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*Research article*

# Fractional Milne-type inequalities for twice differentiable functions

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**Abstract:** In this study, a specific identity was derived for functions that possess two continuous derivatives. Through the utilization of this identity and Riemann-Liouville fractional integrals, several fractional Milne-type inequalities were established for functions whose second derivatives inside the absolute value are convex. Additionally, an example and a graphical representation are included to clarify the core findings of our research.

**Keywords:** Milne-type inequalities; fractional integrals; convexity; differentiable functions
**Mathematics Subject Classification:** 26D15, 26D10, 26D07

## 1. Introduction

Numerous mathematicians have delved into exploring error upper bounds utilizing numerical integration formulas and diverse methodologies. The pursuit of error bounds by numerical integration involves an examination of mathematical inequalities across distinct function categories including convex, bounded, and Lipschitzian functions. This paper focuses specifically on investigating bounds pertaining to functions whose derivatives or second derivatives achieve the convexity condition.

To begin, let's provide an overview of various numerical integration methods along with their associated upper error bounds:

(i) The subsequent expression represents Simpson's quadrature formula, often denoted as Simpson's $1/3$ rule:

$$\int_{\gamma}^{\eta} \mathcal{P}(\psi)\, d\psi \approx \frac{\eta-\gamma}{6}\left[\mathcal{P}(\gamma)+4\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+\mathcal{P}(\eta)\right]. \tag{1.1}$$

(ii) The characterization of Simpson's second formula, also known as the Newton-Cotes quadratic formula or Simpson's 3/8 rule (see [1] ), is given as follows:

$$\int_{\gamma}^{\eta} \mathcal{P}(\psi)\, d\psi \approx \frac{\eta-\gamma}{8}\left[\mathcal{P}(\gamma)+3\mathcal{P}\left(\frac{2\gamma+\eta}{3}\right)+3\mathcal{P}\left(\frac{\gamma+2\eta}{3}\right)+\mathcal{P}(\eta)\right]. \tag{1.2}$$

Equations (1.1) and (1.2) are valid for any function $\mathcal{P}$ that possesses a continuous fourth derivative within the interval $[\gamma, \eta]$.

The traditional statement of the Simpson inequality is presented as follows:

**Theorem 1.1.** *When considering $\mathcal{P} : [\gamma, \eta] \to \mathbb{R}$, a function with four continuous derivatives within the interval $(\gamma, \eta)$, and $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{\psi\in(\gamma,\eta)} \left|\mathcal{P}^{(4)}(\psi)\right| < \infty$, the subsequent inequality holds:*

$$\left|\frac{1}{6}\left[\mathcal{P}(\gamma)+4\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+\mathcal{P}(\eta)\right]-\frac{1}{\eta-\gamma}\int_{\gamma}^{\eta}\mathcal{P}(\psi)d\psi\right| \le \frac{1}{2880}\left\|\mathcal{P}^{(4)}\right\|_{\infty}(\eta-\gamma)^4 .$$

The initial proof of the Simpson-type inequality utilizing convex functions was established by Sarikaya et al. in [2]. Within the domain of Riemann-Liouville fractional integrals, three variations of the Simpson inequality exist, categorized by the representation of fractional integrals. These distinct inequalities were established in the works [3–5]. Moreover, specific attention has been dedicated to Simpson-type inequalities applicable to twice differentiable functions in papers such as [6–8].

The classical Newton inequality is defined as follows:

**Theorem 1.2.** *[See [1]] If $\mathcal{P} : [\gamma, \eta] \to \mathbb{R}$ represents a function with a continuous fourth derivative defined over $(\gamma, \eta)$, and $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{\psi\in(\gamma,\eta)} \left|\mathcal{P}^{(4)}(\psi)\right| < \infty$, then the inequality presented below is valid:*

$$\left|\frac{1}{8}\left[\mathcal{P}(\gamma)+3\mathcal{P}\left(\frac{2\gamma+\eta}{3}\right)+3\mathcal{P}\left(\frac{\gamma+2\eta}{3}\right)+\mathcal{P}(\eta)\right]-\frac{1}{\eta-\gamma}\int_{\gamma}^{\eta}\mathcal{P}(\psi)d\psi\right| \le \frac{1}{6480}\left\|\mathcal{P}^{(4)}\right\|_{\infty}(\eta-\gamma)^4 .$$

The works referenced as [9–11] present Newton-type inequalities utilizing convex functions for local fractional integrals. In the paper [12], the initial proofs of Newton-type inequalities for Riemann-Liouville fractional integrals were established. Subsequently, several papers have focused on deriving Newton-type inequalities for Riemann-Liouville fractional integrals [13, 14]. Additionally, Gao and Shi provided proofs of Newton-type inequalities applicable to twice-differentiable functions in [15].

The classical Milne inequality is formulated as follows:

**Theorem 1.3.** *[See [16]] Let $\mathcal{P} : [\gamma, \eta] \to \mathbb{R}$ be a function with a continuous fourth derivative over $(\gamma, \eta)$, and $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{\psi\in(\gamma,\eta)} \left|\mathcal{P}^{(4)}(\psi)\right| < \infty$. In such a case, the subsequent inequality is valid:*

$$\left|\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]-\frac{1}{\eta-\gamma}\int_{\gamma}^{\eta}\mathcal{P}(\psi)d\psi\right| \le \frac{7(\eta-\gamma)^4}{23040}\left\|\mathcal{P}^{(4)}\right\|_{\infty} .$$

Djenaoui and Meftah initially established Milne-type inequalities using convexity in [17]. Budak et al. expanded upon these inequalities, extending their applicability to Riemann-Liouville fractional integrals in [18]. Within the same study, a diverse array of Milne-type inequalities was introduced, encompassing varied function classes like bounded functions, Lipschitz functions, and functions of bounded variation. Recent research efforts, notably in [19, 20], have introduced novel fractional variations of Milne-type inequalities, utilizing differentiable convex functions and exploring several function classes such as bounded functions, Lipschitz functions, and functions of bounded variation. For further exploration of Milne-type inequalities, references like [21–23] provide additional insights.

This paper aims to derive fractional Milne-type inequalities applicable to mappings characterized by convex second derivatives. To achieve this objective, we begin by outlining the definition of Riemann-Liouville fractional integrals. The widely recognized Riemann-Liouville fractional integrals are defined as follows:

**Definition 1.1.** *[ [24, 25]] The Riemann-Liouville integrals $\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}$ and $\mathcal{J}^{\mu}_{\eta-}\mathcal{P}$, both of order $\mu > 0$ with $\gamma \geq 0$, are expressed as follows*

$$\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\psi) = \frac{1}{\Gamma(\mu)} \int_{\gamma}^{\psi} (\psi - \xi)^{\mu-1} \mathcal{P}(\xi) d\xi, \quad \psi > \gamma,$$

*and*

$$\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\psi) = \frac{1}{\Gamma(\mu)} \int_{\psi}^{\eta} (\xi - \psi)^{\mu-1} \mathcal{P}(\xi) d\xi, \quad \psi < \eta,$$

*respectively. Here, $\mathcal{P}$ belongs to the space $L_1[\gamma, \eta]$, and $\Gamma(\mu)$ denotes the Gamma function, defined as:*

$$\Gamma(\mu) := \int_{0}^{\infty} e^{-u} u^{\mu-1} du.$$

The fractional integrals in Definition 1.1 equate to the classical integral when $\mu = 1$.

The main point of interest here is the investigation of certain fractional Milne-type inequalities that apply to twice-differentiable functions in particular, whose second derivatives exhibit convex properties when contained in absolute value. This implies that an emphasis should be placed on comprehending and measuring the behavior of these functions within the context of fractional calculus, since this could provide light on their characteristics and potential uses in a variety of mathematical settings. To further clarify the key conclusions, the study also provides a graphical representation and an example.

## 2. Main results

Within this section, we introduce multiple fractional Milne-type inequalities applicable to twice-differentiable functions.

**Lemma 2.1.** *If $\mathcal{P} : [\gamma, \eta] \to \mathbb{R}$ is absolutely continuous over $(\gamma, \eta)$ and $\mathcal{P}'' \in L_1([\gamma, \eta])$, then the following holds:*

$$\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}} \left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta) + \mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right] - \frac{1}{3}\left[2\mathcal{P}(\gamma) - \mathcal{P}\left(\frac{\gamma+\eta}{2}\right) + 2\mathcal{P}(\eta)\right] = \frac{(\eta-\gamma)^2}{2(\mu+1)} \sum_{k=1}^{4} I_k, \tag{2.1}$$

*where*

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \left(\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right) \mathcal{P}''\left(\xi\eta + (1-\xi)\gamma\right) d\xi, \\
I_2 &= \int_0^{\frac{1}{2}} \left(\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right) \mathcal{P}''\left(\xi\gamma + (1-\xi)\eta\right) d\xi, \\
I_3 &= \int_{\frac{1}{2}}^{1} \left(\xi^{\mu+1} - \xi\right) \mathcal{P}''\left(\xi\eta + (1-\xi)\gamma\right) d\xi, \\
I_4 &= \int_{\frac{1}{2}}^{1} \left(\xi^{\mu+1} - \xi\right) \mathcal{P}''\left(\xi\gamma + (1-\xi)\eta\right) d\xi.
\end{aligned}
$$

*Proof.* Through the utilization of integration by parts, we derive:

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \left(\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right) \mathcal{P}''\left(\xi\eta + (1-\xi)\gamma\right) d\xi \qquad (2.2)\\
&= \frac{1}{\eta-\gamma}\left(\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right) \mathcal{P}'\left(\xi\eta + (1-\xi)\gamma\right)\Big|_0^{\frac{1}{2}} \\
&\quad -\frac{1}{\eta-\gamma}\int_0^{\frac{1}{2}} \left((\mu+1)\xi^{\mu} - \frac{\mu+4}{3}\right) \mathcal{P}'\left(\xi\eta + (1-\xi)\gamma\right) d\xi \\
&= \frac{1}{\eta-\gamma}\left(\frac{1}{2^{\mu+1}} - \frac{\mu+4}{6}\right)\mathcal{P}'\left(\frac{\gamma+\eta}{2}\right) \\
&\quad -\frac{1}{\eta-\gamma}\left[\frac{1}{\eta-\gamma}\left((\mu+1)\xi^{\mu} - \frac{\mu+4}{3}\right)\mathcal{P}\left(\xi\eta + (1-\xi)\gamma\right)\Big|_0^{\frac{1}{2}} \right.\\
&\quad \left. -\frac{\mu(\mu+1)}{\eta-\gamma}\int_0^{\frac{1}{2}} \xi^{\mu-1}\mathcal{P}\left(\xi\eta + (1-\xi)\gamma\right) d\xi \right] \\
&= \frac{1}{\eta-\gamma}\left(\frac{1}{2^{\mu+1}} - \frac{\mu+4}{6}\right)\mathcal{P}'\left(\frac{\gamma+\eta}{2}\right) \\
&\quad -\frac{1}{(\eta-\gamma)^2}\left(\frac{\mu+1}{2^{\mu}} - \frac{\mu+4}{3}\right)\mathcal{P}\left(\frac{\gamma+\eta}{2}\right) \\
&\quad -\frac{\mu+4}{3(\eta-\gamma)^2}\mathcal{P}(\gamma) + \frac{\mu(\mu+1)}{(\eta-\gamma)^2}\int_0^{\frac{1}{2}} \xi^{\mu-1}\mathcal{P}\left(\xi\eta + (1-\xi)\gamma\right) d\xi.
\end{aligned}
$$

Likewise, we acquire:

$$
\begin{aligned}
I_2 &= \int_0^{\frac{1}{2}} \left(\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right) \mathcal{P}''(\xi\gamma + (1-\xi)\eta)\, d\xi \qquad (2.3)\\
&= -\frac{1}{\eta-\gamma}\left(\frac{1}{2^{\mu+1}} - \frac{\mu+4}{6}\right)\mathcal{P}'\left(\frac{\gamma+\eta}{2}\right)\\
&\quad -\frac{1}{(\eta-\gamma)^2}\left(\frac{\mu+1}{2^{\mu}} - \frac{\mu+4}{3}\right)\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)\\
&\quad -\frac{\mu+4}{3(\eta-\gamma)^2}\mathcal{P}(\eta) + \frac{\mu(\mu+1)}{(\eta-\gamma)^2}\int_0^{\frac{1}{2}} \xi^{\mu-1}\mathcal{P}(\xi\gamma + (1-\xi)\eta)\, d\xi.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_{\frac{1}{2}}^{1} \left(\xi^{\mu+1} - \xi\right) \mathcal{P}''(\xi\eta + (1-\xi)\gamma)\, d\xi \qquad (2.4)\\
&= -\frac{1}{\eta-\gamma}\left(\frac{1}{2^{\mu+1}} - \frac{1}{2}\right)\mathcal{P}'\left(\frac{\gamma+\eta}{2}\right)\\
&\quad -\frac{\mu}{(\eta-\gamma)^2}\mathcal{P}(\eta) + \frac{1}{(\eta-\gamma)^2}\left(\frac{\mu+1}{2^{\mu}} - 1\right)\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)\\
&\quad +\frac{\mu(\mu+1)}{(\eta-\gamma)^2}\int_{\frac{1}{2}}^{1} \xi^{\mu-1}\mathcal{P}(\xi\eta + (1-\xi)\gamma)\, d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_{\frac{1}{2}}^{1} \left(\xi^{\mu+1} - \xi\right) \mathcal{P}''(\xi\gamma + (1-\xi)\eta)\, d\xi \qquad (2.5)\\
&= \frac{1}{\eta-\gamma}\left(\frac{1}{2^{\mu+1}} - \frac{1}{2}\right)\mathcal{P}'\left(\frac{\gamma+\eta}{2}\right)\\
&\quad -\frac{\mu}{(\eta-\gamma)^2}\mathcal{P}(\gamma) + \frac{1}{(\eta-\gamma)^2}\left(\frac{\mu+1}{2^{\mu}} - 1\right)\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)\\
&\quad +\frac{\mu(\mu+1)}{(\eta-\gamma)^2}\int_{\frac{1}{2}}^{1} \xi^{\mu-1}\mathcal{P}(\xi\gamma + (1-\xi)\eta)\, d\xi.
\end{aligned}
$$

Summing (2.2)–(2.5) results in:

$$
\sum_{k=1}^{4} I_k = \frac{\mu(\mu+1)}{(\eta-\gamma)^2}\left[\int_0^1 \xi^{\mu-1}\mathcal{P}(\xi\eta + (1-\xi)\gamma)\, d\xi + \int_0^1 \xi^{\mu-1}\mathcal{P}(\xi\gamma + (1-\xi)\eta)\, d\xi\right] \qquad (2.6)
$$

$$
\begin{aligned}
&-\frac{2(\mu+1)}{3(\eta-\gamma)^2}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\\
=\;&\frac{\mu(\mu+1)\Gamma(\mu)}{(\eta-\gamma)^{\mu+2}}\left[\frac{1}{\Gamma(\mu)}\int_{\gamma}^{\eta}(\psi-\gamma)^{\mu-1}\mathcal{P}(\psi)\,d\psi+\frac{1}{\Gamma(\mu)}\int_{\gamma}^{\eta}(\eta-\psi)^{\mu-1}\mathcal{P}(\psi)\,d\psi\right]\\
&-\frac{2(\mu+1)}{3(\eta-\gamma)^2}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\\
=\;&\frac{(\mu+1)\Gamma(\mu+1)}{(\eta-\gamma)^{\mu+2}}\left[\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)+\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta)\right]\\
&-\frac{2(\mu+1)}{3(\eta-\gamma)^2}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right].
\end{aligned}
$$

By multiplying both sides of (2.6) with $\frac{(\eta-\gamma)^2}{2(\mu+1)}$, we arrive at (2.1). This concludes the proof. □

**Theorem 2.1.** *Assume the conditions of Lemma 2.1 are satisfied. Furthermore, if $|\mathcal{P}''|$ exhibits convexity over $[\gamma,\eta]$, then:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta)+\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right| \qquad (2.7)\\
&\leq\frac{(\eta-\gamma)^2}{48}\left(\frac{\mu^2+15\mu+2}{(\mu+1)(\mu+2)}\right)\left[|\mathcal{P}''(\gamma)|+|\mathcal{P}''(\eta)|\right].
\end{aligned}
$$

*Proof.* On applying the modulus operation to Lemma 2.1, we obtain:

$$
\begin{aligned}
&\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta)+\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right| \qquad (2.8)\\
&\leq\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right||\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|\,d\xi\right.\\
&+\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right||\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|\,d\xi\\
&+\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right||\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|\,d\xi\\
&\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right||\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|\,d\xi\right].
\end{aligned}
$$

Leveraging the convexity property of $|\mathcal{P}''|$, we derive

$$
\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta)+\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|
$$

$$\leq \frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)\left[\xi\left|\mathcal{P}''(\eta)\right|+(1-\xi)\left|\mathcal{P}''(\gamma)\right|\right]d\xi\right.$$

$$+\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)\left[\xi\left|\mathcal{P}''(\gamma)\right|+(1-\xi)\left|\mathcal{P}''(\eta)\right|\right]d\xi$$

$$+\int_{\frac{1}{2}}^{1}\left(\xi-\xi^{\mu+1}\right)\left[\xi\left|\mathcal{P}''(\eta)\right|+(1-\xi)\left|\mathcal{P}''(\gamma)\right|\right]d\xi$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(\xi-\xi^{\mu+1}\right)\left[\xi\left|\mathcal{P}''(\gamma)\right|+(1-\xi)\left|\mathcal{P}''(\eta)\right|\right]d\xi\right]$$

$$=\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)d\xi+\int_{\frac{1}{2}}^{1}\left(\xi-\xi^{\mu+1}\right)d\xi\right]\left(\left|\mathcal{P}''(\gamma)\right|+\left|\mathcal{P}''(\eta)\right|\right)$$

$$=\frac{(\eta-\gamma)^2}{2(\mu+1)}\left(\frac{\mu+13}{24}-\frac{1}{\mu+2}\right)\left(\left|\mathcal{P}''(\gamma)\right|+\left|\mathcal{P}''(\eta)\right|\right).$$

This concludes the proof of Theorem 2.1. □

**Remark 2.1.** *When setting $\mu = 1$ in Theorem 2.1, we derive the midpoint-type inequality.*

$$\left|\frac{1}{\eta-\gamma}\int_\gamma^\eta \mathcal{P}(\xi)\,d\xi-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|\leq\frac{(\eta-\gamma)^2}{16}\left(\left|\mathcal{P}''(\gamma)\right|+\left|\mathcal{P}''(\eta)\right|\right),$$

*which is proved by Demir et al. in [21].*

**Example 2.1.** *Let's consider the interval $[\gamma,\eta]=[1,3]$ and define the function $\mathcal{P}:[1,3]\to\mathbb{R}$ as $\mathcal{P}(\xi)=\xi^4$. This gives us $\mathcal{P}''(\xi)=12\xi^2$ and $|\mathcal{P}''|$ exhibits convexity over the interval $[1,3]$. Under these conditions, we obtain*

$$\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]=\frac{148}{3}.$$

*Employing the definition of the Riemann-Liouville fractional integral, we achieve:*

$$\mathcal{J}^\mu_{\gamma+}\mathcal{P}(\eta)=\mathcal{J}^\mu_{1+}\mathcal{P}(3)=\frac{1}{\Gamma(\mu)}\int_1^3(3-\xi)^{\mu-1}\xi^4d\xi=\frac{2^\mu\left(\mu^4+18\mu^3+155\mu^2+786\mu+1944\right)}{\Gamma(\mu+5)},$$

*and*

$$\mathcal{J}^\mu_{\eta-}\mathcal{P}(\gamma)=\mathcal{J}^\mu_{3-}\mathcal{P}(1)=\frac{1}{\Gamma(\mu)}\int_1^3(\xi-1)^{\mu-1}\xi^4d\xi=\frac{3\cdot2^\mu\left(27\mu^4+198\mu^3+441\mu^2+294\mu+8\right)}{\Gamma(\mu+5)}.$$

*Therefore, the left-hand side of inequality* (2.7) *simplifies to:*

$$\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta)+\mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right| \tag{2.9}$$

$$= \left|\frac{\Gamma(\mu+1)}{2^{\mu+1}}\frac{2^{\mu}\left(82\mu^4+612\mu^3+1478\mu^2+1668\mu+1968\right)}{\Gamma(\mu+5)}-\frac{148}{3}\right|$$

$$= \left|\frac{\left(41\mu^4+306\mu^3+739\mu^2+834\mu+984\right)}{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}-\frac{148}{3}\right|.$$

*Similarly, the right-hand side of inequality* (2.7) *was reduced to:*

$$\frac{(\eta-\gamma)^2}{48}\left(\frac{\mu^2+15\mu+2}{(\mu+1)(\mu+2)}\right)\left[|\mathcal{P}''(\gamma)|+|\mathcal{P}''(\eta)|\right]$$

$$= \frac{10\left(\mu^2+15\mu+2\right)}{(\mu+1)(\mu+2)}.$$

*Thus, from inequality* (2.7)*, we derive the following inequality:*

$$\left|\frac{\left(41\mu^4+306\mu^3+739\mu^2+834\mu+984\right)}{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}-\frac{148}{3}\right| \leq \frac{10\left(\mu^2+15\mu+2\right)}{(\mu+1)(\mu+2)}. \tag{2.10}$$

*Observing Figure 1, it is evident that the left-hand side of (2.10) consistently remains below the corresponding right-hand side across all values of* $\mu \in (0, 10]$ .

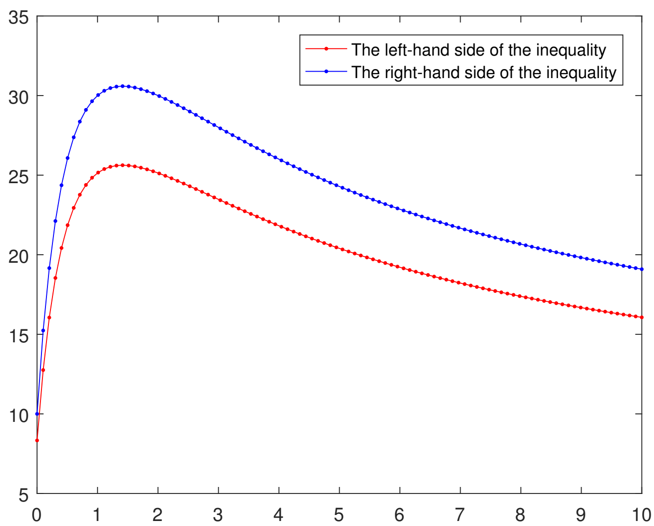


**Figure 1.** MATLAB was utilized for the computation and visualization of both sides of (2.10).

**Theorem 2.2.** *Assuming the conditions of Lemma 2.1 are met and, additionally, if $|\mathcal{P}''|^q$, where $q>1$, exhibits convexity over the interval $[\gamma,\eta]$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}_{\gamma+}^{\mu}\mathcal{P}(\eta)+\mathcal{J}_{\eta-}^{\mu}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|\\
&\leq\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}+\left(\frac{1}{\mu}\mathfrak{B}\left(p+1,\frac{p+1}{\mu},1-\left(\frac{1}{2}\right)^{\mu}\right)\right)^{\frac{1}{p}}\right]\\
&\quad\times\left[\left(\frac{3|\mathcal{P}''(\eta)|^q+|\mathcal{P}''(\gamma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{P}''(\gamma)|^q+|\mathcal{P}''(\eta)|^q}{8}\right)^{\frac{1}{q}}\right]\\
&\leq\frac{(\eta-\gamma)^2}{2^{\frac{3}{q}-1}(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}+\left(\frac{1}{\mu}\mathfrak{B}\left(p+1,\frac{p+1}{\mu},1-\left(\frac{1}{2}\right)^{\mu}\right)\right)^{\frac{1}{p}}\right](|\mathcal{P}''(\eta)|+|\mathcal{P}''(\gamma)|),
\end{aligned}$$

*where $\frac{1}{p}+\frac{1}{q}=1$ and $\mathfrak{B}$ represents the incomplete beta function, defined as:*

$$\mathfrak{B}(\kappa,y,r)=\int_0^r \xi^{\kappa-1}(1-\xi)^{y-1}d\xi.$$

*Proof.* Applying Hölder's inequality to (2.8), we derive:

$$\begin{aligned}
&\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}_{\gamma+}^{\mu}\mathcal{P}(\eta)+\mathcal{J}_{\eta-}^{\mu}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|\\
&\leq\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|^q d\xi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|^q d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\left|\xi^{\mu+1}-\xi\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|^q d\xi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|\xi^{\mu+1}-\xi\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|^q d\xi\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the convexity of $|\mathcal{P}''|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}_{\gamma+}^{\mu}\mathcal{P}(\eta)+\mathcal{J}_{\eta-}^{\mu}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|\\
&\leq\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\xi|\mathcal{P}''(\eta)|^q+(1-\xi)|\mathcal{P}''(\gamma)|^q\right]d\xi\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\xi\left|\mathcal{P}''(\gamma)\right|^q+(1-\xi)\left|\mathcal{P}''(\eta)\right|^q\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\xi\left|\mathcal{P}''(\eta)\right|^q+(1-\xi)\left|\mathcal{P}''(\gamma)\right|^q\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\xi\left|\mathcal{P}''(\gamma)\right|^q+(1-\zeta)\left|\mathcal{P}''(\eta)\right|^q\right]d\xi\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\mu+4}{3}\xi-\xi^{\mu+1}\right)^p d\xi\right)^{\frac{1}{p}}+\left(\frac{1}{\mu}\mathfrak{B}\left(p+1,\frac{p+1}{\mu},1-\left(\frac{1}{2}\right)^{\mu}\right)\right)^{\frac{1}{p}}\right]$$

$$\times\left[\left(\frac{3\left|\mathcal{P}''(\eta)\right|^q+\left|\mathcal{P}''(\gamma)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{P}''(\gamma)\right|^q+\left|\mathcal{P}''(\eta)\right|^q}{8}\right)^{\frac{1}{q}}\right].$$

Regarding the proof of the second inequality, let $\gamma_1=\left|\mathcal{P}''(\gamma)\right|^q$, $\eta_1=3\left|\mathcal{P}''(\eta)\right|^q$, $\gamma_2=3\left|\mathcal{P}''(\gamma)\right|^q$ and $\eta_2=\left|\mathcal{P}''(\eta)\right|^q$. Leveraging the given facts that

$$\sum_{k=1}^{n}(\gamma_k+\eta_k)^s\le\sum_{k=1}^{n}\gamma_k^s+\sum_{k=1}^{n}\eta_k^s,\ 0\le s<1,$$

and $1+3^{\frac{1}{q}}\le 4$, the desired result can be acquired straightforwardly. That concludes the proof. □

**Remark 2.2.** *When setting $\mu=1$ in Theorem 2.2, we arrive at the inequalities*

$$\left|\frac{1}{(\eta-\gamma)}\int_{\gamma}^{\eta}\mathcal{P}(\xi)\,d\xi-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|$$

$$\le\frac{(\eta-\gamma)^2}{4}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\xi-\xi^2\right)^p d\xi\right)^{\frac{1}{p}}+\left(\mathfrak{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right]$$

$$\times\left[\left(\frac{3\left|\mathcal{P}''(\eta)\right|^q+\left|\mathcal{P}''(\gamma)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|\mathcal{P}''(\gamma)\right|^q+\left|\mathcal{P}''(\eta)\right|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$\le\frac{(\eta-\gamma)^2}{2^{\frac{3}{q}}}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\xi-\xi^2\right)^p d\xi\right)^{\frac{1}{p}}+\left(\mathfrak{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right]\left(\left|\mathcal{P}''(\eta)\right|+\left|\mathcal{P}''(\gamma)\right|\right).$$

**Theorem 2.3.** *Assuming the conditions of Lemma 2.1 are satisfied, if $|\mathcal{P}''|^q$, where $q \geq 1$, has convex behavior over $[\gamma, \eta]$, then*

$$\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^\mu}\left[\mathcal{J}_{\gamma+}^{\mu}\mathcal{P}(\eta)+\mathcal{J}_{\eta-}^{\mu}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right|$$
$$\leq \frac{(\eta-\gamma)^2}{2(\mu+1)}\left[(\Phi_5(\mu))^{1-\frac{1}{q}}\left(\Phi_1(\mu)|\mathcal{P}''(\eta)|^q+\Phi_2(\mu)|\mathcal{P}''(\gamma)|^q\right)^{\frac{1}{q}}\right.$$
$$+(\Phi_5(\mu))^{1-\frac{1}{q}}\left(\Phi_1(\mu)|\mathcal{P}''(\gamma)|^q+\Phi_2(\mu)|\mathcal{P}''(\eta)|^q\right)^{\frac{1}{q}}$$
$$+(\Phi_6(\mu))^{1-\frac{1}{q}}\left(\Phi_3(\mu)|\mathcal{P}''(\eta)|^q+\Phi_4(\mu)|\mathcal{P}''(\gamma)|^q\right)^{\frac{1}{q}}$$
$$\left.+(\Phi_6(\mu))^{1-\frac{1}{q}}\left(\Phi_3(\mu)|\mathcal{P}''(\gamma)|^q+\Phi_4(\mu)|\mathcal{P}''(\eta)|^q\right)^{\frac{1}{q}}\right],$$

*where*

$$\begin{aligned}
\Phi_1(\mu) &= \frac{\mu+4}{72}-\frac{1}{(\mu+3)2^{\mu+3}},\\
\Phi_2(\mu) &= \frac{\mu+4}{36}+\frac{1}{(\mu+3)2^{\mu+3}}-\frac{1}{(\mu+2)2^{\mu+2}},\\
\Phi_3(\mu) &= \frac{7\mu-3}{24(\mu+3)}+\frac{1}{(\mu+3)2^{\mu+3}},\\
\Phi_4(\mu) &= \frac{\mu^2+5\mu-6}{12(\mu+2)(\mu+3)}+\frac{1}{(\mu+2)2^{\mu+2}}-\frac{1}{(\mu+3)2^{\mu+3}},\\
\Phi_5(\mu) &= \frac{\mu+4}{24}-\frac{1}{(\mu+2)2^{\mu+2}},\\
\Phi_6(\mu) &= \frac{3\mu-2}{8(\mu+2)}+\frac{1}{(\mu+2)2^{\mu+2}}.
\end{aligned}$$

*Proof.* By applying the power-mean inequality in (2.8), we have

$$\left|\frac{\Gamma(\mu+1)}{2(\eta-\gamma)^\mu}\left[\mathcal{J}_{\gamma+}^{\mu}\mathcal{P}(\eta)+\mathcal{J}_{\eta-}^{\mu}\mathcal{P}(\gamma)\right]-\frac{1}{3}\left[2\mathcal{P}(\gamma)-\mathcal{P}\left(\frac{\gamma+\eta}{2}\right)+2\mathcal{P}(\eta)\right]\right| \tag{2.11}$$
$$\leq \frac{(\eta-\gamma)^2}{2(\mu+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right||\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|^q\,d\xi\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^{\mu+1}-\frac{\mu+4}{3}\xi\right||\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|^q\,d\xi\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right||\mathcal{P}''(\xi\eta+(1-\xi)\gamma)|^q\,d\xi\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\mu+1}-\xi\right||\mathcal{P}''(\xi\gamma+(1-\xi)\eta)|^q\,d\xi\right)^{\frac{1}{q}}\right].$$

Since $|\mathcal{P}''|^q$ is convex, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right| |\mathcal{P}''(\xi\eta + (1-\xi)\gamma)|^q \, d\xi \qquad (2.12)\\
\leq\ & \int_0^{\frac{1}{2}} \left(\frac{\mu+4}{3}\xi - \xi^{\mu+1}\right)\left[\xi\,|\mathcal{P}''(\eta)|^q + (1-\xi)\,|\mathcal{P}''(\gamma)|^q\right] d\xi \\
=\ & \left(\frac{\mu+4}{72} - \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\eta)|^q + \left(\frac{\mu+4}{36} + \frac{1}{(\mu+3)\,2^{\mu+3}} - \frac{1}{(\mu+2)\,2^{\mu+2}}\right)|\mathcal{P}''(\gamma)|^q \\
=\ & \Phi_1(\mu)\,|\mathcal{P}''(\eta)|^q + \Phi_2(\mu)\,|\mathcal{P}''(\gamma)|^q .
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right| |\mathcal{P}''(\xi\gamma + (1-\xi)\eta)|^q \, d\xi \qquad (2.13)\\
\leq\ & \left(\frac{\mu+4}{72} - \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\gamma)|^q + \left(\frac{\mu+4}{36} + \frac{1}{(\mu+3)\,2^{\mu+3}} - \frac{1}{(\mu+2)\,2^{\mu+2}}\right)|\mathcal{P}''(\eta)|^q \\
=\ & \Phi_1(\mu)\,|\mathcal{P}''(\gamma)|^q + \Phi_2(\mu)\,|\mathcal{P}''(\eta)|^q ,
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left|\xi^{\mu+1} - \xi\right| |\mathcal{P}''(\xi\eta + (1-\xi)\gamma)|^q \, d\xi \qquad (2.14)\\
\leq\ & \left(\frac{7\mu-3}{24(\mu+3)} + \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\eta)|^q \\
& + \left(\frac{\mu^2+5\mu-6}{12(\mu+2)(\mu+3)} + \frac{1}{(\mu+2)\,2^{\mu+2}} - \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\gamma)|^q \\
=\ & \Phi_3(\mu)\,|\mathcal{P}''(\eta)|^q + \Phi_4(\mu)\,|\mathcal{P}''(\gamma)|^q ,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left|\xi^{\mu+1} - \xi\right| |\mathcal{P}''(\xi\gamma + (1-\xi)\eta)|^q \, d\xi \qquad (2.15)\\
\leq\ & \left(\frac{7\mu-3}{24(\mu+3)} + \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\gamma)|^q \\
& + \left(\frac{\mu^2+5\mu-6}{12(\mu+2)(\mu+3)} + \frac{1}{(\mu+2)\,2^{\mu+2}} - \frac{1}{(\mu+3)\,2^{\mu+3}}\right)|\mathcal{P}''(\eta)|^q \\
=\ & \Phi_3(\mu)\,|\mathcal{P}''(\gamma)|^q + \Phi_4(\mu)\,|\mathcal{P}''(\eta)|^q .
\end{aligned}$$

Moreover, we also have

$$\int_0^{\frac{1}{2}} \left|\xi^{\mu+1} - \frac{\mu+4}{3}\xi\right| d\xi = \int_0^{\frac{1}{2}} \left(\frac{\mu+4}{3}\xi - \xi^{\mu+1}\right) d\xi = \frac{\mu+4}{24} - \frac{1}{(\mu+2)\,2^{\mu+2}} = \Phi_5(\mu), \tag{2.16}$$

and

$$\int_{\frac{1}{2}}^{1} \left|\xi^{\mu+1} - \xi\right| d\xi = \int_{\frac{1}{2}}^{1} \left(\xi - \xi^{\mu+1}\right) d\xi = \frac{3\mu-2}{8(\mu+2)} + \frac{1}{(\mu+2)\,2^{\mu+2}} = \Phi_6(\mu). \tag{2.17}$$

If we substitute (2.12)–(2.17) in (2.11), then we obtain the desired inequality. □

**Remark 2.3.** *If we let $\mu = 1$ in Theorem 2.3, then we have the midpoint-type inequality*

$$\begin{aligned}&\left|\frac{1}{2(\eta-\gamma)^{\mu}}\left[\mathcal{J}^{\mu}_{\gamma+}\mathcal{P}(\eta) + \mathcal{J}^{\mu}_{\eta-}\mathcal{P}(\gamma)\right] - \frac{1}{3}\left[2\mathcal{P}(\gamma) - \mathcal{P}\left(\frac{\gamma+\eta}{2}\right) + 2\mathcal{P}(\eta)\right]\right| \\ &\leq \frac{(\eta-\gamma)^2}{48}\left[2\left(\frac{31\left|\mathcal{P}''(\eta)\right|^q + 65\left|\mathcal{P}''(\gamma)\right|^q}{96}\right)^{\frac{1}{q}} + 2\left(\frac{31\left|\mathcal{P}''(\gamma)\right|^q + 65\left|\mathcal{P}''(\eta)\right|^q}{96}\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\frac{11\left|\mathcal{P}''(\eta)\right|^q + 2\left|\mathcal{P}''(\gamma)\right|^q}{16}\right)^{\frac{1}{q}} + \left(\frac{11\left|\mathcal{P}''(\gamma)\right|^q + 2\left|\mathcal{P}''(\eta)\right|^q}{16}\right)^{\frac{1}{q}}\right].\end{aligned}$$

## 3. Conclusions

This study established an identity suitable for functions possessing two continuous derivatives, facilitating the validation of multiple Milne-type inequalities applicable to functions showcasing convex second derivatives within Riemann-Liouville fractional integrals. The inclusion of an illustrative example and corresponding graphical representation enhances the comprehension of our primary findings, emphasizing the significance of the derived identity in elucidating these functions' behavior. Future investigations could explore enhancements or extensions of our outcomes by examining various convex function classes or alternative fractional integral operators.

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## Author contributions

Areej A. Almoneef, Abd-Allah Hyder, Hüseyin Budak, Mohamed A. Barakat: Methodology, Conceptualization, Data curation, Writing-original draft, Investigation, Visualization, Validation, Writing-reviewing, Editing. All authors have read and approved the final version of the manuscript for publication.

**Use of AI tools declaration**

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

**Conflict of interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

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