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*Research article*

# New refinements of Becker-Stark inequality

**Suxia Wang**[1] **and Tiehong Zhao**[2,*]

[1] School of Finance and Mathematics, Huainan Normal University, Huainan 232038, China
[2] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China

* **Correspondence:** Email: tiehong.zhao@hznu.edu.cn.

**Abstract:** This paper deals with the well-known Becker-Stark inequality. By using variable replacement from the viewpoint of hypergeometric functions, we provide a new and general refinement of Becker-Stark inequality. As a particular case, the double inequality

$$\frac{\pi^2-(\pi^2-8)\sin^2 x}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2-(4-\pi^2/3)\sin^2 x}{\pi^2-4x^2}$$

for $x\in(0,\pi/2)$ will be established. The importance of our result is not only to provide some refinements preserving the structure of Becker-Stark inequality but also that the method can be extended to the case of generalized trigonometric functions.

**Keywords:** Becker-Stark inequality; power series; Gaussian hypergeometric function; monotonicity
**Mathematics Subject Classification:** 26D05, 26D15, 33B10

## 1. Introduction

It is known in the literature that, for $x\in(0,\pi/2)$, the inequality

$$\frac{8}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2}{\pi^2-4x^2} \tag{1.1}$$

was first established by Becker and Stark [6]. This is always known as Becker-Stark inequality, which has attracted much interest many researchers and has been generalized in many different ways; see [7, 8, 10, 13, 17, 26–28] and the references therein. The importance of Becker-Stark inequality is to find the bounds for $\tan x/x$, which are the rational functions with the same order of infinity near $\pi/2$. In particular, the first of the notable refinements is given by Zhu [27, Theorem 1.3], who proved that, for $x\in(0,\pi/2)$,

$$\frac{\pi^2-\frac{4(\pi^2-8)}{\pi^2}x^2}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2-\left(\frac{\pi^2}{3}-4\right)x^2}{\pi^2-4x^2}. \tag{1.2}$$

As a matter of fact, Zhu [27, Theorem 1.4] gives a general refinement of the Becker-Stark inequality. In view of

$$\lim_{x\to\frac{\pi}{2}^-}\left(\frac{\tan x}{x}-\frac{8}{\pi^2-4x^2}\right)=\frac{2}{\pi^2},$$

the left-hand side of (1.1) becomes a good approximate of $\tan x/x$ near $\pi/2$. Motivated by this remark, Zhu [28, Theorem 3] gives a refinement of (1.1), for $x\in(0,\pi/2)$,

$$\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\frac{\pi^2-9}{6\pi^4}(\pi^2-4x^2)<\frac{\tan x}{x}<\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\frac{10-\pi^2}{\pi^4}(\pi^2-4x^2),$$

where $-(\pi^2-9)/(6\pi^4)$ and $-(10-\pi^2)/\pi^4$ are the best constants. Further, Debnath et al. [13] present two estimates of $\tan x/x$ near $\pi/2$ but not in the whole interval $(0,\pi/2)$; more precisely, the following inequalities hold true

$$\frac{8+\frac{8}{\pi}\left(\frac{\pi}{2}-x\right)+\left(\frac{16}{\pi^2}-\frac{8}{3}\right)\left(\frac{\pi}{2}-x\right)^2}{\pi^2-4x^2}<\frac{\tan x}{x}$$

for $x\in(0.373,\pi/2)$ and

$$\frac{\tan x}{x}<\frac{8+\frac{8}{\pi}\left(\frac{\pi}{2}-x\right)+\left(\frac{16}{\pi^2}-\frac{8}{3}\right)\left(\frac{\pi}{2}-x\right)^2+\left(\frac{32}{\pi^3}-\frac{8}{3\pi}\right)\left(\frac{\pi}{2}-x\right)^3}{\pi^2-4x^2}$$

for $x\in(0,301,\pi/2)$. Recently, alternative good improvements can be found in [10, Equation (2.11)] and [4, Theorem 2.1], where they establish the inequalities

$$\frac{\pi^2+\frac{\pi^2-12}{3}x^2+\frac{384-4\pi^4}{3\pi^4}x^4}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2+\frac{72-8\pi^2}{\pi^2}x^2+\frac{16\pi^2-160}{\pi^4}x^4}{\pi^2-4x^2}$$

and

$$\sqrt{1+\frac{128}{\pi^4}\frac{x^2(5\pi^2-12x^2)}{(\pi^2-4x^2)^2}}<\frac{\tan x}{x}<\sqrt{1+\frac{2\pi^2}{15}\frac{x^2(5\pi^2-12x^2)}{(\pi^2-4x^2)^2}}$$

for $x\in(0,\pi/2)$, where the second inequality had been improved by Zhu [29] to the following inequality

$$\sqrt{1+\frac{(240-17\pi^2)\pi^2}{45}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)}}<\frac{\tan x}{x}<\sqrt{1+\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)}}$$

for $x\in(0,\pi/2)$ with the best constants $\frac{(240-17\pi^2)\pi^2}{45}$ and $\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}$. It is observed that all the above improvements keep the structure of the Becker-Stark inequality, that is to say, the denominator of their approximate functions is $\pi^2-4x^2$.

Very recently, Wu and Bercu [18] approximated $\tan x/x$ by utilizing the cosine polynomials due to the property of even function, and established the inequalities

$$1+\frac{(1-\cos x)(604\cos^2 x-1817\cos x+1843)}{945}<\frac{\tan x}{x}<1+\frac{(1-\cos x)(31\cos x-5\cos^2 x+604)}{945\cos x} \tag{1.3}$$

for $x\in(0,\pi/2)$. Clearly, inequality (1.3) has broken the structure of the Becker-Stark inequality, which leads to the left-hand side of (1.3) being just a bounded function.

The main objective of this paper is to provide new lower and upper bounds for $\tan x/x$ whose forms preserve the structure of the Becker-Stark inequality and numerator is a polynomial of $\sin^2 x$. More precisely, we transform the function $(\pi^2 - 4x^2)\tan x/x$ into the ratio of two hypergeometric functions by changing a variable $t = \sin^2 x$ and use the first few terms of the series expansion to approximate the objective function. This method, as a practice toy, can be used to reprove the Becker-Stark inequality. The importance of our findings is not only illustrated by giving some new refinements of inequality (1.1), but also by the fact that the method can be extended to generalized trigonometric functions.

The rest of this paper is organized as follows: In this section, we give an introduction and highlight the relevant previous results. Section 2 consists of some basic knowledge and two lemmas, and is devoted to the proof of the main result. Diverse complements are offered in Section 3, including a comparison of the obtained bounds by graphical analysis, a conjecture raised from the main result, and a $p$-analogue of Becker-Stark inequality.

## 2. Main results and proofs

### *2.1. Preliminaries and lemmas*

In this section, we first introduce some basic knowledge and present two lemmas that are used to prove the main result.

**Definition 2.1.** For real numbers $a, b$, and $c$ with $-c \notin \mathbb{N} \cup \{0\}$, the *Gaussian hypergeometric function* is defined as

$$F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

for $x \in (-1, 1)$, where $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol or the shifted factorial function for $n \in \mathbb{N}$. In particular, $(a)_0 = 1$ for $a \neq 0$. Here $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the classical Euler gamma function [21, 23].

Recall that the hypergeometric function $F(a, b; c; x)$ has the following properties:

**Property 2.1.** A simple derivative formula

$$\frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x).$$

**Property 2.2.** The behavior of hypergeometric function $F(a, b; c; x)$ near $x = 1$ satisfies the following situations:

◇ $c > a + b$ (cf. [16, p. 49])

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.1}$$

◇ $c = a + b$ (cf. [1, 15.3.10]), the Ramanujan's asymptotic formula

$$B(a, b) F(a, b; c; x) + \log(1-x) = R(a, b) + O\left[(1-x)\log(1-x)\right], \quad (x \to 1). \tag{2.2}$$

$\diamond$ $c < a + b$ (cf. [15, (1.2)]), as $x \to 1$,

$$F(a, b; c; x) = (1 - x)^{c-a-b} F(c - a, c - b; c; x) = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - x)^{c-a-b} [1 + o(1)], \quad (2.3)$$

where $B(a, b) = [\Gamma(a)\Gamma(b)]/\Gamma(a + b)$, $R(a, b) = -2\gamma - \psi(a) - \psi(b)$, $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\gamma$ is the beta function, the Ramanujan constant, the psi function, and the Euler-Mascheroni constant.

In a particular case of $a, b, c$, the inverse trigonometric tangent function can be represented by hypergeometric function.

**Property 2.3.** (see [1, 15.1.5])

$$\arctan(x) = xF\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right) = \frac{x}{\sqrt{1 + x^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{1 + x^2}\right). \quad (2.4)$$

**Remark 2.1.** The second equality of (2.4) can be obtained from the transformation formula $F(a, b; c; z) = (1 - z)^{-a} F(a, c - b; c; z/(z - 1))$ (c.f. [1, 15.3.4]), and also coincides with the case of $p = 2$ in [3, Lemma 1].

**Property 2.4.** (see [31, (3.6)]) An identity

$$(1 - x)F(a, 1; c; x) = 1 - \frac{(c - a)x}{c} F(a, 1; c + 1; x). \quad (2.5)$$

As is known, a real function $\varphi$ is said to be absolutely monotonic on the interval $I$ if the $k$th derivative of $\varphi$, denoted by $\varphi^{(k)}(x)$, exists and is non-negative for each $k \geq 0$ and $x \in I$. In other words, if $\varphi$ can be expressed as a power series on $I$, then all coefficients are non-negative. In particular, a special power series, roughly speaking, whose coefficients are first negative and then positive is said to be a negative-positive type series, of which the name was first proposed formally in [25] although this type of special series has been studied extensively in the literature [11, 22, 30].

**Definition 2.2.** A power series $S(x)$ given by

$$S(x) = -\sum_{k=0}^{m} a_k x^k + \sum_{k=m+1}^{\infty} a_k x^k$$

is called a "Negative-Positive type" (or "NP type" for short) power series, if its coefficients $a_k$ for $k \geq 0$ satisfy

(i) $a_k \geq 0$ for all $k \geq 0$;

(ii) There exist at least two integers $0 \leq k_1 \leq m$ and $k_2 \geq m + 1$ such that $a_{k_1}, a_{k_2} \neq 0$.

Correspondingly, $S(x)$ is called a "Positive-Negative type" (or "PN type" for short) power series if $-S(x)$ is a Negative-Positive type power series.

The following lemma is a simple and efficient tool to determine the sign of an NP (or PN) type power series, which has been proved in [22, 24].

**Lemma 2.1.** *Let $S(x)$ be a Negative-Positive type power series converging on the interval $(0, R)$. Then* *(i) if* $S(R^-) \le 0$*, then* $S(x) < 0$ *for all* $x \in (0, R)$*;*

*(ii) if* $S(R^-) > 0$*, then there is a unique* $\tilde{x} \in (0, R)$ *such that* $S(x) < 0$ *for* $x \in (0, \tilde{x})$ *and* $S(x) > 0$ *for* $x \in (\tilde{x}, R)$*.*

*As a consequence, for a PN-type power series, the inequalities of* $(i)$ *and* $(ii)$ *are reversed.*

We provide a power series expansion of $[F(a, b; a + b + 1/2)]^2$ in the following lemma, which has been demonstrated in [19, Example 14.11] (see also [9]).

**Lemma 2.2.** *For* $c = a + b + 1/2$*, it holds that*

$$\left[F(a, b; c; x)\right]^2 = \frac{\Gamma(c)\Gamma(2c-1)}{\Gamma(2a)\Gamma(2b)\Gamma(a+b)} \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)\Gamma(a+b+n)\Gamma(2b+n)}{n!\Gamma(c+n)\Gamma(2c-1+n)} x^n.$$

*In particular, we have*

$$\left[F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x\right)\right]^2 = \sum_{n=0}^{\infty} \frac{n!}{(n+1)(3/2)_n} x^n. \tag{2.6}$$

*2.2. Statement of Theorem 2.1*

Let $t = \sin^2 x$ for $x \in (0, \pi/2)$, and then $t \in (0, 1)$. This gives $\tan^2 x = t/(1-t)$, which by (2.4) is equivalent to

$$x = \arctan \sqrt{\frac{t}{1-t}} = \sqrt{t} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; t\right). \tag{2.7}$$

By (2.7), it can be rewritten as

$$\frac{(\pi^2 - 4x^2)\tan x}{x} = \frac{\left[\pi^2 - 4tF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; t\right)^2\right]\left(\frac{t}{1-t}\right)^{1/2}}{[t(1-t)]^{1/2} F\left(1, 1; \frac{3}{2}; t\right)} \triangleq \frac{f(t)}{g(t)}, \tag{2.8}$$

where

$$f(t) = \pi^2 - 4tF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; t\right)^2 = \pi^2 - 4\sum_{n=0}^{\infty} u_n t^{n+1}, \tag{2.9}$$

$$g(t) = (1-t)F\left(1, 1; \frac{3}{2}; t\right) = 1 - \frac{t}{3} F\left(1, 1; \frac{5}{2}; t\right) = 1 - \frac{1}{3}\sum_{n=0}^{\infty} v_n t^{n+1} \tag{2.10}$$

by (2.5) and (2.6). Here, $u_n$ and $v_n$ are given by

$$u_n = \frac{n!}{(n+1)(3/2)_n} \quad \text{and} \quad v_n = \frac{n!}{(5/2)_n}.$$

Moreover, by (2.1), we have

$$\lim_{t \to 1^-} \frac{f(t)}{g(t)} = \lim_{t \to 1^-} \frac{f'(t)}{g'(t)} = \lim_{t \to 1^-} \frac{4F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; t\right) F\left(1, 1; \frac{5}{2}; t\right)}{3F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, t\right) - 2F\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; t\right)} = 8. \tag{2.11}$$

Suppose that $\alpha_n$ is the Maclaurin's coefficients of $\pi^2 - f(t)/g(t)$, that is,

$$\pi^2 - \frac{f(t)}{g(t)} = \sum_{n=1}^{\infty} \alpha_n t^n,$$

then it follows from (2.9) and (2.10) that

$$\pi^2 - 4\sum_{n=1}^{\infty} u_{n-1} t^n = \left(\pi^2 - \sum_{n=1}^{\infty} \alpha_n t^n\right)\left(1 - \frac{1}{3}\sum_{n=1}^{\infty} v_{n-1} t^n\right),$$

which deduces $\alpha_1 = 4 - \pi^2/3$ and the recurrence relation

$$3\alpha_n = -\left(\pi^2 - 8 - \frac{4}{n}\right) v_{n-1} + \sum_{k=1}^{n-1} \alpha_k v_{n-k-1}, \qquad (n \geq 2). \tag{2.12}$$

Before stating Theorem 2.1, we can compute a finite number of $\alpha_n$ by (2.12), which are listed numerically in Table 1. Table 1 illustrates that $\alpha_n > 0$ for $1 \leq n \leq 30$. Although we only know a finite $\alpha_n > 0$, it still encourages us to prove the following theorem. These evidence demonstrate that Theorem 2.1 is valid in the case of $2 \leq N \leq 28$.

**Table 1.** The values of $\alpha_n$ with 2-digit precision.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_n$ | 0.71 | 0.25 | 0.14 | 0.090 | 0.064 | 0.049 | 0.039 | 0.032 | 0.027 | 0.023 |
| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $\alpha_n$ | 0.020 | 0.017 | 0.015 | 0.014 | 0.012 | 0.011 | 0.010 | 0.0094 | 0.0087 | 0.0080 |
| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $\alpha_n$ | 0.0075 | 0.0070 | 0.0065 | 0.0061 | 0.0058 | 0.0054 | 0.0051 | 0.0049 | 0.0046 | 0.0044 |

**Theorem 2.1.** *Let $\alpha_n$ be defined as in* (2.12). *If there exists an integer $N \geq 2$ such that $\alpha_n > 0$ for $1 \leq n \leq N + 2$, then the double inequality*

$$\frac{\pi^2 - \sum_{n=1}^{N-1} \alpha_n \sin^{2n} x - \widetilde{\alpha}_N \sin^{2N} x}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 - \sum_{n=1}^{N} \alpha_n \sin^{2n} x}{\pi^2 - 4x^2} \tag{2.13}$$

*holds for all $x \in (0, \pi/2)$ with the best constants $\alpha_N$ and $\widetilde{\alpha}_N$, where*

$$\widetilde{\alpha}_N = \pi^2 - 8 - \sum_{n=1}^{N-1} \alpha_n.$$

*2.3. Proofs*

***Proof of Theorem 2.1.*** In order to obtain inequality (2.13), it suffices to show that

$$\left(\pi^2 - \sum_{n=1}^{N} \alpha_n \sin^{2n} x\right) - \frac{(\pi^2 - 4x)\tan x}{x} > 0,$$

$$\left(\pi^2 - \sum_{n=1}^{N-1} \alpha_n \sin^{2n} x - \widetilde{\alpha}_N \sin^{2N} x\right) - \frac{(\pi^2 - 4x)\tan x}{x} < 0$$

for $x \in (0, \pi/2)$, which by (2.7) and (2.8) is equivalent to

$$\phi_1(t) := \left(\pi^2 - \sum_{n=1}^{N} \alpha_n t^n\right) g(t) - f(t) > 0, \tag{2.14}$$

$$\phi_2(t) := \left(\pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \widetilde{\alpha}_N t^N\right) g(t) - f(t) < 0 \tag{2.15}$$

for $t \in (0, 1)$.

In terms of power series, by (2.9) and (2.10), we can rewrite $\phi_1(t)$ and $\phi_2(t)$ as

$$\phi_1(t) = \left(\pi^2 - \sum_{n=1}^{N} \alpha_n t^n\right)\left(1 - \frac{1}{3}\sum_{n=0}^{\infty} v_n t^{n+1}\right) - \left(\pi^2 - 4\sum_{n=0}^{\infty} u_n t^{n+1}\right) = \sum_{n=N+1}^{\infty} \tau_n t^n, \tag{2.16}$$

$$\begin{aligned}\phi_2(t) &= \left(\pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \widetilde{\alpha}_N t^N\right)\left(1 - \frac{1}{3}\sum_{n=0}^{\infty} v_n t^{n+1}\right) - \left(\pi^2 - 4\sum_{n=0}^{\infty} u_n t^{n+1}\right) \\ &= (\widetilde{\tau}_N - \widetilde{\alpha}_N) t^N + \sum_{n=N+1}^{\infty}\left(\widetilde{\tau}_n + \frac{1}{3}\widetilde{\alpha}_N v_{n-N-1}\right) t^n, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned}\tau_n &= 4u_{n-1} - \frac{1}{3}\left(\pi^2 v_{n-1} - \sum_{k=1}^{N} \alpha_k v_{n-k-1}\right), \\ \widetilde{\tau}_n &= 4u_{n-1} - \frac{1}{3}\left(\pi^2 v_{n-1} - \sum_{k=1}^{N-1} \alpha_k v_{n-k-1}\right).\end{aligned}$$

(i) To prove $\phi_1(t) > 0$ for $t \in (0, 1)$.

We first assert that *if* $\tau_n \le 0$ *for* $n \ge N + 1$*, then* $\tau_{n+1} < 0$. To confirm this, if $\tau_n \le 0$ for $n \ge N + 1$, that is,

$$u_{n-1} \le \frac{1}{12}\left(\pi^2 v_{n-1} - \sum_{k=1}^{N} \alpha_k v_{n-k-1}\right), \tag{2.18}$$

then we deduce by (2.18) that

$$\begin{aligned}\tau_{n+1} &= 4u_n - \frac{1}{3}\left(\pi^2 v_n - \sum_{k=1}^{N} \alpha_k v_{n-k}\right) \\ &= 4\frac{u_n}{u_{n-1}} u_{n-1} - \frac{1}{3}\left(\pi^2 \frac{v_n}{v_{n-1}} v_{n-1} - \sum_{k=1}^{N} \alpha_k \frac{v_{n-k}}{v_{n-k-1}} v_{n-k-1}\right) \\ &< \frac{1}{3}\left[\pi^2\left(\frac{u_n}{u_{n-1}} - \frac{v_n}{v_{n-1}}\right) v_{n-1} + \sum_{k=1}^{N} \alpha_k \left(\frac{v_{n-k}}{v_{n-k-1}} - \frac{u_n}{u_{n-1}}\right) v_{n-k-1}\right] < 0\end{aligned}$$

for $n \geq N+1$, where the last inequality follows from

$$\begin{aligned}
\frac{u_n}{u_{n-1}} - \frac{v_n}{v_{n-1}} &= -\frac{2n}{3+11n+12n^2+4n^3} < 0, \\
\frac{v_{n-k}}{v_{n-k-1}} - \frac{u_n}{u_{n-1}} &= \frac{1+3n}{(1+n)(1+2n)} - \frac{3}{3-2k+2n} \\
&\leq \frac{1+3n}{(1+n)(1+2n)} - \frac{3}{3-2+2n} = -\frac{2}{(1+n)(1+2n)} < 0
\end{aligned}$$

for $1 \leq k \leq N$. This confirms the truth of the assertion.

We now complete the proof in the following two steps:

Step 1: We prove $\tau_{N+1} > 0$. Otherwise, we see from the above assertion that $\tau_n < 0$ for $n \geq N+2$. This, together with (2.16) implies that $\phi_1(t) < 0$ for $t \in (0,1)$. On the other hand, it follows from (2.14) and $\alpha_{N+1} > 0$ that

$$\frac{\phi_1(t)}{g(t)} = \pi^2 - \sum_{n=1}^{N} \alpha_n t^n - \frac{f(t)}{g(t)} = \sum_{n=N+1}^{\infty} \alpha_n t^n > 0$$

for $t \in (0, \epsilon_1)$ with a sufficiently small $\epsilon_1 > 0$, which is a contradiction.

Step 2: There are only two situations:

(a) If all $\tau_n > 0$ for $n \geq N+1$, then $\phi_1(t) > 0$ for $t \in (0,1)$ by (2.16).

(b) If there exists an integer $m \geq N+2$ such that $\tau_m \leq 0$, we may assume that $\tau_m$ is the first non-positive term. Then the above assertion tells us that $\tau_n > 0$ for $N+1 \leq n \leq m-1$ and $\tau_n \leq 0$ for $n \geq m$. That is to say, $\phi_1(t)$ is a PN-type power series on $(0,1)$. Combining this with Lemma 2.1 and $\phi_1(1^-) = 0$, it follows that $\phi_1(t) > 0$ for $t \in (0,1)$.

(ii) To prove $\phi_2(t) < 0$ for $t \in (0,1)$.

Due to $\alpha_{N+2} > 0$, by repeating the above steps, it can also be shown that

$$\pi^2 - \sum_{n=1}^{N+1} \alpha_n t^n > \frac{f(t)}{g(t)} \quad \Longrightarrow \quad \pi^2 - \sum_{n=1}^{N+1} \alpha_n \geq \lim_{t \to 1^-} \frac{f(t)}{g(t)} = 8$$

by (2.11), which gives $\widetilde{\alpha}_N \geq \alpha_N + \alpha_{N+1} > \alpha_N$. Observe that $v_n/v_{n-1} = 1 - 3/(3+2n) < 1$, that is to say, $v_n$ is strictly decreasing for $n \geq 0$. According to this, with $\widetilde{\alpha}_N > 0$ and $\alpha_k > 0$ $(1 \leq k \leq N+1)$, it follows that

$$\begin{aligned}
\widetilde{\tau}_n + \frac{1}{3}\widetilde{\alpha}_N v_{n-N-1} &= 4u_{n-1} - \frac{\pi^2}{3} v_{n-1} + \frac{1}{3}\sum_{k=1}^{N-1} \alpha_k v_{n-k-1} + \frac{1}{3}\widetilde{\alpha}_N v_{n-N-1} \\
&> \frac{1}{3}\left[\left(8 - \pi^2 + \frac{4}{n}\right) v_{n-1} + \left(\sum_{k=1}^{N-1} \alpha_k + \widetilde{\alpha}_N\right) v_{n-2}\right] \\
&= \frac{1}{3}\left[\frac{4}{n} v_{n-1} + (\pi^2 - 8)(v_{n-2} - v_{n-1})\right] > 0
\end{aligned}$$

for $n \geq N+1$. If $\widetilde{\tau}_N - \widetilde{\alpha}_N \geq 0$, then $\phi_2(t) > 0$ for $t \in (0,1)$ by (2.17). This, together with (2.15), implies that

$$\frac{\phi_2(t)}{g(t)} = \pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \widetilde{\alpha}_N t^N - \frac{f(t)}{g(t)} = (\alpha_N - \widetilde{\alpha}_N) t^N + \sum_{n=N+1}^{\infty} \alpha_n t^n < 0$$

for $t \in (0, \epsilon_2)$ with a sufficiently small $\epsilon_2 > 0$. This contradicts $\phi_2(t) > 0$ for $t \in (0,1)$ and thereby $\widetilde{\tau}_N - \widetilde{\alpha}_N < 0$. According to (2.17), we conclude that $\phi_2(t)$ is an NP-type power series on $(0,1)$ and so $\phi_2(t) < 0$ for $t \in (0,1)$ by Lemma 2.1 and $\phi_2(1^-) = 0$.

In this end, the optimality of constants follows from

$$\frac{1}{\sin^{2N} x}\left[\pi^2 - \sum_{n=1}^{N-1} \alpha_n \sin^{2n} x - \frac{(\pi^2 - 4x^2)\tan x}{x}\right] = \frac{1}{t^N}\left[\pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \frac{f(t)}{g(t)}\right]$$

and

$$\lim_{t \to 0^+} \frac{1}{t^N}\left[\pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \frac{f(t)}{g(t)}\right] = \alpha_N, \quad \lim_{t \to 1^-} \frac{1}{t^N}\left[\pi^2 - \sum_{n=1}^{N-1} \alpha_n t^n - \frac{f(t)}{g(t)}\right] = \widetilde{\alpha}_N.$$

This completes the proof of Theorem 2.1. □

**Remark 2.2.** It is worth pointing out that the numerator of (2.13) is just a $N$-order polynomial of $\sin^2 x$, but the condition of Theorem 2.1 still requires $\alpha_{N+1} > 0$ and $\alpha_{N+2} > 0$. This is mainly used to determine the sign of the first terms of the power series in (2.16) and (2.17). As a fact to remember, if a specific integer $N \geq 2$ is given, then it can be directly verified the sign of $\tau_{N+1}$ and $\widetilde{\tau}_N - \widetilde{\alpha}_N$ without the conditions that $\alpha_{N+1} > 0$ and $\alpha_{N+2} > 0$.

**Remark 2.3.** Inequality (2.13) can provide better bounds for larger $N$. First, our remark is obvious on the right side of (2.13). To see the left side, it suffices to verify from $\widetilde{\alpha}_{N+1} > 0$ that

$$\begin{aligned}
&\sum_{n=1}^{N-1} \alpha_n \sin^{2n} x + \widetilde{\alpha}_N \sin^{2N} x - \left(\sum_{n=1}^{N} \alpha_n \sin^{2n} x + \widetilde{\alpha}_N \sin^{2N+2} x\right) \\
&= -\alpha_N \sin^{2N} x + \widetilde{\alpha}_N \sin^{2N} x - \widetilde{\alpha}_{N+1} \sin^{2N+2} x \\
&= (-\alpha_N + \widetilde{\alpha}_N - \widetilde{\alpha}_{N+1}) \sin^{2N} x = 0.
\end{aligned}$$

**Remark 2.4.** It is worth noting that it can be seen from the left side of (2.13) that $N$ must be greater than or equal to 2. Now we can extend the range of $N$ to $N \geq 1$. Indeed, due to $\widetilde{\alpha}_{N+1} = \widetilde{\alpha}_N - \alpha_N$, we can rewrite as

$$\begin{aligned}
\pi^2 - \sum_{n=1}^{N-1} \alpha_n \sin^{2n} x - \widetilde{\alpha}_N \sin^{2N} x &= \pi^2 - \sum_{n=1}^{N-1} \alpha_n \sin^{2n} x - (\widetilde{\alpha}_{N+1} + \alpha_N) \sin^{2N} x \\
&= \pi^2 - \sum_{n=1}^{N} \alpha_n \sin^{2n} x - \widetilde{\alpha}_{N+1} \sin^{2N} x.
\end{aligned}$$

This, together with (2.13), gives

$$\frac{\pi^2 - \sum_{n=1}^{N} \alpha_n \sin^{2n} x - \widetilde{\alpha}_{N+1} \sin^{2N} x}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 - \sum_{n=1}^{N} \alpha_n \sin^{2n} x}{\pi^2 - 4x^2} \tag{2.19}$$

holds for all $x \in (0, \pi/2)$.

Taking $N = 1$ into (2.19), we obtain

**Corollary 2.1.** *For all $x \in (0, \pi/2)$, it holds*

$$\frac{\pi^2 - (\pi^2 - 8)\sin^2 x}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 - (4 - \pi^2/3)\sin^2 x}{\pi^2 - 4x^2} \tag{2.20}$$

*with the sharp constants $\pi^2 - 8$ and $4 - \pi^2/3$.*

*Proof.* The sharp constants follow from

$$\lim_{x\to 0^+} \frac{1}{\sin^2 x}\left[\pi^2 - \frac{(\pi^2 - 4x^2)\tan x}{x}\right] = \lim_{t\to 0^+} \frac{1}{t}\left[\pi^2 - \frac{f(t)}{g(t)}\right] = \alpha_1 = 4 - \frac{\pi^2}{3}$$

and

$$\lim_{x\to \frac{\pi}{2}-} \frac{1}{\sin^2 x}\left[\pi^2 - \frac{(\pi^2 - 4x^2)\tan x}{x}\right] = \lim_{t\to 1^-} \frac{1}{t}\left[\pi^2 - \frac{f(t)}{g(t)}\right] = \pi^2 - 8$$

by (2.11). □

Taking $N = 2$ into Theorem 2.1, we obtain

**Corollary 2.2.** *For all $x \in (0, \pi/2)$, it holds*

$$\frac{\pi^2 - (4 - \frac{\pi^2}{3})\sin^2 x - \frac{4(\pi^2-9)}{3}\sin^4 x}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 - (4 - \frac{\pi^2}{3})\sin^2 x - \frac{120-11\pi^2}{45}\sin^4 x}{\pi^2 - 4x^2}, \tag{2.21}$$

*where the constants $\frac{4(\pi^2-9)}{3}$ and $\frac{120-11\pi^2}{45}$ are sharp.*

**Remark 2.5.** Remark 2.3 enables us to know that the inequality (2.21) is better than inequality (2.20). Further, it is easy to see that inequality (2.20) is better than (1.1). In conclusion, inequality (2.13) completely improves the Becker-Stark inequality. As a matter of fact, Corollaries 2.1 and 2.2 can also be obtained through the method used in [5, 14].

## 3. Complements

In this section, we provide a graphical analysis of the obtained bounds, give a conjecture and propose a $p$-analogue of Becker-Stark inequality.

### 3.1. Graphical analysis

We now provide a graphical analysis of the lower bounds of Theorem 2.1 ($N = 5$) and (1.3) by distinguishing lower bounds.

By (2.12), we can compute the first few $\alpha_n$ as follows:

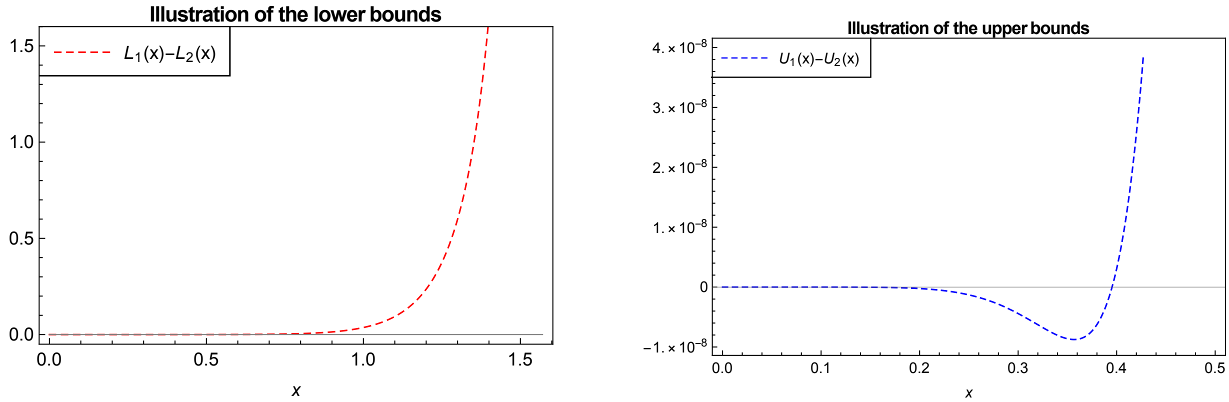
$$\alpha_1 = 4 - \frac{\pi^2}{3}, \quad \alpha_2 = \frac{8}{3} - \frac{11\pi^2}{45}, \quad \alpha_3 = \frac{32}{15} - \frac{191\pi^2}{945}, \quad \alpha_4 = \frac{64}{35} - \frac{2497\pi^2}{14175},$$

$$\alpha_5 = \frac{512}{315} - \frac{14797\pi^2}{93555}, \quad \widetilde{\alpha}_5 = \frac{3961\pi^2}{2025} - \frac{652}{35}.$$

We denote by $L_j(x)$ and $U_j(x)$ ($j = 1, 2$) the lower and upper bounds of Theorem 2.1 ($N = 5$) and (1.3), respectively, as follows:

$$\begin{aligned} L_1(x) &= \frac{\pi^2 - \sum_{n=1}^{4} \alpha_n \sin^{2n} x - \widetilde{\alpha}_5 \sin^{10} x}{\pi^2 - 4x^2}, \\ L_2(x) &= 1 + \frac{(1 - \cos x)(604 \cos^2 x - 1817 \cos x + 1843)}{945}, \\ U_1(x) &= \frac{\pi^2 - \sum_{n=1}^{5} \alpha_n \sin^{2n} x}{\pi^2 - 4x^2}, \\ U_2(x) &= 1 + \frac{(1 - \cos x)(31 \cos x - 5 \cos^2 x + 604)}{945 \cos x}. \end{aligned}$$

Figure 1 presents the graph of the functions $L_1(x)$ and $L_2(x)$ for $x \in (0, \pi/2)$. An immediate remark arising from Figure 1(a) is that the lower bound of Theorem 2.1 ($N = 5$) is better than (1.3). Figure 1(b) illustrates that the upper bound of Theorem 2.1 ($N = 5$) is better than the one of (1.3) near at $x = 0$.




(a) $x \in (0, \pi/2)$ (b) $x \in (0, 1/2)$

**Figure 1.** Plots of "the bounds of Theorem 2.1 ($N = 5$) and (1.3)".

### *3.2. A conjecture*

From Table 1, it can be seen that $\alpha_n > 0$ for $1 \le n \le 30$. This allows us to pose the following conjecture:

**Conjecture 3.1.** *Let $f(t)$ and $g(t)$ be defined as in* (2.9) *and* (2.10) *respectively. Then $\pi^2 - f(t)/g(t)$ is absolute monotonic on* $(0, 1)$.

**Remark 3.1.** If Conjecture 3.1 can be confirmed, then inequality (2.13) can be directly derived from Conjecture 3.1. However, the advantage of Theorem 2.1 is that we only need to know a finite number of $\alpha_n > 0$ to prove inequality (2.13) .

### *3.3. p-analogue of Becker-Stark inequality*

For $p > 1$, the generalized sine function $\sin_p$ is the eigenfunction of the one-dimensional $p$-Laplacian problem [12]

$$-\Delta_p u = -(|u'|^{p-2}u')' = \lambda |u|^{p-2}, \quad u(0) = u(1) = 0,$$

which is also the inverse function of $\arcsin_p : (0,1) \mapsto (0, \pi_p/2)$ defined as

$$\arcsin_p x = \int_0^x (1-t^p)^{-1/p} dt,$$

where

$$\pi_p = \int_0^1 (1-t^p)^{-1/p} dt = \frac{2\pi}{p \sin(\pi/p)}.$$

In this case, $\sin_p x$ is defined on the interval $[0, \pi_p/2]$ and can be extended to the whole $\mathbb{R}$ by symmetry and periodicity. Define $\cos_p : \mathbb{R} \mapsto \mathbb{R}$ by

$$\cos_p x := \frac{d}{dx} \sin_p x, \quad x \in \mathbb{R}.$$

In particular, it holds

$$\sin_p^p x + \cos_p^p x = 1, \quad x \in [0, \pi_p/2],$$

which leads to

$$\frac{d}{dx} \cos_p x = -\sin_p^{p-1} x \cos_p^{2-p} x.$$

Similar to the classical trigonometric function, one can define the generalized tangent function

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad \text{for } x \in \mathbb{R} \setminus \{(\mathbb{Z} + 1/2)\pi_p\}.$$

It is natural to ask whether the $p$-analogue of the Becker–Stark inequality holds for $x \in (0, \pi_p/2)$. Observed that

$$\lim_{x \to \frac{\pi_p}{2}^-} \frac{(\pi_p^2 - 4x^2)\tan_p x}{x} = \lim_{x \to \frac{\pi_p}{2}^-} \frac{8}{\sin_p^{p-1} x \cos_p^{2-p} x} = \begin{cases} \infty & 1 < p < 2, \\ 8, & p = 2, \\ 0, & p > 2, \end{cases}$$

which allows us to pose the following problem:

**Problem 3.1.** To determine the range of $p$ in $[2, \infty)$ (resp. $(1,2)$) such that the inequality

$$\frac{\tan_p x}{x} < (\text{resp.} >) \frac{\pi_p^2}{\pi_p^2 - 4x^2} \tag{3.1}$$

holds for $x \in (0, \pi_p/2)$.

**Remark 3.2.** Inequality (3.1) can be viewed as the $p$-analogue of Becker-Stark inequality. Our method in this paper reveals that it only needs to study a ratio of two hypergeometric functions by changing the variable $t = \sin_p^p x$ in (3.1).

## 4. Conclusions

In this paper, from the viewpoint of hypergeometric function, we study the well-known Becker-Stark inequality by changing a variable $t = \sin^2 x$. Our main result is to approximate the function $\pi - [(\pi^2 - 4x^2)\tan x]/x$ by the first few terms of the Taylor series, even if we only know finitely many positive coefficients. In particular, the double inequality

$$\frac{\pi^2 - (\pi^2 - 8)\sin^2 x}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 - (4 - \pi^2/3)\sin^2 x}{\pi^2 - 4x^2}$$

holds for $x \in (0, \pi/2)$, which improves Becker-Stark inequality (1.1).

## Author contributions

Suxia Wang: Conceptualisation, writing – original draft, formal analysis; Tiehong Zhao: Writing – review & editing, supervision, validation. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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