AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 19859–19887.
DOI: 10.3934/math.2024970
Received: 22 April 2024
Revised: 28 May 2024
Accepted: 31 May 2024
Published: 19 June 2024

Research article

Classification of irreducible based modules over the complex representation ring of S_4

Wenxia Wu and Yunnan Li*

School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

* **Correspondence:** Email: ynli@gzhu.edu.cn.

Abstract: The complex representation rings of finite groups are the fundamental class of fusion rings, categorified by the corresponding fusion categories of complex representations. The category of $\mathbb{Z}_+$-modules of finite rank over such a representation ring is also semisimple. In this paper, we classify the irreducible based modules of rank up to 5 over the complex representation ring $r(S_4)$ of the symmetric group S_4. In total, 16 inequivalent irreducible based modules were obtained. In this process, the MATLAB program was used in order to obtain some representation matrices. Based on such a classification result, we further discuss the categorification of based modules over $r(S_4)$ by module categories over the complex representation category $\mathrm{Rep}(S_4)$ of S_4 arisen from projective representations of certain subgroups of S_4.

Keywords: $\mathbb{Z}_+$-module; $\mathbb{Z}_+$-ring; representation ring; module category; symmetric group
Mathematics Subject Classification: 13C05, 18M20, 19A22

1. Introduction

Tensor categories should be thought as counterparts of rings in the world of categories [1–3], i.e., the categorification of groups and rings [4–6]. They are ubiquitous in noncommutative algebra and representation theory. Tensor categories were introduced by Bénabou [7] in 1963 and Lane [8] as "categories with multiplication", and its related theories are now widely used in many fields of mathematics, including algebraic geometry [9], algebraic topology [10], number theory [11], operator algebraic theory [12], etc. The theory of tensor categories is also seen as a development following from that of Hopf algebras and their representation theory [13, 14]. As an important invariant in the theory of tensor categories, the concept of a $\mathbb{Z}_+$-ring can be traced back to Lusztig's work [15] in 1987. Later, in [16, 17], the notion of a $\mathbb{Z}_+$-module over a $\mathbb{Z}_+$-ring was introduced. Module categories over multitensor categories were first considered in [4, 18], and then the notion of an indecomposable module category was introduced in [17]. As a categorification of irreducible $\mathbb{Z}_+$-modules, it is interesting to classify indecomposable exact module categories over a given tensor category. In this process, it is often necessary to first classify all irreducible $\mathbb{Z}_+$-modules over the Grothendieck ring of a given tensor category.

Typical examples of $\mathbb{Z}_+$-rings are the representation rings of Hopf algebras [19–24]. Another example is the Grothendieck rings of tensor categories [25–28]. It is natural to consider the classification of all irreducible $\mathbb{Z}_+$-modules over them. For example, Etingof and Khovanov classified irreducible $\mathbb{Z}_+$-modules over the group ring $\mathbb{Z}G$, and showed that indecomposable $\mathbb{Z}_+$-modules over the representation ring of SU(2), under certain conditions, correspond to affine and infinite Dynkin diagrams [16]. Also, there is a lot of related research in the context of near-group fusion categories. For instance, Tambara and Yamagami classified semisimple tensor categories with fusion rules of self-duality for finite abelian groups. Evans, Gannon, and Izumi have contributed to the classification of the near-group C^*-categories [29, 30]. Yuan et al. [31] studied irreducible $\mathbb{Z}_+$-modules of the near-group fusion ring $K(\mathbb{Z}_3, 3)$ and so on.

In this paper, we explore the problem of classifying irreducible based modules of rank up to 5 over the complex representation ring $r(S_4)$, and then discuss their categorification. Furthermore, we overcome the technical difficulty of solving a series of non-negative integer equations using MATLAB. In contrast with the representation ring of S_3, $r(S_n)$ is no longer a near-group fusion ring when $n > 3$, and the classification of irreducible $\mathbb{Z}_+$-modules over general $r(S_n)$ seems to be a hopeless task. Hence, our paper attempts to classify irreducible based modules for the non-near-group fusion ring $r(S_4)$. In fact, the fusion rule of $r(S_n)$ is already a highly nontrivial open problem in combinatorics, namely counting the multiplicities of irreducible components of the tensor product of any two irreducible complex representations of S_n (so called the Kronecker coefficients).

The paper is organized as follows. In Section 2, we recall some basic definitions and propositions. In Section 3, we discuss the irreducible based modules of rank up to 5 over $r(S_4)$ and give the classification of all these based modules (Propositions 3.1–3.5). In Section 4, we first show that any $\mathbb{Z}_+$-module over the representation ring $r(G)$ of a finite group G categorified by a module category over the representation category Rep(G) should be a based module (Theorem 4.2), and then determine which irreducible based modules over $r(S_4)$ can be categorified (Theorem 4.12).

2. Preliminaries

Throughout this paper, all rings are assumed to be associative with unit 1. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. First, we recall the definitions of $\mathbb{Z}_+$-rings and $\mathbb{Z}_+$-modules. For more details about these concepts, readers can refer to [17, 32].

2.1. $\mathbb{Z}_+$-rings and $\mathbb{Z}_+$-modules

In this section, we first recall some definitions, and then we exhibit a class of identities for transposed Poisson n-Lie algebras.

Definition 2.1. *Let A be a ring which is free as a $\mathbb{Z}$-module:*

*(i) A $\mathbb{Z}_+$-**basis** of A is a basis*

$$B = \{b_i\}_{i \in I},$$

such that

$$b_i b_j = \sum_{k \in I} c_{ij}^k b_k,$$

where $c_{ij}^k \in \mathbb{Z}_+$.

(ii) *A* $\mathbb{Z}_+$-**ring** *is a ring with a fixed* $\mathbb{Z}_+$*-basis and with unit 1 being a non-negative linear combination of the basis elements.*

(iii) *A* $\mathbb{Z}_+$*-ring is* **unital** *if the unit* 1 *is one of its basis elements.*

Definition 2.2. *Let A be a* $\mathbb{Z}_+$*-ring with basis* $\{b_i\}_{i \in I}$*. A* $\mathbb{Z}_+$-**module** *over A is an A-module M with a fixed* $\mathbb{Z}$*-basis* $\{m_l\}_{l \in L}$ *such that all the structure constants* a_{il}^k*, defined by the equality*

$$b_i m_l = \sum_k a_{il}^k m_k$$

are non-negative integers.

A $\mathbb{Z}_+$-module has the following equivalent definition referring to [32, Section 3.4].

Definition 2.3. *Let A be a* $\mathbb{Z}_+$*-ring with basis* $\{b_i\}_{i \in I}$*. A* $\mathbb{Z}_+$-**module** *M over A means an assignment where each basis* b_i *in A is in one-to-one correspondence with a non-negative integer square matrix* M_i *such that M forms a representation of A:*

$$M_i M_j = \sum_{k \in I} c_{ij}^k M_k, \quad \forall i, j, k \in I,$$

where the unit of A corresponds to the identity matrix. The rank of a $\mathbb{Z}_+$*-module M is equal to the order of the matrix* M_i.

Definition 2.4. *(i) Two* $\mathbb{Z}_+$*-modules* M_1, M_2 *over A with bases* $\left\{m_i^1\right\}_{i \in L_1}$, $\left\{m_j^2\right\}_{j \in L_2}$ *are* **equivalent** *if and only if there exists a bijection* ϕ*:* $L_1 \to L_2$ *such that the induced* $\mathbb{Z}$*-linear map* $\tilde{\phi}$ *of abelian groups* M_1, M_2 *defined by*

$$\tilde{\phi}(m_i^1) = m_{\phi(i)}^2$$

is an isomorphism of A-modules. In other words, for $a \in A$*, let* a_{M_1} *and* a_{M_2} *be the matrices with respect to the bases* $\left\{m_i^1\right\}_{i \in L_1}$ *and* $\left\{m_j^2\right\}_{j \in L_2}$*, respectively. Then, two* $\mathbb{Z}_+$*-modules* M_1, M_2 *of rank n are equivalent if and only if there exists an* $n \times n$ *permutation matrix P such that*

$$a_{M_2} = P a_{M_1} P^{-1}, \quad \forall a \in A.$$

(ii) *The* **direct sum** *of two* $\mathbb{Z}_+$*-modules* M_1, M_2 *over A is the module* $M_1 \oplus M_2$ *over A whose basis is the union of the bases of* M_1 *and* M_2.

(iii) *A* $\mathbb{Z}_+$*-module M over A is* **indecomposable** *if it is not equivalent to a nontrivial direct sum of* $\mathbb{Z}_+$*-modules.*

(iv) *A* $\mathbb{Z}_+$-**submodule** *of a* $\mathbb{Z}_+$*-module M over A with basis* $\{m_l\}_{l \in L}$ *is a subset* $J \subset L$ *such that the abelian subgroup of M generated by* $\left\{m_j\right\}_{j \in J}$ *is an A-submodule.*

(v) *A* $\mathbb{Z}_+$*-module M over A is* **irreducible** *if any* $\mathbb{Z}_+$*-submodule of M is 0 or M. In other words, the* $\mathbb{Z}$*-span of any proper subset of the basis of M is not an A-submodule.*

2.2. Based rings and based modules

Let A be a $\mathbb{Z}_+$-ring with basis $\{b_i\}_{i\in I}$, and let I_0 be the set of $i \in I$ such that b_i occurs in the decomposition of 1. Let τ: $A \to \mathbb{Z}$ denote the group homomorphism defined by

$$\tau(b_i) = \begin{cases} 1, & \text{if } i \in I_0, \\ 0, & \text{if } i \notin I_0. \end{cases}$$

Definition 2.5. *A $\mathbb{Z}_+$-ring with basis $\{b_i\}_{i\in I}$ is called a* **based ring** *if there exists an involution $i \mapsto i^*$ of I such that the induced map*

$$a = \sum_{i\in I} a_i b_i \mapsto a^* = \sum_{i\in I} a_i b_{i^*}, \quad a_i \in \mathbb{Z},$$

is an anti-involution of the ring A, and

$$\tau(b_i b_j) = \begin{cases} 1, & \textit{if } i = j^*, \\ 0, & \textit{if } i \neq j^*. \end{cases}$$

A **fusion ring** *is a unital based ring of finite rank.*

Definition 2.6. *A* **based module** *over a based ring A with basis $\{b_i\}_{i\in I}$ is a $\mathbb{Z}_+$-module M with basis $\{m_l\}_{l\in L}$ over A such that*

$$a_{il}^k = a_{i^*k}^l,$$

where a_{il}^k are defined as in Definition 2.2.

Let A be a unital $\mathbb{Z}_+$-ring of finite rank with basis $\{b_i\}_{i\in I}$, and let M be a $\mathbb{Z}_+$-module over A with $\mathbb{Z}$-basis $\{m_l\}_{l\in L}$. Take

$$b = \sum_{i\in I} b_i.$$

For any fixed m_{l_0}, the $\mathbb{Z}_+$-submodule of M generated by m_{l_0} is the $\mathbb{Z}$-span of $\{m_k\}_{k\in Y}$, where the set Y consists of $k \in L$ such that m_k is a summand of bm_{l_0}. Also, we need the following facts.

Proposition 2.1. *[32, Proposition 3.4.6] Let A be a based ring of finite rank over $\mathbb{Z}$. Then there exist only finitely many irreducible $\mathbb{Z}_+$-modules over A.*

Proposition 2.2. *[17, Lemma 2.1] Let M be a based module over a based ring A. If M is decomposable as a $\mathbb{Z}_+$-module over A, then M is irreducible as a $\mathbb{Z}_+$-module over A.*

As a result, any $\mathbb{Z}_+$-module of finite rank over a fusion ring is completely reducible, and then only irreducible $\mathbb{Z}_+$-modules need to be classified.

In general, the rank of an irreducible $\mathbb{Z}_+$-module over a fusion ring A may be larger than the rank of A; e.g., $A = r(D_5)$ for the dihedral group D_5 ([33, Remark 1]). In this paper, we explore which irreducible based modules over $r(S_4)$ can be categorified by indecomposable exact module categories over the representation category Rep(S_4). Since all these module categories are of rank not greater than 5, we only deal with based modules of rank up to 5 correspondingly.

3. Irreducible based modules over $r(S_4)$

In this section, we will classify the irreducible based modules over the complex representation ring $r(S_4)$ of S_4 up to equivalence. $r(S_4)$ is a commutative fusion ring having a $\mathbb{Z}_+$-basis $\left\{1, V_\psi, V_{\rho_1}, V_{\rho_2}, V_{\rho_3}\right\}$ with the fusion rule.

$$\begin{gathered} V_\psi^2 = 1, \quad V_\psi V_{\rho_1} = V_{\rho_1}, \quad V_\psi V_{\rho_2} = V_{\rho_3}, \quad V_{\rho_1}^2 = 1 + V_\psi + V_{\rho_1}, \\ V_{\rho_1} V_{\rho_2} = V_{\rho_2} + V_{\rho_3}, \quad V_{\rho_2}^2 = 1 + V_{\rho_1} + V_{\rho_2} + V_{\rho_3}, \end{gathered} \tag{3.1}$$

where 1, V_ψ, and V_{ρ_1} denote the trivial representation, sign representation, and 2-dimensional irreducible representation, respectively, while V_{ρ_2} stands for the 3-dimensional standard representation and V_{ρ_3} denotes its conjugate representation. Then we have the following Table 1.

Table 1. The complex character table of S_4.

	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_ψ	1	−1	1	−1	1
χ_{ρ_1}	2	0	−1	0	2
χ_{ρ_2}	3	1	0	−1	−1
χ_{ρ_3}	3	−1	0	1	−1

Let M be a based module of $r(S_4)$ with the basis $\{m_l\}_{l\in L}$. Let T, Q, U, and W be the matrices representing the action of V_ψ, V_{ρ_1}, V_{ρ_2}, and V_{ρ_3} on M respectively. They are all symmetric matrices with nonnegative integer entries by Definition 2.6. Let E be the identity matrix. By the fusion rule of $r(S_4)$, we have

$$T^2 = E, \tag{3.2}$$

$$TQ = QT = Q, \tag{3.3}$$

$$TU = UT = W, \tag{3.4}$$

$$Q^2 = E + T + Q, \tag{3.5}$$

$$QU = U + TU, \tag{3.6}$$

$$U^2 = E + Q + U + TU. \tag{3.7}$$

In particular, since $T^2 = E$ and T has nonnegative integer entries, we know that T is a symmetric permutation matrix.

Convention 3.1. *Let P_n be the group of $n \times n$ permutation matrices. Since there is naturally a group isomorphism between S_n and P_n, we will use the cycle notation of permutations to represent permutation matrices.*

3.1. Irreducible based modules of rank ≤ 3 over $r(S_4)$

We define a $\mathbb{Z}_+$-module $M_{1,1}$ of rank 1 over $r(S_4)$ by letting

$$V_\psi \mapsto 1, \quad V_{\rho_1} \mapsto 2, \quad V_{\rho_2} \mapsto 3, \quad V_{\rho_3} \mapsto 3. \tag{3.8}$$

Proposition 3.1. *Any irreducible based module of rank* 1 *over* $r(S_4)$ *is equivalent to* $M_{1,1}$.

Proof. Note that any integral fusion ring A has the unique character FPdim: $A \to \mathbb{Z}$, which takes non-negative values on the $\mathbb{Z}_+$-basis, so there exists a unique $\mathbb{Z}_+$-module M of rank 1 over it. Clearly, such M is a based module. Now this argument is available for the situation $A = r(S_4)$. □

Next, we consider irreducible based modules of rank 2, 3. According to the fusion rule of $r(S_4)$ given in (3.1), it is sufficient to only list the representation matrices of V_ψ, V_{ρ_1}, and V_{ρ_2} acting on them. For simplicity, we choose to present our result for the cases of small rank 2 and 3 directly, and then analyze the cases of higher rank 4 and 5 with details.

Proposition 3.2. *Let* M *be an irreducible based module of rank* 2 *over* $r(S_4)$. *Then* M *is equivalent to one of the based modules* $M_{2,i}$, $1 \le i \le 3$, *listed in Table 2.*

Table 2. Inequivalent irreducible based modules of rank 2 over $r(S_4)$.

	V_ψ	V_{ρ_1}	V_{ρ_2}
$M_{2,1}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$
$M_{2,2}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$
$M_{2,3}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$

Proposition 3.3. *Let* M *be an irreducible based module of rank* 3 *over* $r(S_4)$. *Then* M *is equivalent to one of the based modules* $M_{3,i}$, $1 \le i \le 3$, *listed in Table 3.*

Table 3. Inequivalent irreducible based modules of rank 3 over $r(S_4)$.

	V_ψ	V_{ρ_1}	V_{ρ_2}
$M_{3,1}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
$M_{3,2}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&1\\1&1&2\end{pmatrix}$
$M_{3,3}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&2\end{pmatrix}$

3.2. Irreducible based modules of rank 4, 5 *over* $r(S_4)$

Proposition 3.4. *Let* M *be an irreducible based module of rank* 4 *over* $r(S_4)$. *Then* M *is equivalent to one of the based modules* $M_{4,i}$, $1 \le i \le 7$, *listed in Table 4.*

Table 4. Inequivalent irreducible based modules of rank 4 over $r(S_4)$.

	V_ψ	V_{ρ_1}	V_{ρ_2}
$M_{4,1}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1&1\\0&2&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0&0\\1&2&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}$
$M_{4,2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$
$M_{4,3}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$
$M_{4,4}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$
$M_{4,5}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$
$M_{4,6}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$
$M_{4,7}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$

Proof. Before giving its detailed proof, we provide the following proof outline first.

(i) The symmetric group S_4 has 3 conjugacy classes of permutations of order ≤ 2, so there are 3 representatives for matrix T up to conjugation as follows:

$$T_1 = E_4, \quad T_2 = (12), \quad T_3 = (12)(34).$$

Consequently, we can take $T = T_r$ for some $r = 1, 2, 3$ as the representation matrix of V_ψ for the based module M up to equivalence.

(ii) Use MATLAB to search all solutions of the representation matrices Q and U in the group of nonnegative integer matrix Eqs (3.3)–(3.7) by constraint satisfaction.

(iii) Distinguish all conjugacy classes of tuples (T, Q, U) without simultaneous block decomposition. They correspond to the equivalence classes of irreducible based modules over $r(S_4)$.

□

Proof. Let M be a based module of rank 4 over $r(S_4)$, with the action of $r(S_4)$ on it given by

$$V_\psi \mapsto T, \quad V_{\rho_1} \mapsto Q = (a_{ij})_{1\leq i,j\leq 4}, \quad V_{\rho_2} \mapsto U = (b_{ij})_{1\leq i,j\leq 4}, \quad V_{\rho_3} \mapsto W = TU,$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$.

The symmetric group S_4 has two conjugacy classes of permutations of order 2. One conjugacy class of 6 permutations includes (12), and the other one of 3 permutations includes (12)(34). As previously seen, T is the unit or an element of order 2 in P_4, so we have 10 candidates for T, and each of them is conjugate to one of the following 3 matrices:

$$T_1 = E_4, \quad T_2 = (12), \quad T_3 = (12)(34).$$

Hence, for the based module M determined by the pair (T, Q, U), there exists a 4×4 permutation matrix P such that

$$T' = PTP^{-1}$$

is one of the above T_r's ($1 \leq r \leq 3$). Correspondingly, let

$$Q' = PQP^{-1}, \quad U' = PUP^{-1}.$$

Then we get a based module M' determined by the pair (T', Q', U') and equivalent to M as based modules by Definition 2.4 (i). So, we have reduced the proof to the situation when $T = T_r$.

Case 1. $T = T_1 = E_4$.

Since Q satisfies Eq (3.5), we obtain the following system of integer equations:

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 = 2 + a_{11}, \\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{23} + a_{14}a_{24} = a_{12}, \\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} + a_{14}a_{34} = a_{13}, \\ a_{11}a_{14} + a_{12}a_{24} + a_{13}a_{34} + a_{14}a_{44} = a_{14}, \\ a_{12}^2 + a_{22}^2 + a_{23}^2 + a_{24}^2 = 2 + a_{22}, \\ a_{12}a_{13} + a_{22}a_{23} + a_{23}a_{33} + a_{24}a_{34} = a_{23}, \\ a_{12}a_{14} + a_{22}a_{24} + a_{23}a_{34} + a_{24}a_{44} = a_{24}, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 + a_{34}^2 = 2 + a_{33}, \\ a_{13}a_{14} + a_{23}a_{24} + a_{33}a_{34} + a_{34}a_{44} = a_{34}, \\ a_{14}^2 + a_{24}^2 + a_{34}^2 + a_{44}^2 = 2 + a_{44}. \end{cases}$$

We use MATLAB to figure out all the solutions of Q as follows:

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$
$$Q_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Next, we calculate U after taking Q as one Q_k ($1 \leq k \leq 5$).

Case 1.1. $Q = Q_1$.

Since U satisfies Eq (3.6), we get

$$b_{12} = b_{23} = b_{24},$$
$$b_{11} = b_{13} = b_{14} = b_{33} = b_{34} = b_{44}.$$

Then, by Eq (3.7), we have

$$\begin{cases} 3b_{11}^2 + b_{12}^2 = 2b_{11} + 1, \\ 3b_{11}b_{12} + b_{12}b_{22} = 2b_{12}, \\ 3b_{12}^2 + b_{22}^2 = 2b_{22} + 3. \end{cases}$$

The solutions of U given by MATLAB are as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check that the based module determined by (T_1, Q_1, U_1) is an irreducible based module denoted as $M_{4,1}$, while the based module determined by (T_1, Q_1, U_2) is reducible.

Note that there exists a permutation matrix $P = (14)(23)$ such that

$$PQ_1P^{-1} = Q_2.$$

Let

$$U_1' = PU_1P^{-1}.$$

There is an irreducible based module N' determined by the pair (T_1, Q_2, U_1') and equivalent to $M_{4,1}$ by Definition 2.4 (i). Conversely, any irreducible based module with representation matrices T_1 and Q_2 is equivalent to $M_{4,1}$. The same analysis tells us that irreducible based modules with representation matrices T_1 and Q_3 (or Q_4) are also equivalent to $M_{4,1}$.

Case 1.2. $Q = Q_5$.

Since U satisfies Eqs (3.6) and (3.7), we get a system of integer equations as follows:

$$\begin{cases} b_{11}^2 + b_{12}^2 + b_{13}^2 + b_{14}^2 = 2b_{11} + 3, \\ b_{11}b_{12} + b_{12}b_{22} + b_{13}b_{23} + b_{14}b_{24} = 2b_{12}, \\ b_{11}b_{13} + b_{12}b_{23} + b_{13}b_{33} + b_{14}b_{34} = 2b_{13}, \\ b_{11}b_{14} + b_{12}b_{24} + b_{13}b_{34} + b_{14}b_{44} = 2b_{14}, \\ b_{12}^2 + b_{22}^2 + b_{23}^2 + b_{24}^2 = 2b_{22} + 3, \\ b_{12}b_{13} + b_{22}b_{23} + b_{23}b_{33} + b_{24}b_{34} = 2b_{23}, \\ b_{12}b_{14} + b_{22}b_{24} + b_{23}b_{34} + b_{24}b_{44} = 2b_{24}, \\ b_{13}^2 + b_{23}^2 + b_{33}^2 + b_{34}^2 = 2b_{33} + 3, \\ b_{13}b_{14} + b_{23}b_{24} + b_{33}b_{34} + b_{34}b_{44} = 2b_{34}, \\ b_{14}^2 + b_{24}^2 + b_{34}^2 + b_{44}^2 = 2b_{44} + 3. \end{cases}$$

Thus, the solutions of U by MATLAB are as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$

$$U_5=\begin{pmatrix}1&0&2&0\\0&3&0&0\\2&0&1&0\\0&0&0&3\end{pmatrix},\quad U_6=\begin{pmatrix}1&0&0&2\\0&3&0&0\\0&0&3&0\\2&0&0&1\end{pmatrix},\quad U_7=\begin{pmatrix}1&2&0&0\\2&1&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix},\quad U_8=\begin{pmatrix}3&0&0&0\\0&3&0&0\\0&0&1&2\\0&0&2&1\end{pmatrix},$$
$$U_9=\begin{pmatrix}3&0&0&0\\0&1&2&0\\0&2&1&0\\0&0&0&3\end{pmatrix},\quad U_{10}=\begin{pmatrix}3&0&0&0\\0&1&0&2\\0&0&3&0\\0&2&0&1\end{pmatrix},\quad U_{11}=\begin{pmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix}.$$

Since T_1 and Q_5 are diagonal and the solutions U_t $(2 \leq t \leq 11)$ are block diagonal with at least two blocks, only the based module determined by (T_1, Q_5, U_1) is irreducible, denoted as $M_{4,2}$.

Case 2. $T = T_2 = (12)$.

Since Q satisfies Eq (3.3), we get

$$Q=\begin{pmatrix}a_{11}&a_{11}&a_{13}&a_{14}\\a_{11}&a_{11}&a_{13}&a_{14}\\a_{13}&a_{13}&a_{33}&a_{34}\\a_{14}&a_{14}&a_{34}&a_{44}\end{pmatrix}.$$

Since Q also satisfies Eq (3.5), we have the following system of integer equations:

$$\begin{cases}2a_{11}^2+a_{13}^2+a_{14}^2=a_{11}+1,\\2a_{11}a_{13}+a_{13}a_{33}+a_{14}a_{34}=a_{13},\\2a_{11}a_{14}+a_{13}a_{34}+a_{14}a_{44}=a_{14},\\2a_{13}^2+a_{33}^2+a_{34}^2=a_{33}+2,\\2a_{13}a_{14}+a_{33}a_{34}+a_{34}a_{44}=a_{34},\\2a_{14}^2+a_{34}^2+a_{44}^2=a_{44}+2.\end{cases}$$

Hence, the solutions of Q by MATLAB are as follows:

$$Q_1=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&2&0\\1&1&0&1\end{pmatrix},\quad Q_2=\begin{pmatrix}0&0&1&0\\0&0&1&0\\1&1&1&0\\0&0&0&2\end{pmatrix},\quad Q_3=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}.$$

Since U satisfies Eq (3.4), we get

$$U=\begin{pmatrix}b_{11}&b_{12}&b_{13}&b_{14}\\b_{12}&b_{11}&b_{13}&b_{14}\\b_{13}&b_{13}&b_{33}&b_{34}\\b_{14}&b_{14}&b_{34}&b_{44}\end{pmatrix}.$$

Next, we calculate U after taking Q as one Q_k $(1 \leq k \leq 3)$.

Case 2.1. $Q = Q_1$.

Since U satisfies Eqs (3.6) and (3.7), the solutions of U given by MATLAB are as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

Since T_2, Q_1 and all the solutions U_t for $t = 1, 2$ are block diagonal with at least two blocks, the based modules determined by each pair (T_2, Q_1, U_t) are reducible.

Note that there exists a permutation matrix $P = (12)(34)$ such that

$$PQ_1P^{-1} = Q_2.$$

Let

$$U_t' = PU_tP^{-1}.$$

Then each based module N_t determined by the pair (T_2, Q_2, U_t') is reducible. Namely, any based module with representation matrices T_2 and Q_2 is reducible.

Case 2.2. $Q = Q_3$.

Since U satisfies Eqs (3.6) and (3.7), we have

$$U_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$
$$U_4 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_5 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad U_6 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since T_2, Q_3 and the solutions U_s ($3 \le s \le 6$) are block diagonal with at least two blocks, only the based module determined by (T_2, Q_3, U_1) and (T_2, Q_3, U_2) are irreducible, denoted as $M_{4,3}$ and $M_{4,4}$, respectively. It is easy to check that $M_{4,3}$ and $M_{4,4}$ are inequivalent based modules.

Case 3. $T = T_3 = (12)(34)$.

Since Q satisfies Eq (3.3), we get

$$Q = \begin{pmatrix} a_{11} & a_{11} & a_{13} & a_{13} \\ a_{11} & a_{11} & a_{13} & a_{13} \\ a_{13} & a_{13} & a_{33} & a_{33} \\ a_{13} & a_{13} & a_{33} & a_{33} \end{pmatrix}.$$

Then, by Eq (3.5), we have the following system of integer equations:

$$\begin{cases} 2a_{11}^2 + 2a_{13}^2 = a_{11} + 1, \\ 2a_{11}a_{13} + 2a_{13}a_{33} = a_{13}, \\ 2a_{13}^2 + 2a_{33}^2 = a_{33} + 1. \end{cases}$$

Q has the following unique solution:

$$Q_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Since U satisfies Eq (3.4), we get

$$U = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{11} & b_{14} & b_{13} \\ b_{13} & b_{14} & b_{33} & b_{34} \\ b_{14} & b_{13} & b_{34} & b_{33} \end{pmatrix}.$$

Since U also satisfies Eqs (3.6) and (3.7), we obtain the solutions of U by MATLAB as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$U_5 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad U_6 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad U_7 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad U_8 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Clearly, T_3, Q_1 and the solutions U_s are block diagonal with at least two blocks, but the based module determined by the pair (T_3, Q_1, U_t) is irreducible, denoted as $M_{4,s}$, where $5 \le s \le 8, 1 \le t \le 4$. Define the $\mathbb{Z}$-module isomorphism ϕ: $M_{4,6} \to M_{4,8}$ by

$$\phi(v_1^1) = v_4^2, \quad \phi(v_2^1) = v_3^2, \quad \phi(v_3^1) = v_2^2, \quad \phi(v_4^1) = v_1^2.$$

It is easy to see that $M_{4,6}$ is equivalent to $M_{4,8}$ as based modules over $r(S_4)$ under ϕ. Then, we can check that $\{M_{4,s}\}_{5 \le s \le 7}$ are inequivalent irreducible based modules. □

Finally, we construct two based modules $M_{5,i}$ $(i = 1, 2)$ over $r(S_4)$ with the actions of $r(S_4)$ on them presented in Table 5.

Table 5. Inequivalent irreducible based modules of rank 5 over $r(S_4)$.

	V_ψ	V_{ρ_1}	V_{ρ_2}
$M_{5,1}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$
$M_{5,2}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Proposition 3.5. *Let M be an irreducible based module of rank 5 over $r(S_4)$. Then M is equivalent to one of the based modules $M_{5,i}$ $(i = 1, 2)$, listed in Table 5.*

Proof. Let M be a based module of rank 5 over $r(S_4)$, with the action of $r(S_4)$ on it given by

$$V_\psi \mapsto T, \quad V_{\rho_1} \mapsto Q = (a_{ij})_{1\le i,j\le 5}, \quad V_{\rho_2} \mapsto U = (b_{ij})_{1\le i,j\le 5}, \quad V_{\rho_3} \mapsto W = TU,$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$.

First, by a similar argument applied in the case of rank 4, we only need to deal with one of the following 3 cases for T:

$$T_1 = E_5, \quad T_2 = (12), \quad T_3 = (12)(34).$$

Case 1. $T = T_1 = E_5$.

There are 11 solutions of Q satisfying Eq (3.5), but only two conjugacy classes by permutation matrices with their representatives given as follows:

$$Q_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Next, we calculate U after taking Q as one Q_k $(k = 1, 2)$.

Case 1.1. $Q = Q_1$.

There are 4 solutions of U satisfying Eqs (3.6) and (3.7) as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Case 1.2. $Q = Q_2$.

There are 31 solutions of U satisfying Eqs (3.6) and (3.7), but only 4 conjugacy classes by permutation matrices and their representatives as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Each pair (T_1, Q_k, U_r) above determines a based module, but is not irreducible for any $1 \le r \le 4$.

Case 2. $T = T_2 = (12)$.

There are 5 solutions of Q satisfying Eq (3.5), but only 3 conjugacy classes with the following representatives:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Next, we calculate U after choosing Q.

Case 2.1. $Q = Q_1$.

There are 4 solutions of U satisfying Eqs (3.6) and (3.7) as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \; U_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \; U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \; U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Case 2.2. $Q = Q_2$.

There are 2 solutions of U satisfying Eqs (3.6) and (3.7) as follows:

$$U_1 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \; U_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Case 2.3. $Q = Q_3$.

There are 14 solutions of U satisfying Eqs (3.6) and (3.7), but only 6 conjugacy classes by permutation matrices with their representatives given as follows:

$$U_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

$$U_4 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_5 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_6 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Through analysis, all based modules derived from Case 2 are reducible.

Case 3. $T = T_3 = (12)(34)$.

There are 3 solutions of Q satisfying Eq (3.5) as follows:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Next, we calculate U after fixing Q.

Case 3.1. $Q = Q_1$.

There are 6 solutions of U satisfying Eqs (3.6) and (3.7) as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix},$$

$$U_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \quad U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \quad U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Each pair (T_3, Q_1, U_r) $(1 \leq r \leq 6)$ above determines a based module, but only the based modules with representation matrices U_1 and U_2 are irreducible. Such two irreducible based modules are denoted by $M_{5,1}$ and $M'_{5,1}$, with the corresponding $\mathbb{Z}$-basis $\left\{v_1^k, v_2^k, v_3^k, v_4^k, v_5^k\right\}$ for $k = 1, 2$, respectively. Define the $\mathbb{Z}$-module isomorphism ϕ: $M_{5,1} \to M'_{5,1}$ by

$$\phi(v_s^1) = v_s^2, \quad \phi(v_3^1) = v_4^2, \quad \phi(v_4^1) = v_3^2, \quad s = 1, 2, 5.$$

Then it is easy to see that $M_{5,1}$ is equivalent to $M'_{5,1}$ as based modules over $r(S_4)$ under ϕ.

Case 3.2. $Q = Q_2$.

There are 10 solutions of U satisfying Eqs (3.6) and (3.7), but only 7 conjugacy classes with their representatives given as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

$$U_5 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_6 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad U_7 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Each pair (T_3, Q_2, U_t) $(2 \leq t \leq 7)$ above determines a based module, but only the based module with representation matrix U_1 is irreducible. We denote it by $M_{5,2}$.

Also, the based modules obtained by taking $Q = Q_3$ are equivalent to the based module $M_{5,1}$ found in Case 3.1. □

4. Categorified based modules by module categories over Rep(S_4)

In this section, we will apply the knowledge of module categories over the complex representation category of a finite group to find which based modules over $r(S_4)$ can be categorified by module categories over the representation category Rep(S_4) of S_4. For the details about module categories over tensor categories, see, e.g., [32, Section 7].

First, we recall the required result for the upcoming discussion. For any finite group G, the second cohomology group $H^2(G, \mathbb{C}^*)$ is known to be a finite abelian group called the **Schur multiplier** and classifies central extensions of G. The notion of a universal central extension of a finite group was first investigated by Schur in [34].

Let $\mathrm{Rep}(G, \alpha)$ denote the semisimple abelian category of projective representations of G with the multiplier $\alpha \in Z^2(G, \mathbb{C}^*)$. Equivalently, $\mathrm{Rep}(G, \alpha)$ is the representation category of the twisted group algebra $\mathbb{C}G_\alpha$ of G with multiplication

$$g \cdot_\alpha h = \alpha(g, h)gh, \quad g, h \in G.$$

In particular,

$$\mathrm{Rep}(G, \alpha) = \mathrm{Rep}(G),$$

when taking $\alpha = 1$.

Let $\alpha \in Z^2(G, \mathbb{C}^*)$ represent an element of order d in $H^2(G, \mathbb{C}^*)$. Define

$$\mathrm{Rep}^\alpha(G) = \bigoplus_{j=0}^{d-1} \mathrm{Rep}(G, \alpha^j).$$

According to the result in [35], we know that $\mathrm{Rep}^\alpha(G)$ becomes a fusion category with the tensor product of two projective representations in $\mathrm{Rep}(G, \alpha^i)$ and $\mathrm{Rep}(G, \alpha^j)$ respectively lying in $\mathrm{Rep}(G, \alpha^{i+j})$, and the dual object in $\mathrm{Rep}(G, \alpha^i)$ lying in $\mathrm{Rep}(G, \alpha^{d-i})$. Correspondingly, we have the fusion ring

$$r^\alpha(G) = \bigoplus_{j=0}^{d-1} r(G, \alpha^j). \tag{4.1}$$

Now let H be a subgroup of G and $\alpha \in Z^2(H, \mathbb{C}^*)$. The category $\mathrm{Rep}(H, \alpha)$ is a module category over $\mathrm{Rep}(G)$ by applying the restriction functor $\mathrm{Res}_H^G\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Theorem 4.1. *[17, Theorem 3.2] The indecomposable exact module categories over the representation category* Rep(G) *are of the form* Rep(H, α) *and are classified by conjugacy classes of pairs* $(H, [\alpha])$.

Consequently, by [32, Proposition 7.7.2], we know the following:

Proposition 4.1. *The Grothendieck group*

$$r(H, \alpha) = \mathrm{Gr}(\mathrm{Rep}(H, \alpha))$$

is an irreducible $\mathbb{Z}_+$-module over $r(G)$.

Next, we show that any $\mathbb{Z}_+$-module over the complex representation ring $r(G)$ of a finite group G categorified in this way is a based module.

Theorem 4.2. *Let G be a finite group, H a subgroup of G, and $\alpha \in Z^2(H, \mathbb{C}^*)$. The $\mathbb{Z}_+$-module $r(H, \alpha)$ over $r(G)$ is a based module.*

Proof. Let $\{\psi_i\}_{i\in I}$ be the $\mathbb{Z}_+$-basis of $r(G)$. Take $r^\alpha(H)$ defined in Eq (4.1) as a $\mathbb{Z}_+$-module over $r(G)$ with the $\mathbb{Z}$-basis $\{\chi_k\}_{k\in J}$ such that

$$\psi_i.\chi_k = \sum_l a_{ik}^l \chi_l, \quad a_{ik}^l \in \mathbb{Z}_+.$$

On the other hand, we write the fusion rule of the fusion ring $r^\alpha(H)$ as follows:

$$\chi_i \chi_j = \sum_{k=1}^s n_{ij}^k \chi_k, \quad n_{ij}^k \in \mathbb{Z}_+.$$

Since the number $n_{ij}^{k^*}$ is invariant under cyclic permutations of i, j, k , we have

$$n_{ij}^k = n_{k^* i}^{j^*} = n_{i^* k}^j.$$

By the restriction rule, we interpret $r(G)$ as a subring of $r^\alpha(H)$ and write down

$$\psi_i = \sum_j r_{ij} \chi_j, \quad r_{ij} \in \mathbb{Z}_+.$$

Then

$$\psi_i.\chi_k = \sum_j r_{ij} \chi_j \chi_k = \sum_{j,l} r_{ij} n_{jk}^l \chi_l.$$

By comparing the coefficients, we see that

$$a_{ik}^l = \sum_j r_{ij} n_{jk}^l = \sum_j r_{ij} n_{j^* l}^k = \sum_j r_{i^* j^*} n_{j^* l}^k = \sum_j r_{i^* j} n_{jl}^k = a_{i^* l}^k,$$

so $r^\alpha(H)$ is a based module over $r(G)$, and $r(H, \alpha)$ is clearly a based submodule of $r^\alpha(H)$. Equivalently, any $\mathbb{Z}_+$-module over $r(G)$ categorified by a module category Rep(H, α) over Rep(G) must be a based module. □

By Theorem 4.2, we only need to focus on those inequivalent irreducible based modules $M_{i,j}$ over $r(S_4)$ collected in Section 3, each of which is possibly categorified by a module category $\mathrm{Rep}(H,\alpha)$ for some $H < S_4$ and $\alpha \in Z^2(H,\mathbb{C}^*)$.

All the non-isomorphic subgroups of the symmetric group S_4 are as follows:

(i) The symmetric group S_3;

(ii) The cyclic groups $\mathbb{Z}_i$, $1 \le i \le 4$;

(iii) The Klein 4-group K_4;

(iv) The alternating group A_4;

(v) The dihedral group D_4;

(vi) The symmetric group S_4 itself.

Correspondingly, the Schur multipliers we consider here are given as follows (see e.g., [36]):

$$H^2(\mathbb{Z}_n,\mathbb{C}^*) \cong H^2(S_3,\mathbb{C}^*) \cong 0,\ n \ge 1, \qquad H^2(K_4,\mathbb{C}^*) \cong H^2(D_4,\mathbb{C}^*) \cong H^2(A_4,\mathbb{C}^*) \cong H^2(S_4,\mathbb{C}^*) \cong \mathbb{Z}_2.$$

As a result, we only need to consider the following two situations:

(1) Module category $\mathrm{Rep}(H)$ for any subgroup $H < S_4$;

(2) Module category $\mathrm{Rep}(H,\alpha)$ for any subgroup $H < S_4$ and nontrivial twist $\alpha \in Z^2(H,\mathbb{C}^*)$.

4.1. The module categories over $\mathrm{Rep}(S_4)$ *with trivial twists*

(i) First, we consider the representation category $\mathrm{Rep}(S_3)$ as a module category over $\mathrm{Rep}(S_4)$.

Theorem 4.3. *$r(S_3) = \mathrm{Gr}(\mathrm{Rep}(S_3))$ is an irreducible based module over $r(S_4) = \mathrm{Gr}(\mathrm{Rep}(S_4))$ equivalent to the based module $M_{3,2}$ in Table 3.*

Proof. According to the branching rule of symmetric groups (see e.g., [37, Theorem 2.8.3]), we have the following restriction rules:

$$\mathrm{Res}^{S_4}_{S_3}(1) = 1,\quad \mathrm{Res}^{S_4}_{S_3}\left(V_\psi\right) = \chi,\quad \mathrm{Res}^{S_4}_{S_3}\left(V_{\rho_1}\right) = V,\quad \mathrm{Res}^{S_4}_{S_3}\left(V_{\rho_2}\right) = 1 + V,\quad \mathrm{Res}^{S_4}_{S_3}\left(V_{\rho_3}\right) = \chi + V,$$

where χ and V denote the sign representation and the standard representation in $\mathrm{Rep}(S_3)$, respectively. Hence, we get the representation matrices of basis elements of $r(S_4)$ acting on $r(S_3)$ as follows:

$$1 \mapsto E_3,\quad V_\psi \mapsto \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad V_{\rho_1} \mapsto \begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix},\quad V_{\rho_2} \mapsto \begin{pmatrix}1&0&1\\0&1&1\\1&1&2\end{pmatrix},\quad V_{\rho_3} \mapsto \begin{pmatrix}0&1&1\\1&0&1\\1&1&2\end{pmatrix}.$$

We see that $r(S_3)$ is an irreducible based module $M_{3,2}$ according to Table 3. In other words, the based module $M_{3,2}$ can be categorified by the module category $\mathrm{Rep}(S_3)$ over $\mathrm{Rep}(S_4)$. □

Remark 4.1. *Since the roles of the standard representation and its dual in $r(S_4)$ are symmetric, we can exchange the notations V_{ρ_2} and V_{ρ_3} for them to get the following restriction rules instead:*

$$\mathrm{Res}^{S_4}_{S_3}\left(V_{\rho_2}\right) = \chi + V,\quad \mathrm{Res}^{S_4}_{S_3}\left(V_{\rho_3}\right) = 1 + V.$$

Therefore, we get another action of $r(S_4)$ *on* $r(S_3)$ *such that* $r(S_3)$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module* $M_{3,3}$ *according to Table 3. In other words, the based module* $M_{3,3}$ *can also be categorified by the module category* $\mathrm{Rep}(S_3)$ *over* $\mathrm{Rep}(S_4)$.

(ii) Second, we consider $\mathrm{Rep}(\mathbb{Z}_4)$ as a module category over $\mathrm{Rep}(S_4)$.

Theorem 4.4. $r(\mathbb{Z}_4) = \mathrm{Gr}(\mathrm{Rep}(\mathbb{Z}_4))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module* $M_{4,5}$ *in Table 4.*

Proof. Let

$$\mathbb{Z}_4 = \left\{1, g, g^2, g^3\right\}$$

be the cyclic group of order 4, with four non-isomorphic 1-dimensional irreducible representations denoted by U_i, $i = 0, 1, 2, 3$. Let $U_0 = 1$ represent the trivial representation, and

$$\chi_{U_1}(g) = \sqrt{-1}, \quad \chi_{U_2}(g) = -1, \quad \chi_{U_3}(g) = -\sqrt{-1}.$$

On the other hand, we consider $\mathbb{Z}_4$ as the subgroup of S_4 generated by $g = (1234)$. Then, by the character table of S_4 (Table 1), we have

$$\begin{aligned} &\chi_{\psi}(g^i) = (-1)^i, \quad \chi_{\rho_1}(g^i) = 1 + (-1)^i, \\ &\chi_{\rho_2}(g^i) = (-1)^i + (\sqrt{-1})^i + (-\sqrt{-1})^i, \\ &\chi_{\rho_3}(g^i) = 1 + (\sqrt{-1})^i + (-\sqrt{-1})^i. \end{aligned}$$

So, the restriction rule of $r(S_4)$ on $r(\mathbb{Z}_4)$ is given as follows:

$$\begin{gathered} \mathrm{Res}^{S_4}_{\mathbb{Z}_4}(1) = 1, \quad \mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\psi}\right) = U_2, \quad \mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\rho_1}\right) = 1 + U_2, \\ \mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\rho_2}\right) = U_1 + U_2 + U_3, \qquad \mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\rho_3}\right) = 1 + U_1 + U_3. \end{gathered}$$

Then, we get the representation matrices of basis elements of $r(S_4)$ acting on $r(\mathbb{Z}_4)$ as follows:

$$1 \mapsto E_4, V_{\psi} \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, V_{\rho_1} \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad V_{\rho_2} \mapsto \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, V_{\rho_3} \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Let $\{w_i\}_{1 \le i \le 4}$ be the stated $\mathbb{Z}$-basis of $M_{4,5}$, and define a $\mathbb{Z}$-linear map φ: $M_{4,5} \to r(\mathbb{Z}_4)$ by

$$\varphi(w_1) = U_3, \quad \varphi(w_2) = U_1, \quad \varphi(w_3) = U_2, \quad \varphi(w_4) = 1.$$

Then, it is easy to check that φ is an isomorphism of $r(S_4)$-modules, so $M_{4,5}$ is equivalent to $r(\mathbb{Z}_4)$ as based modules by Definition 2.4 (i). In other words, the based module $M_{4,5}$ can be categorified by the module category $\mathrm{Rep}(\mathbb{Z}_4)$ over $\mathrm{Rep}(S_4)$. □

Remark 4.2. *By the same argument as in Remark 4.1,* V_{ρ_2} *and* V_{ρ_3} *can be required to satisfy the following restriction rules instead:*

$$\mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\rho_2}\right) = 1 + U_1 + U_3, \quad \mathrm{Res}^{S_4}_{\mathbb{Z}_4}\left(V_{\rho_3}\right) = U_1 + U_2 + U_3.$$

Therefore, we get another action of $r(S_4)$ on $r(\mathbb{Z}_4)$ such that $r(\mathbb{Z}_4)$ is an irreducible based module over $r(S_4)$ equivalent to the based module $M_{4,7}$ according to Table 4. In other words, the based module $M_{4,7}$ can also be categorified by the module category $\mathrm{Rep}(\mathbb{Z}_4)$ *over* $\mathrm{Rep}(S_4)$.

Also, one can similarly check that the module category $\mathrm{Rep}(\mathbb{Z}_2)$ *over* $\mathrm{Rep}(S_4)$ *categorifies the based modules $M_{2,2}$ and $M_{2,3}$, while* $\mathrm{Rep}(\mathbb{Z}_3)$ *over* $\mathrm{Rep}(S_4)$ *categorifies the based module $M_{3,1}$.*

(iii) Now we consider $\mathrm{Rep}(K_4)$ as a module category over $\mathrm{Rep}(S_4)$.

Theorem 4.5. $r(K_4) = \mathrm{Gr}(\mathrm{Rep}(K_4))$ *is an irreducible based module over $r(S_4)$ equivalent to the based module $M_{4,7}$ in Table 4.*

Proof. We consider K_4 as the subgroup of S_4 generated by (12) and (34), and it has four non-isomorphic 1-dimensional irreducible representations $Y_0 = 1$ and Y_1, Y_2, Y_3 such that

$$\chi_{Y_1}((12)) = -1, \quad \chi_{Y_1}((34)) = 1; \quad \chi_{Y_2}((12)) = 1,$$
$$\chi_{Y_2}((34)) = -1; \quad \chi_{Y_3}((12)) = -1, \quad \chi_{Y_3}((34)) = -1.$$

On the other hand, by the character table of S_4 (Table 1), we have

$$\chi_{\psi}((12)) = \chi_{\psi}((34)) = -1, \quad \chi_{\psi}((12)(34)) = 1;$$
$$\chi_{\rho_1}((12)) = \chi_{\rho_1}((34)) = 0, \quad \chi_{\rho_1}((12)(34)) = 2;$$
$$\chi_{\rho_2}((12)) = \chi_{\rho_2}((34)) = 1, \quad \chi_{\rho_2}((12)(34)) = -1;$$
$$\chi_{\rho_3}((12)) = \chi_{\rho_3}((34)) = -1, \quad \chi_{\rho_3}((12)(34)) = -1.$$

So, we have the following restriction rules:

$$\mathrm{Res}^{S_4}_{K_4}(1) = 1, \quad \mathrm{Res}^{S_4}_{K_4}\left(V_{\psi}\right) = Y_3, \quad \mathrm{Res}^{S_4}_{K_4}\left(V_{\rho_1}\right) = 1 + Y_3,$$
$$\mathrm{Res}^{S_4}_{K_4}\left(V_{\rho_2}\right) = 1 + Y_1 + Y_2, \qquad \mathrm{Res}^{S_4}_{K_4}\left(V_{\rho_3}\right) = Y_1 + Y_2 + Y_3.$$

Then we get the representation matrices of basis elements of $r(S_4)$ acting on $r(K_4)$ as follows:

$$1 \mapsto E_4, \quad V_{\psi} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad V_{\rho_1} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$
$$V_{\rho_2} \mapsto \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad V_{\rho_3} \mapsto \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $\{w_i\}_{1 \le i \le 4}$ be the stated $\mathbb{Z}$-basis of $M_{4,7}$ listed in Table 4. Then

$$w_1 \mapsto Y_2, \quad w_2 \mapsto Y_1, \quad w_3 \mapsto 1, \quad w_4 \mapsto Y_3,$$

defines an equivalence of $\mathbb{Z}_+$-modules between $M_{4,7}$ and $r(K_4)$. In other words, the irreducible based module $M_{4,7}$ can be categorified by the module category $\mathrm{Rep}(K_4)$ over $\mathrm{Rep}(S_4)$. □

Remark 4.3. *In a manner analogous to the argument in Remark 4.1, it follows that the irreducible based module $M_{4,5}$ can also be categorified by the module category* $\mathrm{Rep}(K_4)$ *over* $\mathrm{Rep}(S_4)$.

(iv) We consider $\mathrm{Rep}(A_4)$ as a module category over $\mathrm{Rep}(S_4)$.

Theorem 4.6. $r(A_4) = \mathrm{Gr}(\mathrm{Rep}(A_4))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module $M_{4,1}$ in Table 4.*

Proof. We know that A_4 has three non-isomorphic 1-dimensional irreducible representations and one 3-dimensional irreducible representation, denoted by N_0, N_1, N_2, and N_3, respectively, where $N_0 = 1$ represents the trivial representation, and

$$\chi_{N_1}((123)) = \omega, \quad \chi_{N_1}((12)(34)) = 1; \quad \chi_{N_2}((123)) = \omega^2, \quad \chi_{N_2}((12)(34)) = 1;$$
$$\chi_{N_3}((123)) = \chi_{N_3}((132)) = 0, \quad \chi_{N_3}((12)(34)) = -1, \quad \omega = \frac{-1+\sqrt{-3}}{2}.$$

On the other hand, the character table of S_4 (Table 1) tells us that

$$\chi_{\psi}((123)) = 1, \quad \chi_{\psi}((12)(34)) = 1; \quad \chi_{\rho_1}((123)) = -1, \quad \chi_{\rho_1}((12)(34)) = 2;$$
$$\chi_{\rho_2}((123)) = 0, \quad \chi_{\rho_2}((12)(34)) = -1; \quad \chi_{\rho_3}((123)) = 0, \quad \chi_{\rho_3}((12)(34)) = -1.$$

So, we have the following restriction rules:

$$\mathrm{Res}_{A_4}^{S_4}(1) = \mathrm{Res}_{A_4}^{S_4}\left(V_{\psi}\right) = 1, \quad \mathrm{Res}_{A_4}^{S_4}\left(V_{\rho_1}\right) = N_1 + N_2, \quad \mathrm{Res}_{A_4}^{S_4}\left(V_{\rho_2}\right) = \mathrm{Res}_{A_4}^{S_4}\left(V_{\rho_3}\right) = N_3.$$

Hence, we get the representation matrices of basis elements of $r(S_4)$ acting on $r(A_4)$ as follows:

$$1 \mapsto E_4, \quad V_{\psi} \mapsto E_4, \quad V_{\rho_1} \mapsto \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad V_{\rho_2}, V_{\rho_3} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Then, $r(A_4)$ is an irreducible based module over $r(S_4)$ equivalent to $M_{4,1}$ listed in Table 4. In other words, the irreducible based module $M_{4,1}$ can be categorified by the module category $\mathrm{Rep}(A_4)$ over $\mathrm{Rep}(S_4)$. □

(v) Next, we consider $\mathrm{Rep}(D_4)$ as a module category over $\mathrm{Rep}(S_4)$.

Theorem 4.7. $r(D_4) = \mathrm{Gr}(\mathrm{Rep}(D_4))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module $M_{5,2}$ in Table 5.*

Proof. The dihedral group

$$D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle$$

has four 1-dimensional irreducible representations and one 2-dimensional irreducible representation up to isomorphism, denoted by W_0, W_1, W_2, W_3, and W_4, respectively. Let $W_0 = 1$ stand for the trivial representation, and

$$\chi_{W_1}(r) = 1, \quad \chi_{W_1}(s) = -1; \quad \chi_{W_2}(r) = -1, \quad \chi_{W_2}(s) = 1;$$

$$\chi_{W_3}(r) = -1, \quad \chi_{W_3}(s) = -1; \quad \chi_{W_4}(r) = \chi_{W_4}(s) = \chi_{W_4}(rs) = 0.$$

On the other hand, we consider D_4 as the subgroup of S_4 by taking $r = (1234)$ and $s = (12)(34)$. Then $rs = (13)$. By the character table of S_4 (Table 1), we have

$$\begin{aligned}
&\chi_{\psi}((1234)) = -1, \quad \chi_{\psi}((12)(34)) = 1, \quad \chi_{\psi}((13)) = -1;\\
&\chi_{\rho_1}((1234)) = 0, \quad \chi_{\rho_1}((12)(34)) = 2, \quad \chi_{\rho_1}((13)) = 0;\\
&\chi_{\rho_2}((1234)) = -1, \quad \chi_{\rho_2}((12)(34)) = -1, \quad \chi_{\rho_2}((13)) = 1;\\
&\chi_{\rho_3}((1234)) = 1, \quad \chi_{\rho_3}((12)(34)) = -1, \quad \chi_{\rho_3}((13)) = -1.
\end{aligned}$$

So, we have the following restriction rules:

$$\mathrm{Res}^{S_4}_{D_4}(1) = 1, \quad \mathrm{Res}^{S_4}_{D_4}\left(V_{\psi}\right) = W_2, \quad \mathrm{Res}^{S_4}_{D_4}\left(V_{\rho_1}\right) = 1 + W_2,$$
$$\mathrm{Res}^{S_4}_{D_4}\left(V_{\rho_2}\right) = W_3 + W_4, \qquad \mathrm{Res}^{S_4}_{D_4}\left(V_{\rho_3}\right) = W_1 + W_4.$$

Then we get the representation matrices of basis elements of $r(S_4)$ acting on $r(D_4)$ as follows:

$$1 \mapsto E_5, \quad V_{\psi} \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad V_{\rho_1} \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$
$$V_{\rho_2} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad V_{\rho_3} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then $r(D_4)$ is an irreducible based module over $r(S_4)$ equivalent to $M_{5,2}$ listed in Table 5. In other words, the irreducible based module $M_{5,2}$ can be categorified by the module category Rep(D_4) over Rep(S_4). □

(vi) Finally, we consider Rep(S_4) as a module category over itself.

Theorem 4.8. *The regular* $\mathbb{Z}_+$*-module* $r(S_4)$ *over itself is equivalent to the irreducible based module* $M_{5,1}$ *in Table 5.*

Proof. Let $r(S_4)$ be the regular $\mathbb{Z}_+$-module over itself with the $\mathbb{Z}$-basis $\left\{1, V_{\psi}, V_{\rho_1}, V_{\rho_2}, V_{\rho_3}\right\}$, and the action of $r(S_4)$ on it is given as follows:

$$1 \mapsto E_5, \quad V_{\psi} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad V_{\rho_1} \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$V_{\rho_2} \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad V_{\rho_3} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then, the regular $\mathbb{Z}_+$-module $r(S_4)$ over itself is equivalent to $M_{5,1}$ listed in Table 5. In other words, the irreducible based module $M_{5,1}$ over $r(S_4)$ can be categorified by the module category $\mathrm{Rep}(S_4)$ over itself. □

Remark 4.4. *Following the argument presented in Remark 4.1, if we exchange the notations V_{ρ_2} and V_{ρ_3} with their restriction rules given in the proof of Theorems 4.7 and 4.8, we see that $r(D_4)$ and $r(S_4)$ are still equivalent to $M_{5,2}$ and $M_{5,1}$, respectively.*

4.2. *The module categories over* $\mathrm{Rep}(S_4)$ *with nontrivial twists*

Lastly, we consider the module category $\mathrm{Rep}(H, \alpha)$ over $\mathrm{Rep}(S_4)$, where H is a subgroup of S_4 with α representing the unique nontrivial cohomological class in $H^2(H, \mathbb{C}^*)$. All non-isomorphic irreducible projective representations of H with the multiplier α form a $\mathbb{Z}$-basis of $r(H, \alpha)$, whose cardinality is the number of α-regular conjugacy classes by [38, Theorem 6.1.1].

First, we consider the twisted group algebra of K_4. There is only one irreducible projective representation with respect to α up to isomorphism, see, e.g., [39, Appendix D.1]. Hence, $r(K_4, \alpha)$ is a based module of rank 1 over $r(S_4)$ equivalent to $M_{1,1}$ defined in (3.8). Namely, the based module $M_{1,1}$ can also be categorified by $\mathrm{Rep}(K_4, \alpha)$.

Second, we consider the twisted group algebra of D_4.

Theorem 4.9. $r(D_4, \alpha) = \mathrm{Gr}(\mathrm{Rep}(D_4, \alpha))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module* $M_{2,3}$ *in Table 2.*

Proof. Let

$$D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle.$$

Let $\alpha \in Z^2(D_4, \mathbb{C}^*)$ be the 2-cocycle defined by

$$\alpha(r^i s^j, r^{i'} s^{j'}) = (\sqrt{-1})^{ji'}.$$

Here, $i, i' \in \{0, 1, 2, 3\}$, $j, j' \in \{0, 1\}$. As shown in [40, Section 3.7], this is a unitary 2-cocycle representing the unique non-trivial cohomological class in $H^2(D_4, \mathbb{C}^*)$. According to [35, Section 3], there exist two (2-dimensional) non-isomorphic irreducible projective representations of D_4 with respect to α, which are given by

$$\pi_l : D_4 \to \mathrm{GL}_2(\mathbb{C}),$$
$$r^i s^j \mapsto A_l^i B^j,$$

where

$$A_l = \begin{pmatrix} (\sqrt{-1})^l & 0 \\ 0 & (\sqrt{-1})^{1-l} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad l = 1, 2.$$

Also, for irreducible representations W_0–W_4 of D_4 mentioned in the proof of Theorem 4.7, we have

$$W_0 \otimes \pi_l = W_1 \otimes \pi_l = \pi_l, \quad W_2 \otimes \pi_l = W_3 \otimes \pi_l = \pi_{3-l}, \quad W_4 \otimes \pi_l = \pi_1 + \pi_2.$$

Next, using the previous restriction rule of $r(S_4)$ on $r(D_4)$, we get the representation matrices of basis elements of $r(S_4)$ acting on $r(D_4, \alpha)$ as follows:

$$1 \mapsto E_2, \quad V_\psi \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V_{\rho_1} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad V_{\rho_2} \mapsto \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad V_{\rho_3} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then, $r(D_4, \alpha)$ is an irreducible based module over $r(S_4)$ equivalent to $M_{2,3}$ listed in Table 2. In other words, the irreducible based module $M_{2,3}$ can be categorified by the module category $\mathrm{Rep}(D_4, \alpha)$ over $\mathrm{Rep}(S_4)$. □

Remark 4.5. *As discussed in Remark 4.1, it follows that the irreducible based module $M_{2,2}$ can also be categorified by the module category* $\mathrm{Rep}(D_4, \alpha)$ *over* $\mathrm{Rep}(S_4)$*.*

Next, we consider the twisted group algebras of A_4 and S_4. By [38, Theorem 6.1.1], A_4 has three (2-dimensional) non-isomorphic irreducible projective representations, denoted as $V_{\gamma_1}, V_{\gamma_2}$, and V_{γ_3}, respectively. Similarly, S_4 has two (2-dimensional) non-isomorphic irreducible projective representations V_{ξ_1}, V_{ξ_2}, and one (4-dimensional) irreducible projective representation V_{ξ_3}. We give the character table for projective representations of A_4 and S_4 in Tables 6 and 7, respectively, where primes are used to differentiate between the two classes splitting from a single conjugacy class of A_4 in its double cover $\tilde{A}_4$, and the same applies to S_4; subscripts distinguish between the two classes splitting from the conjugacy classes $(31)'$ and $(31)''$ in the double cover $\tilde{S}_4$ of S_4, respectively. For more details, see [41, Section 4].

In Table 6, we denote

$$\omega = e^{2\pi\sqrt{-1}/3} = \frac{-1 + \sqrt{-3}}{2}.$$

Table 6. The character table for irreducible projective representations of A_4.

	$(1^4)'$	$(1^4)''$	(2^2)	$(31)'_1$	$(31)''_1$	$(31)'_2$	$(31)''_2$
χ_{γ_1}	2	−2	0	1	−1	1	−1
χ_{γ_2}	2	−2	0	ω	$-\omega$	ω^2	$-\omega^2$
χ_{γ_3}	2	−2	0	ω^2	$-\omega^2$	ω	$-\omega$

Table 7. The character table for irreducible projective representations of S_4.

	$(1^4)'$	$(1^4)''$	(21^2)	(2^2)	$(31)'$	$(31)''$	$(4)'$	$(4)''$
χ_{ξ_1}	2	−2	0	0	1	−1	$\sqrt{2}$	$-\sqrt{2}$
χ_{ξ_2}	2	−2	0	0	1	−1	$-\sqrt{2}$	$\sqrt{2}$
χ_{ξ_3}	4	−4	0	0	−1	1	0	0

Then we have the following theorems.

Theorem 4.10. $r(A_4, \alpha) = \mathrm{Gr}(\mathrm{Rep}(A_4, \alpha))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module $M_{3,1}$ in Table 3.*

Proof. For the irreducible representations N_0, N_1, N_2, and N_3 of A_4 mentioned in the proof of Theorem 4.6, we obtain the following tensor product rule in $r^{\alpha}(A_4)$ by computing the values of products of characters:

$$N_0 \otimes V_{\gamma_i} = V_{\gamma_i}, \quad N_1 \otimes V_{\gamma_j} = V_{\gamma_{j+1}}, \quad N_1 \otimes V_{\gamma_3} = V_{\gamma_1}; \quad N_2 \otimes V_{\gamma_1} = V_{\gamma_3},$$
$$N_2 \otimes V_{\gamma_2} = V_{\gamma_1}, \quad N_2 \otimes V_{\gamma_3} = V_{\gamma_2}; \quad N_3 \otimes V_{\gamma_i} = V_{\gamma_1} + V_{\gamma_2} + V_{\gamma_3},$$

where $i = 1,2,3$, $j = 1,2$. Next, by combining this with the previous restriction rule of $r(S_4)$ on $r(A_4)$, we obtain

$$1, V_{\psi} \mapsto E_3, \quad V_{\rho_1} \mapsto \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad V_{\rho_2}, V_{\rho_3} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $r(A_4, \alpha)$ is an irreducible based module over $r(S_4)$ equivalent to $M_{3,1}$ listed in Table 3. In other words, the irreducible based module $M_{3,1}$ can be categorified by $\mathrm{Rep}(A_4, \alpha)$. □

Theorem 4.11. $r(S_4, \alpha) = \mathrm{Gr}(\mathrm{Rep}(S_4, \alpha))$ *is an irreducible based module over* $r(S_4)$ *equivalent to the based module* $M_{3,3}$ *in Table 3.*

Proof. Let α be a nontrivial 2-cocycle in $Z^2(S_4, \mathbb{C}^*)$ (see e.g., [42, Section 3.2.4]). By checking products of characters, we get the following tensor product rule in $r^{\alpha}(S_4)$:

$$1 \otimes V_{\xi_i} = V_{\xi_i}; \quad V_{\psi} \otimes V_{\xi_j} = V_{\xi_{3-j}}, \quad V_{\psi} \otimes V_{\xi_3} = V_{\xi_3}; \quad V_{\rho_1} \otimes V_{\xi_j} = V_{\xi_3}, \quad V_{\rho_1} \otimes V_{\xi_3} = V_{\xi_1} + V_{\xi_2} + V_{\xi_3};$$
$$V_{\rho_2} \otimes V_{\xi_j} = V_{\xi_{3-j}} + V_{\xi_3}, \quad V_{\rho_2} \otimes V_{\xi_3} = V_{\rho_3} \otimes V_{\xi_3} = V_{\xi_1} + V_{\xi_2} + 2V_{\xi_3}; \quad V_{\rho_3} \otimes V_{\xi_j} = V_{\xi_j} + V_{\xi_3};$$

where $i = 1,2,3$, $j = 1,2$. Thus, we get

$$1 \mapsto E_3, \quad V_{\psi} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_{\rho_1} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad V_{\rho_2} \mapsto \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad V_{\rho_3} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then $r(S_4, \alpha)$ is an irreducible based module over $r(S_4)$ equivalent to $M_{3,3}$ listed in Table 3. In other words, the irreducible based module $M_{3,3}$ over $r(S_4)$ can be categorified by $\mathrm{Rep}(S_4, \alpha)$. □

In summary, we have the following classification theorem.

Theorem 4.12. *The inequivalent irreducible based modules over* $r(S_4)$ *are*

$$M_{1,1}, \quad \{M_{2,i}\}_{i=1,2,3}, \quad \left\{M_{3,j}\right\}_{j=1,2,3}, \quad \{M_{4,s}\}_{1\leq s\leq 7} \quad \textit{and} \quad \{M_{5,t}\}_{t=1,2},$$

among which

$$M_{1,1}, \quad \{M_{2,i}\}_{i=2,3}, \quad \left\{M_{3,j}\right\}_{j=1,2,3}, \quad \{M_{4,s}\}_{s=1,5,7} \quad \textit{and} \quad \{M_{5,t}\}_{t=1,2}$$

can be categorified by module categories over $\mathrm{Rep}(S_4)$*; see Table 8.*

Table 8. Inequivalent irreducible based modules over $r(S_4)$.

		V_ψ	V_{ρ_1}	V_{ρ_2}	Categorification
Rank 1	$M_{1,1}$	1	2	3	Rep($\mathbb{Z}_1$), Rep(K_4, α)
Rank 2	$M_{2,1}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}2&0\\0&2\end{pmatrix}$	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$	No
	$M_{2,2}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	Rep($\mathbb{Z}_2$), Rep(D_4, α)
	$M_{2,3}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$	Rep($\mathbb{Z}_2$), Rep(D_4, α)
Rank 3	$M_{3,1}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	Rep($\mathbb{Z}_3$), Rep(A_4, α)
	$M_{3,2}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&1&1\\1&1&2\end{pmatrix}$	Rep(S_3)
	$M_{3,3}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&1\\1&1&2\end{pmatrix}$	Rep(S_3), Rep(S_4, α)
Rank 4	$M_{4,1}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1&1\\0&2&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0&0\\1&2&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}$	Rep(A_4)
	$M_{4,2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$	No
	$M_{4,3}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$	No
	$M_{4,4}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$	$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$	No
	$M_{4,5}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$	Rep($\mathbb{Z}_4$), Rep(K_4)
	$M_{4,6}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$	No
	$M_{4,7}$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$	Rep($\mathbb{Z}_4$), Rep(K_4)
Rank 5	$M_{5,1}$	$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&1&1&0\\0&0&1&1&0\\1&1&0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&1&1&1\\1&0&1&1&1\\0&0&1&1&0\end{pmatrix}$	Rep(S_4)
	$M_{5,2}$	$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&0&0&2\end{pmatrix}$	$\begin{pmatrix}0&0&0&1&1\\0&0&1&0&1\\0&1&0&0&1\\1&0&0&0&1\\1&1&1&1&1\end{pmatrix}$	Rep(D_4)

5. Conclusions

The analysis in this paper shows that the classification of the irreducible based modules of rank up to 5 over the complex representation ring $r(S_4)$. We also showed that any $\mathbb{Z}_+$-modules over the representation ring $r(G)$ categorified by a module category over the representation category Rep(G) must be a based module. At the end, we present the categorification of based modules over $r(S_4)$ by module categories over the complex representation category Rep(S_4) of S_4, using projective representations of specific subgroups of S_4. We expect that the studies developed here will be helpful in investigations of the structures of module categories over fusion categories. Our future study will focus on the existence of any irreducible based module of rank $\geq$ 6 over $r(S_4)$ and classifying irreducible $\mathbb{Z}_+$-modules over $r(S_4)$, especially for high-rank cases. Also, some other small finite groups may be interesting to consider, e.g., the dihedral group D_5.

Author contributions

Wenxia Wu: Writing-original draft and editing, conceptualization, software, methodology; Yunnan Li: Topic selection, writing-review and editing, funding acquisition, methodology, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to thank Zhiqiang Yu for helpful discussion. This work is supported by Guangdong Basic and Applied Basic Research Foundation (2022A1515010357).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. J. W. Barrett, B. W. Westbury, Spherical categories, *Adv. Math.*, **143** (1999), 357–375. https://doi.org/10.1006/aima.1998.1800
2. P. Deligne, J. S. Milne, A. Ogus, K. Y. Shih, *Hodge cycles, motives, and Shimura varieties*, Springer, 1982. https://doi.org/10.1007/BFb0091634
3. V. Drinfeld, S. Gelaki, D. Nikshych, V. Ostrik, On braided fusion categories I, *Selecta Math.*, **16** (2010), 1–119. https://doi.org/10.1007/s00029-010-0017-z
4. P. Etingof, D. Nikshych, V. Ostrik, On fusion categories, *Ann. Math.*, **162** (2005), 581–642. https://doi.org/10.4007/annals.2005.162.581

5. P. Etingof, D. Nikshych, V. Ostrik, Weakly group-theoretical and solvable fusion categories, *Adv. Math.*, **226** (2011), 176–205. https://doi.org/10.1016/j.aim.2010.06.009
6. S. Mac Lane, *Categories for the working mathematician*, 2 Eds., Springer, 1998. https://doi.org/10.1007/978-1-4612-9839-7
7. J. Bénabou, Catégories avec multiplication, *C. R. Acad. Sci. Paris*, **256** (1963), 1887–1890.
8. S. M. Lane, Natural associativity and commutativity, *Rice Univ. Stud.*, **49** (1963), 28–46.
9. D. Ben-Zvi, J. Francis, D. Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry, *J. Amer. Math. Soc.*, **23** (2010), 909–966. https://doi.org/10.1090/S0894-0347-10-00669-7
10. P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Fusion categories and homotopy theory, *Quantum Topol.*, **1** (2010), 209–273.https://doi.org/10.4171/QT/6
11. E. Rowell, R. Stong, Z. Wang, On classification of modular tensor categories, *Commun. Math. Phys.*, **292** (2009), 343–389. https://doi.org/10.1007/s00220-009-0908-z
12. J. Böckenhauer, D. E. Evans, Y. Kawahigashi, Chiral structure of modular invariants for subfactors, *Commun. Math. Phys.*, **210** (2000), 733–784. https://doi.org/10.1007/s002200050798
13. M. Bischoff, A. Davydov, Hopf algebra actions in tensor categories, *Transform. Groups*, **26** (2021), 69–80. https://doi.org/10.1007/s00031-020-09560-w
14. P. Etingof, R. Kinser, C. Walton, Tensor algebras in finite tensor categories, *Int. Math. Res. Not.*, **2021** (2021), 18529–18572. https://doi.org/10.1093/imrn/rnz332
15. G. Lusztig, Leading coefficients of character values of Hecke algebras, *Proc. Symp. Pure Math.*, **47** (1987), 235–262. https://doi.org/10.1090/pspum/047.2/933415
16. P. Etingof, M. Khovanov, Representations of tensor categories and Dynkin diagrams, *Int. Math. Res. Not.*, **1995** (1995), 235–235. https://doi.org/10.1155/S1073792895000183
17. V. Ostrik, Module categories, weak Hopf algebras and modular invariants, *Transform. Groups*, **8** (2003), 177–206. https://doi.org/10.1007/s00031-003-0515-6
18. P. Etingof, V. Ostrik, Finite tensor categories, *J. Mosc. Math.*, **4** (2004), 627–654. https://doi.org/10.17323/1609-4514-2004-4-3-627-654
19. H. X. Chen, F. V. Oystaeyen, Y. Zhang, The green rings of Taft algebras, *Proc. Amer. Math. Soc.*, **142** (2014), 765–775. https://doi.org/10.1090/s0002-9939-2013-11823-x
20. H. L. Huang, F. V. Oystaeyen, Y. Yang, Y. Zhang, The green rings of pointed tensor categories of finite type, *J. Pure Appl. Algebra*, **218** (2014), 333–342. https://doi.org/10.1016/j.jpaa.2013.06.005
21. L. Li, Y. Zhang, The green rings of the generalized Taft Hopf algebras, *Contemp. Math.*, **585** (2013), 275–288. https://doi.org/10.1090/conm/585/11618
22. Z. Wang, L. Li, Y. Zhang, Green rings of pointed rank one Hopf algebras of nilpotent type, *Algebras Represent. Theory*, **17** (2014), 1901–1924. https://doi.org/10.1007/s10468-014-9484-9
23. Z. Wang, L. Li, Y. Zhang, Green rings of pointed rank one Hopf algebras of non-nilpotent type, *J. Algebra*, **449** (2016), 108–137. https://doi.org/10.1016/j.jalgebra.2015.11.002
24. S. J. Witherspoon, The representation ring of the quantum double of a finite group, *J. Algebra*, **179** (1996), 305–329. https://doi.org/10.1006/jabr.1996.0014

25. V. Ostrik, Fusion categories of rank 2, *Math. Res. Lett.*, **10** (2003), 177–183. https://doi.org/10.4310/MRL.2003.V10.N2.A5

26. S. Burciu, On the Grothendieck rings of generalized Drinfeld doubles, *J. Algebra*, **486** (2017), 14–35. https://doi.org/10.1016/j.jalgebra.2017.05.019

27. F. Calegari, S. Morrison, N. Snyder, Cyclotomic integers, fusion categories, and subfactors, *Commun. Math. Phys.*, **303** (2011), 845–896. https://doi.org/10.1007/s00220-010-1136-2

28. V. Ostrik, Pivotal fusion categories of rank 3, *J. Mosc. Math.*, **15** (2015), 373–396. https://doi.org/10.17323/1609-4514-2015-15-2-373-396

29. D. E. Evans, T. Gannon, Near-group fusion categories and their doubles, *Adv. Math.*, **255** (2014), 586–640. https://doi.org/10.1016/j.aim.2013.12.014

30. M. Izumi, A Cuntz algebra approach to the classification of near-group categories, *ArXiv*, 2017. https://doi.org/10.48550/arXiv.1512.04288

31. C. Yuan, R. Zhao, L. Li, Irreducible $\mathbb{Z}_+$-modules of near-group fusion ring $K(\mathbb{Z}_3, 3)$, *Front. Math. China*, **13** (2018), 947–966. https://doi.org/10.1007/s11464-018-0709-9

32. P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, *Amer. Math. Soc.*, 2015. https://doi.org/10.1090/surv/205

33. C. Jones, S. Morrison, D. Nikshych, E. Rowell, Rank-finiteness for G-crossed braided fusion categories, *Transform. Groups*, **26** (2021), 915–927. https://doi.org/10.1007/s00031-020-09576-2

34. J. Schur, Über die Darstellung der endlichen Gruppen durch gebrochen lineare substitutionen, *J. Reine Angew. Math.*, **127** (1904), 20–50. https://doi.org/10.1515/crll.1904.127.20

35. C. Cheng, A character theory for projective representations of finite groups, *Linear Algebra Appl.*, **469** (2015), 230–242. https://doi.org/10.1016/j.laa.2014.11.027

36. G. Karpilovsky, *The Schur multiplier*, Oxford University Press, 1987.

37. B. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, 2 Eds., Springer, 2001. https://doi.org/10.1007/978-1-4757-6804-6

38. G. Karpilovsky, *Group representations*, North Holland Publishing Corporation, 1993.

39. D. G. Robbins, E. Sharpe, T. Vandermeulen, A generalization of decomposition in orbifolds, *J. High Energy Phys.*, **2021** (2021), 134. https://doi.org/10.1007/JHEP10(2021)134

40. G. Karpilovsky, *Projective representations of finite groups*, Marcel Dekker, 1985.

41. P. N. Hoffman, J. F. Humphreys, *Projective representations of the symmetric groups: Q-functions and shifted tableaux*, Oxford University Press, 1992. https://doi.org/10.1093/oso/9780198535560.001.0001

42. S. Ramgoolam, E. Sharpe, Combinatoric topological string theories and group theory algorithms, *J. High Energy Phys.*, **2022** (2022), 147. https://doi.org/10.1007/JHEP10(2022)147

AIMS Press