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Research article

Interpolation unitarily invariant norms inequalities for matrices with applications

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Abstract: Let A_j, B_j, P_j, and $Q_j \in M_n(\mathbb{C})$, where $j = 1, 2, \ldots, m$. For a real number $c \in [0, 1]$, we prove the following interpolation inequality:

$$\left|\left|\left|\sum_{j=1}^{m} A_j P_j {Q_j}^* {B_j}^*\right|\right|\right|^2 \leq (\max\{L,\ M\})^4 |||K_c|||\,|||K_{1-c}|||,$$

where

$$L = \left\|\sum_{j=1}^{m} \left|{A_j}^*\right|^2\right\|^{\frac{1}{2}}, M = \left\|\sum_{j=1}^{m} \left|{B_j}^*\right|^2\right\|^{\frac{1}{2}},$$

and

$$K_c = \left(c|P_1|^2 + (1-c)\,|Q_1|^2\right) \oplus \cdots \oplus \left(c|P_m|^2 + (1-c)\,|Q_m|^2\right).$$

Many other related interpolation inequalities are also obtained.

Keywords: singular value; unitarily invariant norm; inequality; positive semidefinite matrix
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1. Introduction

In this paper, $M_n(\mathbb{C})$ stands for the set of all $n \times n$ complex matrices. A symmetric matrix $A \in M_n(\mathbb{C})$ is positive semidefinite, if for every $x \in \mathbb{C}^n$, we have $\langle Ax, x\rangle \geq 0$. For $H, K \in M_n(\mathbb{C})$, The block matrix

$\begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}$ will be denoted by $H \oplus K$, and it is called the direct sum of H and K. We will use $|||\cdot|||$ to denote a unitarily invariant matrix norm, which satisfies the property that $|||A||| = |||UAV|||$ for all $A, U, V \in M_n(\mathbb{C})$, where U and V are "unitary matrices" while we will use $\|\cdot\|$ to denote the usual operator or spectral norm. For $A \in M_n(\mathbb{C})$, $s_i(A)$ will denote *ith* largest singular value of A, which is the *ith* largest eigenvalue of $|A| = (A^*A)^{\frac{1}{2}}$. It is well known that $\left|\left|\left| \begin{bmatrix} 0 & H \\ H^* & 0 \end{bmatrix} \right|\right|\right| = |||H \oplus H|||$, and $\|H \oplus K\| = \max(\|H\|, \|K\|)$ for all $H, K \in M_n(\mathbb{C})$. We refer to [7] for more about unitarily invariant matrix norms and singular values.

The arithmetic–geometric mean inequality for matrices, obtained in [8], states that if $H, K \in M_n(\mathbb{C})$, then

$$|||HK^*||| \leq \left|\left|\left| \frac{|H|^2 + |K|^2}{2} \right|\right|\right|, \tag{1.1}$$

and the Cauchy–Schwarz inequality states that

$$|||HK^*|||^2 \leq \left|\left|\left| |H|^2 \right|\right|\right| \left|\left|\left| |K|^2 \right|\right|\right|. \tag{1.2}$$

The author in [4] obtained two main results; in both of them, he used an increasing convex nonnegative function f defined on an interval I that contains the number 0 with $f(0) \leq 0$. The first result states that if A, B, P, and $Q \in M_n(\mathbb{C})$ with $\max\{\|A\|, \|B\|\} \leq 1$, then

$$2s_i(|APQ^*B^*|) \leq (\max\{\|A\|, \|B\|\})^2 s_i\left(f\left(|P|^2 + |Q|^2\right)\right), \tag{1.3}$$

for all $i = 1, 2, \ldots, n$. In the second result, he obtained that if A_j, B_j, P_j, and $Q_j \in M_n(\mathbb{C})$, where $j = 1, 2, \ldots, m$, then

$$2s_i\left(f\left(\left|\sum_{j=1}^{m} A_j P_j {Q_j}^* {B_j}^*\right|\right)\right) \leq (\max\{L, M\})^2 s_i(K), \tag{1.4}$$

for all $i = 1, 2, \ldots, n$, where

$$L = \left\|\sum_{j=1}^{m} \left|{A_j}^*\right|^2\right\|^{\frac{1}{2}}, M = \left\|\sum_{j=1}^{m} \left|{B_j}^*\right|^2\right\|^{\frac{1}{2}},$$

and

$$K = f\left(|P_1|^2 + |Q_1|^2\right) \oplus \cdots \oplus f\left(|P_m|^2 + |Q_m|^2\right).$$

Letting $f(t) = t$, the norm version of inequality (1.3) is given by

$$|||APQ^*B^*||| \leq \frac{(\max\{\|A\|, \|B\|\})^2}{2} \left|\left|\left| |P|^2 + |Q|^2 \right|\right|\right|, \tag{1.5}$$

while the norm version of inequality (1.4) is given by

$$2\left|\left|\left|\sum_{j=1}^{m} A_j P_j {Q_j}^* {B_j}^*\right|\right|\right| \leq (\max\{L, M\})^2 |||K|||, \tag{1.6}$$

where L and M are the same as given formerly, but

$$K = \left(|P_1|^2 + |Q_1|^2\right) \oplus \cdots \oplus \left(|P_m|^2 + |Q_m|^2\right).$$

For more inequalities related to the inequalities (1.3)–(1.6), we refer to [1,5]. And for some inequalities of interpolation type, we refer to [10].

In this paper, interpolation inequalities that can be considered generalizations of the inequalities (1.5) and (1.6) are introduced, and many other consequences and applications of these generalizations are also presented.

2. Results

We begin this section by introducing three lemmas; these lemmas support the proof of the first main result of this paper. The first lemma can be obtained using "The min-max principle" (see, e.g., [7, p. 75]); in addition, it is a direct consequence of a result introduced in [9, p. 27]. The second lemma can be found in [7, p. 253], and the last lemma was introduced in [1].

Lemma 2.1. *Let H, K, and $L \in M_n(\mathbb{C})$. Then*

$$|||HLK||| \leq \|H\| \, |||L||| \, \|K\|.$$

Lemma 2.2. *Let $P, Q \in M_n(\mathbb{C})$ such that PQ is normal. Then*

$$|||PQ||| \leq |||QP|||.$$

Lemma 2.3. *Let A, B, P, and $Q \in M_n(\mathbb{C})$. Then*

$$|||APQ^*B^*|||^2 \leq \left|\left|\left|cP^*|A|^2P + (1-c)\,Q^*|B|^2Q\right|\right|\right| \times \left|\left|\left|(1-c)\,P^*|A|^2P + cQ^*|B|^2Q\right|\right|\right|,$$

for every real number $c \in [0, 1]$.

It should be mentioned here that the inequality in Lemma 2.3 interpolates between the arithmetic-geometric mean inequality (1.1) $\left(c = \frac{1}{2}, P = Q = I\right)$ and the Cauchy-Schwarz inequality (1.2) ($c = 0$ or $1, P = Q = I$). For more results on interpolation inequalities, we refer to [1–3, 6].

Now, we will give the first main result in this paper.

Theorem 2.4. *Let A, B, P, and $Q \in M_n(\mathbb{C})$. Then*

$$|||APQ^*B^*|||^2 \leq (\max\{\|A\|, \|B\|\})^4 \left|\left|\left|c|P|^2 + (1-c)\,|Q|^2\right|\right|\right| \times \left|\left|\left|c|Q|^2 + (1-c)\,|P|^2\right|\right|\right|, \tag{2.1}$$

for every real number $c \in [0, 1]$.

Proof. Using Lemma 2.3, we get that

$$|||APQ^*B^*|||^2 \leq |||cP^*A^*AP + (1-c)\,Q^*B^*BQ||| \times |||(1-c)\,P^*A^*AP + cQ^*B^*BQ|||. \tag{2.2}$$

Now let F_c be the block 2×2 matrix defined as $F_c = \begin{bmatrix} \sqrt{c}P & 0 \\ \sqrt{1-c}Q & 0 \end{bmatrix}$. Then

$$F_c{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_c = \begin{bmatrix} cP^*A^*AP + (1-c)\,Q^*B^*BQ & 0 \\ 0 & 0 \end{bmatrix},$$

therefore

$$|||cP^*A^*AP + (1-c)\,Q^*B^*BQ||| = \left|\left|\left| F_c{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_c \right|\right|\right|. \tag{2.3}$$

Similarly, we can get the following equality:

$$|||(1-c)\,P^*A^*AP + cQ^*B^*BQ||| = \left|\left|\left| F_{1-c}{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_{1-c} \right|\right|\right|. \tag{2.4}$$

Combining the inequality (2.2) with the equalities (2.3) and (2.4) leads to

$$|||APQ^*B^*|||^2 \le \left|\left|\left| F_c{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_c \right|\right|\right| \times \left|\left|\left| F_{1-c}{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_{1-c} \right|\right|\right|. \tag{2.5}$$

By using Lemmas 2.1 and 2.2, we can get that

$$\begin{aligned} \left|\left|\left| F_c{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_c \right|\right|\right| &\le \left|\left|\left| \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} F_c F_c{}^* \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} \right|\right|\right| \\ &\le \left|\left|\left| \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} \right|\right|\right|^2 |||F_c F_c{}^*||| \\ &= \left|\left|\left| \begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix} \right|\right|\right|^2 |||F_c{}^* F_c|||. \end{aligned}$$

So

$$\begin{aligned} \left|\left|\left| F_c{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_c \right|\right|\right| &\le (\max\{\|A\|, \|B\|\})^2\, |||F_c{}^* F_c||| \\ &= (\max\{\|A\|, \|B\|\})^2 \left|\left|\left| c|P|^2 + (1-c)\,|Q|^2 \right|\right|\right|. \end{aligned} \tag{2.6}$$

Similarly

$$\begin{aligned} \left|\left|\left| F_{1-c}{}^*\begin{bmatrix} |A|^2 & 0 \\ 0 & |B|^2 \end{bmatrix} F_{1-c} \right|\right|\right| &\le (\max\{\|A\|, \|B\|\})^2\, |||F_{1-c}{}^* F_{1-c}||| \\ &= (\max\{\|A\|, \|B\|\})^2 \left|\left|\left| (1-c)\,|P|^2 + c|Q|^2 \right|\right|\right|. \end{aligned} \tag{2.7}$$

The result follows immediately from the inequality (2.5) and the inequalities (2.6) and (2.7). □

Note that substituting $c = \frac{1}{2}$ in the inequality (2.1) leads us directly to the inequality (1.5), which means that Theorem 2.4 generalizes the inequality (1.5).

Corollary 2.5. *Let $A, B, P,$ and $Q \in M_n(\mathbb{C})$. Then*

$$|||APQ^*B^*||| \le (\max\{\|A\|, \|B\|\})^2 \left|\left|\left| |P|^2 \right|\right|\right| \left|\left|\left| |Q|^2 \right|\right|\right|. \tag{2.8}$$

Proof. Substitute $c = 0$ or $c = 1$ in the inequality (2.1) to get the result directly. □

Corollary 2.5 illustrates that the inequality (2.1) can be considered an interpolation inequality between the inequalities (1.5) and (2.8).

Corollary 2.6. *Let A, B, P, and $Q \in M_n(\mathbb{C})$ such that P and Q are positive semidefinite matrices. Then*

$$\left|\left|\left|AP^{\frac{1}{2}}Q^{\frac{1}{2}}B^*\right|\right|\right|^2 \leq (\max\{\|A\|, \|B\|\})^4 \, |||cP + (1-c)\,Q||| \, |||(1-c)\,P + cQ|||, \tag{2.9}$$

for every real number $c \in [0,1]$.

Proof. Replace P and Q in the inequality (2.1) by $P^{\frac{1}{2}}$ and $Q^{\frac{1}{2}}$ respectively to get the result directly. □

Now we are ready to give the second main result in this paper.

Theorem 2.7. *Let A_j, B_j, P_j, and $Q_j \in M_n(\mathbb{C})$, where $j = 1, 2, \ldots, m$. For a real number $c \in [0,1]$, we have*

$$\left|\left|\left|\sum_{j=1}^{m} A_j P_j Q_j^* B_j^*\right|\right|\right|^2 \leq (\max\{L,\ M\})^4 |||K_c||| \, |||K_{1-c}|||, \tag{2.10}$$

where

$$L = \left\|\sum_{j=1}^{m} \left|A_j^*\right|^2\right\|^{\frac{1}{2}}, M = \left\|\sum_{j=1}^{m} \left|B_j^*\right|^2\right\|^{\frac{1}{2}},$$

and

$$K_c = \left(c|P_1|^2 + (1-c)\,|Q_1|^2\right) \oplus \cdots \oplus \left(c|P_m|^2 + (1-c)\,|Q_m|^2\right).$$

Proof. Consider the following $m \times m$ block matrices

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 & \cdots & B_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_m \end{bmatrix}, \text{and } Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_m \end{bmatrix}.$$

Through simple and direct calculations, we get that

$$APQ^*B^* = \begin{bmatrix} \sum_{j=1}^{m} A_j P_j Q_j^* B_j^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus

$$\left|\left|\left|\sum_{j=1}^{m} A_j P_j Q_j^* B_j^*\right|\right|\right| = |||APQ^*B^*|||. \tag{2.11}$$

Also, it is an easy task to see that

$$\|A\| = \left\| \sum_{j=1}^{m} \left|A_j{}^*\right|^2 \right\|^{\frac{1}{2}}, \tag{2.12}$$

$$\|B\| = \left\| \sum_{j=1}^{m} \left|B_j{}^*\right|^2 \right\|^{\frac{1}{2}}, \tag{2.13}$$

$$\left|\left|\left| c|P|^2 + (1-c)\,|Q|^2 \right|\right|\right| = \left|\left|\left| \left((1-c)\,|Q_1|^2 + c|P_1|^2\right) \oplus \cdots \oplus \left((1-c)\,|Q_m|^2 + c|P_m|^2\right) \right|\right|\right|, \tag{2.14}$$

and

$$\left|\left|\left| (1-c)\,|P|^2 + c|Q|^2 \right|\right|\right| = \left|\left|\left| \left(c|Q_1|^2 + (1-c)\,|P_1|^2\right) \oplus \cdots \oplus \left(c|Q_m|^2 + (1-c)\,|P_m|^2\right) \right|\right|\right|. \tag{2.15}$$

We get our result by applying the inequality (2.1) to the block matrices A, B, P, and Q and using the Eqs (2.11) to (2.15). □

Note that substituting $c = \frac{1}{2}$ in the inequality (2.10) leads us directly to the inequality (1.6), which means that Theorem 2.7 is a generalization of the inequality (1.6).

Corollary 2.8. *Let A_j, B_j, P_j, and $Q_j \in M_n(\mathbb{C})$, where $j = 1, 2, \ldots, m$. Then*

$$\left|\left|\left| \sum_{j=1}^{m} A_j P_j Q_j{}^* B_j{}^* \right|\right|\right|^2 \leq (\max\{L,\ M\})^4 |||S|||\,|||T|||, \tag{2.16}$$

where

$$L = \left\| \sum_{j=1}^{m} \left|A_j{}^*\right|^2 \right\|^{\frac{1}{2}}, M = \left\| \sum_{j=1}^{m} \left|B_j{}^*\right|^2 \right\|^{\frac{1}{2}},$$

and

$$S = |P_1|^2 \oplus \cdots \oplus |P_m|^2, T = |Q_1|^2 \oplus \cdots \oplus |Q_m|^2.$$

Proof. Substitute $c = 0$ or $c = 1$ in the inequality (2.10) to get the result directly. □

The inequality (2.10) can be viewed as an interpolation inequality between the inequalities (1.6) and (2.16), as demonstrated by Corollary 2.8.

The following corollary is nothing but constraining the inequality (2.10) for two pairs of matrices.

Corollary 2.9. *Let A_j, B_j, P_j, and $Q_j \in M_n(\mathbb{C})$, where $j = 1, 2$. For a real number $c \in [0, 1]$, we have*

$$\left|\left|\left| \sum_{j=1}^{2} A_j P_j Q_j{}^* B_j{}^* \right|\right|\right|^2 \leq (\max\{L,\ M\})^4 |||K_c|||\,|||K_{1-c}|||, \tag{2.17}$$

where

$$L = \|A_1 A_1{}^* + A_2 A_2{}^*\|^{\frac{1}{2}}, M = \|B_1 B_1{}^* + B_2 B_2{}^*\|^{\frac{1}{2}},$$

and

$$K_c = \left(c|P_1|^2 + (1-c)\,|Q_1|^2\right) \oplus \left(c|P_2|^2 + (1-c)\,|Q_2|^2\right).$$

Proof. Let $A = P = Q = B = 0$ for $3 \le j \le n$ in inequality (2.10) to get the result directly. □

Corollary 2.10. *Let $A, B, P,$ and $Q \in M_n(\mathbb{C})$, where P and Q are positive semidefinite. For a real number $c \in [0,1]$, we have*

$$\left|\left|\left| AP^{\frac{1}{2}}Q^{\frac{1}{2}}B^* + BP^{\frac{1}{2}}Q^{\frac{1}{2}}A^* \right|\right|\right|^2 \le \|AA^* + BB^*\|^2 |||K_c|||\,|||K_{1-c}|||, \tag{2.18}$$

where

$$K_c = (cP + (1-c)\,Q) \oplus (cP + (1-c)\,Q)\,.$$

Proof. Replace A_1 and B_2 by A, replace A_2 and B_1 by B, and let $P_1 = P_2 = P^{\frac{1}{2}}$ and $Q_1 = Q_2 = Q^{\frac{1}{2}}$ in the inequality (2.17) to get the result. □

Now we will present two directly proven inequalities in the subsequent two corollaries.

Corollary 2.11. *Let $A, B, P,$ and $Q \in M_n(\mathbb{C})$, where P and Q are positive semidefinite. Then*

$$2\left|\left|\left| AP^{\frac{1}{2}}Q^{\frac{1}{2}}B^* + BP^{\frac{1}{2}}Q^{\frac{1}{2}}A^* \right|\right|\right| \le \|AA^* + BB^*\|\,|||K|||, \tag{2.19}$$

where

$$K = (P + Q) \oplus (P + Q)\,.$$

Proof. Substitute $c = \frac{1}{2}$ in inequality (2.18) to get the result directly. □

Corollary 2.12. *Let $A, B, P,$ and $Q \in M_n(\mathbb{C})$, where P and Q are positive semidefinite. Then*

$$\left|\left|\left| AP^{\frac{1}{2}}Q^{\frac{1}{2}}B^* + BP^{\frac{1}{2}}Q^{\frac{1}{2}}A^* \right|\right|\right|^2 \le \|AA^* + BB^*\|^2 |||P \oplus P|||\,|||Q \oplus Q|||. \tag{2.20}$$

Proof. Substitute $c = 0$ or $c = 1$ in the inequality (2.18) to get the result directly. □

It can be observed that the inequality (2.18) is an interpolation inequality between the inequalities (2.19) and (2.20).

Remark 2.13. *Substitute $P = Q = X$, where $X \in M_n(\mathbb{C})$ is positive semidefinite, in the inequality* (2.20) *to get that*

$$|||AXB^* + BXA^*||| \le \|AA^* + BB^*\|\,|||X \oplus X|||, \tag{2.21}$$

letting $X = I$ gives the following inequality:

$$|||AB^* + BA^*||| \le \|AA^* + BB^*\|. \tag{2.22}$$

Corollary 2.14. *Let $A, B, P,$ and $Q \in M_n(\mathbb{C})$ be positive semidefinite. Then for a real number $c \in [0,1]$, we have*

$$|||S + T|||^2 \le \|A + B\|^2\,|||K_c|||\,|||K_{1-c}|||, \tag{2.23}$$

where

$$S = A^{\frac{1}{2}}P^{\frac{1}{2}}Q^{\frac{1}{2}}A^{\frac{1}{2}}, T = B^{\frac{1}{2}}P^{\frac{1}{2}}Q^{\frac{1}{2}}B^{\frac{1}{2}},$$

and

$$K_c = (cP + (1-c)\,Q) \oplus (cP + (1-c)\,Q)\,.$$

Letting $P = Q = X$. We get that

$$\left|\left|\left| A^{\frac{1}{2}}XA^{\frac{1}{2}} + B^{\frac{1}{2}}XB^{\frac{1}{2}} \right|\right|\right|^2 \le \|A + B\|^2\,|||X \oplus X|||. \tag{2.24}$$

Proof. Let $A_1 = B_1 = A^{\frac{1}{2}}$, $A_2 = B_2 = B^{\frac{1}{2}}$, $P_1 = P_2 = P^{\frac{1}{2}}$, and $Q_1 = Q_2 = Q^{\frac{1}{2}}$ in the inequality (2.17) to get the inequality (2.23). □

Corollary 2.15. *Let $A, B, P_1, P_2, Q_1, Q_2 \in M_n(\mathbb{C})$. For a real number $c \in [0, 1]$, we have*

$$\left|\left|\left|AP_1Q_1^*A^* - BP_2Q_2^*B^*\right|\right|\right|^2 \leq \|AA^* + BB^*\|^2 \, |||K_c||| \, |||K_{1-c}|||, \tag{2.25}$$

where

$$K_c = \left(c|P_1|^2 + (1-c)\,|Q_1|^2\right) \oplus \left(c|P_2|^2 + (1-c)\,|Q_2|^2\right).$$

Proof. Let $A_1 = B_1 = A$, $A_2 = -B_2 = B$ in the inequality (2.17) to get the inequality (2.25). □

It can be directly deduced that the inequality (2.25) can be considered an interpolation inequality between the inequalities (2.26) and (2.27) that are demonstrated by the subsequent two corollaries.

Corollary 2.16. *Let $A, B, P_1, P_2, Q_1, Q_2 \in M_n(\mathbb{C})$. Then*

$$2\left|\left|\left|AP_1Q_1^*A^* - BP_2Q_2^*B^*\right|\right|\right| \leq \|AA^* + BB^*\| \, |||K|||, \tag{2.26}$$

where

$$K = \left(|P_1|^2 + |Q_1|^2\right) \oplus \left(|P_2|^2 + |Q_2|^2\right).$$

Proof. Substitute $c = \frac{1}{2}$ in inequality (2.25) to get the result directly. □

Corollary 2.17. *Let $A, B, P_1, P_2, Q_1, Q_2 \in M_n(\mathbb{C})$. Then*

$$\left|\left|\left|AP_1Q_1^*A^* - BP_2Q_2^*B^*\right|\right|\right|^2 \leq \|AA^* + BB^*\|^2 \, |||S||| \, |||T|||, \tag{2.27}$$

where

$$S = |P_1|^2 + |P_2|^2, \; T = |Q_1|^2 + |Q_2|^2.$$

Proof. Substitute $c = 0$ or $c = 1$ in the inequality (2.25) to get the result directly. □

Remark 2.18. *Substitute $P_2 = Q_2 = B = 0$ in the inequality* (2.25) *to get that*

$$\left|\left|\left|AP_1Q_1^*A^*\right|\right|\right|^2 \leq \left\||A^*|^2\right\|^2 |||K_c||| \, |||K_{1-c}|||, \tag{2.28}$$

where

$$K_c = c|P_1|^2 + (1-c)\,|Q_1|^2.$$

Corollary 2.19. *Let A, B, and $X \in M_n(\mathbb{C})$, where X is positive semidefinite. Then*

$$|||AXB^* \oplus AXB^*||| \leq (\max\{\|A\|, \|B\|\})^2 \; |||X \oplus X|||. \tag{2.29}$$

Proof. In the inequality (2.25), replace A and B by $\begin{bmatrix} A \\ B \end{bmatrix}$ and $\begin{bmatrix} A \\ -B \end{bmatrix}$ respectively and let $P_1 = P_2 = Q_1 = Q_2 = X^{\frac{1}{2}}$, to get that the left-hand side of this inequality equals

$$\left|\left|\left| 2 \begin{bmatrix} 0 & AXB^* \\ BXA^* & 0 \end{bmatrix} \right|\right|\right|^2 = 4|||AXB^* \oplus AXB^*|||^2,$$

while the right-hand side of this inequality equals

$$\left\| 2 \begin{bmatrix} AA^* & 0 \\ 0 & BB^* \end{bmatrix} \right\|^2 |||X \oplus X|||^2 = 4(\max\{\|AA^*\|, \|BB^*\|\})^2 |||X \oplus X|||^2$$
$$= 4\left(\max\left\{\|A\|^2, \|B\|^2\right\}\right)^2 |||X \oplus X|||^2$$
$$= 4(\max\{\|A\|, \|B\|\})^4 |||X \oplus X|||^2.$$

The result follows directly from the above discussion. □

3. Conclusions

Wasim Audeh latterly obtained two matrix singular values inequalities. The norm versions of these inequalities are provided in this paper, and we utilize a recent result by Mohammad Al-khlyleh to derive an interpolation inequalities that are related to Audeh's inequalities.

Author contributions

Mohammad Al-Khlyleh: Writing-original draft, Methodology; Mohammad Abdel Aal: Funding acquisition, Supervision; Mohammad F. M. Naser: Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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