma_i}(s), s_i) < \gamma.$$

An *IFS* $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ is said

a) To have *SP* if $\forall$ $\gamma > 0$, $\exists$ $\eta > 0$ such that every η-pseudo-orbit is γ-shadowed by a point in M.

b) To be *CT* if $\forall$ $\eta > 0$, and for any pair of points $s, t \in M$, $\exists$ an η-chain joining s and t.

c) To be *CM* if $\forall$ $\eta > 0$ and for any pair of points $s, t \in M$, $\exists$ $N > 0$ such that $\forall$ $n \ge N$, $\exists$ an η-chain joining s and t of length n.

d) To have mean shadowing property (*MSP*) if $\forall$ $\gamma > 0$, $\exists$ $\eta > 0$ such that every η-mean-pseudo-orbit is γ-mean shadowed by a point in M.

Mean chain properties and almost mean shadowing properties in dynamical systems have been introduced in [24]. The main aim of this paper is to extend these notions in *IFS*s. Therefore, we recall the following definitions in dynamical systems:

Let $\eta > 0$, a finite sequence $\{s_0, s_1, \ldots, s_n\}$ is an η-mean chain of length n if $\exists$ an integer $0 < P \le n$ such that $\forall$ $P \le m \le n$,

$$\frac{1}{m}\sum_{i=0}^{m-1} d(g(s_i), s_{i+1}) < \eta.$$

(M, g) is said to be mean chain transitive (*MCT*) if, for every $\eta > 0$ and for any pair of points $s, t \in M$, $\exists$ an η-mean chain joining s and t. It is said to be totally mean chain transitive (*TMCT*) if g^k is *MCT* for each $k > 0$. And, it is said to be mean chain mixing (*MCM*) if for every $\eta > 0$ and for any pair of points $s, t \in M$, $\exists$ an integer $N > 0$ such that $\forall$ $n \ge N$, $\exists$ an η-mean chain joining s and t of length n.

Let $\eta > 0$, a sequence $\{s_i\}_{i\in\mathbb{Z}_+}$ is an almost η-mean pseudo-orbit of (M, g) if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(g(s_i), s_{i+1}) < \eta.$$

An almost η-mean pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-mean shadowed by $s \in M$ if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(g^i(s), s_i) < \gamma.$$

(M, g) has almost mean shadowing property (*AMSP*) if for every $\gamma > 0$, $\exists$ an $\eta > 0$ such that every almost η-mean pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-mean shadowed by a point in M. Throughout the paper, we consider (M, d) to be a compact metric space and $g_\alpha : X \to X$ to be a continuous self-map in X for any $\alpha \in \Lambda$.

3. Mean chain properties

This section introduces the notions of mean chain transitivity (*MCT*), mean chain mixing (*MCM*), and totally mean chain transitivity (*TMCT*) to *IFS*s, and proves some preliminary results.

Definition 3.1. *Let $\eta > 0$, a finite sequence $\{s_0, s_1, \ldots, s_n\}$ is an η-mean chain of length n if $\exists$ $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ and an integer $0 < P \leq n$ such that $\forall$ integer m with $P \leq m \leq n$,*

$$\frac{1}{m} \sum_{i=0}^{m-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \eta.$$

Definition 3.2. *An IFS $\mathfrak{F}$ is considered to be MCT if for any $\eta > 0$ and for any pair of points $s, t \in M$, $\exists$ an η-mean chain joining s and t.*

Definition 3.3. *An IFS $\mathfrak{F}$ is considered to be TMCT if $\mathfrak{F}^k$ is MCT for each $k > 0$.*

Definition 3.4. *An IFS $\mathfrak{F}$ is considered to be MCM if for any $\eta > 0$ and for any pair of points $s, t \in M$, $\exists$ an integer $N > 0$ such that $\forall$ $n \geq N$, $\exists$ an η-mean joining s and t of length n.*

Following, we give an example of an *IFS* that is not *CT* but *MCM*.

Example 3.5. *Consider (M, d) to be a metric space with more than two elements, and let $a, b \in M$. Let g_1, g_2 be two self-maps in M defined by $g_1(s) = a$ and $g_2(s) = b$ for every $s \in M$. Then, the IFS, $\mathfrak{F} = \{M; g_1, g_2\}$ is not CT but MCM.*

Proof. Clearly, $\mathfrak{F}$ is not *CT*, indeed for any pair $s, t \in M$ with $t \notin \{a, b\}$, there is no η-chain joining s and t with $\eta < \min\{d(t, a), d(t, b)\}$. Now, we claim that $\mathfrak{F}$ is *MCM*. Take $\eta > 0$ and $s, t \in M$. For $t \in \{a, b\}$, it is obvious. Suppose $t \notin \{a, b\}$ and, let $\max\{d(t, a), d(t, b)\} = \gamma > 0$. Choose an integer $N > 0$ for which $N > \frac{\gamma}{\eta}$. For every $n \geq N$ and a finite sequence $\{\alpha_i\}_{i=0}^{n-1}$ where $\alpha_i \in \{1, 2\}$, define $s_i = \mathfrak{F}_{\sigma_i}(s)$ for $0 \leq i \leq n-1$ and $s_n = t$. Then, for every integer m with $N \leq m \leq n$, we have

$$\frac{1}{m} \sum_{i=0}^{m-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \eta.$$

Thus, $\{s_i\}_{i=0}^{n}$ is an η-mean chain joining s and t of length n. Hence, $\mathfrak{F} = \{M; g_1, g_2\}$ is not *CT* but *MCM*. □

Theorem 3.6. *Let $\mathfrak{F}$ be an IFS. Then, $\mathfrak{F}$ is MCT if $\mathfrak{F}^k$ is MCT for some $k > 1$.*

Proof. Let $\eta > 0$ and let $s, t \in M$ be two points. Let $k > 1$ be fixed such that $\mathfrak{F}^k$ is *MCT*. Then, there exists an η-mean chain $\{s_i\}_{i=0}^n$ of $\mathfrak{F}^k$ joining s and t. Therefore, there exists a finite sequence $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ and an integer $0 < P \leq n$ such that $\forall$ integer m with $P \leq m \leq n$,

$$\frac{1}{m}\sum_{i=0}^{m-1} d(f_{\mu_i}(s_i), s_{i+1}) < \eta \tag{3.1}$$

where $f_{\mu_i} = g_{\alpha^i_{k-1}} \circ g_{\alpha^i_{k-2}} \circ \cdots \circ g_{\alpha^i_0}$ and $\mu_i = \{\alpha^i_0, \alpha^i_1, \ldots, \alpha^i_{k-1}\}$ for $0 \leq i \leq n-1$.

For $0 \leq i \leq n$, let

$$t_j = \begin{cases} s_i, & \text{if } j = ki, \\ g_{\alpha^i_{j-ki-1}} \circ g_{\alpha^i_{j-ki-2}} \circ \cdots \circ g_{\alpha^i_0}(s_i), & \text{if } ki < j < (i+1)k, \end{cases}$$

i.e.,

$$\{t_j\}_{j=0}^{nk} = \{t_0 = s, t_1 = g_{\alpha^0_0}(s), t_2 = g_{\alpha^0_1} \circ g_{\alpha^0_0}(s), \ldots, t_k = s_1, t_{k+1} = g_{\alpha^1_0}(s_1),$$
$$t_{k+2} = g_{\alpha^1_1} \circ g_{\alpha^1_0}(s_1), \ldots, t_{nk-1} = g_{\alpha^{n-1}_{k-2}} \circ \cdots \circ g_{\alpha^{n-1}_0}(s_{n-1}), t_{nk} = t\}.$$

Again, let

$$\{\alpha'_j\}_{j=0}^{nk} = \{\alpha^0_0, \alpha^0_1, \ldots, \alpha^1_0, \alpha^1_1, \ldots, \alpha^1_{k-1}, \ldots, \alpha^{n-1}_{k-2}, \alpha^{n-1}_{k-1}\}.$$

Then, $\forall$ integer l with $n \leq l \leq nk$, we have

$$\frac{1}{l}\sum_{j=0}^{l-1} d(g_{\alpha'_j}(t_j), t_{j+1}) \leq \frac{1}{n}\sum_{j=0}^{nk-1} d(g_{\alpha'_j}(t_j), t_{j+1}).$$

For $j \neq ik$, the term vanishes, therefore

$$\frac{1}{l}\sum_{j=0}^{l-1} d(g_{\alpha'_j}(t_j), t_{j+1}) < \frac{1}{n}\sum_{i=0}^{n-1} d(f_{\mu_i}(s_i), s_{i+1}).$$

By using (3.1), we have

$$\frac{1}{l}\sum_{j=0}^{l-1} d(g_{\alpha'_j}(t_j), t_{j+1}) < \eta.$$

Thus, $\{t_j\}_{j=0}^{nk}$ is an η-mean chain joining s and t of length nk. Hence $\mathfrak{F}$ is *MCT*. □

Theorem 3.7 shows that a *TMCT IFS* is *MCM* if the constituent maps are Lipschitz. A self-continuous function g on a metric space M is a Lipschitz function if $\exists\, L > 0$ such that $d(g(s), g(t)) \leq Ld(s, t)$, $\forall\, s, t \in M$.

Theorem 3.7. *Let $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ be an IFS where each g_α is a Lipschitz function. If $\mathfrak{F}$ is MCM, then $\mathfrak{F}$ is TMCT.*

Proof. Let $k > 1$ be an integer, let $\eta > 0$ and let $s, t \in M$ be any pair of points. For each $\alpha \in \Lambda$, as g_α is Lipschitz, $\exists\, L_\alpha > 0$ such that $d(g_\alpha(u), g_\alpha(v)) \leq L_\alpha d(u, v)$, $\forall\, u, v \in M$. Let $L = \max\{L_\alpha : \alpha \in \Lambda\}$. Then, $d(g_\alpha(u), g_\alpha(v)) \leq Ld(u, v)$, $\forall\, \alpha \in \Lambda$ and $\forall\, u, v \in M$. Without loss of generality, let $L \geq 1$ and take $\gamma = \frac{\eta}{kL^{k-1}}$. Since $\mathfrak{F}$ is *MCM*, $\exists\, N > 0$ such that $\forall\, n \geq N$, $\exists$ a γ-mean chain of $\mathfrak{F}$ joining s and t

of length n. Take an integer $r > 0$ such that $rk \geq N$. Then, we can get a γ-mean chain of $\mathfrak{F}$ joining s and t of length rk, say $\{s_0 = s, s_1, \ldots, s_{rk} = t\}$. Therefore, there exists an integer $0 < P \leq rk$ and a finite sequence, say $\{\alpha_i\}_{i=0}^{rk-1} = \{\alpha_0^0, \alpha_1^0, \ldots, \alpha_{k-2}^0, \alpha_{k-1}^0, \alpha_0^1, \alpha_1^1, \ldots, \alpha_{k-2}^1, \alpha_{k-1}^1, \ldots, \alpha_{k-2}^{r-1}, \alpha_{k-1}^{r-1}\}$ such that $\forall$ integer m with $P \leq m \leq rk$,

$$\frac{1}{m}\sum_{i=0}^{m-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \gamma. \tag{3.2}$$

Put $\gamma_i = d(g_{\alpha_i(s_i)}, s_{i+1})$ for $0 \leq i \leq rk-1$. Then, from Eq (3.2), we get

$$\frac{1}{rk}\sum_{i=0}^{rk-1} \gamma_i < \gamma. \tag{3.3}$$

Define $t_i = s_{ik}$ for $0 \leq i \leq r$. We claim that $\{t_0, t_1, \ldots, t_r\}$ is an η-mean chain of $\mathfrak{F}^k$ joining s and t.

Let $f_{\mu_i} = g_{\alpha_{k-1}^i} \circ g_{\alpha_{k-2}^i} \circ \cdots \circ g_{\alpha_0^i}$ where $\mu_i = \{\alpha_0^i, \alpha_1^i, \ldots, \alpha_{k-1}^i\}$, $\forall\ 0 \leq i \leq r-1$. Then, $\forall\ 0 \leq i \leq r-1$, we have

$$\begin{aligned}
d(f_{\mu_i}(t_i), t_{i+1}) &= d(f_{\mu_i}(s_{ik}), s_{(i+1)k}) \\
&= d(g_{\alpha_{k-1}^i} \circ g_{\alpha_{k-2}^i} \circ \cdots \circ g_{\alpha_0^i}(s_{ik}), (s_{(i+1)k})) \\
&\leq d\left(g_{\alpha_{k-1}^i} \circ g_{\alpha_{k-2}^i} \circ \cdots \circ g_{\alpha_0^i}(s_{ik}), g_{\alpha_{k-1}^i} \circ g_{\alpha_{k-2}^i} \circ \cdots \circ g_{\alpha_1^i}(s_{ik+1})\right) \\
&\quad + \cdots + d\left(g_{\alpha_{k-1}^i} \circ g_{\alpha_{k-2}^i}(s_{(i+1)k-2}), g_{\alpha_{k-1}^i}(s_{(i+1)k-1})\right) \\
&\quad + d\left(g_{\alpha_{k-1}^i}(s_{(i+1)k-1}), s_{(i+1)k}\right) \\
&\leq L^{k-1}\gamma_{ik} + \cdots + L^2\gamma_{(i+1)k-2} + L\gamma_{(i+1)k-1} \\
&< L^{k-1}(\gamma_{ik} + \cdots + \gamma_{(i+1)k-2} + \gamma_{(i+1)k-1}).
\end{aligned}$$

Therefore, using Eq (3.3), it is clear that

$$\frac{1}{r}\sum_{i=0}^{r-1} d(f_{\mu_i}(t_i), t_{i+1}) < \frac{1}{r}L^{k-1}\sum_{i=0}^{rk-1}\gamma_i < L^{k-1}k\gamma = \eta.$$

Thus, $\{t_0, t_1, \ldots, t_r\}$ is an η-mean chain of $\mathfrak{F}^k$ joining s and t. Hence, $\mathfrak{F}$ is *TMCT*. □

Given two compact metric spaces (M, d) and (M', d'), we take the metric space $M \times M'$ with metric

$$d''((s_1, t_1), (s_2, t_2)) = \max\{d(s_1, s_2), d'(t_1, t_2)\}$$

and let $\mathfrak{F} = \{M, g_\alpha | \alpha \in \Lambda\}$ and $\mathfrak{G} = \{M'; g_\beta | \beta \in \Gamma\}$ be two *IFS*s.

Then, we define the *IFS*, $\mathfrak{F} \times \mathfrak{G}$ as

$$\mathfrak{F} \times \mathfrak{G} = \{M \times M'; h_{\alpha,\beta} | \alpha \in \Lambda, \beta \in \Gamma\},$$

where $h_{\alpha,\beta}(s, t) = (g_\alpha(s), g_\beta(t))$, $\forall\ s \in M$ and $t \in M'$.

Theorem 3.8. *If $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ is a MCM IFS, then $\mathfrak{F} \times \mathfrak{F}$ is MCT.*

Proof. Let $\eta > 0$ and let $(s, t), (u, v) \in M \times M$ be any two points.

Since $\mathfrak{F}$ is *MCM* and as $\frac{\eta}{2} > 0$, there exist integers N_1, $N_2 > 0$ such that for any $n_1 \geq N_1$ and $n_2 \geq N_2$, there are $\frac{\eta}{2}$-mean chains joining s and u, and joining t and v respectively.

Put N =max$\{N_1, N_2\}$. Then, we can find two $\frac{\eta}{2}$-mean chains joining s and u, t and v respectively; say $\{s_0 = s, s_1, \ldots, s_N = u\}$ and $\{t_0 = t, t_1, \ldots, t_N = v\}$. Therefore, there exist finite sequences $\{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\}$ and $\{\alpha_0^{'}, \alpha_1^{'}, \ldots, \alpha_{N-1}^{'}\}$, and integers $0 < P_1,\ P_2 \leq N$ such that $\forall$ integers m', m'' with $P_1 \leq m' \leq N$ and $P_2 \leq m'' \leq N$, we have

$$\frac{1}{m'} \sum_{i=0}^{m'-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \frac{\eta}{2}$$

and

$$\frac{1}{m''} \sum_{i=0}^{m''-1} d(g_{\alpha_i^{'}}(t_i), t_{i+1}) < \frac{\eta}{2}.$$

Take P =max$\{P_1,\ P_2\}$ and consider $\{(s_i, t_i)\}_{i=0}^{N}$. Then, $\forall$ integer m with $P \leq m \leq N$, we have

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^{m-1} d^{''} \left((g_{\alpha_i}(s_i), g_{\alpha_i^{'}}(t_i)), (s_{i+1}, t_{i+1}) \right) &\leq \frac{1}{m} \sum_{i=0}^{m-1} d(g_{\alpha_i}(s_i), s_{i+1}) + \frac{1}{m} \sum_{i=0}^{m-1} d(g_{\alpha_i^{'}}(t_i), t_{i+1}) \\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned}$$

Thus, $\{(s_i, t_i)\}_{i=0}^{N}$ is an η-mean chain of $\mathfrak{F} \times \mathfrak{F}$ joining (s, t) and (u, v) of length N. Hence, $\mathfrak{F} \times \mathfrak{F}$ is *MCT*. □

Theorem 3.9. *If $\mathfrak{F}$ is a TMCT IFS, then $\mathfrak{F} \times \mathfrak{F}$ is MCT.*

Proof. Let $\eta > 0$ and let $(s, t), (u, v) \in M \times M$ be any two points. By definition of *TMCT*, we have $\mathfrak{F}$ is *MCT*. Suppose, $\{s_0 = s, s_1, \ldots, s_n = u\}$ and $\{u_0 = u, u_1, \ldots, u_k = u\}$ are two $\frac{\eta}{4}$-mean chains respectively, joining s and u, and joining u to itself. Then, there exist finite sequences $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ and $\{\alpha_0^{'}, \alpha_1^{'}, \ldots, \alpha_{k-1}^{'}\}$ and integers $0 < P_1 \leq n$ and $0 < P_2 \leq k$ such that

$$\tfrac{1}{m} \sum_{i=0}^{m-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \tfrac{\eta}{4}, \forall \text{ integer } m \text{ with } P_1 \leq m \leq n,$$

and

$$\tfrac{1}{m'} \sum_{i=0}^{m'-1} d(g_{\alpha_i^{'}}(u_i), u_{i+1}) < \tfrac{\eta}{4}, \forall \text{ integer } m' \text{ with } P_2 \leq m' \leq k.$$

In particular,

$$\tfrac{1}{n} \sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \tfrac{\eta}{4},$$

and

$$\tfrac{1}{k} \sum_{i=0}^{k-1} d(g_{\alpha_i^{'}}(u_i), u_{i+1}) < \tfrac{\eta}{4}.$$

From the definition of *TMCT*, $\mathfrak{F}^k$ is *MCT*. Let $\{t_0 = g_{\alpha_{n-1}} \circ \cdots \circ g_{\alpha_0}(t), t_1, \ldots, t_p = v\}$ be an $\frac{\eta}{2}$-mean chain of $\mathfrak{F}^k$ joining $g_{\alpha_{n-1}} \circ \cdots \circ g_{\alpha_0}(t)$ and v of length p. Therefore, we can find a finite sequence $\{\mu_0, \mu_1, \ldots, \mu_{p-1}\}$ and an integer $0 < P \leq p$ such that

$$\frac{1}{m''}\sum_{i=0}^{m''-1} d(f_{\mu_i}(t_i), t_{i+1}) < \frac{\eta}{2}, \forall \text{ integer } m'' \text{ with } P \le m'' \le p,$$

where $f_{\mu_i} = g_{\alpha^i_{k-1}} \circ g_{\alpha^i_{k-2}} \circ \cdots \circ g_{\alpha^i_0}$ and $\mu_i = \{\alpha^i_0, \alpha^i_1, \ldots, \alpha^i_{k-1}\}$ for $0 \le i \le p-1$.

Consider

$$\begin{aligned}\{z_i\}_{i=0}^{n+pk} &= \{t, g_{\alpha_0}(t), \ldots, g_{\alpha_{n-1}} \circ \cdots \circ g_{\alpha_0}(t)\\ &= t_0, g_{\alpha^0_0}(t_0), \ldots, g_{\alpha^0_{k-2}} \circ \cdots \circ g_{\alpha^0_0}(t_0), t_1, g_{\alpha^1_0}(t_1), \ldots, g_{\alpha^1_{k-2}} \circ \cdots \circ g_{\alpha^1_0}(t_1), t_2, g_{\alpha^2_0}(t_2),\\ &\ldots, t_{p-1}, g_{\alpha^{p-1}_0}(t_{p-1}), \ldots, g_{\alpha^{p-1}_{k-2}} \circ \cdots \circ g_{\alpha^{p-1}_0}(t_{p-1}), t_p = v\}\end{aligned}$$

with respect to the finite sequence

$$\{\alpha''_i\}_{i=0}^{n+pk} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha^0_0, \alpha^0_1, \ldots, \alpha^0_{k-1}, \alpha^1_0, \ldots, \alpha^1_{k-1}, \ldots, \alpha^{p-1}_0, \ldots, \alpha^{p-1}_{k-1}\}.$$

Then, it is clear that the term $d(g_{\alpha''_i}(z_i), z_{i+1})$ vanishes whenever $i \neq n + jk$ where $0 < j \le p-1$. Therefore,

$$\begin{aligned}\frac{1}{n+pk}\sum_{i=0}^{n+pk-1} d(g_{\alpha''_i}(z_i), z_{i+1}) &= \frac{1}{n+pk}\sum_{i=0}^{p-1} d(g_{\alpha^i_{k-1}} \circ g_{\alpha^i_{k-2}} \circ \cdots \circ g_{\alpha^i_0}(t_i), t_{i+1})\\ &= \frac{1}{n+pk}\sum_{i=0}^{p-1} d(f_{\mu_i}(t_i), t_{i+1})\\ &< \frac{1}{p}\sum_{i=0}^{p-1} d(f_{\mu_i}(t_i), t_{i+1})\\ &< \frac{\eta}{2}.\end{aligned}$$

Thus, $\{z_i\}_{i=0}^{n+pk}$ is an $\frac{\eta}{2}$-mean chain of $\mathfrak{F}$ joining t and v.

Again, consider

$$\{w_i\}_{i=0}^{n+pk} = \{s_0 = s, s_1, \ldots, s_n = u = u_0, \underbrace{u_1, \ldots, u_k = u, u_1, \ldots, u_k = u, \ldots, u_1, \ldots, u_k = u}_{p \text{ times}}\}$$

with respect to the finite sequence

$$\{\alpha'''_i\}_{i=0}^{n+pk} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \underbrace{\alpha'_0, \alpha'_1, \ldots, \alpha'_{k-1}, \ldots, \alpha'_0, \alpha'_1, \ldots, \alpha'_{k-1}}_{p \text{ times}}\}.$$

Now,

$$\begin{aligned}\frac{1}{n+pk}\sum_{i=0}^{n+pk-1} d(g_{\alpha'''_i}(w_i), w_{i+1}) &= \frac{1}{n+pk}\left[\sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) + p\sum_{i=0}^{k-1} d(g_{\alpha'_i}(u_i), u_{i+1})\right]\\ &= \frac{1}{n+pk}\sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) + \frac{p}{n+pk}\sum_{i=0}^{k-1} d(g_{\alpha'_i}(u_i), u_{i+1})\\ &< \frac{1}{n}\sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) + \frac{p}{pk}\sum_{i=0}^{k-1} d(g_{\alpha'_i}(u_i), u_{i+1})\end{aligned}$$

$$
\begin{aligned}
&< \frac{\eta}{4}+\frac{\eta}{4}\\
&= \frac{\eta}{2}.
\end{aligned}
$$

Thus, $\{w_i\}_{i=0}^{n+pk}$ is an $\frac{\eta}{2}$-mean chain of $\mathfrak{F}$ joining s and u. This implies that $\{(w_i, z_i)\}_{i=0}^{n+pk}$ is an $\frac{\eta}{2}$-mean chain of $\mathfrak{F}\times\mathfrak{F}$ joining (s,t) and (u,v) with respect to d''. Hence, $\mathfrak{F}\times\mathfrak{F}$ is *MCT*. □

4. Almost mean shadowing property

This section introduces the notion of almost mean shadowing property (*AMSP*) to *IFS*s.

Definition 4.1. *Let $\eta > 0$, a sequence $\{s_i\}_{i\in\mathbb{Z}_+}$ is an almost η-mean pseudo-orbit of an IFS $\mathfrak{F}$ if $\exists\ \sigma \in \Lambda^{\mathbb{Z}_+}$ such that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(g_{\alpha_i}(s_i), s_{i+1}) < \eta.$$

An almost η-mean pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ of an IFS $\mathfrak{F}$ is γ-mean shadowed by $s \in M$ if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\mathfrak{F}_{\sigma_i}(s), s_i) < \gamma.$$

Definition 4.2. *An IFS $\mathfrak{F}$ has AMSP if for any $\gamma > 0$, $\exists$ an $\eta > 0$ such that every almost η-mean pseudo-orbit $\{s_i\}_{i\in\mathbb{Z}_+}$ is γ-mean shadowed by a point in M.*

Remark 4.3. *From the definition, it is clear that* AMSP *implies* MSP, *but the converse may not be true.*

In the following, we give an example of an *IFS* that has *MSP* but does not have the *AMSP*.

Example 4.4. *Let (M,d) be the metric space as defined in [29, Example 9.1]. Let $g_1,\ g_2$ be self maps on M defined as*

$$g_1(p) = p,\ g_1(a_n) = a_{n+1},\ g_1(b_n) = b_{n+1},$$
$$g_2(p) = p,\ g_2(a_n) = a_n,\ g_2(b_n) = b_{n+1}.$$

Then, the IFS, $\mathfrak{F} = \{M; g_1, g_2\}$ has MSP but does not have the AMSP.

Proof. Proceeding similarly, as in the proof of [29, Theorem 9.2], it is clear that $\mathfrak{F}$ has *MSP*.

Also, in [24], it is given that (M, g_1) does not have the *AMSP*. So, for any $\epsilon > 0$, we can find a $\delta > 0$ and an almost δ-pseudo orbit with respect to $\sigma = \{g_1, g_1, g_1, \cdots\}$ which is not ϵ-shadowed in average by any point in M. Hence, the *IFS* $\mathfrak{F} = \{M; g_1, g_2\}$ does not have the *AMSP*. □

Example 4.5. *Let $\mathfrak{F}$ be the IFS as defined in [30, Example 3.5]. Then, $\mathfrak{F}$, does not have MSP and AMSP.*

Proof. In [31, Remark 4.5], it has been given that $\mathfrak{F}$ does not have *MSP*. Using, the above Remark 4.3, it is clear that $\mathfrak{F}$ does not have the *AMSP*. □

Theorem 4.6. *If $\mathfrak{F}$ is an IFS with AMSP, then so does $\mathfrak{F}^k$ for every $k \geq 2$.*

Proof. Let $k \geq 2$ and $\gamma > 0$. By hypothesis, $\exists$ an $\eta > 0$ such that every almost η-mean pseudo-orbit is $\frac{\gamma}{k}$-mean shadowed by a point in M.

Let $\{t_i\}_{i\in\mathbb{Z}_+}$ be an almost η-mean pseudo-orbit of $\mathfrak{F}^k$. Then, $\exists\ \sigma = \langle\mu_i\rangle$ such that $\forall\ \mu_i \in \sigma$

$$\limsup_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} d(f_{\mu_i}(t_i), t_{i+1}) < \eta,$$

where $f_{\mu_i} = g_{\alpha^i_{k-1}} \circ g_{\alpha^i_{k-2}} \circ \cdots \circ g_{\alpha^i_0}$ and $\mu_i = \{\alpha^i_0, \alpha^i_1, \ldots, \alpha^i_{k-1}\} \in \Lambda^k$, $\forall\ i \in \mathbb{Z}_+$.

Now, for some $\sigma' = \langle\alpha'_j\rangle = \{\alpha^0_0, \alpha^0_1, \ldots, \alpha^0_{k-1}, \alpha^1_0, \ldots\}$, consider a sequence $\{s_j\}_{j\in\mathbb{Z}_+}$ defined by

$$s_j = \begin{cases} t_i, & \text{if } j = ki, \\ g_{\alpha^i_{j-ki-1}} \circ g_{\alpha^i_{j-ki-2}} \circ \cdots \circ g_{\alpha^i_0}(t_i), & \text{if } ki < j < (i+1)k. \end{cases}$$

For $ki < j < (i+1)k$, we have $0 < l \leq k-1$ such that $s_j = s_{ik+l} = g_{\alpha^i_{l-1}} \circ g_{\alpha^i_{l-2}} \circ \cdots \circ g_{\alpha^i_0}(t_i)$. Also, for any integer $n > 0$, we can get some $i \geq 0$ and $0 < l \leq k-1$ for which $n = ik + l$. Thus,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} d(g_{\alpha'_j}(s_j), s_{j+1}) = \limsup_{i\to\infty} \frac{1}{ik+l} \sum_{j=0}^{ik+l-1} d(g_{\alpha'_j}(s_j), s_{j+1}).$$

For $j \neq ki - 1$, the term vanishes. Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} d(g_{\alpha'_j}(s_j), s_{j+1}) &= \limsup_{i\to\infty} \frac{1}{ik+l} \sum_{j=0}^{i-1} d(f_{\mu_j}(t_j), t_{j+1}) \\ &\leq \limsup_{i\to\infty} \frac{1}{i} \sum_{j=0}^{i-1} d(f_{\mu_j}(t_j), t_{j+1}) \\ &< \eta. \end{aligned}$$

This implies that $\{s_j\}_{j\in\mathbb{Z}_+}$ is an almost η-mean pseudo-orbit of $\mathfrak{F}$ with respect to σ'. Therefore, $\exists$ $z \in M$ such that

$$\limsup_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} d(\mathfrak{F}_{\sigma'_j}(z), s_j) < \tfrac{\gamma}{k}.$$

Now,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathfrak{F}^k_{\sigma_i}(z), t_i) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathfrak{F}_{\sigma'_{ki}}(z), s_{ki}) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{l=0}^{k-1} d(\mathfrak{F}_{\sigma'_{ki+l}}(z), s_{ki+l}) \\ &= k \limsup_{n\to\infty} \frac{1}{nk} \sum_{j=0}^{nk-1} d(\mathfrak{F}_{\sigma'_j}(z), s_j) \\ &< \gamma. \end{aligned}$$

Hence, $\mathfrak{F}^k$ has*AMSP* for every $k \geq 2$. □

Theorem 4.7. *Let $\mathfrak{F} = \{M; g_\alpha | \alpha \in \Lambda\}$ be an IFS, where one of the g_α is surjective. If $\mathfrak{F}$ has AMSP, then it is CT.*

Proof. Let $\gamma > 0$ and let $s, t \in M$ be two points. Since $\{g_\alpha : \alpha \in \Lambda\}$ is a family of uniformly continuous mappings, it is uniformly equicontinuous. Thus, $\exists\ 0 < \eta < \gamma$ such that $\forall\ u, v \in M$ and $\forall\ \alpha \in \Lambda$, $d(g_\alpha(u), g_\alpha(v)) < \gamma$ whenever $d(u, v) < \eta$. By hypothesis, $\mathfrak{F}$ has *AMSP*. Therefore, $\exists\ \delta > 0$ such that every almost δ-mean pseudo-orbit of $\mathfrak{F}$ is $\frac{\eta}{2}$-mean shadowed by a point in M.

Let D =diam(M) and let $K > 0$ be an integer such that $\frac{D}{K} < \delta$. Suppose for a fixed $\alpha^* \in \Lambda$, g_{α^*} is surjective. Then, we can easily see that $g_{\alpha^*}^{-l}(t)$ exists $\forall$ integer l with $0 \leq l \leq K - 1$.

For $i \in \mathbb{Z}_+$, fix an infinite sequence $\sigma = \langle \alpha_i \rangle \in \Lambda^{\mathbb{Z}_+}$. Again, for $j \in \mathbb{Z}_+$, we consider an infinite sequence $\{s_j\}_{j \in \mathbb{Z}_+}$, where

$$s_j = \begin{cases} \mathfrak{F}_{\sigma_{j-2iK}}, & 2iK \leq j \leq (2i+1)K - 1, \\ g_{\alpha^*}^{j-2(i+1)K+1}(t), & (2i+1)K \leq j \leq 2(i+1)K - 1. \end{cases}$$

For any $\alpha' \in \Lambda$, let us define $\sigma' = \langle \alpha'_j \rangle \in \Lambda^{\mathbb{Z}_+}$, where

$$\alpha'_j = \begin{cases} \alpha_{j-2iK}, & 2iK \leq j \leq (2i+1)K - 2, \\ \alpha', & j = (2i+1)K - 1, \\ \alpha^*, & (2i+1)K \leq j \leq 2(i+1)K - 2, \\ \alpha' & j = 2(i+1)K - 1. \end{cases}$$

Now, for any $n > 0$ with $iK \leq n \leq (i+1)K$, we have

$$\tfrac{1}{n} \sum_{j=0}^{n-1} d(g_{\alpha'_j}(s_j), s_{j+1}) \leq \tfrac{iD}{iK} = \tfrac{D}{K} < \eta.$$

Therefore,

$$\limsup_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} d(g_{\alpha'_j}(s_j), s_{j+1}) < \eta.$$

This implies that $\{s_j\}_{j \in \mathbb{Z}_+}$ is an almost δ-mean pseudo-orbit of $\mathfrak{F}$. By hypothesis, $\exists\ z \in M$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} d(\mathfrak{F}_{\sigma'_j}(z), s_j) < \frac{\eta}{2}. \tag{4.1}$$

Notice that, there exist infinitely many $i \in \mathbb{Z}_+$ for which there is some l with $2iK \leq l \leq (2i+1)K - 1$, i.e., $s_l \in \{s, \mathfrak{F}_{\sigma_1}(s), \mathfrak{F}_{\sigma_2}(s), \ldots, \mathfrak{F}_{\sigma_{K-1}}(s)\}$ such that $d\Big(\mathfrak{F}_{\sigma'_l}(z), s_l\Big) < \frac{\eta}{2}$. Otherwise,

$$\limsup_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} d(\mathfrak{F}_{\sigma'_j}(z), s_j) \geq \tfrac{\eta}{2}$$

which contradicts (4.1).

Similarly, the above statement holds when $(2i+1)K \leq l \leq 2(i+1)K - 1$, i.e., $s_l \in \{g_{\alpha^*}^{-(K-1)}(t), g_{\alpha^*}^{-(K-2)}(t), \ldots, g_{\alpha^*}^{-1}(t), t\}$.

Thus, we can find two integers l_1 and l_2 with $0 < l_1 < l_2$ such that $s_{l_1} = \mathfrak{F}_{\sigma_{p_1}}(s)$ for some $0 \leq p_1 \leq K - 1$ satisfying $d\Big(\mathfrak{F}_{\sigma'_{l_1}}(z), s_{l_1}\Big) < \frac{\eta}{2}$ and $s_{l_2} = g_{\alpha^*}^{-p_2}(t)$ for some $0 \leq p_2 \leq K - 1$ satisfying

$d\left(\mathfrak{F}_{\sigma'_{l_2}}(z), s_{l_2}\right) < \frac{\eta}{2}$. Using the condition of equicontinuity, we have, $d\left(g_{\alpha'_{l_1}}(\mathfrak{F}_{\sigma'_{l_1}}(z)), g_{\alpha'_{l_1}}(s_{l_1})\right) < \gamma$. This implies that $d\left(\mathfrak{F}_{\sigma'_{l_1+1}}(z), g_{\alpha'_{l_1}}(s_{l_1})\right) < \gamma$ and $d\left(\mathfrak{F}_{\sigma'_{l_2}}(z), s_{l_2}\right) < \gamma$. Therefore,

$$\{s, \mathfrak{F}_{\sigma_1}(s), \mathfrak{F}_{\sigma_2}(s), \ldots, \mathfrak{F}_{\sigma_{p_1-1}}(s), \mathfrak{F}_{\sigma_{p_1}}(s) = s_{l_1}, \mathfrak{F}_{\sigma'_{l_1+1}}(z),$$

$\mathfrak{F}_{\sigma'_{l_1+2}}(z), \ldots, \mathfrak{F}_{\sigma'_{l_2-1}}(z), g_{\alpha^*}^{-p_2}(t) = s_{l_2}, g_{\alpha^*}^{-(p_2-1)}(t), \ldots, g_{\alpha^*}^{-1}(t), t\}$ is a γ-chain joining s and t with respect to the finite sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_{p_1-1}, \alpha'_{l_1}, \alpha'_{l_1+1}, \ldots, \alpha'_{l_2-1}, \underbrace{\alpha^*, \alpha^*, \ldots \alpha^*}_{p_2 \text{ times}}\}$. Hence, $\mathfrak{F}$ is *CT*. □

5. Conclusions

In this work, we have introduced the notions of *MCT, TMCT, MCM,* and *AMSP* to *IFS*s and studied their interrelations. In Example 3.5, we have given an example of an *IFS* that is not *CT* but *MCM*. In Theorem 3.7, we proved that a *TMCT IFS* is *MCM* if the constituent maps are Lipschitz. For an iterated function system $\mathfrak{F}$, we show that $\mathfrak{F} \times \mathfrak{F}$ is *MCT* if $\mathfrak{F}$ is *MCM*. We also showed that $\mathfrak{F} \times \mathfrak{F}$ is *MCT* if $\mathfrak{F}$ is *TMCT*. Lastly, we prove that an *IFS* $\mathfrak{F}$, one of whose constituent maps g_α is surjective and has *AMSP*, is *CT*.

Author contributions

Thiyam Thadoi Devi: ideas, states, proof, and examples; Khundrakpam Binod Mangang: ideas, conceptualization, states, examples, and submission; Sonika Akoijam: states, proofs, and first draft; Lalhmangaihzuala: states, proofs, and edition; Phinao Ramwungzan: states, proofs, and examples; Jay Prakash Singh: revision and draft of the manuscript. All authors have read and approved the final version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conficts of interest.

References

1. A. Allison, D. Abbott, Control systems with stochastic feedback, *Chaos*, **11** (2001), 715–724. https://doi.org/10.1063/1.1397769

2. J. M. R. Parrondo, P. G. Harmer, D. Abbott, New paradoxical games based on Brownian ratchets, *Phys. Rev. Lett.*, **85** (2000), 5226. https://doi.org/10.1103/PhysRevLett.85.5226
3. A. Z. Bahabadi, Shadowing and average shadowing properties of for iterated function systems, *Georgian Math. J.*, **22** (2015), 179–184. https://doi.org/10.1515/gmj-2015-0008
4. R. Devaney, *An introduction to chaotic dynamical systems*, 2 Eds., CRC Press, 2003. https://doi.org/10.4324/9780429502309
5. T. Y. Li, J. Yorke, Period three implies chaos, *Amer. Math. Mon.*, **82** (1975), 985–992. https://doi.org/10.2307/2318254
6. E. T. Whittaker, D. Birkhoff's collected papers, *Nature*, **167** (1951), 250–251. https://doi.org/10.1038/167250a0
7. W. H. Gottschalk, G. A. Hedlund, *Topological dynamics*, American Mathematical Society Colloquium Publications, 1995.
8. K. E. Petersen, *Ergodic Theory*, Cambridge University Press, 1983. https://doi.org/10.1017/CBO9780511608728
9. A. M. Bruckner, T. Hu, On scrambled sets for chaotic functions, *Trans. Am. Math. Soc.*, **301** (1987), 289–297. https://doi.org/10.2307/2000339
10. J. Vries, *Elements of topological dynamics*, Springer Dordrecht, 1993. https://doi.org/10.1007/978-94-015-8171-4
11. Y. A. Sidorov, Topologically indecomposable transformations of the n-dimensional space, *Math. Notes Acad. Sci. USSR*, **4** (1968), 939–943. https://doi.org/10.1007/BF01110833
12. T. T. Devi, K. B. Mangang, On equicontinuity, transitivity and distality of iterated function systems, *J. Dyn. Syst. Geom. Theor.*, **18** (2020), 223–239. https://doi.org/10.1080/1726037X.2020.1847766
13. K. B. Mangang, Mean equicontinuity, sensitivity, expansiveness and distality of product dynamical systems, *J. Dyn. Syst. Geom. Theor.*, **13** (2015), 27–33. https://doi.org/10.1080/1726037X.2015.1027106
14. W. Brian, J. Meddaugh, B. Raines, Chain transitivity and variations of the shadowing property, *Ergod. Theory Dyn. Syst.*, **35** (2015), 2044–2052. https://doi.org/10.1017/etds.2014.21
15. D. V. Anosov, On a class of invariant sets of smooth dynamical systems, *Proc. 5th Int. Conf. on Nonlin. Oscill*, **2** (1970), 39–45.
16. R. Bowen, ω-limit sets for Axiom A diffeomorphisms, *J. Differ. Equations*, **18** (1975), 333–339. https://doi.org/10.1016/0022-0396(75)90065-0
17. D. Kwietniak, P. Oprocha, A note on the average shadowing property for expansive maps, *Topol. Appl.*, **159** (2012), 19–27. https://doi.org/10.1016/j.topol.2011.04.016
18. A. D. Barwell, C. Good, P. Oprocha, Shadowing and expansivity in subspaces, *Fund. Math.*, **219** (2012), 223–243. https://doi.org/10.4064/fm219-3-2
19. A. Fakhari, F. H. Ghane, On shadowing: ordinary and ergodic, *J. Math. Anal. Appl.*, **349** (2010), 151–155. https://doi.org/10.1016/j.jmaa.2009.11.004
20. D. A. Dastjerdi, M. Hosseini, Sub-shadowings, *Nonlinear Anal.*, **72** (2010), 3759–3766. https://doi.org/10.1016/j.na.2010.01.014

21. A. Darabi, Periodic shadowing in iterated function systems, *Asian-Eur. J. Math.*, **15** (2022), 2250064. https://doi.org/10.1142/S1793557122500644

22. J. Jiang, L. Wang, Y. Zhao, The d-shadowing property and average shadowing property for iterated function systems, *Complexity*, **2020** (2020), 4374508. https://doi.org/10.1155/2020/4374508

23. H. Wang, Q. Liu, Ergodic shadowing properties of iterated function systems, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 767–783. https://doi.org/10.1007/s40840-020-00976-x

24. R. Das, M. Garg, Average chain transitivity and the almost average shadowing property, *Commun. Korean Math. Soc.*, **32** (2017), 201–214. https://doi.org/10.4134/CKMS.c160066

25. M. Garg, R. Das, Chaotic behaviour of maps possesing the almost average shadowing property, *Hacet. J. Math. Stat.*, **50** (2021), 1371–1383.

26. J. E. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.*, **30** (1981), 713–747.

27. M. F. Barnsley, *Fractals everywhere*, 2 Eds., Academic Press, 1993. https://doi.org/10.1016/C2013-0-10335-2

28. M. F. Barnsley, S. Demko, Iterated function systems and the global construction of fractals, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Sci.*, **399** (1985), 243–275. https://doi.org/10.1098/rspa.1985.0057

29. M. Kulczycki, D. Kwietniak, P. Oprocha, On almost specification and average shadowing properties, *Fund. Math.*, **224** (2014), 241–278. https://doi.org/10.4064/fm224-3-4

30. M. F. Nia, Parameterized IFS with the asymptotic average shadowing property, *Qual. Theory Dyn. Syst.*, **15** (2016), 367–381. https://doi.org/10.1007/s12346-015-0184-6

31. X. Wu, L. Wang, J. Liang, The chain properties and average shadowing property of iterated function systems, *Qual. Theory Dyn. Syst.*, **17** (2018), 219–227. https://doi.org/10.1007/s12346-016-0220-1