Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 20763–20772.
DOI: 10.3934/math.20241009
Received: 10 May 2024
Revised: 18 June 2024
Accepted: 20 June 2024
Published: 26 June 2024

Research article

Least energy solutions to a class of nonlocal Schrödinger equations

Yong-Chao Zhang*

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Taishan Road 143, Qinhuangdao 066004, China

* **Correspondence:** Email: zhangyc@neuq.edu.cn.

Abstract: We explore a class of nonlocal Schrödinger equations that include not only fractional Schrödinger equations but also other nonlocal Schrödinger equations studied in the literature. We prove the existence of least energy solutions to this class of equations by the variational method, which extends the results obtained by Gu et al. (2018) and Xiang et al. (2019).

Keywords: least energy solution; nonlocal Schrödinger equation; critical point; Nehari manifold
Mathematics Subject Classification: 35Q55, 35J60, 35A15

1. Introduction

The nonlinear fractional Schrödinger equation like

$$(-\Delta)^{\alpha/2}u + u = |u|^{p-2}u \text{ on } \mathbb{R}^N, \tag{1.1}$$

which is driven by a rotationally invariant stable process of index $\alpha \in (0, 2)$, where p satisfies some conditions, was studied by many authors. There are so many references about Eq (1.1) that we only list some of them [1–9].

Nonlocal Eq (1.1) appeals to many authors, as just mentioned above, so people are interested in studying other nonlocal equations. Since the generator $(-\Delta)^{\alpha/2}$ of some stable process has Lévy kernel $c|x-y|^{-N-\alpha}\mathrm{d}y$ for some positive number c, Gu et al. [10] studied an equation involving L_K given by $L_K u(x) := \int_{\mathbb{R}^N}(u(y)-u(x))K(x-y)\mathrm{d}y$ for some function K and Xiang et al. [11] investigated an equation driven by $L_s u(x) := \int_{\mathbb{R}^N}(u(y) - u(x))|x-y|^{-N-s(x,y)}\mathrm{d}s$ for a symmetric function s.

Thus, it is reasonable to explore the nonlocal Schrödinger equation

$$L_A u + Vu = f(x, u) \text{ on } \mathbb{R}^N, \tag{1.2}$$

where $L_A u(x) := 2\int_{\mathbb{R}^N}(u(y) - u(x))A(x, y)\mathrm{d}y$.

Some assumptions are given as follows:

$(A1)$ The function $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a symmetric function such that $A(x, y)\mathrm{d}y$ is a Lévy kernel and $A(x, y) \geq a|x - y|^{-N-\alpha}$ for some positive number a, where $\alpha \in (0, 2)$. The following assumption is satisfied: $A(x + \tau_j, y + \tau_j) = A(x, y)$, where $\tau_j := (0, 0, \cdots, \underbrace{1}_{j\text{th}}, \cdots, 0) \in \mathbb{R}^N$, $j = 1, 2 \cdots, N$.

$(A2)$ The 1-periodic function $V : \mathbb{R}^N \to \mathbb{R}$ fulfills $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

$(A3)$ The function f is continuous on $\mathbb{R}^N \times \mathbb{R}$, 1-periodic in the variable in $\mathbb{R}^N$ along with $f(x, u)u \geq 0$ and, for some $p \in (2, 2^*_\alpha)$ and $c > 0$,

$$|f(x, u)| \leq c(|u| + |u|^{p-1}),$$

where $2 < p < 2^*_\alpha$ with $2^*_\alpha := +\infty$ if $N \leq \alpha$, and $2^*_\alpha := 2N/(N - \alpha)$ if $N > \alpha$.

$(A4)$ There exists $\mu > 2$ such that $\mu F(x, u) \leq u f(x, u)$ for every $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, where $F(x, u) := \int_0^u f(x, s)\mathrm{d}s$.

$(A5)$ $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \mathbb{R}^N$.

$(A6)$ $f(x, u)|u|^{-1}$ is increasing as a function of u on $\mathbb{R} \setminus \{0\}$ for any $x \in \mathbb{R}^N$.

Example 1.1. Let h be a positive continuous function on $\mathbb{R}^N$ such that $h(\cdot + \tau_j) = h(\cdot)$, and $\gamma : \mathbb{R}^N \to (0, 2)$ be a continuous function such that $\gamma(\cdot)^2 < 2$ and $\gamma(\cdot + \tau_j) = \gamma(\cdot)$. Define a Lévy kernel $A(x, y)\mathrm{d}y$ satisfying $(A1)$ by $A(x, y) := h(x)h(y)|x - y|^{-N-\gamma(x)\gamma(y)}$.

Example 1.2. Let h be a positive continuous function on $\mathbb{R}^N$ such that $h(\cdot + \tau_j) = h(\cdot)$, and $\gamma : \mathbb{R}^N \to (0, 2)$ be a continuous function such that $\gamma(\cdot + \tau_j) = \gamma(\cdot)$. Define a Lévy kernel $A(x, y)\mathrm{d}y$ satisfying $(A1)$ by $A(x, y) := h(x)|x - y|^{-N-\gamma(y)} + h(y)|x - y|^{-N-\gamma(x)}$.

Equation (1.2) is an extension of the equations appearing in the papers [10, 11].

Equation (1.2) has a variational structure. In light of $(A1)$ and $(A2)$, we define the space $\mathcal{H}$ to be the completion of the space $\mathcal{S}$ of all tempered functions under the inner product, for $u, v \in \mathcal{S}$,

$$(u, v) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(y) - u(x))(v(y) - v(x))A(x, y)\mathrm{d}y\mathrm{d}x + \left(\sqrt{V}u,\ \sqrt{V}v\right)_{L^2}.$$

The corresponding induced norm is denoted by $\|\cdot\|$.

Define a functional $E : \mathcal{H} \to \mathbb{R}$ by

$$E(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x))\mathrm{d}x. \tag{1.3}$$

Then, by Lemma 2.1 and [12, Corollary 3.10], E is in $C^1(\mathcal{H}, \mathbb{R})$. Thus,

$u \in \mathcal{H}$ solves Eq (1.2) if and only if u is a critical point of the functional E.

The main results are summarized in the following theorem.

Theorem 1.3. *There is a nonzero function $w \in \mathcal{H}$ such that*

$$L_A w + Vw = f(x, w) \quad \text{on } \mathbb{R}^N$$

in the distribution sense. Moreover,

$$E(w) = \inf \left\{ E(u) : u \in \mathcal{H} \setminus \{0\} \text{ and } \|u\|^2 = \int_{\mathbb{R}^N} f(x, u(x))u(x)\mathrm{d}x \right\}. \tag{1.4}$$

Remark 1.4. Here w is known as a least energy critical point of functional E [12, p. 71]. So we call w a least energy solution to Eq (1.2).

The rest of the paper is organized as follows. In Section 2, we list some facts that will be used in the proof of Theorem 1.3. In Section 3, drawing inspiration from [13], we provide a proof of Theorem 1.3. We end the paper with some conclusions in Section 4.

2. Preliminaries

In order to prove Theorem 1.3, we make in this section the necessary preparations.

Lemma 2.1. (i) *The following embeddings are continuous:*

$$\mathcal{H} \hookrightarrow L^q(\mathbb{R}^N), \quad N \le \alpha \text{ and } q \ge 2,$$

$$\mathcal{H} \hookrightarrow L^q(\mathbb{R}^N), \quad N > \alpha \text{ and } 2 \le q \le 2^*_\alpha.$$

(ii) *Let Ω be a bounded domain of $\mathbb{R}^N$. If $2 \le q < 2^*_\alpha$, then every bounded sequence in $\mathcal{H}$ has a convergent subsequence in $L^q(\Omega)$.*

Proof. Ad (i). It follows from $(A1)$ and $(A2)$ that the embedding $\mathcal{H} \hookrightarrow H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ is continuous. Then, by [14, Theorem 7.63], we have the continuous embeddings in (i).

Ad (ii). The conclusion follows from (i) and [6, Lemma 2.1]. □

Lemma 2.2. *Let $r > 0$ and $2 \le q < 2^*_\alpha$. If $\{u_n\}_{n=1}^\infty$ is bounded in $\mathcal{H}$, and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n(x)|^q \mathrm{d}x = 0,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*_\alpha$.

Proof. It follows from $(A1)$ and $(A2)$ that the embedding $\mathcal{H} \hookrightarrow H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ is continuous and then $\{u_n\}_{n=1}^\infty$ is bounded in $H^{\frac{\alpha}{2}}(\mathbb{R}^N)$. The rest of the proof is similar to that of [12, Lemma 1.21] or [6, Lemma 2.2]. □

3. Proof of Theorem 1.3

We prove that Eq (1.2) possesses the least energy solution. To this end, we show that functional (1.3) has a nontrivial critical point in Theorem 3.1, and this critical point is the least energy solution in Theorem 3.2, respectively.

Recall functional (1.3)

$$E(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} F(x, u(x))\mathrm{d}x, \qquad u \in \mathcal{H}.$$

Theorem 3.1. *The functional E has a nontrivial critical point.*

Proof. (1) Thanks to $(A4)$, there exists $r > 0$ and a positive function k such that

$$F(x, u) \geq k(x)|u|^{\mu} \qquad \text{for } x \in \mathbb{R}^N \text{ and } |u| \geq r. \tag{3.1}$$

By $(A3)$, we also have, for some positive number c',

$$F(x, u) \geq -c'|u|^2 \qquad \text{for } x \in \mathbb{R}^N \text{ and } |u| \leq r. \tag{3.2}$$

Set

$$\Gamma := \{\gamma : \gamma \in C([0, 1], \mathcal{H}) \text{ such that } \gamma(0) = 0 \text{ and } E(\gamma(1)) < 0\}. \tag{3.3}$$

As $\mu > 2$, we have, for T large enough,

$$\begin{aligned}
E(T\exp(-|\cdot|^2)) &= \frac{T^2}{2}\|\exp(-|\cdot|^2)\|^2 - \int_{\mathbb{R}^N} F(x, T\exp(-|x|^2))\mathrm{d}x \\
&= \frac{T^2}{2}\|\exp(-|\cdot|^2)\|^2 - \int_{T\exp(-|x|^2)\geq r} F(x, T\exp(-|x|^2))\mathrm{d}x \\
&\quad - \int_{T\exp(-|x|^2)\leq r} F(x, T\exp(-|x|^2))\mathrm{d}x \\
&\leq \frac{T^2}{2}\|\exp(-|\cdot|^2)\|^2 - T^{\mu}\int_{T\exp(-|x|^2)\geq r} k(x)\exp(-\mu|x|^2)\mathrm{d}x \\
&\quad + T^2 c'\int_{T\exp(-|x|^2)\leq r} \exp(-2|x|^2)\mathrm{d}x \qquad \text{by (3.1) and (3.2)} \\
&< 0.
\end{aligned}$$

Thus $\Gamma \neq \emptyset$.

(2) Define

$$\delta := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E(\gamma(t)). \tag{3.4}$$

Thanks to Lemma 2.1, there is a positive constant c_1 such that

$$\|u\|_{L^2} \leq c_1\|u\| \text{ and } \|u\|_{L^p} \leq c_1\|u\| \qquad \text{for all } u \in \mathcal{H}. \tag{3.5}$$

In light of $(A3)$ and $(A5)$, there exists a positive number c_2 such that

$$F(x, u) \leq \frac{1}{4c_1^2}|u|^2 + c_2|u|^p. \tag{3.6}$$

Then it follows from the definition of the functional E that

$$E(u) \geq \frac{1}{4}\|u\|^2 - c_1^p c_2\|u\|^p.$$

Setting $d := (8c_1^p c_2)^{1/(2-p)}$, we have

$$\min_{\|u\|\leq d} E(u) = 0 \quad \text{and} \quad \min_{\|u\|=d} E(u) \geq \frac{1}{8}(8c_1^p c_2)^{2/(2-p)} > 0. \tag{3.7}$$

It follows from the above fact that $\delta \geq \frac{1}{8}(8c_1^p c_2)^{2/(2-p)} > 0$. Therefore, by [12, Theorem 2.9], there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}$ satisfying

$$E(u_n) \to \delta \quad \text{and} \quad E'(u_n) \to 0 \qquad \text{as } n \to \infty. \tag{3.8}$$

(3) By (3.8) and $(A4)$, we have, for n large enough,

$$\delta + 1 + \|u_n\| \geq E(u_n) - \frac{1}{\mu}\langle E'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2.$$

It follows that $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{H}$. Thus, $\{u_n\}_{n=1}^{\infty}$ possesses a subsequence, again denoted by $\{u_n\}_{n=1}^{\infty}$, such that

$$u_n \rightharpoonup u \qquad \text{in } \mathcal{H} \tag{3.9}$$

for some $u \in \mathcal{H}$, and, by Lemma 2.1,

$$u_n \to u \qquad \text{in } L^p_{\text{loc}}(\mathbb{R}^N). \tag{3.10}$$

Therefore, by (3.8)–(3.10), we get

$$E'(u)\psi = \lim_{n\to\infty} E'(u_n)\psi = 0 \qquad \text{for any } \psi \in C_0^{\infty}(\mathbb{R}^N),$$

namely, u is a critical point of E.
(4) It follows from $(A3)$ and $(A5)$ that, for any natural number m, there is a positive number k_m, such that

$$|f(x,u)| \leq \frac{|u|}{m} + k_m|u|^{p-1}. \tag{3.11}$$

We claim

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n(x)|^2 \mathrm{d}x > 0 \tag{3.12}$$

by contradiction.
Otherwise, by Lemma 2.2, we have

$$u_n \to 0 \qquad \text{in } L^p(\mathbb{R}^N). \tag{3.13}$$

For n large enough, we have, by (3.8),

$$\frac{1}{2}\delta \leq E(u_n) - \frac{1}{2}E'(u_n)u_n = \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n(x))u_n(x)\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u_n(x))\mathrm{d}x.$$

In light of (3.11), we get

$$\frac{1}{2}\delta \leq \frac{1}{2}\int_{\mathbb{R}^N} \frac{|u_n(x)|^2}{m} + k_m|u_n(x)|^p\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|u_n(x)|^2}{2m} + \frac{k_m|u_n(x)|^p}{p}\mathrm{d}x.$$

Thus, together with (3.13) and the boundedness of $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, it follows that $\delta \leq 0$ by taking the limits first as $n \to \infty$ and then as $m \to \infty$, which is contradictory to $\delta > 0$.

(5) Thanks to (3.12), there is a subsequence of $\{u_n\}_{n=1}^{\infty}$, also denoted by $\{u_n\}_{n=1}^{\infty}$, such that

$$\int_{B(z_n,1)} u_n(x)^2 \mathrm{d}x > \varepsilon$$

for some positive number ε and a sequence $\{z_n\}_{n=1}^{\infty}$ with $z_n \in \mathbb{R}^N$. Then there are integral lattices $\{z_n'\}_{n=1}^{\infty}$ satisfying

$$\int_{B(z_n',2)} u_n(x)^2 \mathrm{d}x > \varepsilon.$$

Define $w_n(\cdot) := u_n(\cdot + z_n')$, $n = 1, 2, \cdots$. Then

$$\int_{B(0,2)} w_n(x)^2 \mathrm{d}x > \varepsilon, \tag{3.14}$$

and

$$E(w_n) \to \delta \quad \text{and} \quad E'(w_n) \to 0 \qquad \text{as } n \to \infty. \tag{3.15}$$

By repeating Step 3, $\{w_n\}_{n=1}^{\infty}$ possesses a subsequence, again denoted by $\{w_n\}_{n=1}^{\infty}$, such that

$$w_n \rightharpoonup w \qquad \text{in } \mathcal{H} \tag{3.16}$$

for some $w \in \mathcal{H}$,

$$w_n \to w \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N), \tag{3.17}$$

and w is a critical point of E. Moreover, by (3.14) and (3.17), w is nontrivial. □

We prove in the following theorem that the function w in (3.17) is the least energy solution to Eq (1.2).

Theorem 3.2. *Define Nehari manifold $\mathcal{N}$ by*

$$\mathcal{N} := \{u : u \in \mathcal{H} \setminus \{0\} \text{ and } E'(u)u = 0\}.$$

Then the number δ defined in (3.4) *fulfills $\delta = \inf_{u \in \mathcal{N}} E(u)$. Moreover, the function w in* (3.17) *is a critical point of the critical value δ.*

Proof. (1) Taking a function $u \in \mathcal{N}$, we have $u \neq 0$. Thus, for $n \in \mathbb{N}$ large enough, we get, as $\mu > 2$,

$$\begin{aligned} E(nu) &= \frac{n^2}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, nu(x))\mathrm{d}x \\ &\le \frac{n^2}{2}\|u\|^2 - n^\mu \int_{|nu(x)|>r} k(x)|u(x)|^\mu \mathrm{d}x + c' n^2 \int_{|nu(x)|\le r} |u(x)|^2 \mathrm{d}x \quad \text{(by (3.1) and (3.2))} \\ &< 0. \end{aligned} \tag{3.18}$$

Define a path $\widetilde{\gamma}(t) := tnu$, where $t \in [0, 1]$. Then, on account of (3.18), $\widetilde{\gamma} \in \Gamma$ (for the definition of Γ, see (3.3)). Consequently, we get

$$\delta \le \sup_{t\in[0,1]} E(\widetilde{\gamma}(t)). \tag{3.19}$$

(2) It follows from (3.7) that

$$\min_{\|v\|=d} E(v) \geq \frac{1}{8}(8c_1^p c_2)^{2/(2-p)} > 0. \tag{3.20}$$

Take $n > 1$ large enough such that $n\|u\| > d$. Then, by (3.18) and (3.20),

$$\begin{aligned}&\text{the function } E(\widetilde{\gamma}(\cdot)) : [0,1] \to \mathbb{R} \text{ reaches}\\ &\text{its maximum at some point } t \in (0,1).\end{aligned} \tag{3.21}$$

(3) Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{\gamma}(t)) = tn^2\|u\|^2 - \int_{\mathbb{R}^N} f(x, tnu(x))nu(x)\mathrm{d}x.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{\gamma}(t)) = 0 \text{ iff } \|u\|^2 = \int_{u(x)\neq 0} \frac{f(x, tnu(x))u(x)^2}{tnu(x)}\mathrm{d}x. \tag{3.22}$$

By $(A6)$, as a function of t, $\int_{u(x)\neq 0} \frac{f(x, tnu(x))u(x)^2}{tnu(x)}\mathrm{d}x$ is increasing, and then the equation $\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{\gamma}(t)) = 0$ has at most one solution in $(0, 1)$.

Since $u \in \mathcal{N}$, we have

$$\|u\|^2 - \int_{\mathbb{R}^N} f(x, u(x))u(x)\mathrm{d}x = 0.$$

Inserting the above equality into (3.22), we get

$$\int_{\mathbb{R}^N} f(x, u(x))tnu(x)\mathrm{d}x = \int_{\mathbb{R}^N} f(x, tnu(x))u(x)\mathrm{d}x.$$

Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{\gamma}(t)) = 0$ has a unique solution $t = n^{-1}$ on $(0, 1)$. Consequently, the function $E(\widetilde{\gamma}(\cdot)) : [0, 1] \to \mathbb{R}$ reaches its maximum at the point $t = n^{-1}$. Thus, by noting (3.19), we obtain

$$\delta \leq E(\widetilde{\gamma}(n^{-1})) = E(u) \text{ for any } u \in \mathcal{N}. \tag{3.23}$$

(4) Let $\gamma \in \Gamma$. Then $E(\gamma(1)) < 0$, i.e.,

$$\frac{1}{2}\|\gamma(1)\|^2 - \int_{\mathbb{R}^N} F(x, \gamma(1))\mathrm{d}x < 0.$$

Noting $(A4)$, we have

$$\frac{1}{2}\|\gamma(1)\|^2 - \frac{1}{\mu}\int_{\mathbb{R}^N} f(x, \gamma(1))\gamma(1)\mathrm{d}x < 0.$$

As $\mu > 2$, we get

$$\|\gamma(1)\|^2 - \int_{\mathbb{R}^N} f(x, \gamma(1))\gamma(1)\mathrm{d}x < 0. \tag{3.24}$$

(5) Set $\tau := \sup\{t : E'(\gamma(t))\gamma(t) \geq 0,\ t \in [0, 1]\}$.

By (3.5) and (3.11), choosing m such that $\frac{c_1^2}{m} \leq \frac{1}{2}$, we get

$$\begin{aligned}E'(u)u &= \|u\|^2 - \int_{\mathbb{R}^N} f(x, u(x))u(x)\mathrm{d}x\\ &\geq \frac{1}{2}\|u\|^2 - c_1^p k_m \|u\|^p.\end{aligned}$$

Taking a positive number d_1 less than $\min\left\{\|\gamma(1)\|, (4c_1^p k_m)^{1/(2-p)}\right\}$, we have

$$\min_{\|u\|=d_1} E'(u)u \geq \frac{1}{4}d_1^2,$$

which and (3.24) give us that there is a point $t_0 \in [\tau, 1)$ such that $E'(\gamma(t_0))\gamma(t_0) = 0$.

We prove that $\gamma(t_0) \neq 0$ by contradiction. If $\gamma(t_0) = 0$, then, by the same argument as above, there exists a number τ' such that $\tau < \tau' < 1$ and $E'(\gamma(\tau'))\gamma(\tau') \geq 0$, which is contradictory to the definition of τ.

In summary, $\gamma(t_0) \in \mathcal{N}$, i.e., $\gamma([0,1]) \cap \mathcal{N} \neq \emptyset$.

(6) It follows from $\gamma([0,1]) \cap \mathcal{N} \neq \emptyset$ that $\delta \geq \inf_{\mathcal{N}} E(u)$. This and (3.23) show us that $\delta = \inf_{u\in\mathcal{N}} E(u)$.

(7) We have proven in Theorem 3.1 that w is a nontrivial critical point of E. In particular, it follows that $w \in \mathcal{N}$. In this step, we prove $E(w) = \delta$. First, we have $E(w) \geq \delta$ since $w \in \mathcal{N}$ and $\delta = \inf_{u\in\mathcal{N}} E(u)$. In the following, we show that $E(w) \leq \delta$.

Noting that $(A4)$ and

$$E(w_n) - \frac{1}{2}E'(w_n)w_n = \frac{1}{2}\int_{\mathbb{R}^N} f(x, w_n(x))w_n(x)\mathrm{d}x - \int_{\mathbb{R}^N} F(x, w_n(x))\mathrm{d}x,$$

for any positive number R, noting $(A3)$ and $(A4)$, we have

$$E(w_n) - \frac{1}{2}E'(w_n)w_n \geq \frac{1}{2}\int_{B(0,R)} f(x, w_n(x))w_n(x)\mathrm{d}x - \int_{B(0,R)} F(x, w_n(x))\mathrm{d}x.$$

Thanks to $(A3)$, (3.1), (3.2), (3.6), and (3.15)–(3.17), taking limits in the above inequality, we find

$$\delta \geq \frac{1}{2}\int_{B(0,R)} f(x, w)w\mathrm{d}x - \int_{B(0,R)} F(x, w)\mathrm{d}x,$$

i.e., as R is arbitrary,

$$\delta \geq \frac{1}{2}\int_{\mathbb{R}^N} f(x, w)w\mathrm{d}x - \int_{\mathbb{R}^N} F(x, w)\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\delta &\geq \int_{\mathbb{R}} f(x, w)w\mathrm{d}x - \int_{\mathbb{R}^N} F(x, w)\mathrm{d}x + \frac{1}{2}\|w\|^2 - \frac{1}{2}\|w\|^2 \\ &= E(w) - \frac{1}{2}E'(w)w = E(w),\end{aligned}$$

where we have used the fact that w is a critical point of E in the last identity. □

4. Conclusions

In the present paper, we study a class of nonlinear Schrödinger equations that are driven by a kind of nonlocal operator. This kind of operator generalizes the fractional Laplacian and some nonlocal operators in the literature [10, 11]. We prove by the variational method that Eq (1.2) possesses a nontrivial least energy solution, which extends the results in [10, 11]. The results may apply to fractional quantum mechanics [15].

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Many thanks are due to the editors and reviewers for their constructive suggestions and valuable comments.

Conflict of interest

No potential conflict of interest exists regarding the publication of this paper.

References

1. V. Ambrosio, H. Hajaiej, Multiple solutions for a class of nonhomogeneous fractional Schrödinger equations in $\mathbb{R}^N$, *J. Dynam. Differ. Equ.*, **30** (2018), 1119–1143. https://doi.org/10.1007/s10884-017-9590-6
2. G. Autuori, P. Pucci, Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$, *J. Differ. Equations*, **255** (2013), 2340–2362. https://doi.org/10.1016/j.jde.2013.06.016
3. G. M. Bisci, R. Servadei, A Brezis-Nirenberg splitting approach for nonlocal fractional equations, *Nonlinear Anal.*, **119** (2015), 341–353. https://doi.org/10.1016/j.na.2014.10.025
4. J. Dávila, M. D. Pino, J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation, *J. Differ. Equations*, **256** (2014), 858–892. https://doi.org/10.1016/j.jde.2013.10.006
5. S. Dipierro, G. Palatucci, E. Valdinoci, Existence and symmetry results for a Schödinger type problem involving the fractional Laplacian, *Matematiche*, **68** (2013), 201–216.
6. P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *P. Roy. Soc. Edinb. A*, **142** (2012), 1237–1262. https://doi.org/10.1017/S0308210511000746
7. B. Ge, Multiple solutions of nonlinear Schrödinger equation with the fractional Laplacian, *Nonlinear Anal.-Real*, **30** (2016), 236–247. https://doi.org/10.1016/j.nonrwa.2016.01.003
8. Y. Hua, X. Yu, On the ground state solution for a critical fractional Laplacian equation, *Nonlinear Anal.*, **87** (2013), 116–125. https://doi.org/10.1016/j.na.2013.04.005
9. X. Shang, J. Zhang, Concentrating solutions of nonlinear fractional Schrödinger equation with potentials, *J. Differ. Equations*, **258** (2015), 1106–1128. https://doi.org/10.1016/j.jde.2014.10.012
10. G. Gu, W. Zhang, F. Zhao, Infinitely many sign-changing solutions for a nonlocal problem, *Ann. Mat. Pur. Appl.*, **197** (2018), 1429–1444. https://doi.org/10.1007/s10231-018-0731-2
11. M. Xiang, B. Zhang, D. Yang, Multiplicity results for variable-order fractional Laplacian equations with variable growth, *Nonlinear Anal.*, **178** (2019), 190–204. https://doi.org/10.1016/j.na.2018.07.016

12. M. Willem, *Minimax theorems*, Boston: Birkhäuser, 1996. https://doi.org/10.1007/978-1-4612-4146-1

13. Y. C. Zhang, Existence, regularity and positivity of ground states for nonlocal nonlinear Schrödinger equations, *Electron. J. Differ. Eq.*, **2019** (2019), 1–11.

14. R. A. Adams, *Sobolev spaces*, New York: Academic Press, 1975.

15. N. Laskin, *Fractional quantum mechanics*, Hackensack: World Scientific, 2018. https://doi.org/10.1142/10541