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Research article

Some new characterizations of boundedness of commutators of p-adic maximal-type functions on p-adic Morrey spaces in terms of Lipschitz spaces

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Abstract: In this note, we investigate some new characterizations of the p-adic version of Lipschitz spaces via the boundedness of commutators of the p-adic maximal-type functions, including p-adic sharp maximal functions, p-adic fractional maximal functions, and p-adic fractional maximal commutators on p-adic Morrey spaces, when a symbol function b belongs to the Lipschitz spaces.

Keywords: p-adic sharp maximal function; p-adic fractional maximal function; p-adic Morrey spaces; Lipschitz space; commutators
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1. Introduction

In the contemporary era, p-adic analysis is so significant that there is a lot of research being done on theories that are only concerned with p-adic objects, a case in point is the p-adic Hodge theory [1], Coleman's theory of p-adic integration [2], p-adic geometry [3], the theory of p-adic differential equations [4], the p-adic Langlands correspondence [5], study of p-adic cohomologies [6], and the study of p-adic modular forms [7]. In this connection, numerous of these concepts and advancements are present in the proof of Fermet's last theorem [8]. Recently, they have been applied in mathematical physics [9] and harmonic analysis [10–16].

Ostrowski's theorem states that any nontrivial norm on the field of rational numbers $\mathbb{Q}$ is either the p-adic norm $|\cdot|_p$, or the real norm $|\cdot|$, where p is a prime number. The former norm is stated as follows, if any rational number r is represented as $r = p^{\theta}\frac{m}{n}$, where $\theta = \theta(\mathbf{x}) \in \mathbb{Z}$, $(p, m, n) = 1$ and $m, n \in \mathbb{Z}$, then

$$|0|_p = 0, \quad |r|_p = p^{-\theta} \quad r \neq 0.$$

This norm exhibits an ultrametric property

$$|r + s|_p \leq \max\{|r|_p, |s|_p\}.$$

A symbol $\mathbb{Q}_p$ is the field of p-adic numbers, and is the completion of field of rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. Any $\mathbb{Q}_p \ni r \neq 0$ is uniquely represented as, see [9]

$$r = p^{\theta} \sum_{k=0}^{\infty} \gamma_k p^k, \tag{1.1}$$

where $\gamma_k, \theta \in \mathbb{Z}$, $\gamma_k \in \frac{\mathbb{Z}}{p\mathbb{Z}_p}$, $\gamma_0 \neq 0$. It is eminent that the series in (1.1) is convergent as $|\gamma_k p^k| = p^{-k}$.

The n-dimensional field $\mathbb{Q}_p^n$ is defined as n-tuples of p-adic numbers, $(r_1, r_2, \cdots, r_n)$, where $\mathbb{Q}_p \ni r_k$, $k = 1, 2, \cdots, n$. The n-dimensional p-adic numbers inherit many properties from the p-adic numbers. They form a complete metric space with respect to the n-dimensional p-adic metric $d(\mathbf{r}, \mathbf{s}) = |\mathbf{r} - \mathbf{s}|_p$, which measures the divisibility of n-tuples by powers of p. The n-dimensional p-adic metric induces a topology on $\mathbb{Q}_p^n$, allowing for the study of continuity, convergence, and limit concepts in this space. The norm on $\mathbb{Q}_p^n$ is

$$|\mathbf{r}|_p = \max_{1 \leq k \leq n} |r_k|_p.$$

The p-adic ball $B_\theta(\mathbf{x})$ and p-adic sphere $S_\theta(\mathbf{x})$ with radius p^θ and center $\mathbf{x}$ are defined by

$$B_\theta(\mathbf{x}) = \{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r} - \mathbf{x}|_p \leq p^\theta\}, \; S_\theta(\mathbf{x}) = \{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r} - \mathbf{x}|_p = p^\theta\}.$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group, then there exists a Haar measure $d\mathbf{y}$ on the additive group $\mathbb{Q}_p^n$, which is normalized by

$$\int_{B_0(\mathbf{0})} d\mathbf{y} = 1.$$

From standard analysis, we get $|B_\theta(\mathbf{x})|_h = p^{n\theta}$ and $|S_\theta(\mathbf{x})|_h = p^{n\theta}(1 - p^{-n})$, for any $\mathbf{x} \in \mathbb{Q}_p^n$.

A measurable function b defined on $\mathbb{Q}_p^n$ is in $L^p(\mathbb{Q}_p^n)$ $(1 \leq p \leq \infty)$, if it satisfies

$$\|b\|_{L^p(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |b(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|b\|_{L^\infty(\mathbb{Q}_p^n)} = ess \sup_{\mathbf{x} \in \mathbb{Q}_p^n} |b(\mathbf{x})| < \infty.$$

The commutators of harmonic analysis are vital integral operators and play a crucial role in examining the regularity characteristics of solutions to various partial differential equations, for instance [17–20]. Suppose T is a classical singular integral operator with an another function b, then the commutator $[b, T]$ generated by T is defined as follows:

$$[b, T](f) = bT(f) - T(bf). \tag{1.2}$$

In [21], authors have studied the L^p boundedness of (1.2) with $b \in B\dot{M}O(\mathbb{R}^n)$. These results were extended with $b \in \Lambda_\delta(\mathbb{R}^n)$ in [22]. Since then, a great attention has been paid with studying the commutators of operators; see for instance, [17, 23–25].

In what follows, for $f \in L^1_{loc}(\mathbb{Q}_p^n)$, we define the p-adic sharp maximal function $M^{p,\sharp}$ and p-adic fractional maximal function M^p_α as

$$M^{p,\sharp} f(\mathbf{x}) = \sup_{\theta\in\mathbb{Z}} \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} |f(\mathbf{t}) - f_{B_\theta(\mathbf{x})}| d\mathbf{t} \tag{1.3}$$

and

$$M^p_\alpha f(\mathbf{x}) = \sup_{\theta\in\mathbb{Z}} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |f(\mathbf{t})| d\mathbf{t}, \tag{1.4}$$

where $f_{B_\theta(\mathbf{x})} = \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} f(\mathbf{t}) d\mathbf{t}$. When $\alpha = 0$, we get the Hardy Littlewood maximal function M^p, which is defined as:

$$M^p f(\mathbf{x}) = \sup_{\theta\in\mathbb{Z}} \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} |f(\mathbf{t})| d\mathbf{t}.$$

Significant work has been done intensively in the past on M^p by many researchers; see for example, [26–28] and the references therein.

The p-adic fractional commutator of M^p_α with $b \in L^1_{loc}(\mathbb{Q}_p^n)$ is defined by

$$M^p_{\alpha,b} f(\mathbf{x}) = \sup_{\theta\in\mathbb{Z}} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{t})||f(\mathbf{t})| d\mathbf{t}.$$

On the other hand, nonlinear commutators of respectively $M^{p,\sharp}$ and M^p_α with a locally integrable function b are defined by

$$[b, M^{p,\sharp}](f)(\mathbf{x}) = b(\mathbf{x}) M^{p,\sharp}(f)(\mathbf{x}) - M^{p,\sharp}(bf)(\mathbf{x}) \tag{1.5}$$

and

$$[b, M^p_\alpha](f)(\mathbf{x}) = b(\mathbf{x}) M^p_\alpha(f)(\mathbf{x}) - M^p_\alpha(bf)(\mathbf{x}). \tag{1.6}$$

When $\alpha = 0$, $[b, M^p_\alpha]$ reduces to $[b, M^p]$, see [29]. In p-adic setting, boundedness of commutators of p-adic maximal function is a new area, and we only found some work in [29]. In that paper, the authors acquired the boundedness of commutators of M^p on p-adic function spaces with $b \in B\dot{M}O(\mathbb{Q}_p^n)$. However, in the case of Euclidean, commutators of maximal-type functions have spotlighted many researchers. For example, in [17], Bastero et al. obtained the boundedness of commutators of maximal and sharp functions on Lebesgue spaces with $b \in B\dot{M}O(\mathbb{R}^n)$. Furthermore, the results of [17] are extended in [30]. Zhang [31] further obtained the characterizations of nonlinear commutators of the Hardy Littlewood maximal function and sharp maximal function in variable exponent Lebesgue spaces with $b \in \Lambda_\delta(\mathbb{R}^n)$. Recently, Xuechun et al. [32] established new characterizations of Lipschitz space in terms of the boundedness of $[b, M^\sharp]$ and $[b, M^\alpha]$ in the context of variable Lipschitz space.

As we observed in the above work, the characterization of nonlinear commutators of $M^{p,\sharp}$ and M^p_α remains widely open. Therefore, we obtain some characterizations of p-adic versions of Lipschitz spaces via the boundedness of $M^{p,\sharp}$ and M^p_α on p-adic Morrey spaces, by considering b from Lipschitz spaces under certain assumptions. Throughtout this article, a letter C represents a constant with different or the same values at different places, and χ_{B_θ} is the characteristic function of $B_\theta(\mathbf{x})$.

Definition 1.1. Let $1 \le p < \infty$ and $0 \le \lambda \le n$. The p-adic Morrey space $L^{p,\lambda}(\mathbb{Q}_p^n)$ is defined as follows:

$$L^{p,\lambda}(\mathbb{Q}_p^n) = \{b \in L^p_{loc}(\mathbb{Q}_p^n) : \|b\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} < \infty\}, \tag{1.7}$$

where

$$\|b\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} = \sup_{\substack{\theta\in\mathbb{Z}\\ \mathbf{x}\in\mathbb{Q}_p^n}} \left(\frac{1}{|B_\theta(\mathbf{x})|_h^{\lambda/n}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}.$$

Remark 1.1. It is evident that $L^{p,-1/p}(\mathbb{Q}_p^n) = L^p(\mathbb{Q}_p^n)$ and $L^{p,0}(\mathbb{Q}_p^n) = L^\infty(\mathbb{Q}_p^n)$.

Definition 1.2. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$, $(\delta \in \mathbb{R}^+)$ is the space of all measurable functions b on $\mathbb{Q}_p^n$ such that

$$\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{\mathbf{t},\mathbf{h}\in\mathbb{Q}_p^n,\mathbf{h}\neq 0} \frac{|b(\mathbf{t}+\mathbf{h}) - b(\mathbf{t})|}{|\mathbf{h}|_p^\delta} < \infty.$$

Next, we have the p-adic version of the Lipschitz space $\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)$, which is the space of all measurable functions b on $\mathbb{Q}_p^n$ with the following norm:

$$\|b\|_{\tilde{\Lambda}_\delta}(\mathbb{Q}_p^n) = \sup_{x\in\mathbb{Q}_p^n,\theta\in\mathbb{Z}} \frac{1}{|B_\theta(\mathbf{x})|_h^{1+\frac{\delta}{n}}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| d\mathbf{t} < \infty,$$

where $b_{B_\theta(\mathbf{x})} = \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} b(\mathbf{t}) d\mathbf{t}$.

In the following section, we state some characterizations of the p-adic version of Lipschitz spaces via the boundedness of the commutators of $[b, M^{p,\sharp}]$, $M^\alpha_{\alpha,b}$, and $[b, M^p_\alpha]$.

2. Some characterizations of p-adic version of Lipschitz spaces

Theorem 2.1. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $0 < \lambda < n - q\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta}{n-\lambda} = \frac{1}{q}$. Then, $[b, M^{p,\sharp}] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ with $b \ge 0$.*

Theorem 2.2. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $0 < \lambda < n - q\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta+\alpha}{n-\lambda} = \frac{1}{q}$. Then, $M^p_{\alpha,b} : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$.*

Theorem 2.3. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $0 < \lambda < n - q\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta+\alpha}{n-\lambda} = \frac{1}{q}$. Then, $[b, M^p_\alpha] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ with $b \ge 0$.*

Since $L^{p,-1/p}(\mathbb{Q}_p^n) = L^p(\mathbb{Q}_p^n)$. So, we have the characterizations in terms of the boundedness of operators $[b, M^{p,\sharp}]$, M^p_b, and $[b, M^p]$ on Lebesgue spaces.

Corollary 2.1. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta}{n} = \frac{1}{q}$. Then, $[b, M^{p,\sharp}] : L^q(\mathbb{Q}_p^n) \to L^p(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ with $b \ge 0$.*

Corollary 2.2. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $0 < \lambda < n - q\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta+\alpha}{n} = \frac{1}{q}$. Then, $M^p_{\alpha,b} : L^q(\mathbb{Q}_p^n) \to L^p(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$.*

Corollary 2.3. *Suppose b is a locally integrable function, $1 < q < n/\delta$, $0 < \lambda < n - q\delta$, $\delta \in (0,1)$, and $\frac{1}{p} + \frac{\delta+\alpha}{n} = \frac{1}{q}$. Then, $[b, M_\alpha^p]$:$L^q(\mathbb{Q}_p^n) \to L^p(\mathbb{Q}_p^n)$ if and only if $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ with $b \geq 0$.*

In order to prove the above results, we need some lemmas and remarks. We begin with a very useful result.

Lemma 2.1. *The p-adic space $\Lambda_\delta(\mathbb{Q}_p^n)$ coincides with $\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)$, for $0 < \delta < 1$.*

Proof. Consider a ball $B_\theta(\mathbf{x})$ and $\mathbf{t} \in B_\theta(\mathbf{x})$, then from the definition (1.2), we have

$$\begin{aligned}|b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| \leq& \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b(\mathbf{z})| d\mathbf{z} \\ \leq& ||b||_{\Lambda_\delta(\mathbb{Q}_p^n)} \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} |\mathbf{t} - \mathbf{z}|_h^\delta d\mathbf{z} \\ \leq& C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)} \frac{1}{|B_\theta(\mathbf{x})|_h} |B_\theta(\mathbf{x})|_h^{\frac{\delta}{n}} \int_{B_\theta(\mathbf{x})} d\mathbf{z} \\ \leq& C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)} |B_\theta(\mathbf{x})|_h^{\frac{\delta}{n}}.\end{aligned}$$

We further proceed as

$$\begin{aligned}\int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| d\mathbf{t} \leq C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)} \int_{B_\theta(\mathbf{x})} |B_\theta(\mathbf{x})|_h^{\frac{\delta}{n}} d\mathbf{t} \\ \leq& C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)} |B_\theta(\mathbf{x})|_h^{1+\delta/n},\end{aligned}$$

which implies that

$$\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| d\mathbf{t} \leq C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)}.$$

Therefore,

$$||b||_{\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)} \leq C||b||_{\Lambda_\delta(\mathbb{Q}_p^n)}. \tag{2.1}$$

On the other hand, let $b \in \tilde{\Lambda}_\delta(\mathbb{Q}_p^n)$. For any $\mathbf{t}, \mathbf{z} \in \mathbb{Q}_p^n$ with $\mathbf{t} \neq \mathbf{z}$. We set $B = B(\mathbf{t}, |\mathbf{t} - \mathbf{z}|_p)$ and $B' = B'(\mathbf{z}, |\mathbf{t} - \mathbf{z}|_p)$. Then we have

$$|b(\mathbf{t}) - b(\mathbf{z})| \leq |b(\mathbf{t}) - b_B| + |b(\mathbf{z}) - b_{B'}| + |b_B - b_{B'}|. \tag{2.2}$$

Estimates of all terms on the right-hand side of (2.2) are more or less the same. So, we will estimate the first term. Let $B_j = B(t, p^{-j}|t - \mathbf{x}|_p)$ for $j \geq 1$ and $B_0 = B$. We proceed as

$$\begin{aligned}|b(\mathbf{t}) - b_B| \leq& \lim_{\theta\to\infty} \left(|b(\mathbf{t}) - b_{B_\theta}| + \sum_{j=0}^{\theta-1} |b_{B_{j+1}} - b_{B_j}| \right) \\ \leq& C \sum_{j=1}^{\infty} \frac{1}{|B_j|_h} \int_{B_j} |b(\mathbf{z}) - b_{B_j}| d\mathbf{z} \\ \leq& C||b||_{\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)} \sum_{j=1}^{\infty} |B_j|_h^{\delta/n} \\ \leq& C||b||_{\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)} \sum_{j=1}^{\infty} p^{-\delta j + \log_p |\mathbf{t}-\mathbf{z}|_p^\delta} \\ \leq& C|\mathbf{t} - \mathbf{z}|_p^\delta ||b||_{\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)}.\end{aligned}$$

Consequently,

$$\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} \le C\|b\|_{\tilde{\Lambda}_\delta(\mathbb{Q}_p^n)}. \tag{2.3}$$

From (2.1) and (2.3), we have completed the proof. □

In what follows, taking into account the characteristic function $\chi_{B_\theta(\mathbf{x})}$, we have the following property:

Lemma 2.2. *Suppose* $1 \le q < \infty$ *and* $0 < \lambda < n$, *then*

$$\|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = |B_\theta(\mathbf{x})|_h^{\frac{n-\lambda}{nq}} = p^{\frac{\theta(n-\lambda)}{q}}.$$

Next, the fractional integral operator on $\mathbb{Q}_p^n$ is introduced by Taibleson [33] and is defined by

$$T_\alpha^p f(\mathbf{x}) = \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\mathbb{Q}_p^n} \frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{\alpha-n}} d\mathbf{t}, \quad 0 < \alpha < n.$$

The following lemma shows the boundedness of T_α^p on p-adic Morrey spaces, which is proved in a book [33].

Lemma 2.3. *Suppose* $1 < q < n/\alpha$, $0 < \alpha < n$, $0 < \lambda < n-q$, *and* $\frac{1}{p} + \frac{\alpha}{n-\lambda} = \frac{1}{q}$, *then* T_α^p *is bounded from* $L^{q,\lambda}(\mathbb{Q}_p^n)$ *to* $L^{p,\lambda}(\mathbb{Q}_p^n)$.

Remark 2.1. From the condition of Lemma 2.3, we get

$$\begin{aligned} |T_\alpha^p(|f|)(\mathbf{x})| &= \left| \int_{\mathbb{Q}_p^n} \frac{|f(\mathbf{t})|}{|\mathbf{x}-\mathbf{t}|_p^\alpha} d\mathbf{t} \right| \\ &\ge \int_{B_\theta(\mathbf{x})} \frac{|f(\mathbf{t})|}{|\mathbf{x}-\mathbf{t}|_p^\alpha} d\mathbf{t} \\ &\ge \frac{1}{p^{\theta(n-\alpha)}} \int_{B_\theta(\mathbf{x})} |f(\mathbf{t})| d\mathbf{t}. \end{aligned}$$

Therefore,

$$|M_\alpha^p(f)(\mathbf{x})| \le C T_\alpha^p(|f|)(\mathbf{x}).$$

From here, we deduce that M_α^p is bounded from $L^{q,\lambda}(\mathbb{Q}_p^n)$ to $L^{p,\lambda}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.1. Consider any $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ with $b \ge 0$, we prove that $[b, M^{p,\sharp}] : L^{q,\lambda}(\mathbb{Q}_p^n) \to$

$L^{p,\lambda}(\mathbb{Q}_p^n)$. Let $f\in L^{q,\lambda}(\mathbb{Q}_p^n)$. From definition (1.2), we deduce

$$\begin{aligned}
&|[b, M^{p,\sharp}](f)(\mathbf{x})|\\
=&\left|\sup_{\theta\in Z}\frac{b(\mathbf{x})}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|f(\mathbf{t})-f_{B_\theta(\mathbf{x})}|d\mathbf{t}-\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|b(\mathbf{t})f(\mathbf{t})-(bf)_{B_\theta(\mathbf{x})}|d\mathbf{t}\right|\\
\le&\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}(|b(\mathbf{t})-b(\mathbf{x})||f(\mathbf{t})|+|b(\mathbf{x})f_{B_\theta(\mathbf{x})}-(bf)_{B_\theta(\mathbf{x})}|)d\mathbf{t}\\
\le&\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|\mathbf{t}-\mathbf{x}|_p^{\delta(\mathbf{x})}|f(\mathbf{t})|d\mathbf{t}\\
&+\sup_{\theta\in Z}\left|\frac{b(\mathbf{x})}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}f(\mathbf{y})d\mathbf{y}-\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}b(\mathbf{y})f(\mathbf{y})d\mathbf{y}\right|\\
\le&\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|\mathbf{t}-\mathbf{x}|_p^{\delta(\mathbf{x})}|f(\mathbf{t})|d\mathbf{t}\\
&+\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|b(\mathbf{x})-b(\mathbf{y})||f(\mathbf{y})|d\mathbf{y}\\
\le&\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|\mathbf{t}-\mathbf{x}|_p^{\delta(\mathbf{x})}|f(\mathbf{t})|d\mathbf{t}\\
&+\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}|\mathbf{x}-\mathbf{y}|_p^{\delta(\mathbf{x})}|f(\mathbf{y})|d\mathbf{y}\\
\le&C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\sup_{\theta\in Z}\frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\delta(\mathbf{x})}{n}}}\int_{B_\theta(\mathbf{x})}|f(\mathbf{t})|d\mathbf{t}\\
\le&C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}M_{\delta(\mathbf{x})}f(\mathbf{x}).
\end{aligned}\tag{2.4}$$

From Remark 2.1 and equation (2.4), we obtain

$$\|[b, M^{p,\sharp}](f)\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\le C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Hence, $[b, M^{p,\sharp}]: L^{q,\lambda}(\mathbb{Q}_p^n)\to L^{p,\lambda}(\mathbb{Q}_p^n)$.

Conversely, suppose that $[b, M^{p,\sharp}]: L^{q,\lambda}(\mathbb{Q}_p^n)\to L^{p,\lambda}(\mathbb{Q}_p^n)$. We need to show $b\in\Lambda_\delta(\mathbb{Q}_p^n)$ and $b\ge 0$. Consider any fixed p-adic ball $B_\theta(\mathbf{x})$, and $\mathbf{t}\in B_\theta(\mathbf{x})$. We see in [29] that

$$M^{p,\sharp}(\chi_{B_\theta}(\mathbf{x}))(\mathbf{t})=\frac{2(p-1)}{p^2}.$$

By above expression, Eq (1.5) and the boundedness of $[b, M^{p,\sharp}]$, we reach at

$$\begin{aligned}
&\left\|\left(b-\frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})\right)\chi_{B_\theta(\mathbf{x})}\right\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\\
=&\left\|\frac{p^2}{2(p-1)}\left(\frac{2(p-1)}{p^2}b-M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})\right)\chi_{B_\theta(\mathbf{x})}\right\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\\
=&\left\|\frac{p^2}{2(p-1)}\left(bM^{p,\sharp}(\chi_{B_\theta(\mathbf{x})})-M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})\right)\chi_{B_\theta(\mathbf{x})}\right\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{p^2}{2(p-1)}\|[b, M^{p,\sharp}](\chi_{B_\theta(\mathbf{x})})\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\\
\leq& C\|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)},
\end{aligned}
$$

which implies that

$$
\frac{\|(b-\frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\theta(\mathbf{x})})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C. \tag{2.5}
$$

Now, consider a p-adic ball $B_\theta(\mathbf{x}) \subset \mathbb{Q}_p^n$. From [29], we see that for any $\mathbf{t} \in B_\theta(\mathbf{x})$,

$$
|b_{B_\theta(\mathbf{x})}| \leq \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t}). \tag{2.6}
$$

Now to achieve $b \in \Lambda_\delta(\mathbb{Q}_p^n)$, we let $A = \{\mathbf{t} \in B_\theta(\mathbf{x}) : b(\mathbf{t}) \leq b_{B_\theta(\mathbf{x})}\}$. Moreover, consider any $\mathbf{t} \in A$ and we get $b(\mathbf{t}) \leq b_{B_\theta(\mathbf{x})} \leq |b_{B_\theta(\mathbf{x})}| \leq 2M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t})$, then

$$
|b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| \leq |b(\mathbf{t}) - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta}(\mathbf{x}))(\mathbf{t})|. \tag{2.7}
$$

Since $\frac{1}{p} = \frac{1}{q} - \frac{\delta}{n-\lambda}$, then using (2.7) along with Hölder's inequality, Lemma 2.2, and (2.5), we ultimately have

$$
\begin{aligned}
&\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}|b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}|d\mathbf{t}\\
=&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{A}|b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}|d\mathbf{t}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}|b(\mathbf{t}) - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t})|d\mathbf{t}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\left(\int_{B_\theta(\mathbf{x})}|b(\mathbf{t}) - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t})|^p d\mathbf{t}\right)^{1/p}\\
&\times\left(\int_{B_\theta(\mathbf{x})}\chi_{B_\theta(\mathbf{x})}(\mathbf{t})d\mathbf{t}\right)^{1/p'}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}.|B_\theta(\mathbf{x})|_h^{\lambda/np}\left(\frac{1}{|B_\theta(\mathbf{x})|_h^{\lambda/n}}\int_{B_\theta(\mathbf{x})}|b(\mathbf{t}) - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t})|^p d\mathbf{t}\right)^{1/p}\\
&\times\left(\int_{B_\theta(\mathbf{x})}\chi_{B_\theta(\mathbf{x})}(\mathbf{t})d\mathbf{t}\right)^{1/p'}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n-\lambda/np}}\|(b - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})}))(\chi_{B_\theta(\mathbf{x})})\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\\
&\times\|\chi_{B_\theta(\mathbf{x})}\|_{L^{p'}(\mathbb{Q}_p^n)}\\
=&\frac{2}{\|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}\|(b - \frac{p^2}{2(p-1)}M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})}))(\chi_{B_\theta(\mathbf{x})})\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\\
\leq& C.
\end{aligned} \tag{2.8}
$$

This shows that $b \in \widetilde{\Lambda}_\delta(\mathbb{Q}_p^n)$. This, along with Lemma 2.1, shows $b \in \Lambda_\delta(\mathbb{Q}_p^n)$.

The final task is to show that $b \geq 0$. For this its suffices to show $b^- = 0$, where $b^- = \min\{b, 0\}$ and $b^+ = |b| - b^-$. Consider a p-adic ball $B_\theta(\mathbf{x})$. Using (2.6), we observe that

$$\frac{p^2}{2(p-1)} M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) - b(\mathbf{t}) \geq |b_{B_\theta(\mathbf{x})}| - b^+(\mathbf{t}) + b^-(\mathbf{t}),$$

for any $\mathbf{t} \in B_\theta(\mathbf{x})$.

Now averaging on a ball $B_\theta(\mathbf{x})$, we deduce that

$$\begin{aligned}
&\frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} \left|\frac{p^2}{2(p-1)} M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) - b(\mathbf{t})\right| d\mathbf{t} \\
\geq &\frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} (|b_{B_\theta(\mathbf{x})}| - b^+(\mathbf{t}) + b^-(\mathbf{t})) d\mathbf{t} \\
= &|b_{B_\theta(\mathbf{x})}| - \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} b^-(\mathbf{t}) d\mathbf{t} + \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} b^-(\mathbf{t}) d\mathbf{t}.
\end{aligned} \tag{2.9}$$

On the other hand, from (2.8), we have

$$\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}} \int_{B_\theta(\mathbf{x})} \left|\frac{p^2}{2(p-1)} M^{p,\sharp}(b\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) - b(\mathbf{t})\right| d\mathbf{t} \leq C. \tag{2.10}$$

From this and (2.9), we get

$$\begin{aligned}
&\left(|b_{B_\theta(\mathbf{x})}| - \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} b^+(\mathbf{t}) d\mathbf{t} + \frac{1}{|B_\theta(\mathbf{x})|_h} \int_{B_\theta(\mathbf{x})} b^-(\mathbf{t}) d\mathbf{t}\right) \\
&\leq C|B_\theta(\mathbf{x})|_h^{\delta/n}.
\end{aligned} \tag{2.11}$$

By letting $\theta \to \infty$ with $t \in B_\theta(\mathbf{x})$, the Lebesgue differentiation theorem in the p-adic field ensures that

$$0 = |b_{B_\theta(\mathbf{x})}| - b^+(\mathbf{t}) + b^-(\mathbf{t}) = 2b^-(\mathbf{t}) = 2|b^-(\mathbf{t})|.$$

Consequently, $b^- = 0$, and hence $b \geq 0$ holds true, which complete the proof of theorem.

Proof of Theorem 2.2. Suppose $b \in \Lambda_\delta(\mathbb{Q}_p^n)$. We show that $M_{\alpha,b} : L^q(\mathbb{Q}_p^n) \to L^p(\mathbb{Q}_p^n)$. From the definition of (1.2) and Eq (1.4), we deduce

$$\begin{aligned}
|M_{\alpha,b}(f)(\mathbf{x})| = &\sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{t})||f(\mathbf{t})| d\mathbf{t} \\
\leq &\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} \sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |\mathbf{t} - \mathbf{x}|_p^{\delta(\mathbf{x})} |f(\mathbf{t})| d\mathbf{t} \\
\leq &\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} \sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha+\delta(\mathbf{x})}{n}}} \int_{B_\theta(\mathbf{x})} |f(\mathbf{t})| d\mathbf{t} \\
\leq &\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} M_{\alpha+\delta x}(f)(\mathbf{x}).
\end{aligned}$$

By this and boundedness of $M_{\alpha+\delta}$ from $L^{q,\lambda}(\mathbb{Q}_p^n)$ to $L^{p,\lambda}(\mathbb{Q}_p^n)$ (see Remark 2.1), we eventually have

$$\|M_{\alpha,b}(f)\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} \|M_{\alpha+\delta}(f)\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)} \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Conversely, suppose that $[M_{\alpha,b}] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$, we show that $b \in \Lambda_\delta(\mathbb{Q}_p^n)$. For this, consider a p-adic ball $B_\theta(\mathbf{x})$, we are down to

$$|(b(\mathbf{t}) - b_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}(\mathbf{t})| \leq |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,b}(\chi_{B_\theta(\mathbf{x})})(\mathbf{t}). \tag{2.12}$$

From (2.12) and that $[M_{\alpha,b}] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$, we obtain

$$\|(b(\mathbf{t}) - b_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,b}(\chi_{B_\theta(\mathbf{x})})\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq C|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} \|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)},$$

which implies that

$$\frac{\|(b(\mathbf{t}) - b_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}. \tag{2.13}$$

Since $\frac{1}{p} = \frac{1}{q} - \frac{\delta+\alpha}{n}$, making use of Hölder's inequality, Lemma 2.2, and (2.13), we have

$$\begin{aligned}
&\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| d\mathbf{t} \\
\leq& \frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n-\lambda/np}} \left(\frac{1}{|B_\theta(\mathbf{x})|_h^{\lambda/n}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}|^p d\mathbf{t} \right)^{1/p} \\
&\times \left(\int_{B_\theta(\mathbf{x})} \chi_{B_\theta(\mathbf{x})}(\mathbf{t}) d\mathbf{t} \right)^{1/p'} \\
\leq& \frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n-\lambda/np}} \|(b - b_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \|\chi_{B_\theta(\mathbf{x})}\|_{L^{p'}(\mathbb{Q}_p^n)} \\
=& \frac{1}{\|\chi_{B_\theta(\mathbf{x})}\|_{L^q(\mathbb{Q}_p^n)}} \|(b - b_{B_\theta(\mathbf{x})})\chi_{B_\theta(\mathbf{x})}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} |B_\theta(\mathbf{x})|_h^{\frac{\alpha}{n}} \\
\leq& C.
\end{aligned} \tag{2.14}$$

This shows that $b \in \tilde{\Lambda}_\delta(\mathbb{Q}_p^n)$. From this and Lemma 2.1, we have $b \in \Lambda_\delta(\mathbb{Q}_p^n)$, which finishes the proof.

Before proving Theorem 2.3, we define the p-adic fractional maximal operator $M^p_{B_\theta(\mathbf{x})}$ with respect to a p-adic ball as follows:

$$M^p_{B_\theta(\mathbf{x})}(f)(\mathbf{t}) = \sup_{B_{\theta_0}(\mathbf{t}) \subseteq B_\theta(\mathbf{x})} \frac{1}{|B_{\theta_0}(\mathbf{t})|_h^{1-\frac{\theta}{n}}} \int_{B_{\theta_0}(\mathbf{t})} |f(\mathbf{t})| d\mathbf{t}, \quad \theta \geq 0,$$

where supremum is taken over all balls $B_{\theta_0}(\mathbf{t})$ such that $B_{\theta_0}(\mathbf{t}) \subseteq B_\theta(\mathbf{x})$.

Proof of Theorem 2.3. Assume that $b \in \Lambda_\delta(\mathbb{Q}_p^n)$ and $b \geq 0$. We show that $[b, M_\alpha] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$. Let $f \in L^{q,\lambda}(\mathbb{Q}_p^n)$. From definitions of (1.2), we reach at

$$\begin{aligned}
|[b, M_\alpha](f)(\mathbf{x})| =& \left| \sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} b(\mathbf{x})|f(\mathbf{t})| d\mathbf{t} - \sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{t}) f(\mathbf{t})| d\mathbf{t} \right| \\
\leq& \sup_{\theta \in Z} \frac{1}{|B_\theta(\mathbf{x})|_h^{1-\frac{\alpha}{n}}} \int_{B_\theta(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{t})||f(\mathbf{t})| d\mathbf{t} \\
=& M^p_{\alpha,b}(f)(\mathbf{x}).
\end{aligned} \tag{2.15}$$

From (2.15) and Theorem 2.2, we acquire

$$\|[b, M_\alpha^p](f)\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq \|M_{\alpha,b}^p(f)\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Consequently, $[b, M_\alpha^p] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$.

Conversely, suppose $[b, M_\alpha^p] : L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{p,\lambda}(\mathbb{Q}_p^n)$. We need to show that

$$b \in \Lambda_\delta(\mathbb{Q}_p^n) \quad and \quad b \geq 0. \tag{2.16}$$

First, we opt for the former one, and in order to do so, we need the following preparation:

Consider a p-adic ball $B_\theta(\mathbf{x})$. For all $\mathbf{t} \in B_\theta(\mathbf{x})$, we have

$$M_\alpha(\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) = M_{\alpha,B_\theta(\mathbf{x})}(\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) = |B_\theta(\mathbf{x})|_h^{\frac{\alpha}{n}}$$

and

$$M_\alpha(b_{\chi_{B_\theta(\mathbf{x})}})(\mathbf{t}) = M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t}).$$

Then, from this and (1.6), we have

$$\begin{aligned} b(\mathbf{t}) - |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t}) =& |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} [b(\mathbf{t})|B_\theta(\mathbf{x})|_h^{\frac{\alpha}{n}} - M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})] \\ =& |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} [b(\mathbf{t})M_\alpha(\chi_{B_\theta(\mathbf{x})})(\mathbf{t}) - M_\alpha(b_{\chi_{B_\theta(\mathbf{x})}})(\mathbf{t})] \\ =& |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_{B_\theta(\mathbf{x})})(\mathbf{t}). \end{aligned}$$

which implies that

$$\left(b(\mathbf{t}) - |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})\right)\chi_{B_\theta(\mathbf{x})}(\mathbf{t}) = |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_{B_\theta(\mathbf{x})})(\mathbf{t})\chi_{B_\theta(\mathbf{x})}(\mathbf{t}). \tag{2.17}$$

From (2.17) and the boundedness of $[b, M_\alpha^p]$, we obtain

$$\left\|(b - |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b))(\chi_{B_\theta(\mathbf{x})})\right\|_{L^{p,\lambda}(\mathbb{Q}_p^n)} \leq C|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} \|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)},$$

which implies that

$$\frac{\left\|(b - |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b))(\chi_{B_\theta(\mathbf{x})})\right\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\theta(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \leq C|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}. \tag{2.18}$$

Furthermore, consider a p-adic ball $B_\theta(\mathbf{x})$, suppose $A = \{\mathbf{t} \in B_\theta(\mathbf{x}) : b(\mathbf{t}) \leq B_\theta(\mathbf{x})\}$. Now, for any $\mathbf{t} \in A$, we have

$$b(\mathbf{t}) \leq b_{B_\theta(\mathbf{x})} \leq |b_{B_\theta(\mathbf{x})}| \leq |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t}).$$

Thus,

$$|b(\mathbf{t}) - b_{B_\theta(\mathbf{x})}| \leq \left|b(\mathbf{t}) - |B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})\right|. \tag{2.19}$$

Since $\frac{1}{p}=\frac{1}{q}-\frac{\delta+\alpha}{n}$, from (2.19), Hölder's inequality, Lemma 2.2, and (2.18), we sum up that

$$\begin{aligned}
&\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}|b(\mathbf{t})-b_{B_\theta(\mathbf{x})}|d\mathbf{t}\\
=&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{A}|b(\mathbf{t})-b_{B_\theta(\mathbf{x})}|d\mathbf{t}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}\left|b(\mathbf{t})-|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})|\chi_{B_\theta(\mathbf{x})}(\mathbf{t})\right|d\mathbf{t}\\
\leq&\frac{2}{|B_\theta(\mathbf{x})|_h^{1+\delta/n-\lambda/np}}\|(b-|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b))\chi_{B_\theta}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}\|\chi_{B_\theta}\|_{L^{p'}(\mathbb{Q}_p^n)}\\
\leq&\frac{C}{\|\chi_{B_\theta(\mathbf{x})}\|_{L^q(\mathbb{Q}_p^n)}}\|(b-|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b))\chi_{B_\theta}\|_{L^{p,\lambda}(\mathbb{Q}_p^n)}|B_\theta(\mathbf{x})|^{\frac{\alpha}{n}}\\
\leq&C.
\end{aligned}\tag{2.20}$$

which implies that $b\in\widetilde{\Lambda}_\delta(\mathbb{Q}_p^n)$, so, it follows from this and Lemma 2.1 that $b\in\Lambda_\delta(\mathbb{Q}_p^n)$.

Next, we show the latter one in equation (2.16). For this, its suffices to show $b^-=0$, where $b^-=\min\{b,0\}$ and $b^+=|b|-b^-$. Consider any fixed p-adic ball $B_\theta(\mathbf{x})$ and for any $\mathbf{t}\in B_\theta(\mathbf{x})$, we have

$$0\leq b^+(\mathbf{t})\leq|b(\mathbf{x})|\leq B_\theta(\mathbf{x})|^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t}).$$

Therefore, for $\mathbf{t}\in B_\theta(\mathbf{x})$, we obtain

$$0\leq b^-(\mathbf{t})\leq|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})-b^+(\mathbf{t})+b^-(\mathbf{t})=|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})-b(\mathbf{t}).$$

Then, by this and (2.20), we have

$$\begin{aligned}
&\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}b^-(\mathbf{t})d\mathbf{t}\\
\leq&\frac{1}{|B_\theta(\mathbf{x})|_h^{1+\delta/n}}\int_{B_\theta(\mathbf{x})}|B_\theta(\mathbf{x})|_h^{-\frac{\alpha}{n}}M_{\alpha,B_\theta(\mathbf{x})}(b)(\mathbf{t})-b(\mathbf{t})\\
\leq&C.
\end{aligned}$$

Therefore,

$$\frac{1}{|B_\theta(\mathbf{x})|_h}\int_{B_\theta(\mathbf{x})}b^-(\mathbf{t})d\mathbf{t}\leq C|B_\theta(\mathbf{x})|_h^{\delta/n}.$$

By letting $\theta\to\infty$ together with the Lebesgue differentiation theorem in p-adic field, we have $b^-=0$. Hence $b\geq0$, which finishes the proof of theorem.

3. Conclusions

Necessary and sufficient conditions for the boundedness of commutators of p-adic sharp maximal functions, p-adic fractional maximal functions, and p-adic fractional maximal commutators on p-adic Morrey spaces are studied by considering the symbol function as a Lipschitz spaces. Wavelet characterization of p-adic Lebesgue spaces can be obtained as a future prospect.

Author contributions

Naqash Sarfraz: Conceptualization, data curation, investigation, methodology, writing-original draft; Muhammad Bilal Riaz: Formal analysis, methodology, writing-original draft; project management, funding acquisition, supervision; Qasim Ali Malik: Validation, visualization, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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