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Research article

The minimal degree Kirchhoff index of bicyclic graphs

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Abstract: The degree Kirchhoff index of graph G is defined as $Kf^*(G) = \sum_{u,v\subseteq V(G)} d(u)d(v)r_G(u,v)$, where $d(u)$ is the degree of vertex u and $r_G(u,v)$ is the resistance distance between the vertices u and v. In this paper, we characterize bicyclic graphs with exactly two cycles having the minimum degree Kirchhoff index of order $n \geq 5$. Moreover, we obtain the minimum degree Kirchhoff index on bicyclic graphs of order $n \geq 4$ with exactly three cycles, and all bicyclic graphs of order $n \geq 4$ where the minimum degree Kirchhoff index has been obtained.

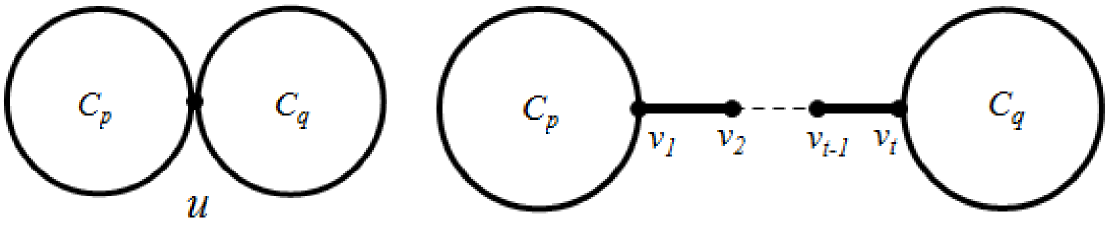
Keywords: bicyclic graph; degree Kirchhoff index; resistance distance; Θ-graph
Mathematics Subject Classification: 05C09

1. Introduction

Let G be a simple connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. Let $d(v_i)$ be the degree of vertex v_i, for $i = 1, 2, \ldots, n$. The distance $d(u,v)$ between the vertices u and v of the graph G is defined as the length of a shortest path between u and v. The resistance distance between the vertices u and v in G is denoted by $r_G(u,v)$.

In 1993, Klein and Randić [1] proposed a new distance function, resistance distance, based on electrical circuit theory. Similar to the long recognized shortest-path distance, the resistance distance is also intrinsic to the graph, and with some nice purely mathematical properties [2]. The effective resistance is mainly used in electronic networks for which nodes correspond to vertices of G and each edge of G is replaced by a resistor of unit resistance. The resistance distance is very sensitive to small changes in the conductances, and it is suitable to discriminate between networks with similar structure [3, 4]. The resistance distance has been studied in mathematical, physical, and chemical papers [5–7], and it has important applications in chemistry.

A bicyclic graph is a connected graph which satisfies $|V(G)| + 1 = |E(G)|$. Let $\mathcal{B}(n)$ be the set of connected bicyclic graphs of order n. It is well known that any graph in $\mathcal{B}(n)$ contains either two cycles or three cycles. There are two basic types: ∞-type graphs and Θ-type graphs; for the basic structure of ∞-type bicyclic graphs and Θ-type bicyclic graphs, see Figure 1. An ∞-type graph is obtained by attaching trees T_i to some vertices of $B_1(p,q)$ or $B_2(p,q)$, where $B_1(p,q)$ can be constructed by two vertex-disjoint cycles C_p and C_q by identifying a vertex u, and $B_2(p,q)$ can be constructed by two vertex-disjoint cycles C_p and C_q connected by a new path $v_1v_2\cdots v_t$ with length $t-1$. Further, the graph $S_n^{p,q}$ is obtained from the graph $B_1(p,q)$ by attaching t pendant vertices at vertex u, where $t = n-p-q+1$.



(a) $B_1(p,q)$ (b) $B_2(p,q)$

Figure 1. The basic structures of ∞-type bicyclic Graphs.

A Θ-graph is the union of three internally disjoint paths with two common end vertices (see Figure 2). A Θ-type bicyclic graph, denoted by $\Theta_n^{p,q,m}$, is a union of three internally disjoint paths $P_p : v_0v_1\ldots v_p$, $P_q : u_0(=v_p)u_1u_2\ldots u_q(=v_0)$, $P_{m+1} : v_0w_1\ldots w_mu_0$, of length $p,q,m+1$, respectively, with common end vertices, and the trees $T_{v_i}(0 \le i \le p-1, p \ge 2)$, $T_{u_j}(0 \le j \le q-1, q \ge 2)$, $T_{w_k}(0 \le k \le m)$ are rooted at v_i,u_j,w_k, respectively; see Figure 3 for an example of such a graph. Let a Θ-type graph $S_n^{p,q,m}$ denote the graph obtained from a Θ-type graph $\Theta_n^{p,q,m}$ by attaching all of its pendent vertices to one of the two common end vertices.

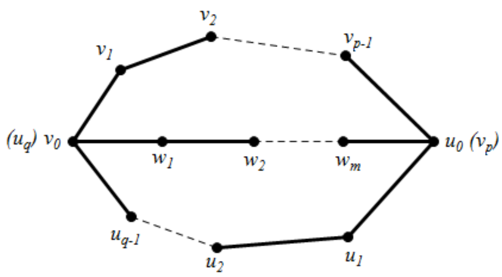


Figure 2. An example of a Θ-graph.

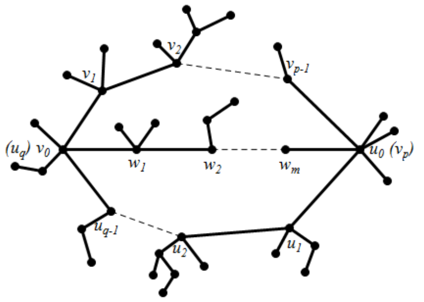


Figure 3. An example of a $\Theta_n^{p,q,m}$-graph.

In 1994, Bonchev et al. put forward the concept of the Kirchhoff index in [7]. After that, many scholars began to study the maximal and minimal Kirchhoff index of the extremal graphs of unicyclic graphs [8–11]. In 2016, Liu et al. completely characterized the bicyclic graphs of order $n \geq 4$ with minimal Kirchhoff index and determined bounds on the Kirchhoff index of bicyclic graphs [12].

The Kirchhoff index of G is defined similarly to the Wiener index

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} r_G(u, v).$$

We define $Kf_v(G)$ as follows:

$$Kf_v(G) = \sum_{u\in V(G)} r_G(u, v).$$

The Kirchhoff index has been widely studied in the literature [13–15].

In 2005, the degree Kirchhoff index was proposed by Chen and Zhang in [16]. The degree Kirchhoff index is defined

$$Kf^*(G) = \sum_{\{u,v\}\subseteq V(G)} d(u)d(v)r_G(u, v) = \frac{1}{2} \sum_{u\in V(G)} d(u)Kf_u^*(G),$$

where $Kf_u^*(G) = \sum_{v\in V(G)} d(v)r_G(v, u)$. More results for the degree Kirchhoff index of graphs can be found in [17–25].

Many scholars have studied the extreme values of degree Kirchhoff index in unicyclic graphs in the literature. In 2013, Feng et al. completely characterized unicyclic graphs having maximum, second-maximum, minimum, and second-minimum degree Kirchhoff index [20]. In 2014, Feng et al. gave some upper and lower bounds for the degree Kirchhoff index of graphs under certain conditions and characterized fully loaded unicyclic graphs having minimum and maximum degree Kirchhoff index [23,24]. In 2020, Qi et al. determined the maximum degree Kirchhoff index of n-vertex unicyclic graphs with fixed maximum degree [25].

The extreme values of degree Kirchhoff index in bicyclic graphs has also been widely studied in the literature. In 2017, Tang, et al. obtained graphs having maximum and minimum degree Kirchhoff index among all n-vertex bicyclic graphs with exactly two cycles [21]. In 2018, Fei et al. completely determine bicyclic graphs of order $n \geq 6$ having the maximum degree Kirchhoff index, and bicyclic graphs of order $n \geq 7$ with the second-maximum degree Kirchhoff index [18].

Let $L^* = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ be the normalized Laplacian matrix of a graph G, and let its eigenvalues be $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} \geq \rho_n = 0$. Then, a known result for the degree Kirchhoff index is [16]

$$Kf^*(G) = 2m\sum_{i=1}^{n-1}\frac{1}{\rho_i}.$$

It is well-known that $r_G(u,v) = r_G(v,u)$, $d(u,v) \geq r_G(u,v)$, and this equation holds if and only if there is a unique path connecting the vertices u and v.

As usual, P_n, C_n, and S_n, denote, respectively, the path, cycle, and star on n vertices. Let U_k^n consist of a cycle size k to which a path with $n-k$ vertices is attached. Let $H_{n,k}$ be obtained from C_k by attaching $n-k$ pendent vertices to a vertex of C_k. For other undefined notations and terminologies from graph theory, the readers are referred to [20].

This paper is organized as follows. In Section 2, we list some previously known results. In Section 3, we characterize the minimal degree Kirchhoff index among all the bicyclic graphs of order $n \geq 5$ with exactly two cycles. In Section 4, we obtain a lower bound on the degree Kirchhoff index of Θ-type graphs. In Section 5, we deduce the degree Kirchhoff index of Θ-graphs, and show the minimal degree Kirchhoff index of the bicyclic graphs.

2. Preliminaries

This section lists some known results to be used in this paper.

Let C_n be the cycle on $n \geq 3$ vertices, and v be the central vertex of S_n. Then [26]

$$r_{C_n}(v_i v_j) = \frac{(j-i)\cdot[n-(j-i)]}{n}, \tag{2.1}$$

where $1 \leq i < j \leq n$. And [20]

$$Kf_{v_i}(C_n) = \frac{n^2-1}{6}, \quad Kf^*_{v_i}(C_n) = \frac{n^2-1}{3}, \ \ for\ 1 \leq i \leq n. \tag{2.2}$$

$$Kf^*(C_n) = 4Kf(C_n) = \frac{n^3-n}{3}\ . \tag{2.3}$$

$$Kf^*_v(S_n) = n-1, \ \ Kf^*(S_n) = (n-1)(2n-3). \tag{2.4}$$

Lemma 2.1. *[1] Let x be a cut vertex of a graph G, and let a and b be vertices occurring in different components which arise upon deletion of x. Then $r_G(a,b) = r_G(a,x) + r_G(x,b)$.*

Lemma 2.2. *[19] Let x be a cut vertex of a graph G such that $G - \{x\}$ consists of vertex-disjoint subgraphs G_1' and G_2'. Let G_i be the subgraph of G induced by $V(G_i') \cup \{x\}$, where $i = 1, 2$. Then:*
(1)

$$Kf^*(G) = Kf^*(G_1) + Kf^*(G_2) + 2|E(G_1)|Kf^*_x(G_2) + 2|E(G_2)|Kf^*_x(G_1);$$

(2) in particular, for $G_2 = P_2$,

$$Kf^*(G) = Kf^*(G_1) + 2Kf^*_x(G_1) + 2|E(G_1)| + 1.$$

Lemma 2.3. *[20] Let $G \cong H_{n,3}$ be a unicyclic graph of order $n \geq 5$. Then $Kf^*(G) = \frac{1}{3}(6n^2 - 5n - 15)$.*

Lemma 2.4. *[21] Let G be a bicyclic graph of order n with exactly two cycles. Then*

$$Kf^*(G) \geq Kf^*(S_n^{3,3}).$$

Lemma 2.5. *[5] Let r_G and $r_{G'}$ be resistance distance functions for edge-weighted connected graphs G and G′, which are the same except for the weights w and w′ on an edge e with end vertices i and j. Then, for any $p, q \in V(G) = V(G')$,*

$$r_G(p,q) = r_{G'}(p,q) - \frac{\delta \cdot [r_{G'}(p,i) + r_{G'}(q,j) - r_{G'}(p,j) - r_{G'}(q,i)]^2}{4[1 + \delta \cdot r_{G'}(i,j)]},$$

where $\delta = w - w'$.

Lemma 2.6. *[20] Let the graph $G \cong U_k^n$ consist of a cycle of size k to which a path with $n-k$ vertices is attached. Then*

$$Kf^*(U_k^n) = \frac{1}{3}(3k^3 - 4nk^2 + 2n^3 - n).$$

Let v be a vertex of degree $p+1$ in a graph G, such that $vv_1, vv_2, \ldots, vv_p$ are pendent edges incident with v, and u is the neighbor of v distinct from $v_1, v_2, \ldots, v_p$. We say that G' is a σ-transform of G if the graph $G' = \sigma(G, v)$ is obtained by removing the edges $vv_1, vv_2, \ldots, vv_p$ and adding new edges $uv_1, uv_2, \ldots, uv_p$ (see Figure 4).

Figure 4. The σ-transformation $\sigma(G, v)$.

Lemma 2.7. *[20] Let $G' = \sigma(G, v)$ be a σ-transform of G. Then $Kf^*(G) \geq Kf^*(G')$. This equation holds if and only if G is a star with v as its center.*

3. The minimum degree Kirchhoff index of bicyclic graphs with exactly two cycles

Theorem 3.1. *Let B_n be a bicyclic graph of order $n \geq 5$ with exactly two cycles. Then $Kf^*(B_n) \geq 2n^2 + \frac{5}{3}n - \frac{31}{3}$.*

Proof. By Lemma 2.4, we have $Kf^*(B_n) \geq Kf^*(S_n^{3,3})$.

Let G_1 and G_2 be the subgraphs of $S_n^{3,3}$ induced by $V(S_n^{3,3} \setminus V(C_3)) \cup \{u\}$ and $V(C_3)$, respectively. By Lemma 2.3 and Eqs (2.2) and (2.3), we get

$$Kf^*(U_k^n) = \frac{1}{3}(3k^3 - 4nk^2 + 2n^3 - n).$$

$$Kf^*(G_1) = \frac{1}{3}(6(n-2)^2 - 5(n-2) - 15) = \frac{1}{3}(6n^2 - 29n + 19);$$

$$Kf^*(G_2) = \frac{q^3 - q}{3} = 8;$$

$$Kf_u^*(G_1) = \frac{p^2 - 1}{3} + n - 5 = n - \frac{7}{3};$$

$$Kf_u^*(G_2) = \frac{q^2 - 1}{3} = \frac{8}{3}.$$

So, by Lemma 2.2(1),

$$\begin{aligned} Kf^*(S_n^{3,3}) &= \frac{1}{3}(6n^2 - 29n + 19) + 8 + 2(n-2)\cdot\frac{8}{3} + 2\cdot 3\cdot(n - \frac{7}{3}) \\ &= 2n^2 + \frac{5}{3}n - \frac{31}{3}. \end{aligned} \tag{3.1}$$

Therefore, we have $Kf^*(B_n) \geq 2n^2 + \frac{5}{3}n - \frac{31}{3}$. ■

This is just to obtain the minimum degree Kirchhoff index for bicyclic graphs with exactly two cycles, which does not account for all bicyclic graphs. Next, we will consider bicyclic graphs which contain three cycles, that is, Θ-type graphs.

4. Lower bounds on the degree Kirchhoff index of Θ-type graphs

Lemma 4.1. *[12] Let G be Θ-graph of order n. Then*

$$Kf_{v_k}(G) - Kf_{v_0}(G) = \frac{k(p-k)(q^2 + qm + m^2 + m)}{(1+m)(p+q) + pq}. \tag{4.1}$$

Lemma 4.2. *Let G be Θ-graph of order n. Then*

$$Kf_{v_0}^*(G) < Kf_{v_k}^*(G), \ \ 1 \leq k \leq p-1.$$

Proof. Let $i = w_m$, $j = u_0$, and G' be attained from G by deleting the edge $w_m u_0$. For convenience, $r(v_k, v_j)$ denotes the $r_{G'}(v_k, v_j)$. Based on Eq (2.1), we can get

$$r(v_0, w_m) = m, \quad r(v_0, u_0) = \frac{pq}{p+q}, \quad r(v_k, u_0) = \frac{(p-k)(q+k)}{p+q},$$

$$r(v_k, v_0) = \frac{k(p+q-k)}{p+q}, \quad r(w_m, u_0) = r(v_0, w_m) + r(v_0, u_0) = m + \frac{pq}{p+q},$$

$$r(w_m, v_k) = r(v_0, w_m) + r(v_0, v_k) = m + \frac{k(p+q-k)}{p+q}.$$

Since every vertex in $V(G) \setminus \{v_0, u_0\}$ has degree 2 and $d(v_0) = d(u_0) = 3$, by using Lemma 2.5 we get

$$Kf_{v_0}^*(G) = \sum_{v \in V(G)} d(v) r_G(v_0, v)$$

$$
\begin{aligned}
&=2\sum_{v\in V(G)}\left\{r(v_0,v)-\frac{[r(v_0,w_m)+r(v,u_0)-r(v_0,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\right\}\\
&\quad+r(v_0,u_0)-\frac{[r(v_0,w_m)+r(u_0,u_0)-r(v_0,u_0)-r(u_0,w_m)]^2}{4[1+r(w_m,u_0)]}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&Kf^*_{v_k}(G)=\sum_{v\in V(G)}d(v)r_G(v_k,v)\\
&=2\sum_{v\in V(G)}\left\{r(v_k,v)-\frac{[r(v_k,w_m)+r(v,u_0)-r(v_k,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\right\}\\
&\quad+r(v_0,v_k)-\frac{[r(v_k,w_m)+r(v_0,u_0)-r(v_k,u_0)-r(v_0,w_m)]^2}{4[1+r(w_m,u_0)]}\\
&\quad+r(u_0,v_k)-\frac{[r(v_k,w_m)+r(u_0,u_0)-r(v_k,u_0)-r(u_0,w_m)]^2}{4[1+r(w_m,u_0)]}.
\end{aligned}
$$

Therefore, based on the Eq (4.1), we can obtain

$$
\begin{aligned}
&Kf^*_{v_k}(G)-Kf^*_{v_0}(G)\\
&=2\sum_{v\in V(G)}[r(v_k,v)-r(v_0,v)]\\
&\quad+2\sum_{v\in V(G)}\frac{[r(v_0,w_m)+r(v,u_0)-r(v_0,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\\
&\quad-2\sum_{v\in V(G)}\frac{[r(v_k,w_m)+r(v,u_0)-r(v_k,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\\
&\quad+\frac{k(p+q-k)}{p+q}-\frac{\left[m+\frac{k(p+q-k)}{p+q}+\frac{pq}{p+q}-\frac{(p-k)(q+k)}{p+q}-m\right]^2}{4[1+m+\frac{pq}{p+q}]}\\
&\quad+\frac{(p-k)(q+k)}{p+q}-\frac{\left[m+\frac{k(p+q-k)}{p+q}-\frac{(p-k)(q+k)}{p+q}-m-\frac{pq}{p+q}\right]^2}{4\left[1+m+\frac{pq}{p+q}\right]}\\
&\quad-\frac{pq}{p+q}+\frac{\left[m-\frac{pq}{p+q}-m-\frac{pq}{p+q}\right]^2}{4\left[1+m+\frac{pq}{p+q}\right]}\\
&=2(Kf_{v_k}(G)-Kf_{v_0}(G))+\frac{2pk-2k^2}{p+q}+\frac{2kpq^2-2k^2q^2}{((1+m)(p+q)+pq)(p+q)}\\
&=2\cdot\frac{k(p-k)(q^2+qm+m^2+m)}{(1+m)(p+q)+pq}+2\cdot\frac{k(p-k)(m+q+1)}{(1+m)(p+q)+pq}\\
&=2\cdot\frac{k(p-k)(q^2+q(m+1)+m^2+2m+1)}{(1+m)(p+q)+pq}>0. \qquad (4.2)
\end{aligned}
$$

■

Lemma 4.3. *Let G be a Θ-type bicyclic graph of order n, s pendent vertices $x_1, x_2, \ldots, x_s$ attached at v_0, and t pendent vertices $y_1, y_2, \ldots, y_t$ attached at v_k. Let*

$$G_1 = G - \{v_k y_1, v_k y_2, \ldots, v_k y_t\} + \{v_0 y_1, v_0 y_2, \ldots, v_0 y_t\},$$

and

$$G_2 = G - \{v_0 x_1, v_0 x_2, \ldots, v_0 x_s\} + \{v_k x_1, v_k x_2, \ldots, v_k x_s\}.$$

Then, either $Kf^*(G) > Kf^*(G_1)$ or $Kf^*(G) > Kf^*(G_2)$.

Proof. Let $A = \{x_1, x_2, \ldots, x_s\}$, $B = \{y_1, y_2, \ldots, y_t\}$, and H be the subgraph induced by Θ-graphs. For convenience, we write $r_G(v_0, v_k) = l$. For $G \to G_1$ and any pair of vertices u, v satisfying $u, v \in V(H - v_0 - v_k)$, $u, v \in A$, $u, v \in B$, or $u \in A, v \in V(H - v_0 - v_k)$, $\sum\limits_{u,v} d(u)d(v)r_G(u, v)$ does not change. Then, by Lemma 2.1, we have

$$\begin{aligned}
&Kf^*(G)\\
=&\left[\sum_{u,v\in V(H-v_0-v_k)} + \sum_{u,v\in A} + \sum_{u,v\in B} + \sum_{\substack{u\in A\\ v\in V(H-v_0-v_k)}}\right] d(u)d(v)r_G(u,v)\\
&+\sum_{\substack{u\in A\\ v\in B}} d(u)d(v)r_G(u,v) + \sum_{\substack{u\in V(H-v_0-v_k)\\ v\in B}} d(u)d(v)r_G(u,v)\\
&+d(v_0)d(v_k)r_G(v_0,v_k)\\
&+d(v_0)\left[\sum_{u\in V(H-v_0-v_k)} + \sum_{u\in A} + \sum_{u\in B}\right] d(u)r_G(u,v_0)\\
&+d(v_k)\left[\sum_{u\in V(H-v_0-v_k)} + \sum_{u\in A} + \sum_{u\in B}\right] d(u)r_G(u,v_k)\\
=&\left[\sum_{u,v\in V(H-v_0-v_k)} + \sum_{u,v\in A} + \sum_{u,v\in B} + \sum_{\substack{u\in A\\ v\in V(H-v_0-v_k)}}\right] d(u)d(v)r_G(u,v)\\
&+(l+2)st + t\sum_{u\in V(H-v_0-v_k)} (d(u)r_G(u,v_k) + d(u)) + (s+3)(t+2)l\\
&+(3+s)\left[\sum_{u\in V(H-v_0-v_k)} d(u)r_G(u,v_0) + s + (l+1)t\right]\\
&+(t+2)\left[\sum_{u\in V(H-v_0-v_k)} d(u)r_G(u,v_k) + (l+1)s + t\right],
\end{aligned}$$

and similarly

$$\begin{aligned}
&Kf^*(G_1)\\
=&\left[\sum_{u,v\in V(H-v_0-v_k)}+\sum_{u,v\in A}+\sum_{u,v\in B}+\sum_{\substack{u\in A\\ v\in V(H-v_0-v_k)}}\right]d(u)d(v)r_G(u,v)\\
&+2st+t\sum_{u\in V(H-v_0-v_k)}(d(u)r_G(u,v_0)+d(u))+2(s+t+3)l\\
&+(3+s+t)\left[\sum_{u\in V(H-v_0-v_k)}d(u)r_G(u,v_0)+s+t\right]\\
&+2\left[\sum_{u\in V(H-v_0-v_k)}d(u)r_G(u,v_k)+(l+1)s+(l+1)t\right].
\end{aligned}$$

So, we have

$$Kf^*(G)-Kf^*(G_1)=2t\left[\sum_{u\in V(H-v_0-v_k)}d(u)[r_G(u,v_k)-r_G(u,v_0)]+2sl+l\right].$$

And analogously, for $G\to G_2$ and any pair of vertices u,v satisfying $u,v\in V(H-v_0-v_k)$, $u,v\in A$, $u,v\in B$ or $u\in B, v\in V(H-v_0-v_k)$, $\sum\limits_{u,v}d(u)d(v)r_G(u,v)$ does not change. Then, by Lemma 2.1, we have

$$\begin{aligned}
&Kf^*(G)\\
=&\left[\sum_{u,v\in V(H-v_0-v_k)}+\sum_{u,v\in A}+\sum_{u,v\in B}+\sum_{\substack{u\in B\\ v\in V(H-v_0-v_k)}}\right]d(u)d(v)r_G(u,v)\\
&+\sum_{\substack{u\in A\\ v\in B}}d(u)d(v)r_G(u,v)+\sum_{\substack{u\in V(H-v_0-v_k)\\ v\in A}}d(u)d(v)r_G(u,v)\\
&+d(v_0)d(v_k)r_G(v_0,v_k)\\
&+d(v_0)\left[\sum_{u\in V(H-v_0-v_k)}+\sum_{u\in A}+\sum_{u\in B}\right]d(u)r_G(u,v_0)\\
&+d(v_k)\left[\sum_{u\in V(H-v_0-v_k)}+\sum_{u\in A}+\sum_{u\in B}\right]d(u)r_G(u,v_k)\\
=&\left[\sum_{u,v\in V(H-v_0-v_k)}+\sum_{u,v\in A}+\sum_{u,v\in B}+\sum_{\substack{u\in B\\ v\in V(H-v_0-v_k)}}\right]d(u)d(v)r_G(u,v)\\
&+(l+2)st+s\sum_{u\in V(H-v_0-v_k)}(d(u)r_G(u,v_0)+d(u))+(s+3)(t+2)l\\
&+(3+s)\left[\sum_{u\in V(H-v_0-v_k)}d(u)r_G(u,v_0)+s+(l+1)t\right]
\end{aligned}$$

$$+ (t+2)\left[\sum_{u\in V(H-v_0-v_k)} d(u)r_G(u,v_k) + (l+1)s + t\right],$$

and similarly

$$\begin{aligned}
&Kf^*(G_2)\\
&=\left[\sum_{u,v\in V(H-v_0-v_k)} + \sum_{u,v\in A} + \sum_{u,v\in B} + \sum_{\substack{u\in B\\ v\in V(H-v_0-v_k)}}\right] d(u)d(v)r_G(u,v)\\
&\quad + 2st + s\sum_{u\in V(H-v_0-v_k)} (d(u)r_G(u,v_k) + d(u)) + 3(s+t+2)l\\
&\quad + 3\left[\sum_{u\in V(H-v_0-v_k)} d(u)r_G(u,v_0) + (l+1)s + (l+1)t\right]\\
&\quad + (2+s+t)\left[\sum_{u\in V(H-v_0-v_k)} d(u)r_G(u,v_k) + s + t\right].
\end{aligned}$$

So, we have

$$Kf^*(G) - Kf^*(G_2) = 2s\left[\sum_{u\in V(H-v_0-v_k)} d(u)[r_G(u,v_0) - r_G(u,v_k)] + 2tl - l\right].$$

If $Kf^*(G) - Kf^*(G_i) > 0$ for $i = 1,2$, then the result follows. If at least one of the differences is negative, say, $Kf^*(G) - Kf^*(G_1) < 0$, then $\sum_{u\in V(H-v_0-v_k)} d(u)[r_G(u,v_k) - r_G(u,v_0)] < -(2ls + l)$. So $Kf^*(G) - Kf^*(G_2) > 2s(2ls + l + 2lt - l) > 0$. ■

Theorem 4.1. *Let $G \in \Theta_n^{p,q,m}$ and $G \ncong S_n^{p,q,m}$. Then*

$$Kf^*(G) > Kf^*(S_n^{p,q,m}).$$

Proof. Recall the description of $\Theta_n^{p,q,m}$-graphs given in Section 1. Let T_v be the tree found at vertex v. Let G' be the graph obtained from G through a sequence of transformations. so that, instead of T_v, G' has a star graph S_v at v, where $|S_v| = |T_v|$. Then, by Lemma 2.7, $Kf^*(G') \leq Kf^*(G)$. Further, by Lemma 4.2, there exists a G'' obtained from G' by moving the stars S_v to a common vertex, so that $Kf^*(G')$ is minimal with respect to these transformations.

Next, we consider where all the pendent vertices should be located in order to attain the minimal degree Kirchhoff index of the graph.

Let G be a Θ-graph of order n. We attach an additional t pendant vertices $S = \{a_1, a_2, \dots, a_t\}$ attached at v_0 or v_k. So, let

$$G_1 = G + \{v_0a_1, v_0a_2, \dots, v_0a_t\},$$

and

$$G_2 = G + \{v_ka_1, v_ka_2, \dots, v_ka_t\}.$$

Then, by Lemma 2.1, we have

$$
\begin{aligned}
Kf^*(G_1) =& \left[\sum_{u,v\in S} + \sum_{u,v\in V(G-v_0-v_k)}\right] d(u)d(v)r_G(u,v) \\
&+ \sum_{\substack{u\in V(G)\\ v\in S}} d(u)d(v)r_G(u,v) + d(v_0)\sum_{u\in V(G)} d(u)r_G(u,v_0) \\
&+ d(v_k)\sum_{u\in V(G)} d(u)r_G(u,v_k) - d(v_0)d(v_k)r_G(v_0,v_k) \\
=& \left[\sum_{u,v\in S} + \sum_{u,v\in V(G-v_0-v_k)}\right] d(u)d(v)r_G(u,v) \\
&+ t\sum_{u\in V(G)} [d(u)r_G(u,v_0) + d(u)] \\
&+ (3+t)Kf^*_{v_0}(G) + 2Kf^*_{v_k}(G) - 2(3+t)r_G(v_0,v_k),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
Kf^*(G_2) =& \left[\sum_{u,v\in S} + \sum_{u,v\in V(G-v_0-v_k)}\right] d(u)d(v)r_G(u,v) \\
&+ t\sum_{u\in V(G)} [d(u)r_G(u,v_k) + d(u)] \\
&+ 3Kf^*_{v_0}(G) + (2+t)Kf^*_{v_k}(G) - 3(2+t)r_G(v_0,v_k).
\end{aligned}
$$

Based on Lemma 2.5, we can have

$$
\begin{aligned}
r_G(v_0,v_k) =& \left\{ r(v_0,v_k) - \frac{[r(v_0,w_m) + r(v_k,u_0) - r(v_0,u_0) - r(v_k,w_m)]^2}{4[1 + r(w_m,u_0)]} \right\} \\
=& \frac{k(p+q-k)}{p+q} - \frac{[m + \frac{(p-k)(q+k)}{p+q} - \frac{pq}{p+q} - m - \frac{k(p+q-k)}{p+q}]^2}{4[1 + m + \frac{pq}{p+q}]} \\
=& \frac{k(p+q-k)}{p+q} - \frac{(-2kq)^2}{4(p+q)\cdot[(1+m)(p+q)+pq]} \\
=& \frac{k(p+q-k)\,[(1+m)(p+q)+pq] - k^2q^2}{(p+q)\cdot[(1+m)(p+q)+pq]}.
\end{aligned}
$$

Therefore, based on Eq (4.2), we can obtain that

$$
\begin{aligned}
& Kf^*(G_2) - Kf^*(G_1) \\
=& 2t(Kf^*_{v_k}(G) - Kf^*_{v_0}(G)) - t\cdot r_G(v_0,v_k) \\
=& t\left\{ 2\cdot \frac{2\cdot k(p-k)(q^2 + q(m+1) + m^2 + 2m + 1)}{(1+m)(p+q)+pq} \right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{k(p+q-k)\left[(1+m)(p+q)+pq\right]-k^2q^2}{(p+q)\cdot\left[(1+m)(p+q)+pq\right]}\Bigg\} \\
=&tk\Bigg\{\frac{4\cdot(p-k)(p+q)[q^2+q(m+1)+(m+1)^2]}{[(1+m)(p+q)+pq]\cdot(p+q)} \\
&-\frac{(p+q-k)\left[(1+m)(p+q)+pq\right]-kq^2}{[(1+m)(p+q)+pq]\cdot(p+q)}\Bigg\}.
\end{aligned}
$$

We can obtain that

$$
\begin{aligned}
z=&4(p-k)(p+q)[q^2+q(m+1)+(m+1)^2] \\
&-(p+q-k)[(1+m)(p+q)+pq]+kq^2 \\
=&(p-k)\left\{2(p+q)[q^2+q(m+1)+(m+1)^2]-(1+m)(p+q)+pq\right\} \\
&+2(p-k)(p+q)[q^2+q(m+1)+(m+1)^2] \\
&-q[(1+m)(p+q)+pq]+kq^2 \\
>&0.
\end{aligned}
$$

Then $Kf^*(G_1) < Kf^*(G_2)$, so by Lemma 4.1, we can attain the minimal value when all pendent vertices are attached to v_0 or u_0, that is, $Kf^*(G) > Kf^*(S_n^{p,q,m})$. ■

5. Minimal degree Kirchhoff index of bicyclic graphs

Lemma 5.1. *[12] Let G be a Θ-graph of order n. Then*

$$
\begin{aligned}
\sum_{u\in V(G)}[r(u,u_0)-r(u,w_m)]^2=&\frac{pq(pq+2)}{3(p+q)}+\frac{m(m^2+2)}{3} \\
&+(p+q)m^2+\frac{2mpq}{p+q}+\frac{p^2q^2m}{(p+q)^2}.
\end{aligned}
\tag{5.1}
$$

$$
\sum_{u\in V(G)} r(u,u_0)-\sum_{u\in V(G)} r(u,w_m)=m-(p+q)m+\frac{mpq}{p+q}. \tag{5.2}
$$

$$
\begin{aligned}
Kf_{v_0}(G)=&\frac{(p+q)^2-1}{6}+\frac{m(m+1)}{2} \\
&-\frac{p+q}{6\left[(1+m)(p+q)+pq\right]}\cdot\left[\frac{pq(2pq+1)}{p+q}+m(m+1)(2m+1)\right].
\end{aligned}
\tag{5.3}
$$

Now we precisely deduce $Kf^*_{v_0}(G)$ and $Kf^*(G)$ for any Θ-graph G.

Let $i = w_m$, $j = u_0$ and G' be attained from G by deleting the edge w_mu_0. For convenience, $r(v_k, v_j)$ indicates the $r_{G'}(v_k, v_j)$. The set C indicates vertices in cycles and the set P indicates other vertices, that is, $P = V(G) \setminus C$. Based on Eq (5.3), we now can obtain

$$
\begin{aligned}
&Kf^*_{v_0}(G) \\
=&2\sum_{v\in V(G)}\left\{r(v_0,v)-\frac{[r(v_0,w_m)+r(v,u_0)-r(v_0,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{pq}{p+q} - \frac{p^2q^2}{(p+q)\left((m+1)(p+q)+pq\right)}\\
=&2Kf_{v_0}(G) + \frac{pq(1+m)}{(1+m)(p+q)+pq}\\
=&\frac{(p+q)^2-1}{3} + m(m+1) + \frac{pq(1+m)}{(1+m)(p+q)+pq}\\
&- \frac{p+q}{3\left[(1+m)(p+q)+pq\right]} \cdot \left[\frac{pq(2pq+1)}{p+q} + m(m+1)(2m+1)\right].
\end{aligned} \tag{5.4}
$$

Then we calculate $Kf^*(G)$ by Lemma 2.5.

$$
\begin{aligned}
Kf^*(G) =& \sum_{u,v\subseteq V(G)} d(u)d(v)r_G(u,v) = \frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u)d(v)r_G(u,v)\\
=&\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u)d(v)\Big\{r(u,v)\\
&\qquad - \frac{[r(u,w_m)+r(v,u_0)-r(u,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}\Big\}\\
=&\underbrace{\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u)d(v)r(u,v)}_{M}\\
&\underbrace{-\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u)d(v)\cdot\frac{[r(u,w_m)+r(v,u_0)-r(u,u_0)-r(v,w_m)]^2}{4[1+r(w_m,u_0)]}}_{N}.
\end{aligned}
$$

Completely expanding M, among $d(u)$ and $d(v)$ are degrees of vertex for graph G. Since every vertex in $V(G')\setminus\{u_0, w_m, v_0\}$ has degree 2 and $d_{G'}(v_0) = 3$, we can obtain

$$
\begin{aligned}
M =&\frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u)d(v)r(u,v)\\
=&\sum_{\{u,v\}\subseteq V(G')} d_{G'}(u)d_{G'}(v)r(u,v) + 2\sum_{v\in V(G')} r(u_0,v)\\
&+ 2\sum_{v\in V(G')} r(w_m,v) + r(u_0,v_0) + r(w_m,v_0)\\
=&Kf^*(G') + 2Kf_{u_0}(G') + 2Kf_{w_m}(G') + r(u_0,v_0) + r(w_m,v_0).
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
Kf_{u_0}(G') =&Kf_{u_0}(C) + Kf_{u_0}(P)\\
=&\frac{(p+q)^2-1}{6} + \left(\frac{pq}{p+q} + 1 + \cdots + \frac{pq}{p+q} + m\right)\\
=&\frac{(p+q)^2-1}{6} + m\frac{pq}{p+q} + \frac{m(m+1)}{2},
\end{aligned} \tag{5.5}
$$

and

$$\begin{aligned}Kf_{w_m}(G') =&Kf_{w_m}(C) + Kf_{w_m}(P)\\ =&\Big(m + m + r(v_0, v_1) + \cdots + m + r(v_0, u_{q-1})\Big)\\ &+ (m - 1 + m - 2 + \cdots + 1)\\ =&(p + q)m + \frac{(p + q)^2 - 1}{6} + \frac{m(m - 1)}{2}.\end{aligned} \tag{5.6}$$

By Lemma 2.6, we know

$$\begin{aligned}Kf^*(G') =&\frac{2m^3}{3} + 2(p + q)m^2 + \frac{(2(p + q)^2 - 1)m}{3}\\ &+ \frac{(p + q)(p + q - 1)(p + q + 1)}{3}.\end{aligned} \tag{5.7}$$

Therefore, based on Eqs (5.5)–(5.7), we can get

$$\begin{aligned}M =&\frac{2m^3}{3} + 2(p + q)m^2 + \frac{(2(p + q)^2 - 1)m}{3} + \frac{(p + q)(p + q - 1)(p + q + 1)}{3}\\ &+ \frac{(p + q)^2 - 1}{3} + \frac{2mpq}{p + q} + m(m + 1) + 2(p + q)m + \frac{(p + q)^2 - 1}{3}\\ &+ m(m - 1) + \frac{pq}{p + q} + m.\end{aligned}$$

Then we calculate N:

$$\begin{aligned}N =&\frac{1}{8[1 + r(w_m, u_0)]} \sum_{u\in V(G)} \sum_{v\in V(G)} d(u)d(v)\Big\{ \Big[r^2(u, u_0) + r^2(u, w_m)\\ &-2r(u, u_0)r(u, w_m)] + \Big[r^2(v, u_0) + r^2(v, w_m) - 2r(v, u_0)r(v, w_m)\Big]\\ &+ [2r(u, u_0)r(v, w_m) - 2r(u, u_0)r(v, u_0) - 2r(v, w_m)r(u, w_m)\\ &+2r(u, w_m)r(v, u_0)] \Big\},\\ =&\frac{1}{8[1 + r(w_m, u_0)]} \Bigg\{2(m + p + q + 1) \sum_{u\in V(G)} d(u) [r(u, u_0) - r(u, w_m)]^2\\ &+ 2(m + p + q + 1) \sum_{v\in V(G)} d(v) [r(v, u_0) - r(v, w_m)]^2\\ &-2\Bigg[\sum_{u\in V(G)} d(u)r(u, u_0) - \sum_{v\in V(G)} d(v)r(v, w_m)\Bigg]^2\Bigg\}\\ =&\frac{1}{4[1 + r(w_m, u_0)]} \cdot \Big\{2(m + p + q + 1)\{2 \sum_{u\in V(G)} [r(u, u_0) - r(u, w_m)]^2\\ &+ [r(v_0, u_0) - r(v_0, w_m)]^2 + [r(u_0, u_0) - r(u_0, w_m)]^2\Big\}\\ &- \Bigg[2 \sum_{u\in V(G)} r(u, u_0) - 2 \sum_{u\in V(G)} r(u, w_m) + r(v_0, u_0)\end{aligned}$$

$$- r(u_0, w_m) - r(v_0, w_m)\Bigg]^2\Bigg\}.$$

By Eqs (5.1) and (5.2), we can obtain that

$$\begin{aligned}N =&\frac{p+q}{4[(1+m)(p+q)+pq]}\cdot\Bigg\{2(m+p+q+1)\Bigg[2\bigg(\frac{pq(pq+2)}{3(p+q)}\\&+\frac{m(m^2+2)}{3}+(p+q)m^2+\frac{2mpq}{p+q}+\frac{p^2q^2m}{(p+q)^2}\bigg)+\frac{2p^2q^2}{(p+q)^2}+2m^2\Bigg]\\&-4m^2\left[\frac{pq}{p+q}-(p+q)\right]^2\Bigg\}.\end{aligned}$$

Thus,

$$\begin{aligned}Kf^*(G) =&\frac{2m^3}{3}+2(p+q)m^2+\frac{(2(p+q)^2-1)m}{3}\\&+\frac{(p+q)(p+q-1)(p+q+1)}{3}+\frac{2(p+q)^2}{3}-\frac{2}{3}\\&+\frac{(2m+1)pq}{p+q}+m(m+1)+2m(p+q)+m^2\\&-\frac{p+q}{(1+m)(p+q)+pq}\cdot\Bigg[\frac{pq(pq+2)}{3}+\frac{m(m^2+2)(p+q)}{3}\\&+2mpq+\frac{p^2q^2(m+1)}{p+q}+m^2(p+q)(m+2)+\frac{(m+1)(pq+2)pq}{3(p+q)}\\&+\frac{m(m+1)(m^2+2)}{3}+\frac{2mpq(m+1)}{p+q}+\frac{p^2q^2(2m+1)}{(p+q)^2}\\&+m^2(m+1)+2m^2pq\Bigg].\end{aligned}\tag{5.8}$$

According to the symmetry, we know $S_n^{p,q,m}\cong S_n^{q,p,m}, S_n^{p,q,m}\cong S_n^{m+1,q,p-1}$, and $S_n^{p,q,m}\cong S_n^{p,m+1,q-1}$. Letting $m+1\le q\le p$,

$$\mathcal{K}_n := \{(p,q,m)|1\le m+1\le q\le p, p+q+m\le n\}.$$

Therefore, we claim the following result.

Lemma 5.2. *Let* $(p_0,q_0,m_0)\in\mathcal{K}_n$, p_0-m_0, *or* $q_0-m_0 = \min_{(p,q,m)\in\mathcal{K}}\{(p-m)or(q-m)|(p,q,m)\in\mathcal{K}\}$. *When* $\max\{p_0-m_0, q_0-m_0\} = \min\{\max(p-m,q-m)|(p,q,m)\in\mathcal{K}_n\}$, $Kf^*(S_n^{p_0,q_0,m_0})$ *can obtain the minimum value.*

Proof. Let G_0 have minimal degree Kirchhoff index among the graphs $S_n^{p,q,m}$ and assume that $p_0 > q_0+1$. Then, we construct the graph $G_0'\in S_n^{p,q,m}$, where $p_0' = p_0-1, q_0' = q_0+1, m_0' = m_0$. By Eq (5.8), we obtain

$$Kf^*(G_0) - Kf^*(G_0')$$

$$
\begin{aligned}
=&\frac{(p_0-q_0-1)}{3(m_0p_0+m_0q_0+p_0q_0+2p_0-1)(m_0p_0+m_0q_0+p_0q_0+p_0+q_0)}\\
&\cdot\{-(p_0+q_0)m_0^4+[2(p_0+q_0)(p_0+q_0-2)]\,m_0^3\\
&+[2(p_0+q_0)(4p_0q_0+5p_0+q_0-5)]\,m_0^2\\
&+\Big[(2q_0+1)p_0^3+(8q_0^2+21q_0+13)p_0^2\\
&+(2q_0^3+11q_0^2+6q_0-12)p_0-q_0^3-3q_0^2-12q_0\Big]m_0\\
&+p_0^3q_0^2+p_0^2q_0^3+3p_0^3q_0+8p_0^2q_0^2+p_0q_0^3+p_0^3+12p_0^2q_0+2p_0q_0^2\\
&-q_0^3+5p_0^2-2p_0q_0-3q_0^2-5p_0-5q_0\}.
\end{aligned}
$$

Obviously,

$$3(m_0p_0+m_0q_0+p_0q_0+2p_0-1)(m_0p_0+m_0q_0+p_0q_0+p_0+q_0)>0.$$

And

$$
\begin{aligned}
&[2(p_0+q_0)(p_0+q_0-2)]\,m_0^3+[2(p_0+q_0)(4p_0q_0+5p_0+q_0-5)]\,m_0^2\\
&+\Big[(2q_0+1)p_0^3+(8q_0^2+21q_0+13)p_0^2+(2q_0^3+11q_0^2+6q_0-12)p_0\\
&-q_0^3-3q_0^2-12q_0\Big]m_0>0.
\end{aligned}
$$

Due to $m_0+1\le q_0$, then the remaining part of the numerator is

$$
\begin{aligned}
&p_0^3q_0^2+p_0^2q_0^3+3p_0^3q_0+8p_0^2q_0^2+p_0q_0^3+p_0^3+12p_0^2q_0+2p_0q_0^2\\
&\ -q_0^3+5p_0^2-2p_0q_0-3q_0^2-5p_0-5q_0-(p_0+q_0)m_0^4\\
>&p_0^3q_0^2+p_0^2q_0^3+3p_0^3q_0+8p_0^2q_0^2+p_0q_0^3+p_0^3+12p_0^2q_0+2p_0q_0^2-q_0^3\\
&\ +5p_0^2-2p_0q_0-3q_0^2-5p_0-5q_0-(p_0+q_0)(q_0-1)^4\\
=&[{p_0}^3{q_0}^2+{p_0}^2{q_0}^3-p_0{q_0}^4-{q_0}^5]+[8\,{p_0}^2{q_0}^2-7\,{q_0}^3]+[5p_0q_0^3+4q_0^4+q_0^2-6q_0]\\
&\ +[p_0^3+5p_0^2+2p_0q_0-6p_0]+[12p_0^2q_0-4p_0q_0^2]+3p_0^3q_0\\
=&q_0^2(p_0-q_0)(p_0+q_0)^2+q_0^2(8p_0^2-7q_0)+q_0(5p_0q_0^2+4q_0^3+q_0-6)\\
&\ +p_0(p_0^2+5p_0+2q_0-6)+4p_0q_0(3p_0-q_0)+3p_0^3q_0\\
>&0.
\end{aligned}
$$

Since $p_0\ge q_0\ge 2$, we can see that each part is greater than 0, so $Kf^*(G_0)-Kf^*(G_0')>0$. Then, this contradicts the choice of G_0. ■

Next, we will consider the graphs $S_n^{p,q,m}$ for small n.

(1) If $n=4$, there is only one case.

$$p=2,q=2,m=0,Kf^*_{v_0}(S_4^{2,2,0})=4,Kf^*(S_4^{2,2,0})=\frac{47}{2}.$$

(2) If $n=5$, by Lemma 5.1, there are two cases. By Lemma 2.2 (2), we have

(i) $p=2,q=2,m=0,$

$$Kf^*(S_5^{2,2,0})=Kf^*(S_4^{2,2,0})+2Kf^*_{v_0}(S_4^{2,2,0})+2|E(S_4^{2,2,0})|+1=\frac{85}{2}.$$

(ii) $p = 2, q = 2, m = 1$,

$$Kf^*_{v_0}(S_5^{2,2,1}) = 6, Kf^*(S_5^{2,2,1}) = 42.$$

If $n \geq 6$, we can get

$$\begin{aligned}
&Kf^*(S_n^{2,2,0}) - Kf^*(S_n^{2,2,1})\\
=&\frac{47}{2} + (n-4)(2(n-3)-3) + 2\cdot 5(n-4) + 2(n-4)\cdot 4\\
&-[42 + (n-5)(2(n-4)-3) + 2\cdot 6(n-5) + 2(n-5)\cdot 6]\\
=&\frac{21}{2} - 2n.
\end{aligned} \tag{5.9}$$

Therefore, when $n \geq 6$, we know $Kf^*(S_n^{2,2,0}) - Kf^*(S_n^{2,2,1}) < 0$, so $S_n^{2,2,1}$ will not be considered.
(3) If $n = 6$, by Lemma 5.1, there are two cases. By Lemma 2.2 (2), we have
(i) $p = 2, q = 2, m = 0$,

$$Kf^*(S_6^{2,2,0}) = Kf^*(S_5^{2,2,0}) + 2Kf^*_{v_0}(S_5^{2,2,0}) + 2|E(S_5^{2,2,0})| + 1 = \frac{131}{2}.$$

(ii) $p = 3, q = 2, m = 1$,

$$Kf^*_{v_0}(S_6^{3,2,1}) = \frac{17}{2}, Kf^*(S_6^{3,2,1}) = \frac{281}{4}.$$

Motivated by the above discussion, we give the following result.

Lemma 5.3. $\min\limits_{(p,q,m)\in\mathcal{K}}\{Kf^*(S_n^{p,q,m})|(p,q,m)\in\mathcal{K}_n\} = Kf^*(S_n^{2,2,0})$ *as* $n \geq 6$.

Proof. We will consider the following two cases.
Case 1. When $p + q + m < n$, adopt mathematical induction. Obviously, $n = 6$ satisfies this. Next, we will prove $n = n + 1$ satisfies this as well. Let G_1 and G_2 be subgraphs of G induced by $V(G \setminus S)$ and $V(S) \cup \{v_0\}$, in which $V(S)$ represents all the pendant vertices on v_0 of the graph G. Let G' be obtained by deleting a pendent vertex from G. According to Lemma 2.2 (1) and Eq (2.4), we get

$$\begin{aligned}
Kf^*(G) =&Kf^*(G_1) + Kf^*(G_2) + 2|E(G_1)|Kf^*_{v_0}(G_2) + 2|E(G_2)|Kf^*_{v_0}(G_1)\\
=&Kf^*(G_1) + (n-p-q-m)(2(n-p-q-m+1)-3)\\
&+ 2(p+q+m+1)(n-p-q-m) + 2(n-p-q-m)Kf^*_{v_0}(G_1)\\
=&Kf^*(G_1) + (n-p-q-m-1)(2(n-p-q-m)-3)\\
&+ 2(p+q+m+1)(n-p-q-m-1)\\
&+ 2(n-p-q-m-1)Kf^*_{v_0}(G_1)\\
&+ 4(n-m-p-q) - 3 + 2(m+p+q+1) + 2Kf^*_{v_0}(G_1)\\
=&Kf^*(G') + 2Kf^*_{v_0}(G_1) + 4n - 2m - 2p - 2q - 1.
\end{aligned}$$

Therefore, based on Lemma 5.2, we get

$$\begin{aligned}
Kf^*(S_n^{p,q,m+1}) =&Kf^*(S_{n-1}^{p,q,m+1}) + 2Kf^*_{v_0}(S_{p+q+m+1}^{p,q,m+1})\\
&+ 4n - 2(m+1) - 2p - 2q - 1.
\end{aligned}$$

$$Kf^*(S_n^{p,q,m}) = Kf^*(S_{n-1}^{p,q,m}) + 2Kf^*_{v_0}(S_{p+q+m}^{p,q,m+1}) + 4n - 2m - 2p - 2q - 1.$$

Obviously, for $n \geq 6$,

$$Kf^*(S_n^{p,q,m+1}) - Kf^*(S_n^{p,q,m}) = Kf^*(S_{n-1}^{p,q,m+1}) - Kf^*(S_{n-1}^{p,q,m}) + 2(Kf^*_{v_0}(S_{p+q+m+1}^{p,q,m+1}) - Kf^*_{v_0}(S_{p+q+m}^{p,q,m}) - 1).$$

Then, according to Eq (5.4),

$$\begin{aligned} &Kf^*_{v_0}(S_{p+q+m}^{p,q,m+1}) - Kf^*_{v_0}(S_{p+q+m+1}^{p,q,m+1}) - 1 \\ &= \frac{2}{3} \cdot \frac{m^3p^2 + 2\,m^3pq + m^3q^2 + 3\,m^2p^2q + 3\,m^2pq^2 + 3\,mp^2q^2 + p^3q^2}{(mp + mq + pq + p + q)\,(mp + mq + pq + 2\,p + 2\,q)} \\ &+ \frac{2}{3} \cdot \frac{1}{(mp + mq + pq + p + q)(mp + mq + pq + 2\,p + 2\,q)}(p^2q^3 + 3m^2p^2 \\ &+ 6m^2pq + 3m^2q^2 + 6mp^2q + 6mpq^2 + 3p^2q^2 + 2mp^2 + 4mpq + 2mq^2 \\ &+ 2p^2q + 2pq^2), \end{aligned}$$

where $m \geq 0$. So $Kf^*(S_n^{p,q,m+1}) - Kf^*(S_n^{p,q,m}) > 0$. Thus, $Kf^*(S_n^{p,q,0}) < Kf^*(S_n^{p,q,m})$ for any positive m. Then, Lemma 5.2 implies that $Kf^*(S_n^{2,2,0}) < Kf^*(S_n^{p,q,m})$.

Case 2. When $p + q + m = n$, based on Eqs (5.8) and (5.9), we know

$$Kf^*(S_n^{2,2,0}) = \frac{47}{2} + (n-4)(2n-9) + 10n - 40 + 8n - 32 = 2n^2 + n - \frac{25}{2},$$

$$Kf^*(S_n^{p,q,m}) - Kf^*(S_n^{2,2,0}) = Kf^*(S_n^{p,q,m}) - (2n^2 + n - \frac{25}{2}).$$

Subcase 1: $n = 3k (k \geq 2)$.

Let $p = k + 1, q = k$, and $m = k - 1$. We get

$$Kf^*(S_n^{p,q,m}) - Kf^*(S_n^{2,2,0}) = \frac{108\,k^4 - 144\,k^3 - 208\,k^2 + 171\,k + 142}{18\,k + 12} > 0.$$

Subcase 2: $n = 3k + 1 (k \geq 2)$.

Let $p = k + 1, q = k + 1$, and $m = k - 1$. We get

$$Kf^*(S_n^{p,q,m}) - Kf^*(S_n^{2,2,0}) = \frac{108\,k^4 - 72\,k^3 - 208\,k^2 + 115\,k + 57}{18\,k + 6} > 0.$$

Subcase 3: $n = 3k + 2 (k \geq 2)$.

Let $p = k + 1, q = k + 1$, and $m = k$. We get

$$Kf^*(S_n^{p,q,m}) - Kf^*(S_n^{2,2,0}) = 6\,k^3 - 12\,k + \frac{11}{2} > 0.$$

■

Combining the above lemmas, we can get the following result.

Theorem 5.1.

$$\min_{(p,q,m)\in\mathcal{K}}\{Kf^*(S_n^{p,q,m})\} = \begin{cases} Kf^*(S_n^{2,2,0}), & if\ n = 4\ or\ n \geq 6, \\ Kf^*(S_n^{2,2,1}), & if\ n = 5. \end{cases}$$

So, according to Eq (3.1), we can known that, for $n \geq 6$,

$$Kf^*(S_n^{3,3}) - Kf^*(S_n^{2,2,0}) = 2n^2 + \frac{5}{3}n - \frac{31}{3} - (2n^2 + n - \frac{25}{2}) = \frac{2n}{3} + \frac{13}{6} > 0.$$

And, for $n = 5$, $Kf^*(S_5^{2,2,1}) = 42, Kf^*(S_5^{3,3}) = 48$.

To sum up, we can obtain the following results.

Theorem 5.2. *Let $G \in \mathcal{B}_n$. Then*

$$\min_{G\in\mathcal{B}_n}\{Kf^*(G)\} = \begin{cases} Kf^*(S_n^{2,2,0}), & if\ n = 4\ or\ n \geq 6, \\ Kf^*(S_n^{2,2,1}), & if\ n = 5. \end{cases}$$

6. Conclusions

In this paper, we completely characterize the ∞-type graphs having the minimum degree Kirchhoff index, and show that $S_n^{3,3}$ has the minimum degree Kirchhoff index among all the bicyclic graphs with exactly two cycles. Then, we prove $S_n^{p,q,m}$ has the minimum degree Kirchhoff index for the Θ-type graphs, and precisely calculate $Kf^*_{v_0}(G)$ and $Kf^*(G)$ of Θ-graphs. Finally, we give that all bicyclic graphs attain the minimum degree Kirchhoff index, that is, $S_n^{2,2,0}(n = 4\ and\ n \geq 6)$ and $S_5^{2,2,1}$.

Author contributions

Y. M.: Conceptualization; Y. M.: Methodology; C. G.: Software; Y. M. and C. G.: Formal analysis; Y. M.: Writing-original draft; Y. M.: Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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