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Research article

Sharp estimate for starlikeness related to a tangent domain

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Abstract: In the recent years, the study of the Hankel determinant problems have been widely investigated by many researchers. We were essentially motivated by the recent research going on with the Hankel determinant and other coefficient bounds problems. In this research article, we first considered the subclass of analytic starlike functions connected with the domain of the tangent function. We then derived the initial four sharp coefficient bounds, the sharp Fekete-Szegö inequality, and the sharp second and third order Hankel determinant for the defined class. Also, we derived sharp estimates like sharp coefficient bounds, Fekete-Szegö estimate, and sharp second order Hankel determinant for the functions having logarithmic coefficient and for the inverse coefficient, respectively, for the defined functions class.

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1. Introduction

We present some important and basic concepts in this study section to help you better understand the main findings. To begin with the most basic definition, we employ the sign $\mathcal{A}$. The family $\mathcal{A}$ of all holomorphic functions defined in $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = \{\kappa : \kappa \in \mathbb{C} \quad \text{and} \quad |\kappa| < 1\}.$$

Furthermore, if $\mathcal{A}$ contains $\omega(\kappa)$, the relations

$$\omega(0) = 0 \quad \text{and} \quad \omega'(0) = 1,$$

are satisfied. Taylor and Maclaurin's series for functions belonging to family $\mathcal{A}$ is

$$\omega(\kappa) = \kappa + \sum_{n=2}^{\infty} d_n \kappa^n \qquad (\kappa \in \Omega). \tag{1.1}$$

Furthermore, the distinct family $\mathcal{S}$ contains all of the univalent functions of family $\mathcal{A}$. Even though function theory was first introduced in 1851, Biberbach [1] established the coefficient conjecture in 1916, which helped the subject become more well-known as a possible field for future research. De-Branges [2] proved this conjecture in 1985. From 1916 until 1985, a large number of mathematicians tried to prove or disprove the Bieberbach conjecture. Consequently, they identified multiple subfamilies within the class $\mathcal{S}$ of normalized univalent functions, each of which is associated with a different image domain. Families of starlike and convex functions, $\mathcal{S}^*$ and $\mathcal{K}$, respectively, are the most fundamental and significant subclasses of the functions class $\mathcal{S}$. From an analytical perspective,

$$\mathcal{S}^* = \left\{ \omega \in \mathcal{S} : Re\left(\frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \right) > 0 \right\},$$

and

$$\mathcal{K} = \left\{ \omega \in \mathcal{S} : Re\left(\frac{(\kappa\omega'(\kappa))'}{\omega(\kappa)} \right) > 0 \right\}.$$

In 1992, Ma and Minda defined [3]

$$\mathcal{S}^*(\phi) = \left\{ \omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec \phi(\kappa) \right\}. \tag{1.2}$$

With $Re(\phi) > 0$ in Ω, additionally, with $\phi'(0) > 0$, the image domain is starlike with regard to $\phi(0) = 1$, and the under unit disc the function ϕ maps Ω onto a star-shaped region. The image domain is symmetric about the real axis. Several subfamilies of the function class $\mathcal{A}$ are generalized by the set $\mathcal{S}^*(\phi)$, for example:

i. If

$$\phi(\kappa) = \frac{1 + O_1\kappa}{1 + O_2\kappa},$$

with $-1 \le O_2 < O_1 \le 1$, then

$$\mathcal{S}^*[O_1, O_2] \equiv \mathcal{S}^*\left(\frac{1 + O_1\kappa}{1 + O_2\kappa} \right),$$

where $\mathcal{S}^*[O_1, O_2]$ is the family of Janowski starlike functions, for further details, see [4].

ii. Select $\phi(\kappa) = \sqrt{1+\kappa}$, and we get the class $\mathcal{S}^*_{\mathcal{L}}$, defined and studied by Sokol et al. [5].

iii. Regarding the function

$$\phi(\kappa) = 1 + \sinh^{-1}\kappa,$$

we obtain the family $\mathcal{S}^*_{\rho}$, initialized by Kumar and Arora [6].

iv. If

$$\phi(\kappa) = e^{\kappa},$$

then the family $\mathcal{S}^*(\phi)$ turns into $\mathcal{S}_e^*$, which is examined and defined by Mendiratta [7].

v. In the case where $\phi(\kappa) = 1 + \sin(\kappa)$, the class $\mathcal{S}_{\sin}^*$ is the reduction of the class $\mathcal{S}^*(\phi)$. The $\mathcal{S}_{\sin}^*$ family, presented by Cho et al. [8] as:

$$\mathcal{S}_{\sin}^* = \left\{\omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec 1 + \sin(\kappa), \qquad (\kappa \in \Omega)\right\}, \tag{1.3}$$

indicates that the ration $\frac{\kappa\omega'(\kappa)}{\omega(\kappa)}$ lies in an eight-shaped region.

vi.The family $\mathcal{S}_{\cos}^*$ is obtained if we choose $\phi(\kappa) = \cos(\kappa)$, which was first proposed by Bano and Raza [9].

vii. Choose $\phi(\kappa) = \sec h(\kappa)$, and we derive a class $\mathcal{S}_{\sec h}^*$, which Al-Shbeil et al. [10] introduced.

For the given parameters $n, r \in \mathbb{N}$, the r^{th} Hankel determinant $\mathcal{H}_{r,n}$ was defined by [11] as follows:

$$\mathcal{H}_{r,n}(\omega) = \begin{vmatrix} d_n & d_{n+1} & . & . & . & d_{n+r-1} \\ d_{n+1} & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ d_{n+r-1} & . & . & . & . & d_{n+2(r-1)} \end{vmatrix}.$$

For the given values of n, r and $d_1 = 1$ the second and third Hankel determinants are defined as

$$\mathcal{H}_{2,1}(\omega) = \begin{vmatrix} 1 & d_2 \\ d_2 & d_3 \end{vmatrix} = d_3 - d_2^2, \ \mathcal{H}_{2,2}(\omega) = \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} = d_2 d_4 - d_3^2, \tag{1.4}$$

and

$$\mathcal{H}_{3,1}(\omega) = \begin{vmatrix} 1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{vmatrix} = d_3\left(d_2 d_4 - d_3^2\right) - d_4\left(d_4 - d_2 d_3\right) + d_5\left(d_3 - d_2^2\right). \tag{1.5}$$

This technique has shown to be successful when examining power series with integral coefficients and singularities by taking the Hankel determinant into account; see [12]. Bounds of $\mathcal{H}_{r,n}(\omega)$ for several kinds of univalent functions have been examined recently. Noonan and Thomas [13], Hayman [14], and Ohran et al. [15] evaluated the boundaries related to the third Hankel determinant. For the details the study of the Hankel determinant, we refer the reader to see [16–22].

In this study, we consider the subsequent subclass of analytic functions:

$$\mathcal{S}_{\tan}^* = \left\{\omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec \frac{2 + \tan(\kappa)}{2}\right\} \quad (\kappa \in \mathbf{\Omega}). \tag{1.6}$$

Khan et al. [16] introduced and examined the class $\mathcal{S}_{\tan}^*$. Also the Hankel determinant problem was derived.

2. Set of Lemmas

The following is a list of useful lemmas that we use in our main finding.

The set of all analytic functions p with a positive real component is denoted by $\mathcal{P}$, and its series representation is seen below:

$$p(\kappa) = 1 + \sum_{n=1}^{\infty} c_n \kappa^n, \ \kappa \in \Omega. \tag{2.1}$$

Lemma 2.1. *If $p \in \mathcal{P}$, then the forgoing estimations hold:*

$$|c_k| \leq 2, \ k \geq 1, \tag{2.2}$$

$$|c_{k+n} - \mu c_k c_n| < 2, \ \ 0 < \mu \leq 1, \tag{2.3}$$

and for $\eta \in \mathbb{C}$, we have

$$\left|c_2 - \eta c_1^2\right| < 2 \max\{1, |2\eta - 1|\}. \tag{2.4}$$

For the inequalities (2.2), (2.3), see [11], and (2.4) is provided in [23].

Lemma 2.2. *[24] If $p \in \mathcal{P}$ and has the form* (2.1)*, then*

$$|\alpha_1 c_1^3 - \alpha_2 c_1 c_2 + \alpha_3 c_3| \leq 2|\alpha_1| + 2|\alpha_2 - 2\alpha_1| + 2|\alpha_1 - \alpha_2 + \alpha_3|, \tag{2.5}$$

where α_1, α_2, and α_3 are real numbers.

Lemma 2.3. *[25] Let a_1, b_1, d_1, and e_1 address the approximations $a_1, e_1 \in (0, 1)$, and*

$$\begin{aligned} &8e_1(1 - e_1)\left[(a_1 b_1 - 2d_1)^2 + (a_1(e_1 + a_1) - b_1)^2\right] \\ &+ a_1(1 - a_1)(b_1 - 2e_1 a_1)^2 \\ &\leq 4a_1^2(1 - a_1)^2 e_1(1 - e_1). \end{aligned}$$

If $h \in \mathcal{P}$ and is of the form (2.1)*, then*

$$\left|d_1 c_1^4 + e_1 c_2^2 + 2a_1 c_1 c_3 - \frac{3}{2} b_1 c_1^2 c_2 - c_4\right| \leq 2.$$

Lemma 2.4. *[26] Let $p \in \mathcal{P}$ and x, δ, and ρ with $|x| \leq 1, |\delta| \leq 1$ and $|\rho| \leq 1$ belong to Ω, then we have*

$$\begin{aligned} 2c_2 &= c_1^2 + x\left(4 - c_1^2\right), \\ 4c_3 &= 2x\left(4 - c_1^2\right)c_1 - x^2\left(4 - c_1^2\right)c_1 + 2\delta\left(1 - |x|^2\right)\left(4 - c_1^2\right) + c_1^3, \\ 8c_4 &= c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right)\left(1 - |x|^2\right) \\ &\quad \cdot \left[c_1(x - 1) + \overline{x}\delta^2 - \left(1 - |\delta|^2\right)\rho\right]. \end{aligned}$$

3. Coefficient bounds for the family $\mathcal{S}^*_{\tan}$

Theorem 3.1. *Let $\omega \in \mathcal{S}^*_{\tan}$. Then the following estimates hold:*

$$\left|d_3 - \lambda d_2^2\right| \leq \frac{1}{4}\left\{1, \left|\frac{2\lambda - 1}{2}\right|\right\}, \tag{3.1}$$

and

$$|d_2d_3 - d_4| \leq \frac{1}{6}.$$

The bounds are sharp for functions defined below for $n = 2$ *and* 3, *respectively:*

$$\omega_n(\kappa) = \kappa\left(\exp \int_0^\kappa \left(\frac{\tan(t^n)}{2t}\right)\right)dt = 1 + \frac{1}{2n}\kappa^n + \cdots, \ for\ n = 1, 2, 3, 4. \tag{3.2}$$

Proof. Let $\omega \in \mathcal{S}^*_{\tan}$. Then, by property of Schwarz function $u(\kappa)$, so that $u(0) = 0$ and $|u(\kappa)| \leq |\kappa|$, we have

$$\frac{\kappa\omega'(\kappa)}{\omega(\kappa)} = \frac{2 + \tan(u(\kappa))}{2}.$$

Given that the function with positive real $p(\kappa)$ and $u(\kappa)$ has a one-to-one relationship, we have

$$u(\kappa) = \frac{p(\kappa) - 1}{p(\kappa) + 1}.$$

Now,

$$\begin{aligned} \frac{2 + \tan(u(\kappa))}{2} &= 1 + \frac{1}{4}c_1\kappa + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)\kappa^2 \\ &+ \left(\frac{1}{12}c_1^3 - \frac{1}{4}c_2c_1 + \frac{1}{4}c_3\right)\kappa^3 \\ &+ \left(-\frac{1}{16}c_1^4 + \frac{1}{4}c_1^2c_2 - \frac{1}{4}c_3c_1 - \frac{1}{8}c_2^2 + \frac{1}{4}c_4\right)\kappa^4 + \cdots. \end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned} \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} &= 1 + d_2\kappa + \left(2d_3 - d_2^2\right)\kappa^2 + \left(d_2^3 - 3d_2d_3 + 3d_4\right)\kappa^3 \\ &+ \left(-d_2^4 + 4d_2^2d_3 - 4d_2d_4 - 2d_3^2 + 4d_5\right)\kappa^4 + \cdots. \end{aligned} \tag{3.4}$$

On comparing (3.3) and (3.4), we have

$$d_2 = \frac{1}{4}c_1, \tag{3.5}$$

$$d_3 = \frac{1}{8}c_2 - \frac{1}{32}c_1^2, \tag{3.6}$$

$$d_4 = \frac{17}{1152}c_1^3 - \frac{5}{96}c_2c_1 + \frac{1}{12}c_3, \tag{3.7}$$

$$d_5 = -\frac{157}{18\,432}c_1^4 + \frac{29}{768}c_1^2c_2 - \frac{1}{24}c_3c_1 - \frac{3}{128}c_2^2 + \frac{1}{16}c_4. \tag{3.8}$$

From (3.5) and (3.6), we have

$$\left|d_3 - \lambda d_2^2\right| = \frac{1}{8}\left|c_2 - \frac{2\lambda + 1}{4}c_1^2\right|.$$

By applying (2.4) to the above equation, we get

$$\left|d_3 - \lambda d_2^2\right| \leq \frac{1}{4}\left\{1, \left|\frac{2\lambda - 1}{2}\right|\right\}.$$

Also, from (3.5), (3.6), and (3.7) , we have

$$|d_2 d_3 - d_4| = \frac{1}{12}\left|\frac{13}{48}c_1^3 - c_2 c_1 + c_3\right|.$$

Then, by Lemma 2.2, we have

$$|d_2 d_3 - d_4| \leq \frac{1}{6}.$$

□

Corollary 3.2. *Let* $\omega \in \mathcal{S}^*_{\tan}$. *Then,*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{4}.$$

The result is sharp for the function defined in (3.2)*, for* $n = 2$.

Theorem 3.3. *Let* $\omega \in \mathcal{S}^*_{\tan}$. *Then,*

$$\left|d_2 d_4 - d_3^2\right| \leq \frac{1}{16}.$$

The result is sharp for the function defined in (3.2)*, for* $n = 2$.

Proof. From (3.5), (3.6), and (3.7) , we have

$$\left|d_2 d_4 - d_3^2\right| = \left|\frac{25}{9216}c_1^4 - \frac{1}{192}c_1^2 c_2 + \frac{1}{48}c_3 c_1 - \frac{1}{64}c_2^2\right|.$$

Now, using Lemma 2.4, with $c_1 = c$ and $|x| = y$, we have

$$\begin{aligned}\left|d_2 d_4 - d_3^2\right| &\leq \frac{13}{9216}c^4 + \frac{1}{192}c^2\left(4 - c^2\right)y^2 + \frac{1}{96}c\left(4 - c^2\right)\left(1 - y^2\right) + \frac{1}{256}\left(4 - c^2\right)^2 y^2 \\ &= F(c, y).\end{aligned}$$

Now, partially differentiating with regard to y, the following can be found

$$\frac{\partial F(c, y)}{\partial y} = \frac{1}{384}y(c - 2)^2\left(-c^2 + 4c + 12\right).$$

Clearly we see that $\frac{\partial F(c,y)}{\partial y} > 0$ for all $y \in [0, 1]$ and $c \in [0, 2]$, and we have

$$\begin{aligned}F(c, 1) &= \frac{1}{9216}c^4 - \frac{1}{96}c^2 + \frac{1}{16} \\ &= \Gamma(c).\end{aligned}$$

$\Gamma''(c) < 0$ for $c = 0$, so the maximum attained at $c_1 = 0$ is

$$\left|d_2 d_4 - d_3^2\right| \leq \frac{1}{16}.$$

□

Theorem 3.4. *Let $\omega \in \mathcal{S}^{*}_{\tan}$. Then,*

$$\left|\mathcal{H}_{3,1}(\omega)\right| \leq \frac{1}{64}.$$

The result is sharp for the function defined in (3.2)*, for $n = 3$.*

Proof. From (1.5), we have

$$\mathcal{H}_{3,1}(\omega) = -d_5 d_2^2 + 2d_2 d_3 d_4 - d_3^3 + d_3 d_5 - d_4^2.$$

Taking $c_1 = c$ in the identities (3.5), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}\mathcal{H}_{3,1}(\omega) = \frac{1}{5308416}\Big[&-31104c_1^2c_4 + 768c_1^3c_3 - 9012c_1^4c_2 - 25920c_2^3\\ &-36864c_3^2 + 2021c_1^6 + 12816c_1^2c_2^2 + 41472c_2c_4 + 46080c_1c_2c_3\Big].\end{aligned} \tag{3.9}$$

Furthermore, by using $4 - c^2 = t$ in Lemma 2.4, we have

$$\begin{aligned}\mathcal{H}_{3,1}(\omega) = \frac{1}{5308416}\Big[&-169c^6 - 1296c^4tx^3 + 2544c^4tx^2 - 666c^4tx + 5184c^3tx\left(1-|x|^2\right)\delta\\ &-2496c^3t\left(1-|x|^2\right)\delta + 288c^2t^2x^4 - 4320c^2t^2x^3 + 3564c^2t^2x^2\\ &-5184c^2tx^2 + 5184\overline{x}c^2t\left(1-|x|^2\right)\delta^2\\ &-5184\left(1-|\delta|^2\right)\rho c^2t\left(1-|x|^2\right) - 1152ct^2x^2\left(1-|x|^2\right)\delta\\ &+3456ct^2x\left(1-|x|^2\right)\delta - 3240t^3x^3 + 10368t^2x^3 - 10368\overline{x}t^2x\left(1-|x|^2\right)\delta^2\\ &+10368\left(1-|\delta|^2\right)\rho t^2x\left(1-|x|^2\right) - 9216t^2\left(1-|x|^2\right)^2\delta^2\Big].\end{aligned}$$

Let

$$\mathcal{H}_{3,1}(\omega) = \frac{1}{5308416}\left(d_1(c,x) + d_2(c,x)\delta + d_3(c,x)\delta^2 + \Phi(c,x,\delta)\rho\right),$$

where

$$\begin{aligned}d_1(c,x) &= -169c^6 + \left(4-c^2\right)\Big\{-1296c^4x^3 + 2544c^4x^2 - 666c^4x - 5184c^2x^2\\ &\quad+\left(4-c^2\right)\left(288c^2x^4 - 1080c^2x^3 + 3564c^2x^2 - 2592x^3\right)\Big\},\\ d_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\\ &\quad\cdot\left(5184c^3x - 2496c^3 + \left(4-c^2\right)\left(-1152cx^2 + 3456cx\right)\right),\\ d_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(5184\overline{x}c^2 - \left(4-c^2\right)\left(1152x^2 + 9216\right)\right),\\ \Phi(c,x,\delta) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left(10368\left(4-c^2\right)x - 5184c^2\right).\end{aligned}$$

Let $|x| = \tau$, $|\delta| = y$ and $|\rho| \leq 1$, then

$$\begin{aligned}\mathcal{H}_{3,1}(\omega) &= \frac{1}{5308416}\left(|d_1(c,\tau)| + |d_2(c,\tau)|\,y + |d_3(c,\tau)|\,y^2 + |\Phi(c,\tau,\delta)|\right)\\ &\leq \frac{1}{5308416}\Gamma(c,\tau,y),\end{aligned} \tag{3.10}$$

where

$$\Gamma(c,\tau,y) = h_1(c,\tau) + h_2(c,\tau)\,y + h_3(c,\tau)\,y^2 + h_4(c,\tau)\left(1-y^2\right),$$

with

$$\begin{aligned} h_1(c,\tau) &= 169c^6 + \left(4-c^2\right)\Big\{1296c^4\tau^3 + 2544c^4\tau^2 + 666c^4\tau + 5184c^2\tau^2 \\ &\quad + \left(4-c^2\right)\left(288c^2\tau^4 + 1080c^2\tau^3 + 3564c^2\tau^2 + 2592\tau^3\right)\Big\} \\ h_2(c,\tau) &= 192\left(4-c^2\right)\left(1-\tau^2\right)\left(27c^3\tau + 13c^3 + \left(4-c^2\right)\left(6c\tau^2 + 18c\tau\right)\right), \\ h_3(c,\tau) &= 576\left(4-c^2\right)\left(1-\tau^2\right)\left(9\tau c^2 + \left(4-c^2\right)\left(2\tau^2+16\right)\right), \\ h_4(c,\tau) &= 5184\left(4-c^2\right)\left(1-\tau^2\right)\left(1-|\delta|^2\right)\left(2\left(4-c^2\right)\tau + c^2\right). \end{aligned}$$

In order to determine, the value which is maximum for the function $\Gamma(c,\tau,y)$ in the closed cuboid $[0,2]\times[0,1]\times[0,1]$, we need to show it in the following three manners.

I. Interior points of cuboid

We now determine $\Gamma(c,\tau,y)$ greatest value inside the cuboid. Assume $(c,\tau,y)\in[0,2)\times[0,1)\times(0,1)$. By differentiating $\Gamma(c,\tau,y)$ with regard to y, we get

$$\begin{aligned} \frac{\partial\Gamma(c,\tau,y)}{\partial y} &= 36\left(4-c^2\right)\left(1-\tau^2\right)\Big\{\left(192\left(27c^3\tau + 13c^3 + \left(4-c^2\right)\left(6c\tau^2 + 18c\tau\right)\right)\right) \\ &\quad +1152y(\tau-1)\left(9c^2 + 2\left(4-c^2\right)(\tau-8)\right)\Big\}. \end{aligned}$$

Putting $\frac{\partial\Gamma(c,\tau,\delta)}{\partial y} = 0$, gives

$$y = \frac{27c^3\tau + 13c^3 + \left(4-c^2\right)\left(6c\tau^2 + 18c\tau\right)}{6(\tau-1)\left(-9c^2 + 2\left(4-c^2\right)(8-\tau)\right)} = y_1,$$

if y_1 is a critical point inside Δ, then $y_1\in(0,1)$, this is only feasible if

$$27c^3\tau + 13c^3 + \left(4-c^2\right)\left(6c\tau^2 + 18c\tau\right) - 12\left(4-c^2\right)(\tau-1)(8-\tau) < 54c^2(1-\tau), \tag{3.11}$$

and

$$c^2 > \frac{8(8-\tau)}{(25-2\tau)}. \tag{3.12}$$

In order to determine the critical point, we must come up with a solution that meets both of the (3.11) and (3.12) inequalities. Assume $\Gamma(\tau) = \frac{8(8-\tau)}{(25-2\tau)}$, implies $\Gamma'(\tau) = \frac{-72}{(25-2\tau)^2} < 0$, is decreasing function, so

$$c^2 > \frac{64}{25}.$$

The straightforward computations show that, for $\tau\in\left[\frac{7}{27},1\right)$, (3.11) is not held. Hence, it may be said that the cuboid $[0,2)\times\left[\frac{7}{27},1\right)\times(0,1)$ is devoid of critical points for the function $\Gamma(c,\tau,y)$. Assume that (c,τ,y) is a critical point of Γ in the cuboid's interior that satisfies the requirements $\tau\in\left[0,\frac{7}{27}\right)$ and $y\in(0,1)$, which lead us to $c^2 > g\left(\frac{7}{27}\right) = \frac{1672}{661}$. Furthermore, it is evident that

$$h_1(c,\tau) \le h_1\left(c,\frac{7}{27}\right) = v_1(c).$$

Since $1-\tau^2 \le 1$ and $0<\tau<\frac{7}{27}$, we have

$$\begin{aligned} h_2(c,\tau) &\le 192\left(4-c^2\right)\left(27c^3\left(\frac{7}{27}\right)+13c^3+\left(4-c^2\right)\left(6c\left(\frac{7}{27}\right)^2+18c\left(\frac{7}{27}\right)\right)\right) \\ &= \frac{729}{680}h_2\left(c,\frac{7}{27}\right)=v_2(c). \end{aligned}$$

Similarly, we obtain

$$h_j(c,\tau) \le \frac{729}{680}h_j\left(c,\frac{7}{27}\right)=v_j(c),\ \ j=3,4.$$

Consequently,

$$\Gamma(c,\tau,y) \le v_1(c)+v_4(c)\,y+v_2(c)\,y+\left(v_3(c)-v_4(c)\right)y^2=\Psi(c,y).$$

Differentiating with reference to "y", we get

$$\frac{\partial\Psi(c,y)}{\partial y}=v_2(c)+2\left(v_3(c)-v_4(c)\right)y.$$

Consider

$$h_3(c)-h_4(c)=\left(\frac{846\,080}{81}c^4-\frac{5524\,480}{81}c^2+\frac{8560\,640}{81}\right)\le 0,\ \text{for } c\in\left(\sqrt{\frac{1672}{661}},2\right).$$

Next, for all $c\in\left(\sqrt{\frac{1672}{661}},2\right)$ and $y\in(0,1)$, we get

$$\begin{aligned} \frac{\partial\Psi}{\partial y} &= v_2(c)+2\left(v_3(c)-v_4(c)\right)y \\ &\ge v_2(c)+2\left(v_3(c)-v_4(c)\right) \\ &= -\frac{232\,192}{81}c^5+\frac{1692\,160}{81}c^4+\frac{613\,376}{81}c^3 \\ &\quad -\frac{11\,048\,960}{81}c^2+\frac{1261\,568}{81}c+\frac{17\,121\,280}{81} \\ &\ge 0. \end{aligned}$$

Thus, we obtain where

$$\begin{aligned} \Upsilon(c) \;=\; & -\frac{232\,192}{81}c^5+\frac{1692\,160}{81}c^4+\frac{613\,376}{81}c^3 \\ & -\frac{11\,048\,960}{81}c^2+\frac{1261\,568}{81}c+\frac{17\,121\,280}{81}. \end{aligned}$$

For each point c, where $c\in\left(\sqrt{\frac{1672}{661}},2\right)$, it should be seen that

$$\Upsilon'(c)\neq 0.$$

Moreover, it showing that the the maximum value 116139 can be obtained as $c = 1$ for the function $\Upsilon(c)$.

$$\Psi(c, y) \leq \Psi(c, 1) = v_1(c) + v_2(c) + v_3(c) = \Upsilon(c).$$

II. Cuboid all six faces

Now, we have to determine the given function $\Gamma(c, \tau, y)$ maximum value on all six faces of the cuboid Δ.

(i). Upon letting $c = 0 : \Gamma(0, \tau, y)$, we have

$$\begin{aligned} h_1(\tau, y) &= -18\,432\tau^4 y^2 + 165\,888\tau^3 y^2 - 124\,416\tau^3 - 129\,024\tau^2 y^2 \\ &\quad -165\,888\tau y^2 + 165\,888\tau + 147\,456 y^2. \end{aligned}$$

Then

$$\frac{\partial h_1(\tau, y)}{\partial y} = 36\,864 y(\tau - 1)^2\left(-\tau^2 + 7\tau + 8\right) \neq 0 \qquad (y \in (0, 1)).$$

It indicates that inside the interval $(0, 1) \times (0, 1)$, h_1 has no optimal points.

(ii). If we take $c = 2$, we have

$$\Gamma(2, \tau, y) = 10\,816.$$

(iii). If we take $\tau = 0, \Gamma(c, 0, y)$ turns into

$$\begin{aligned} h_2(c, y) &= 169c^6 - 2496c^5 y + 14\,400c^4 y^2 - 5184c^4 \\ &\quad +9984c^3 y - 94\,464c^2 y^2 + 20\,736c^2 + 147\,456y^2. \end{aligned}$$

Then $\frac{\partial h_2(c,y)}{\partial y} = 0$, gives

$$y = \frac{13c^3}{6(25c^2 - 64)} = y_0. \tag{3.13}$$

For the provided range of $y, y_0 \in (0, 1)$, if $c > c_0 \approx 1.6$.

Also, $\frac{\partial h_2(c,y)}{\partial c} = 0$, gives

$$6c\left(-2080c^3 y + 9600c^2 y^2 + 4992cy - 3456c^2 + 169c^4 - 31\,488y^2 + 6912\right) = 0. \tag{3.14}$$

Putting (3.13) in (3.14), we obtain

$$228\,150c^9 - 13\,738\,752c^7 + 92\,275\,200c^5 - 217\,645\,056c^3 + 169\,869\,312c = 0.$$

When we solve for c inside the interval $(0, 2)$, we get $c \approx 1.325\,1$. According to this, $\Gamma(c, 0, y)$ does not have an ideal solution.

(iv). If we take $\tau = 1 : \Gamma(c, 1, y)$ turns into

$$h_3(c) = 595c^6 - 24\,024c^4 + 78\,912c^2 + 41\,472,$$

then $\frac{\partial h_3}{\partial c} = 0$ gives a critical point $c \approx 1.325\,5$, where h_3 attains its maximum value, that is,

$$h_3(c) \leq 1091\,80.$$

($\mathfrak{v}$). If we take $y = 0 : \Gamma(c, \tau, 0)$ becomes

$$\begin{aligned} h_4(c,\tau) &= 288c^6\tau^4 - 216c^6\tau^3 + 1020c^6\tau^2 - 666c^6\tau + 169c^6 \\ &\quad -2304c^4\tau^4 - 11\,232c^4\tau^3 - 18\,336c^4\tau^2 + 13\,032c^4\tau \\ &\quad -5184c^4 + 4608c^2\tau^4 + 79\,488c^2\tau^3 + 57\,024c^2\tau^2 \\ &\quad -82\,944c^2\tau + 20\,736c^2 - 124\,416\tau^3 + 165\,888\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial h_4}{\partial c} &= 1728c^5\tau^4 - 1296c^5\tau^3 + 6120c^5\tau^2 - 3996c^5\tau + 1014c^5 \\ &\quad -9216c^3\tau^4 - 44\,928c^3\tau^3 - 73\,344c^3\tau^2 + 52\,128c^3\tau \\ &\quad -20\,736c^3 + 9216c\tau^4 + 158\,976c\tau^3 + 114\,048c\tau^2 \\ &\quad -165\,888c\tau + 41\,472c \end{aligned}$$

and

$$\frac{\partial h_4}{\partial \tau} = 6\left(4 - c^2\right)\left(\begin{array}{c} -192c^4\tau^3 + 108c^4\tau^2 - 340c^4\tau + 111c^4 + 768c^2\tau^3 \\ +6048c^2\tau^2 + 4752c^2\tau - 1728c^2 - 15\,552\tau^2 + 6912 \end{array}\right).$$

The computational analysis of the system of equations $\frac{\partial h_4}{\partial c} = 0$ and $\frac{\partial h_4}{\partial \tau} = 0$ reveals that there are no solutions in $(0, 2) \times (0, 1)$.

($\mathfrak{vi}$). Finally, If we take $y = 1 : \Gamma(c, \tau, 1)$, transforms to

$$\begin{aligned} h_5(c,\tau) &= 288c^6\tau^4 - 216c^6\tau^3 + 1020c^6\tau^2 - 666c^6\tau + 169c^6 \\ &\quad -1152c^5\tau^4 + 1728c^5\tau^3 + 3648c^5\tau^2 - 1728c^5\tau \\ &\quad -2496c^5 - 3456c^4\tau^4 + 4320c^4\tau^3 - 31\,584c^4\tau^2 \\ &\quad -2520c^4\tau + 9216c^4 + 9216c^3\tau^4 + 6912c^3\tau^3 \\ &\quad -19\,200c^3\tau^2 - 6912c^3\tau + 9984c^3 + 13\,824c^2\tau^4 \\ &\quad -24\,192c^2\tau^3 + 142\,272c^2\tau^2 + 20\,736c^2\tau - 73\,728c^2 \\ &\quad -18\,432c\tau^4 - 55\,296c\tau^3 + 18\,432c\tau^2 + 55\,296c\tau - 18\,432\tau^4 \\ &\quad +41\,472\tau^3 - 129\,024\tau^2 + 147\,456 \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial h_5}{\partial c} &= 1728c^5\tau^4 - 1296c^5\tau^3 + 6120c^5\tau^2 - 3996c^5\tau \\ &\quad +1014c^5 - 5760c^4\tau^4 + 8640c^4\tau^3 + 18\,240c^4\tau^2 \\ &\quad -8640c^4\tau - 12\,480c^4 - 13\,824c^3\tau^4 + 17\,280c^3\tau^3 \\ &\quad -126\,336c^3\tau^2 - 10\,080c^3\tau + 36\,864c^3 + 27\,648c^2\tau^4 \\ &\quad +20\,736c^2\tau^3 - 57\,600c^2\tau^2 - 20\,736c^2\tau + 29\,952c^2 \\ &\quad +27\,648c\tau^4 - 48\,384c\tau^3 + 284\,544c\tau^2 + 41\,472c\tau \\ &\quad -147\,456c - 18\,432\tau^4 - 55\,296\tau^3 + 18\,432\tau^2 + 55\,296\tau \end{aligned}$$

and

$$\begin{aligned}\frac{\partial h_5}{\partial \tau} &= 6\left(4-c^2\right)\left[-192c^4\tau^3+108c^4\tau^2-340c^4\tau+111c^4+768c^3\tau^3\right.\\&\quad -864c^3\tau^2-1216c^3\tau+288c^3+1536c^2\tau^3-1728c^2\tau^2\\&\quad +9168c^2\tau+864c^2-3072c\tau^3-6912c\tau^2+1536c\tau\\&\quad \left.+2304c-3072\tau^3+5184\tau^2-10\,752\tau\right].\end{aligned}$$

The computation also shows that there are no solutions in $(0,2)\times(0,1)$ for the system of equations

$$\frac{\partial h_5}{\partial c}=0 \quad \text{and} \quad \frac{\partial h_5}{\partial \tau}=0.$$

III. On the twelve edges of the cuboid

The last task is to determine $\Gamma(c,\tau,y)$ maximum values along each of the twelve edges.

(i). $\tau=0$ and $y=0:\Gamma(c,0,0)$ transform to

$$\Gamma(c,0,0)=169c^6-5184c^4+20\,736c^2=h_6(c),$$

then

$$h_6'(c)=0,$$

showing that the maximum value

$$h_6(c)\leq 22337,$$

can be obtained for the critical point $c\approx 1.499$.

(ii). On $\tau=0$ and $y=1$: $\Gamma(c,0,1)$ becomes

$$\Gamma(c,0,1)=\left(13c^3-96c^2+384\right)^2=h_7(c),$$

then

$$h_7'(c)=0,$$

showing that the maximum value

$$h_7(c)\leq 147\,456,$$

can be obtained for the critical point $c=0$.

(iii). On $\tau=0$ and $c=0:\Gamma(0,0,y)$ becomes

$$h_8(y)=147\,456y^2.$$

Clearly, $\frac{\partial\Gamma(0,0,y)}{\partial y}>0$ is an increasing function at $[0,1]$, so the maximum attained at $\tau=1$

$$h_8(1)\leq 147\,456.$$

Since $\Gamma(c,1,1)$ and $\Gamma(c,1,0)$ are the function where the term τ is not involved, that is

$$h_9(c)=\Gamma(c,1,1)=\Gamma(c,1,0)=595c^6-24\,024c^4+78\,912c^2+41\,472.$$

Putting

$$\frac{\partial h_9(c)}{\partial c} = 0,$$

we have the critical point $c \approx 1.3255$ and the maximum value of $h_9(c)$ is

$$h_9(c) \leq 109180.$$

(iv). Let $\tau = 1$ and $c = 0 : \Gamma(0, 1, y)$, we have

$$\Gamma(0, 1, y) = h_{10}(y) = 41472.$$

(v). Putting $c = 2$, we have

$$\Gamma(2, \tau, y) = \Gamma(2, 1, y) = \Gamma(2, q, 1) = \Gamma(2, q, 0) = 10816.$$

(vi). If $c = 0$ and $y = 0 : \Gamma(0, \tau, 0)$, we have

$$h_{11}(\tau) = -41472\tau\left(3\tau^2 - 4\right).$$

Clearly,

$$h'_{11}(\tau) = 165888 - 373248\tau^2.$$

Thus, we know that $h'_{11}(\tau) = 0$ gives $\tau \approx 0.6667$, at which $h_{11}(\tau)$ obtain its maximum value, which is given by

$$h_{11}(\tau) \leq 73728.$$

(vii). If $c = 0$ and $y = 1$: $\Gamma(0, \tau, 1)$ we have

$$h_{12}(\tau) = -18432\tau^4 + 41472\tau^3 - 129024\tau^2 + 147456.$$

Then,

$$h'_{12}(\tau) = -73728\tau^3 + 124416\tau^2 - 258048\tau.$$

Thus, we know that $h'_{12}(\tau) = 0$ gives $\tau = 0$, at which $h_{12}(\tau)$ obtain its maximum value, which is given by

$$h_{12}(\tau) \leq 147456.$$

Thus, it can be seen that

$$\Gamma(c, \tau, y) \leq 147456.$$

According to (3.10), we thus obtain the following inequality.

$$\left|\mathcal{H}_{3,1}(\omega)\right| \leq \frac{1}{16}.$$

□

4. Logarithmic coefficient functionals for the class $\mathcal{S}^*_{\tan}$

The logarithmic coefficients of a given function ω , represented by $\gamma_n = \gamma_n(\omega)$, are defined by

$$\frac{1}{2}\log\left(\frac{\omega(\kappa)}{\kappa}\right) = \sum_{n=1}^{\infty}\gamma_n\kappa^n.$$

The Hankel determinant, whose entries are the logarithmic coefficients, is naturally taken into consideration. In [27, 28], Kowalczyk first introduced the Hankel determinant containing logarithmic coefficients as the elements, which is given by

$$H_{n,q}(\omega) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}.$$

In particular, it is noted that

$$\begin{aligned} H_{n,q}(\omega) &= \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} \\ &= \left|\gamma_1\gamma_3 - \gamma_2^2\right|. \end{aligned}$$

The logarithmic coefficients of ω, if it is represented by (1.1), are as follows:

$$\gamma_1 = \frac{1}{2}d_2, \tag{4.1}$$

$$\gamma_2 = \frac{1}{2}\left(d_3 - \frac{1}{2}d_2^2\right), \tag{4.2}$$

$$\gamma_3 = \frac{1}{2}\left(d_4 - d_2d_3 + \frac{1}{3}d_2^3\right), \tag{4.3}$$

$$\gamma_4 = \frac{1}{2}\left(d_5 - d_2d_4 + d_2^2d_3 - \frac{1}{2}d_3^2 - \frac{1}{4}d_2^4\right). \tag{4.4}$$

Theorem 4.1. *Let ω called member of $\mathcal{S}^*_{\tan}$. Then*

$$|\gamma_1| \le \frac{1}{4},$$
$$|\gamma_2| \le \frac{1}{8},$$
$$|\gamma_3| \le \frac{1}{12},$$
$$|\gamma_4| \le \frac{1}{16}.$$

These outcomes are sharp for functions defined in (3.2) *for* $n = 1, 2, 3, 4$ *respectively.*

Proof. Applying (3.5) to (3.8), then to (4.1) , then to (4.4) , we have

$$\gamma_1 = \frac{1}{8}c_1, \tag{4.5}$$

$$\gamma_2 = \frac{1}{16}c_2 - \frac{1}{32}c_1^2, \tag{4.6}$$

$$\gamma_3 = \frac{1}{72}c_1^3 - \frac{1}{24}c_2c_1 + \frac{1}{24}c_3, \tag{4.7}$$

$$\gamma_4 = -\frac{1}{128}c_1^4 + \frac{1}{32}c_1^2c_2 - \frac{1}{32}c_3c_1 - \frac{1}{64}c_2^2 + \frac{1}{32}c_4. \tag{4.8}$$

Applying (2.2) to (4.5) , we have

$$|\gamma_1| \leq \frac{1}{4}.$$

To find bound of γ_2, apply (2.3) to (4.6) , we have

$$|\gamma_2| \leq \frac{1}{8}.$$

Applying (2.5) to (4.7) , we get

$$|\gamma_3| \leq \frac{1}{12},$$

and

$$\begin{aligned} |\gamma_4| &= \frac{1}{32}\left|\frac{1}{4}c_1^4 - c_1^2c_2 + c_3c_1 + \frac{1}{2}c_2^2 - c_4\right| \\ &\leq \frac{1}{16} \quad \text{(using Lemma (2.3))}. \end{aligned}$$

□

Theorem 4.2. *Let $\omega \in \mathcal{S}^*_{\tan}$. Then for complex number λ, we have*

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \frac{1}{8}\max\left\{1, \frac{|\lambda|}{2}\right\}.$$

The function defined in (3.2) *, for $n = 2$ yields a sharp result.*

Proof. From (4.5) and (4.6) , we have

$$\left|\gamma_2 - \lambda\gamma_1^2\right| = \frac{1}{16}\left|c_2 - \frac{\lambda+2}{4}c_1^2\right|.$$

Applying (2.4) to the preceding equation yields the desired outcome. □

Theorem 4.3. *Let $\omega \in \mathcal{S}^*_{\tan}$. Then*

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{12}.$$

The outcome is sharp for function defined in (3.2) *for $n = 3$.*

Proof. From (4.5) ,(4.6) and (4.7) , we have

$$|\gamma_1\gamma_2 - \gamma_3| = \frac{1}{24}\left|\frac{41}{96}c_1^3 - \frac{19}{16}c_2c_1 + c_3\right|,$$

thus, by Lemma 2.2, we have

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{12}.$$

□

Theorem 4.4. *Let* $\omega \in \mathcal{S}^*_{\tan}$. *Then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{64}.$$

The outcome is sharp for function defined in (3.2) *for* $n = 2$.

Proof. From (4.5) ,(4.6) and (4.7) , we have

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \left|\frac{7}{9216}c_1^4 - \frac{1}{768}c_1^2c_2 + \frac{1}{192}c_3c_1 - \frac{1}{256}c_2^2\right|.$$

Now using Lemma 2.4, with $c_1 = c$, $|\kappa| = 1$ and $|x| = y$, we have

$$\begin{aligned}\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq &\frac{1}{2304}c^4 + \frac{1}{768}c^2\left(4 - c^2\right)y^2 + \frac{1}{384}c\left(4 - c^2\right)\left(1 - y^2\right)\\ &+ \frac{1}{1024}\left(4 - c^2\right)^2 y^2 = G(y, c) \ \text{(say)}.\end{aligned}$$

Further,

$$\frac{\partial G(y, c)}{\partial y} = \frac{1}{1536}y(c - 2)^2\left(-c^2 + 4c + 12\right),$$

clearly the $\frac{\partial G(y,c)}{\partial y} > 0$ in $y \in [0, 1]$ so maximum attained at $y = 1$, i.e.,

$$G(1, c) = \frac{1}{9216}c^4 - \frac{1}{384}c^2 + \frac{1}{64} = H(c).$$

Further,

$$H'(c) = \frac{1}{2304}c\left(c^2 - 12\right),$$

since $H'(c) = 0$ has three roots namely $c = 0, -2\sqrt{3}$ and $2\sqrt{3}$. The only root lies in the interval $[0, 2]$ is 0. Also, one may check easily that $H''(c) \leq 0$ for $c = 0$, thus maximum attained at $c = 0$, that is

$$H(0) \leq \frac{1}{64}.$$

□

5. Inverse coefficient functionals for the class $\mathcal{S}^{*}_{\tan}$

For every univalent function ξ defined in Λ, the well-known Köebe 1/4-theorem guarantees that its inverse ξ^{-1} exists at least on a disc of radius 1/4 with Taylor's series representation form

$$\xi^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad \left(|w| < \frac{1}{4}\right). \tag{5.1}$$

Using the representation $\xi\left(\xi^{-1}(w)\right) = w$, we obtain

$$A_2 = -d_2, \tag{5.2}$$

$$A_3 = -d_3 + 2d_2^2, \tag{5.3}$$

$$A_4 = -d_4 + 5d_2d_3 - 5d_2^3. \tag{5.4}$$

In recent years, scholars have shown a strong fascination in comprehending the geometric behavior of the inverse function. For instance, Krzyz et al. [29] derived the upper bounds of the initial coefficient for inverse function ξ^{-1} when $\xi \in \mathcal{S}^{*}(\beta)$ with $0 \le \beta \le 1$. Furthermore, Ali [30] determined the first four initial sharp coefficients bounds and sharp Fekete–Szegö inequality for the class $\mathcal{SS}^{*}(\zeta)$ $(0 < \zeta \le 1)$ of a strongly starlike function of the inverse function. The papers [31, 32], provide further information on the research of inverse coefficients.

Theorem 5.1. *Let $\omega \in \mathcal{A}$ member of $\mathcal{S}^{*}_{\tan}$. Then*

$$|A_2| \le \frac{1}{4},$$

$$|A_3| \le \frac{1}{6},$$

$$|A_4| \le \frac{1}{8}.$$

These outcomes are sharp for functions defined in (3.2) *for $n = 1, 2, 3$ respectively.*

Proof. Applying (3.5) to (3.7), then to (5.2), then to (5.4), we have

$$A_2 = -\frac{1}{4}c_1, \tag{5.5}$$

$$A_3 = \frac{5}{32}c_1^2 - \frac{1}{8}c_2, \tag{5.6}$$

$$A_4 = -\frac{19}{144}c_1^3 + \frac{5}{24}c_2c_1 - \frac{1}{12}c_3. \tag{5.7}$$

Applying (2.2) to (5.5), we have

$$|A_2| \le \frac{1}{2}.$$

To find bound of A_3, apply (2.3) to (5.6), we have

$$|A_3| \le \frac{1}{4}.$$

Applying (2.5) to (5.7) , we get

$$|A_4| \leq \frac{1}{6}.$$

□

Theorem 5.2. *Let $\omega \in \mathcal{S}^*_{\tan}$. Then for complex number λ, we have*

$$\left|A_3 - \lambda A_2^2\right| \leq \frac{1}{4} \max\left\{1, \frac{|2\lambda - 11|}{16}\right\}.$$

The function defined in (3.2) *, for $n = 2$, yields a sharp result.*

Proof. From (5.5) and (5.6) , we have

$$\left|A_3 - \lambda A_2^2\right| = \frac{1}{8}\left|c_2 - \frac{5-2\lambda}{4}c_1^2\right|.$$

Applying (2.4) to the preceding equation yields the desired outcome. □

Theorem 5.3. *Let $\omega \in \mathcal{S}^*_{\tan}$. Then*

$$|A_2A_3 - A_4| \leq \frac{1}{6}.$$

The outcome is sharp for function defined in (3.2) *for $n = 3$.*

Proof. From (5.5) ,(5.6) and (5.7) , we have

$$|A_2A_3 - A_4| = \left|\frac{107}{1152}c_1^3 - \frac{17}{96}c_2c_1 + \frac{1}{12}c_3\right|,$$

applying Lemma 2.2, we achieve the intended outcomes. □

Author contributions

Mohammad Faisal Khan, Jongsuk Ro, Muhammad Ghaffar Khan: Writing – original draft, Writing – review & editing, Methodology, Funding acquisition. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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