Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 20627–20640.
DOI: 10.3934/math.20241002
Received: 27 March 2024
Revised: 07 June 2024
Accepted: 19 June 2024
Published: 25 June 2024

Research article

An overdetermined problem for elliptic equations

Tynysbek Kalmenov* **and Nurbek Kakharman**

Institute of Mathematics and Mathematical Modeling, Pushkin Str. 125, Almaty 050010, Kazakhstan

* **Correspondence:** Email: kalmenov.t@mail.ru.

Abstract: This paper is devoted to finding a necessary and sufficient condition for the solvability of the overdetermined problem for Poisson's equation with both the Dirichlet and Neumann conditions on the entire boundary. The proof is based on the boundary condition formula for the Newton potential. The obtained results are also extended to general second-order linear elliptic equations. As a byproduct, we present a characterization of the Schiffer property. It gives a definitive answer to the Schiffer problem.

Keywords: overdetermined problem; Schiffer conjecture; Dirichlet condition; Neumann condition; Newton potential
Mathematics Subject Classification: 35J25, 35N25

1. Introduction

The Cauchy problem for the Laplace equation can only have a solution if the given (initial) data has strong compatibility or smoothness conditions. Hadamard proved that if the compatibility relationship among the Cauchy data is not satisfied, then there cannot be a global solution. He also demonstrated that even if the data satisfies the conditions for a classical solution to exist, this solution will not depend continuously on the data. We refer to [1] and the references therein for detailed discussions in the field. In many studies, Cauchy data for Poisson's equation can be posed on a part of the boundary rather than on the whole boundary. Probably one of the best-known results on Cauchy-type overdetermined problems for Poisson's equation in $\Omega \subset \mathbb{R}^n$ with both the Dirichlet and Neumann conditions on the entire boundary $\partial\Omega$ goes back to [2]:

$$\Delta u(x) = 1, \quad x \in \Omega \subset \mathbb{R}^n \tag{1.1}$$

with

$$u(x)\Big|_{x\in\partial\Omega} = 0, \quad \frac{\partial u(x)}{\partial n_x}\Big|_{x\in\partial\Omega} = const, \tag{1.2}$$

where $\frac{\partial}{\partial n_x}$ is an outer normal derivative on the boundary. Serrin proved that if this overdetermined boundary value problem admits a solution, then Ω must be a ball and u is radially symmetric about its center. Shortly later, Weinberger in [3] introduced an alternative proof based on the analysis of the subharmonic function. These two methods have been instrumental in generalizing Serrin's theorem to various settings and nonlinearities. See [4, 5] and the literature cited therein for other delicate issues related to symmetry problems in general. For further discussions, see also e.g., [6–9] and references therein.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with a smooth connected boundary $\partial\Omega$. In this paper, we analyse the following overdetermined problem:

$$\Delta u = f(x), \quad x \in \Omega \subset \mathbb{R}^n \tag{1.3}$$

with

$$u(x)\Big|_{x\in\partial\Omega} = 0, \quad \frac{\partial u(x)}{\partial n_x}\Big|_{x\in\partial\Omega} = 0. \tag{1.4}$$

The main aim of this paper is to establish a criterion for the solvability of the overdetermined Cauchy-type problems (1.3) and (1.4). We will also discuss some consequences and extensions. Note that overdetermined boundary value problems are crucial, for example, for solving inverse boundary value problems on finite networks since they provide the theoretical foundations for the recovery algorithm (see, e.g., [10]).

As usual, a minimal Laplace operator

$$\Delta_0 : D(\Delta_0) \to R(\Delta_0)$$

is the closure of the differential operator Δ on a subset of the functions $u \in C^{2+\alpha}(\overline{\Omega})$, $\alpha > 0$, with

$$u\Big|_{x\in\partial\Omega} = \frac{\partial u}{\partial n_x}\Big|_{x\in\partial\Omega} = 0.$$

It is known that if $u_0 \in D(\Delta_0)$, then $u_0 \in \overset{\circ}{W}{}_2^2(\Omega)$, and the inequality

$$||\Delta_0 u_0||_{L_2(\Omega)} \geq c||u_0||_{\overset{\circ}{W}{}_2^2(\Omega)} \tag{1.5}$$

holds. Here and in the sequel, we denote the standard Sobolev spaces by W (with corresponding indexes).

It is a natural question to find the function $f \in R(\Delta_0)$, such that

$$\Delta_0 u_0 = f(x), \quad u_0\Big|_{x\in\partial\Omega} = \frac{\partial u_0}{\partial n_x}\Big|_{x\in\partial\Omega} = 0. \tag{1.6}$$

By Δ_0^*, we denote the adjoint operator to the operator Δ_0 in the space $L_2(\Omega)$, and its kernel is denoted by $ker\Delta_0^*$.

Further, the operators Δ_0 and Δ_0^* are called the minimal and maximal operators, respectively, generated by the Laplacian. Using the properties of Δ_0 and $\ker \Delta_0^*$ by the method of regular extension of the operator Δ_0, Vishik [11] described all (regular) boundary value problems for Poison's equation (1.3) in the Hilbert space $L_2(\Omega)$. An operator Δ_K is called a regular extension of the operator Δ_0, if

$$\Delta_0 \subset \Delta_K \subset \Delta_0^* \quad \text{and} \quad ||\Delta_k^{-1}|| < \infty,$$

$$u \in D(\Delta_k) \Leftrightarrow \ u = u_0 + Kv + L_Q^{-1}v, \quad \Delta_K u = \Delta_0^* u = \Delta_0 u_0 + v,$$

$$v \in \ker \Delta_0^*, \quad K : \ker \Delta_0^* \to \ker \Delta_0^*,$$

where K is a linear bounded operator and L_Q is a fixed differential operator generated by Eq (1.3) and the regular boundary conditions, see [12, 13].

Otelbaev et al. [12] extended Vishik's result to Banach spaces. They also described the correct restriction of the maximal operator Δ_0^*, which can handle not only boundary value problems but also problems with internal "boundary" conditions. Such problems include the Bitsadze–Samarskii problem [14], which arises in the study of liquid plasma motion.

The boundary conditions of problems (1.3) and (1.4) are overdetermined along the entire boundary $\partial\Omega$, making it an ill-posed problem. Therefore, the main objective of this paper is to identify conditions that ensure the solvability of problems (1.3) and (1.4).

We begin by noting that since

$$D(\Delta_0) = \overset{\circ}{W}{}_2^2(\Omega)$$

is dense in $L_2(\Omega)$, we have the following equality:

$$L_2(\Omega) = R(\Delta_0) \oplus \ker \Delta_0^*. \tag{1.7}$$

Thus, the condition for the operator Δ_0 to be invertible coincides with the condition for f to be orthogonal to the whole $\ker \Delta_0^*$, that is, to all harmonic functions. This viewpoint from the perspective of operator theory provides us with an understanding of the solvability of the overdetermined problems (1.3) and (1.4). Nevertheless, ensuring the fulfillment of this condition can pose significant challenges.

Alternatively, suppose $L_D^{-1} f$ is the solution of the Dirichlet problem. Then, we have

$$u = L_D^{-1} f = \int_\Omega G(x, \xi) f(\xi) d\xi,$$

where $G(x, \xi)$ is the Green's function of the Dirichlet problem, satisfying:

$$L_D^{-1} f \Big|_{x \in \partial\Omega} = 0.$$

Hence, one of the necessary conditions for the overdetermined problem is satisfied, so that $u \in D(\Delta_0)$ is necessary and sufficient for

$$\frac{\partial}{\partial n_x} \int_{\partial\Omega} G(x, \xi) f(\xi) d\xi \Big|_{x \in \partial\Omega} = 0. \tag{1.8}$$

However, it is typically challenging to verify the above condition (1.8) due to the lack of an explicit formula for Green's function $G(x, \xi)$ in Ω, which is only known for some specific domains.

Let $\varepsilon(x)$ denote the fundamental solution of Eq (1.3), which satisfies:

$$\Delta_x \varepsilon(x) = \delta(x), \tag{1.9}$$

where $\delta(x)$ is the Dirac delta function. We use the representations of $\varepsilon(x)$ given by:

$$\begin{aligned} \varepsilon(x) &= \frac{1}{2\pi} \ln |x|, \quad x \in \mathbb{R}^2, \\ \varepsilon(x) &= -\frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, \quad x \in \mathbb{R}^n, \quad n \geq 3, \end{aligned} \tag{1.10}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

We define the Newton potential in the following form

$$u(x) = \int_{\Omega} \varepsilon(x-y) f(y) dy. \tag{1.11}$$

In the present paper, we establish a solvability criterion for the overdetermined problems (1.3) and (1.4) in terms of the Newton potential. Our approach not only addresses this specific problem but also offers potential extensions to more general elliptic operators. Additionally, we demonstrate the applicability of our method in solving the Schiffer problem from [15].

This short paper has a simple structure: In Section 2, we present a solvability criterion for the problems (1.3) and (1.4) in $L_2(\Omega)$. That is, we provide a necessary and sufficient condition for the problems (1.3) and (1.4) to be uniquely solvable in $L_2(\Omega)$. The proof of this criterion relies on the boundary condition of the Newton potential, which was constructed in [16]. We also refer to [17,18] for more general cases. In Section 3, we demonstrate the consequences of our result with one-dimensional examples that involve explicit computations. In Section 4, we apply our result to provide a novel characterization of the Schiffer property of sets. By employing our findings, we offer new insights into the Schiffer problem. Finally, in Section 5, we discuss some extensions of our results to general elliptic equations.

2. Main result

We state the main result of this paper below.

Theorem 2.1. *The Cauchy problem for Poisson's equation* (1.3) *with the condition* (1.4) *on the entire boundary $\partial\Omega$, that is, the minimal operator Δ_0 is invertible in $L_2(\Omega)$ if and only if the following condition holds*

$$\left. \int_{\Omega} \varepsilon(x-y) f(y) dy \right|_{x \in \partial\Omega} = 0, \tag{2.1}$$

where the kernel ε is the fundamental solution of the Laplacian.

It is important to note that the Newton potential of a ball of constant density is constant on the surface of the ball. Interestingly, this property in fact uniquely characterizes the balls for any dimension $n \geq 2$, as was shown by Fraenkel [19] (see also [20]). In essence, the Newton potential of constant mass density is constant on the boundary $\partial\Omega$ if and only if Ω is a ball.

Observing Theorem 2.1, it is straightforward to discern that if the density f is a constant other than zero, there exists no solution for the overdetermined boundary value problems (1.3) and (1.4) for any

$\Omega \subset \mathbb{R}^n,\ n \geq 3$. This is because the fundamental solution is negative when $n \geq 3$. As an example, the overdetermined torsion problem, where $f = -2$, also lacks a solution in any $\Omega \subset \mathbb{R}^n,\ n \geq 3$.

The proof of Theorem 2.1 is based on the boundary property of the Newton potential $u(x)$ given by Eq (1.11), which was obtained in [16, Theorem 1]:

Theorem 2.2. *For any $f \in L_2(\Omega)$, the Newton potential defined by the formula* (1.11) *belongs to $W_2^2(\Omega)$ and satisfies the following boundary condition:*

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\varepsilon(x-y)\frac{\partial u(y)}{\partial n_y} - \frac{\partial \varepsilon(x-y)}{\partial n_y} u(y) \right) dy = 0, \quad x \in \partial\Omega. \tag{2.2}$$

Conversely, if $u \in W_2^2(\Omega)$ satisfies Eq (1.3) *and the boundary condition* (2.2)*, then it coincides with the Newton potential in Ω.*

Note that the special boundary condition (2.2) can be called the boundary condition of the Newton potential.

Proof of Theorem 2.1. **Necessity.** Let $u \in D(\Delta_0)$ and $\Delta_0 u = f$, then

$$u\Big|_{x\in\partial\Omega} = 0$$

and

$$\frac{\partial u}{\partial n_x}\Big|_{x\in\partial\Omega} = 0.$$

Hence, $u(x)$ satisfies the boundary condition (2.2). According to Theorem 2.2, the function $u(x)$ is the Newton potential and satisfies

$$u(x)\Big|_{x\in\partial\Omega} = \int\limits_{\Omega} \varepsilon(x-y) f(y) dy\Big|_{x\in\partial\Omega} = 0. \tag{2.3}$$

Thus, the necessity condition (2.1) is proven.

Sufficiency. If the condition (2.1) is satisfied, then we seek the solution of $\Delta_0 u = f$ in the form

$$u(x) = \int\limits_{\Omega} \varepsilon(x-y) f(y) dy, \quad x \in \Omega.$$

By Theorem 2.2, this Newton potential satisfies the boundary condition (2.2), and it also satisfies the Dirichlet boundary condition

$$u|_{x\in\partial\Omega} = 0$$

according to (2.1).

Therefore, combining (2.2) with (2.1), we have

$$\int\limits_{\partial\Omega} \varepsilon(x-y)\frac{\partial}{\partial n_y} u(y) dy = 0, \quad x \in \partial\Omega. \tag{2.4}$$

Hence, the function

$$v(x) = \int\limits_{\partial\Omega} \varepsilon(x-y)\frac{\partial}{\partial n_y}u(y)dy, \quad x \in \Omega \tag{2.5}$$

is a solution to the Laplace equation

$$\Delta_x v(x) = 0, \qquad x \in \Omega \tag{2.6}$$

with the Dirichlet boundary condition

$$v(x)\big|_{x\in\partial\Omega} = 0. \tag{2.7}$$

From the uniqueness of the solution of the Dirichlet problem, it follows that $v(x) \equiv 0$ in $\overline{\Omega}$. If we continue $v(x)$ throughout $\mathbb{R}^n$ by 0, and use the property of the simple-layer potential, then we arrive at

$$0 = \frac{\partial v}{\partial n_x}\Big|_{x\in\partial\Omega^+} - \frac{\partial v}{\partial n_x}\Big|_{x\in\partial\Omega^-} = \frac{\partial u}{\partial n_x}\Big|_{x\in\partial\Omega}.$$

Here, $\partial\Omega^+$ denotes an exterior domain, while $\partial\Omega^-$ denotes an interior domain. Indeed, the simple-layer potential $v(x)$ is harmonic in Ω and $\mathbb{R}^n \setminus \overline{\Omega}$, and it has a jump discontinuity across the boundary $\partial\Omega$ (see [21, 22])

$$\frac{\partial v}{\partial n_x}\Big|_{x\in\partial\Omega^+} - \frac{\partial v}{\partial n_x}\Big|_{x\in\partial\Omega^-} = \frac{\partial u}{\partial n_x}\Big|_{x\in\partial\Omega}.$$

Since $v(x) = 0$ inside $\overline{\Omega}$, then we have

$$\frac{\partial v}{\partial n_x}\Big|_{x\in\partial\Omega^-} = 0,$$

and its normal derivative from outside $\partial\Omega^+$ is zero.

The proof is now complete. □

3. One-dimensional case

In the one-dimensional case, the Newton potential $u(x)$ is given by the formula

$$u(x) = \frac{1}{2}\int\limits_0^1 |x-\xi| f(\xi)d\xi. \tag{3.1}$$

Let us find the boundary condition for the integral (3.1).

Substituting $f(\xi)$ by $\frac{d^2}{d\xi^2}u(\xi)$ in (3.1) and integrating by part, we obtain

$$\begin{aligned} u(x) &= \frac{1}{2}\int_0^1 |x-\xi| f(\xi)d\xi = \frac{1}{2}\int_0^1 |x-\xi|\frac{d^2}{d\xi^2}u(\xi)d\xi \\ &= \frac{1}{2}\int_0^x (x-\xi)\frac{d^2}{d\xi^2}u(\xi)d\xi + \frac{1}{2}\int_x^1 (\xi-x)\frac{d^2}{d\xi^2}u(\xi)d\xi \\ &= \frac{1}{2}\left[(x-\xi)\frac{d}{d\xi}u(\xi)\Big|_0^x + u(\xi)|_0^x + (\xi-x)\frac{d}{d\xi}u(\xi)\Big|_x^1 - u(\xi)|_x^1\right] \\ &= u(x) + \frac{1}{2}\left[-xu'(0) - u(0) + (1-x)u'(1) - u(1)\right]. \end{aligned} \tag{3.2}$$

Thus,

$$x(-u'(0) - u'(1)) - u(0) + u'(1) - u(1) = 0, \quad x \in (0, 1).$$

Since $x \in (0, 1)$ is arbitrary, it follows that

$$u'(0) + u'(1) = 0, \quad u'(1) = u(0) + u(1). \tag{3.3}$$

Thus, the condition (3.3) is the boundary condition for the one-dimensional Newton potential (3.1), that is, it is the one-dimensional analogue of (2.2).

Now, we rewrite the condition

$$\int_\Omega \varepsilon(x-y) f(y) dy\Big|_{x\in\partial\Omega} = 0$$

in the one-dimensional case. A direct calculation gives

$$\begin{aligned} u(0) &= \frac{1}{2}\int_0^1 |x-\xi| f(\xi) d\xi\Big|_{x=0} \\ &= \frac{1}{2}\int_0^1 \xi f(\xi) d\xi \\ &= 0, \end{aligned}$$

hence,

$$\begin{aligned} u(1) &= \frac{1}{2}\int_0^1 |x-\xi| f(\xi) d\xi\Big|_{x=1} \\ &= \frac{1}{2}\int_0^1 f(\xi) d\xi \\ &= 0. \end{aligned}$$

That is, in the one-dimensional case, the condition (2.1) is equivalent to $f(x)$ being orthogonal to both 1 and x. To show the sufficiency of this condition, let us assume that $f(x)$ is orthogonal to both 1 and x. Taking into account (3.3) and

$$u(0) = u(1) = 0,$$

it follows that

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

It means that the solution

$$u \in \overset{\circ}{W}{}_2^2(0, 1)$$

defined by the formula (3.1) satisfies the (one-dimensional) condition (2.1).

Now let us show necessity. We need to show that if the problem

$$v''(x) = f(x), \quad x \in (0, 1) \tag{3.4}$$

with the overdetermined conditions

$$v(0) = v(1) = v'(0) = v'(1) = 0 \tag{3.5}$$

has a solution, then $f(x)$ is orthogonal to both 1 and x. If a solution exists, it can be written in the form

$$v(x) = \frac{1}{2}\int_0^1 |x-\xi| f(\xi)d\xi + C_1 x + C_2. \tag{3.6}$$

According to the boundary conditions, we have

$$\begin{aligned} v(0) &= \frac{1}{2}\int_0^1 \xi f(\xi)d\xi + C_2 = 0, \\ v(1) &= \frac{1}{2}\int_0^1 f(\xi)d\xi - \frac{1}{2}\int_0^1 \xi f(\xi)d\xi + C_1 + C_2 = 0, \\ v'(0) &= u'(0) + C_1 = 0, \\ v'(1) &= u'(1) + C_1 = 0. \end{aligned}$$

This confirms Theorem 2.1, where

$$\int_0^1 \frac{1}{2}|x-y| f(y)dy\Big|_{x=0,1} = 0$$

or equivalently, f is orthogonal to 1 and x. That is, the overdetermined problem

$$u''(x) = f(x), \quad x \in (0,1) \tag{3.7}$$

with the boundary condition

$$u(0) = u(1) = u'(0) = u'(1) = 0 \tag{3.8}$$

has a solution if and only if $f(x)$ is orthogonal to both 1 and x. For example, set

$$f(x) = x^2 - x + \frac{1}{6}.$$

Since

$$\int_0^1 f(x)dx = 0$$

and

$$\int_0^1 f(x)xdx = 0,$$

in this case, the overdetermined problems (3.4)–(3.8) must have a solution. Indeed, the solution is

$$u(x) = \frac{1}{12}x^2(1-x)^2.$$

4. Characterization of the Schiffer property

A domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with a smooth connected boundary is said to have the Schiffer property if there is no $\lambda > 0$ such that the overdetermined boundary value problem

$$\Delta u + \lambda u = -1$$

in Ω, with

$$u = \frac{\partial u}{\partial n_x} = 0$$

on $\partial\Omega$ where n_x is the exterior normal to $\partial\Omega$, has a solution. For further details on the original statement of this problem, we refer to [15] and [23, Problem 80].

Let us consider the following: non-homogeneous Helmholtz equation

$$\Delta u(x) + \lambda u(x) = f(x), \qquad x \in \Omega, \tag{4.1}$$

with

$$u(x)\Big|_{x\in\partial\Omega} = 0, \qquad \frac{\partial u(x)}{\partial n_x}\Big|_{x\in\partial\Omega} = 0. \tag{4.2}$$

The analog of Theorem 2.2 for the non-homogeneous Helmholtz equation was proved in [24]. Imitating the proof of Theorem 2.1, we obtain that the overdetermined boundary value problems (4.1) and (4.2) have a solution if and only if

$$\int\limits_{\Omega} \varepsilon_\lambda(x-y)f(y)dy\Bigg|_{x\in\partial\Omega} = 0, \tag{4.3}$$

where ε_λ is the fundamental solution of the Helmholtz equation satisfying the Sommerfield radiation condition

$$\lim_{|x|\to\infty} |x|^{(n-1)/2}\left(\frac{\partial \varepsilon_\lambda}{\partial |x|} + i\sqrt{\lambda}\varepsilon_\lambda\right) = 0.$$

That is, Ω has the Schiffer property if and only if

$$\int\limits_{\Omega} \varepsilon_\lambda(x-y)dy\Bigg|_{x\in\partial\Omega} \neq 0 \tag{4.4}$$

for all $\lambda > 0$.

Thus, it gives a definitive answer to the Schiffer problem, which consists of deciding which sets Ω have the Schiffer property. Note that it is also related to the so-called Pompeiu property (see, e.g., [25]).

5. On general elliptic case

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with a smooth connected boundary. Consider the following elliptic (Newton) potential

$$u(x) = \int\limits_{\Omega} \varepsilon(x,\xi)\rho(\xi)d\xi, \tag{5.1}$$

where $\varepsilon(x,\xi)$ is the fundamental solution of the second order linear elliptic equation, i.e.,

$$\begin{aligned}Lu(x) :&= -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}u(x)+a(x)u\\&=\rho(x),\end{aligned}\tag{5.2}$$

where

$$\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\quad \delta>0,$$

$$|\xi|^2=\sum_{i,j=1}^{n}\xi_i^2,$$

$a_{ij}(x)\in C^3(\bar{\Omega})$, $a(x)\in C^2(\bar{\Omega})$, and $a(x)\geq 0$.

Now we recall briefly a method for constructing the fundamental solution of Eq (5.2) according to the classical approach proposed by Bitsadze [14].

Let us denote by A_{ij} the division ratios of the algebraic complement of the elements a_{ij} of the matrix $\|a_{ij}\|$ of the leading coefficients of Eq (5.2) in the determinant

$$a=\det\left\|a_{ij}\right\|.$$

We introduce the function:

$$\sigma(x,\xi)=\sum_{i,j,\xi}^{n}A_{ij}(x)(x_i-\xi_i)(x_j-\xi_j),$$

where x and ξ are arbitrary points in Ω.

Suppose

$$a_{ij}(x)\in C^3(\bar{\Omega})\quad\text{and}\quad a(x)\in C^1(\bar{\Omega}).$$

It is known that (5.1) is uniformly elliptic, and there are positive constants k_0 and k_1 such that

$$k_0|x-\xi|^2\leq\sigma(x,\xi)\leq k_1|x-\xi|^2.$$

For $x\neq\xi$ we define the function

$$\varepsilon(x,\xi)=\begin{cases}\sigma_0(\xi)\sigma(x,\xi)^{\frac{2-n}{2}}, & n>2,\\ -\frac{1}{2\pi\sigma_0\sqrt{\overline{a}(\xi)}}\ln\sigma(x,\xi), & n=2,\end{cases}\tag{5.3}$$

where for $n>2$,

$$\sigma_0(\xi)=\left[\omega_n(n-2)\sqrt{|\overline{a}(\xi)|}\right]^{-1},$$

ω_n is the area of an n-dimensional unit sphere, and $\overline{a}(\xi)$ is the determinant of the matrix $\{a_{i,j}\}$.

The following theorem is analogous to Theorem 2.2, and the proof is similar (see [16, Theorem 1]).

Theorem 5.1. *Let $\rho \in L_2(\Omega)$, then the elliptic potential defined by formula* (5.1) *satisfies the following boundary condition:*

$$-\frac{u(x)}{2} + \int_{\partial\Omega} u(\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} \varepsilon(x,\xi) d\xi - \int_{\partial\Omega} \varepsilon(x,\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} u(\xi) d\xi = 0, \ x \in \partial\Omega. \quad (5.4)$$

Conversely, if a function $u \in W_2^2(\Omega)$ satisfies Eq (5.2) *and the boundary condition* (5.4)*, then $u(x)$ coincides with elliptic potential* (5.1).

Now, we have the following (elliptic) extensions of the results stated in Section 2.

Theorem 5.2. *For all $f \in L_2(\Omega)$, the following Cauchy problem*

$$Lu = f, \quad x \in \Omega,$$
$$u\big|_{x\in\partial\Omega} = \frac{\partial u}{\partial n_x}\big|_{x\in\partial\Omega} = 0,$$

has a unique solution if and only if

$$\int_{\Omega} \varepsilon(x,y) f(y) dy\big|_{x\in\partial\Omega} = 0,$$

where $\varepsilon(x,y)$ is the fundamental solution of the elliptic operator L.

Proof. **Necessity.** Let $u \in D(\Delta_0)$ and $\Delta_0 u = f$, then

$$u\big|_{x\in\partial\Omega} = 0$$

and

$$\frac{\partial u}{\partial n_x}\big|_{x\in\partial\Omega} = 0.$$

Hence, $u(x)$ satisfies the boundary condition (5.4). According to Theorem 5.1, the function $u(x)$ is the volume potential and satisfies

$$u(x)\big|_{x\in\partial\Omega} = \int_{\Omega} \varepsilon(x,y) f(y) dy\big|_{x\in\partial\Omega} = 0. \quad (5.5)$$

Thus, the necessity condition is proved.

Sufficiency. Assuming that the condition (5.5) is fulfilled, we propose the solution to $\Delta_0 u = f$ as follows:

$$u(x) = \int_{\Omega} \varepsilon(x-y) f(y)\, dy, \quad x \in \Omega.$$

By Theorem 5.1, this Newton potential satisfies the boundary condition (5.4). Now sufficiency can be established using the same method as demonstrated in the proof of Theorem 2.1. □

6. Conclusions

In this paper, we establish a criterion for the solvability of the overdetermined problem for Poisson's equation

$$\Delta u = f(x), \quad x \in \Omega \subset \mathbb{R}^n$$

with both the Dirichlet and Neumann conditions on the entire boundary

$$u(x)\Big|_{x\in\partial\Omega} = 0, \quad \frac{\partial u(x)}{\partial n_x}\Big|_{x\in\partial\Omega} = 0.$$

The proof is based on the boundary condition formula for the Newton potential

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\varepsilon(x-y)\frac{\partial u(y)}{\partial n_y} - \frac{\partial \varepsilon(x-y)}{\partial n_y} u(y) \right) dy = 0, \quad x \in \partial\Omega.$$

The obtained results are also extended to general second-order linear elliptic equations. As a byproduct, we present a characterization of the Schiffer property. It gives a definitive answer to the Schiffer problem.

Author contributions

Tynysbek Kalmenov: completing the main study, carring out the results of this article, drafting the paper; Nurbek Kakharman: writing-original draft, editing. All authors have read and approved the final version of the manuscript for publication. Both authors read and approved the final version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank Professor Durvudkhan Suragan for valuable discussions to complete this paper. This research was funded by the Science Committee of the Ministry of Science and Higher Education of Kazakhstan (Grant No. AP14871460). This work was also supported by the Nazarbayev University program 20122022CRP1601.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. D. Hào, T. Van, R. Gorenflo, Towards the Cauchy problem for the Laplace equation, *Banach Center Publ.*, **27** (1992), 111–128. https://doi.org/10.4064/-27-1-111-128

2. J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.*, **43** (1971), 304–318. https://doi.org/10.1007/BF00250468
3. H. F. Weinberger, Remark on the preceding paper of Serrin, *Arch. Ration. Mech. Anal.*, **43** (1971), 319–320, https://doi.org/10.1007/BF00250469
4. I. Fragalà, F. Gazzola, J. Lamboley, M. Pierre, Counterexamples to symmetry for partially overdetermined elliptic problems, *Analysis*, **29** (2009), 85–93. https://doi.org/10.1524/anly.2009.1016
5. A. Greco, F. Pisanu, Improvements on overdetermined problems associated to the p-Laplacian, *Math. Eng.*, **4** (2022), 1–14. https://doi.org/10.3934/mine.2022017
6. A. Farina, E. Valdinoci, Partially and globally overdetermined problems of elliptic type, *Adv. Nonlinear Anal.*, **1** (2012), 27–45. https://doi.org/10.1515/ana-2011-0002
7. S. Dipierro, G. Poggesi, E. Valdinoci, A Serrin-type problem with partial knowledge of the domain, *Nonlinear Anal.*, **208** (2021), 112330. https://doi.org/10.1016/j.na.2021.112330
8. M. Onodera, Dynamical approach to an overdetermined problem in potential theory, *J. Math. Pures Appl.*, **106** (2016), 768–796. https://doi.org/10.1016/j.matpur.2016.03.011
9. M. Onodera, Linear stability analysis of overdetermined problems with non-constant data, *Math. Eng.*, **5** (2023), 1–18. https://doi.org/10.3934/mine.2023048
10. C. Araúz, A. Carmona, A. M. Encinas, Overdetermined partial boundary value problems on finite networks, *J. Math. Anal. Appl.*, **423** (2015), 191–207. https://doi.org/10.1016/j.jmaa.2014.09.025
11. M. I. Višik, On general boundary problems for elliptic differential equations, *Tr. Mosk. Mat. Obs.*, **1** (1952), 187–246.
12. B. K. Kokebayev, M. Otelbaev, A. N. Shynybekov, On questions of expansion and restriction of operators, *Dokl. Akad. Nauk SSSR*, **271** (1983), 1307–1310.
13. N. Kakharman, K. Tulenov, L. Zhumanova, On hyponormal and dissipative correct extensions and restrictions, *Math. Methods Appl. Sci.*, **45** (2022), 9049–9060. https://doi.org/10.1002/mma.8292
14. A. V. Bitsadze, A. A. Samarskii, *On some simple generalizations of linear elliptic boundary value problems*, Russian Academy of Sciences, 1969.
15. R. Dalmasso, A note on the Schiffer conjecture, *Hokkaido Math. J.*, **28** (1999), 373–383. https://doi.org/10.14492/hokmj/1351001220
16. T. Sh. Kal'menov, D. Suragan, To spectral problems for the volume potential, *Dokl. Math.*, **80** (2009), 646–649. https://doi.org/10.1134/S1064562409050032
17. T. Sh. Kal'menov, D. Suragan, Boundary conditions for the volume potential for the polyharmonic equation, *Differ. Equations*, **48** (2012), 604–608. https://doi.org/10.1134/S0012266112040155
18. M. Ruzhansky, D. Suragan, Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups, *Adv. Math.*, **308** (2017), 483–528. https://doi.org/10.1016/j.aim.2016.12.013
19. L. E. Fraenkel, *Introduction to maximum principles and symmetry in elliptic problems*, Cambridge University Press, 2000. https://doi.org/10.1017/cbo9780511569203

20. H. Shahgholian, A characterization of the sphere in terms of single-layer potentials, *Proc. Amer. Math. Soc.*, **115** (1992), 1167–1168. https://doi.org/10.1090/S0002-9939-1992-1162956-7
21. C. Miranda, *Partial differential equations of elliptic type*, Springer, 1970. https://doi.org/10.1007/978-3-642-87773-5
22. A. N. Tikhonov, A. A. Samarskii, *Equations of mathematical physics*, Courier Corporation, 1963.
23. S. T. Yau, Problem section, *Semin. Differ. Geom.*, **102** (1982), 669–706. https://doi.org/10.1515/9781400881918-035
24. T. Sh. Kal'menov, D. Suragan, Transfer of Sommerfeld radiation conditions to the boundary of a bounded domain, *Zh. Vychisl. Mat. Mat. Fiz.*, **52** (2012), 1063–1068.
25. C. A. Berenstein, D. Khavinson, *Do solid tori have the Pompeiu property*? Libraries University Press, 1996.