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Research article

Moser-Trudinger inequalities on 2-dimensional Hadamard manifolds

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Abstract: We derive two types of sharp Moser-Trudinger inequalities on complete, simply connected, two-dimensional Riemannian manifolds whose sectional curvatures K satisfy the bounds $-b^2 \leq K \leq -a^2 < 0$.

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1. Introduction

A pinched Hadamard manifold is a complete, simply connected Riemannian manifold (M, g) whose Gaussian curvature K satisfies $-b^2 \leq K \leq -a^2$ for some $a, b > 0$. Let μ denote the Riemannian measure of M, induced by the metric g. Below, $W^{1,2}(M)$ denotes the usual Sobolev space on M.

In this note, we establish the following theorem:

Theorem 1. *If (M, g) is a two-dimensional pinched Hadamard manifold, then there exists a constant C such that for all $u \in W^{1,2}(M)$ with $\|\nabla u\|_2 \leq 1$, we have*

$$\int_M (e^{4\pi u^2} - 1) d\mu \leq C, \tag{1.1}$$

and

$$\int_M \frac{e^{4\pi u^2} - 1}{1 + |u|^2} d\mu \leq C\|u\|_2^2. \tag{1.2}$$

The exponential constant 4π is sharp in both inequalities.

On pinched Hadamard manifolds of dimension $n \geq 3$, a sharp version of estimate (1.1) was derived by Bertrand-Sandeep [1] for the operators $D^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ if α is even, and $D^\alpha = \nabla(-\Delta)^{\frac{\alpha-1}{2}}$ if α is odd

$(1 \le \alpha < n)$. A sharp version of (1.2) on the same manifolds and for the same operators was obtained by Morpurgo-Qin [14], also for $n \ge 3$. Estimate (1.2) is known as "Moser-Trudinger inequality with exact growth condition", and was obtained first on $\mathbb{R}^2$ by Ibrahim-Masmoudi-Nakanishi [6], later extended in higher dimensions and for higher order operators in Masmoudi-Sani [11–13], and also in Lu-Tang-Zhu [8].

The first step of the strategy used in [1] was to write $u = T(D^\alpha u)$, where T is an integral operator with a kernel K_α, given in terms of the Green function $G(x,y)$ of the Laplace-Beltrami operator on M; in particular, $K_2 = G$ and $K_1 = \nabla_y G$. The second step was to derive sharp asymptotic estimates and critical integrability estimates on K_α, which allowed the authors to apply general results in [5] regarding sharp Adams type inequalities on measure spaces.

The same ideas were then used in [14] where further estimates on K_α were obtained, allowing the authors to apply general results in [14] on Adams inequalities with exact growth conditions on metric measure spaces.

The case $n = 2$ was somehow left out in the above works, due to small technical reasons. The purpose of this note is to fill in the gap and to establish the required estimates on $K_1 = \nabla_y G$ even when $n = 2$.

2. Proof of Theorem 1

It is enough to prove the inequalities of the theorem for $u \in C_c^\infty(M)$. Let $G(x,y)$ be the minimal positive Green function on M. Then, for each $x \in M$ we can write

$$u(x) = \int_M \langle \nabla_y G(x,y),\ \nabla_y u(y) \rangle d\mu(y), \tag{2.1}$$

where $\langle Z, W \rangle = g(Z,W)$ for $Z, W \in T_y M$, and $\nabla_y G$ is the gradient of $G(x,y)$ with respect to the y variable. Also, below we will denote the open ball centered at x and with radius r as $B(x,r) = \{y \in M : d(x,y) < r\}$ and its volume as $V_x(r) = \mu(B(x,r))$.

For a measurable function f on M, we define its nonincreasing rearrangement as

$$f^*(t) = \inf\{s > 0 :\ \mu\{x : |f(x)| > s\}) \le t\}, \qquad t > 0. \tag{2.2}$$

In view of (2.1) and Theorem 1 in [5] (see also (10.8), (10.10), and related remarks in [14]), to prove (1.1) it is enough to show that

$$|\nabla_y G(x,\cdot)|^*(t) \le \frac{1}{2\sqrt{\pi}} t^{-1/2} + C, \qquad 0 < t \le 1, \quad x \in M \tag{2.3}$$

$$|\nabla_y G(\cdot,y)|^*(t) \le C t^{-1/2}, \qquad t > 0, \quad y \in M \tag{2.4}$$

and

$$\int_1^\infty (|\nabla_y G(x,\cdot)|^*(t))^2 dt \le C, \qquad x \in M. \tag{2.5}$$

On the other hand, to prove (1.2), it is enough to verify the following additional conditions:

$$\int_{r_1 \le d(x,y) \le r_2} |\nabla_y G(x,y)|^2 d\mu(y) \le \frac{1}{4\pi} \log \frac{V_x(r_2)}{V_x(r_1)} + C, \qquad 0 < V_x(r_1) < V_x(r_2) \le 1, \tag{2.6}$$

$$\int_{d(x,y)\geq r} |\nabla_y G(x,y)|^2 d\mu(y) \leq C, \qquad V_x(r) \geq 1, \tag{2.7}$$

$$|\nabla_y G(x,\cdot)|^*(t) \leq Ct^{-1/2}, \qquad t > 0, \tag{2.8}$$

$$|\nabla_y G(x,y)| \leq CV_x(d(x,y))^{-1/2}, \qquad V_x(d(x,y)) \leq 1, \tag{2.9}$$

and for each $\delta > 0$, there is $B_\delta > 0$ such that

$$\int_{d(x,y)>R} |\nabla_y G(x',y) - \nabla_y G(x,y)|^2 d\mu(y) \leq B_\delta, \quad V_x(R) \geq (1+\delta)V_x(r),\ V_x(r) \leq 1,\ x' \in B(x,r). \tag{2.10}$$

We first show that for any given $R > 0$, there exists C such that

$$|\nabla_y G(x,y)| \leq C \begin{cases} d(x,y)^{-1} & \text{if } d(x,y) \leq R \\ 1 & \text{if } d(x,y) \geq R \end{cases} \tag{2.11}$$

with C independent of x, y (but depending on R). First, let us recall this consequence of the well-known Li-Yau gradient estimate ((3.14) in [1], Thm. 6.1 in [9], Lemma 2.1 in [10]):

$$|\nabla_y G(x,y)| \leq G(b + Cd(x,y)^{-1}), \qquad x, y \in M. \tag{2.12}$$

Observe that such an estimate is valid in any dimension n, and that it was used in [1] when $n \geq 3$ to derive the bound $|\nabla_y G(x,y)| \leq Cd(x,y)^{1-n}$, given the bound $G(x,y) \leq Cd(x,y)^{2-n}$. However, in dimension 2 this does not work for small distances, since $G(x,y)$ behaves like $-\log d(x,y)$.

Instead, we use the fact that

$$G(x,y) = \int_0^\infty H(t,x,y)dt \tag{2.13}$$

where $H(t,x,y)$ is the heat kernel for the Laplace-Beltrami operator on M, and give some good (even if rough) estimates on $|\nabla_y H|$. First, we recall that on any $n-$dimensional Hadamard manifold satisfying $K \leq -a^2$, $(a \geq 0)$, we have the comparison theorem

$$H(t,x,y) \leq H_a(t, d(x,y)), \qquad t > 0,\ x, y \in M, \tag{2.14}$$

where $H_a(t, d_a(\tilde{x},\tilde{y}))$ is the heat kernel on the space form of constant curvature $-a^2$, and where $d_a(\tilde{x},\tilde{y})$ denotes the distance of two points $\tilde{x}, \tilde{y}$ in such space (see Théorème 1 in [4]). When $a > 0$, the heat kernel H_a is well-known and somewhat explicit. In particular, we have that $H_a(t,r) = a^n H_1(a^2 t, ar)$, for all $t, r > 0$, where $H_1(t,r)$ yields the heat kernel on the hyperbolic space $\mathbb{H}^n$, and satisfies the estimate

$$\begin{aligned} H_1(t,r) &\leq C_n t^{-n/2}(1+r)(1+r+t)^{\frac{n-3}{2}} e^{-\frac{(n-1)^2}{4}t - \frac{n-1}{2}r - \frac{r^2}{4t}} \\ &\leq C_n \begin{cases} (1+r)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}r}\ t^{-n/2} e^{-\frac{r^2}{4t}} & \text{if } 0 < t \leq 1 \\ (1+r)^{\frac{n}{2}} e^{-\frac{n-1}{2}r}\ t^{-3/2} e^{-\frac{(n-1)^2}{4}t - \frac{r^2}{4t}} & \text{if } t \geq 1 \end{cases} \end{aligned} \tag{2.15}$$

for some C_n depending on n (see [3], Thm. 3.1]). Hence, there are some $c_1, c_2, c_3 > 0$ depending on n, a such that

$$H(t,x,y) \leq c_1 e^{-c_2 d(x,y)} \begin{cases} t^{-n/2} e^{-\frac{d(x,y)^2}{4t}} & \text{if } 0 < t \leq 1 \\ e^{-c_3 t - \frac{d(x,y)^2}{4t}} & \text{if } t \geq 1. \end{cases} \tag{2.16}$$

Now, we can appeal to a result by E. B. Davies [2], Theorem 6, which, under the additional condition $\mathrm{Ric} \geq -(n-1)b^2$, gives

$$|\nabla_y H(t,x,y)| \leq c_4 \begin{cases} t^{-n/2-1} e^{-\frac{d(x,y)^2}{8t}} & \text{if } 0 < t \leq 1 \\ e^{-c_3 t} & \text{if } t \geq 1, \end{cases} \tag{2.17}$$

for some c_4 depending on a, b, n. From (2.13) we then easily get

$$|\nabla_y G(x,y)| \leq C \begin{cases} d(x,y)^{1-n} & \text{if } d(x,y) \leq R \\ 1 & \text{if } d(x,y) \geq R \end{cases} \tag{2.18}$$

in any dimension $n \geq 2$ (and, hence, (2.11)).

We remark that (2.17) can be refined somewhat by replacing c_4 with $c_4 e^{-c_2 d(x,y)}$, which also gives $|\nabla_y G(x,y)| \leq C e^{-c_5 d(x,y)}$ for $d(x,y) \geq R$, some $c_5 > 0$. This can be done using the same method as in [2], using the the gradient estimate for the heat kernel (see, e.g., [9], Thm. 12.2)

$$|\nabla_y H|^2 \leq \alpha H \partial_t H + H^2\Big(\frac{n\alpha^2}{2t} + C_n \frac{\alpha^2}{\alpha - 1} b^2\Big), \tag{2.19}$$

for any $\alpha > 1$, combined with (2.16) and the time derivative estimate

$$\left|\frac{\partial H}{\partial t}(t,x,y)\right| \leq c_6 e^{-c_2 d(x,y)} \begin{cases} t^{-n/2-1} e^{-\frac{d(x,y)^2}{8t}} & \text{if } 0 < t \leq 1 \\ e^{-c_3 t - \frac{d(x,y)^2}{8t}} & \text{if } t \geq 1. \end{cases} \tag{2.20}$$

The latter estimate can be obtained using the method in [2], Theorem 4.

Using (2.11) and following the same argument in [1], Theorem 3.2, with some minor changes, we can now obtain

$$|\nabla_y G(x,y)| \leq \frac{1}{2\pi} d(x,y)^{-1} + C, \quad d(x,y) \leq 1, \tag{2.21}$$

which implies (2.3), (2.9), and (2.6) by the volume comparison theorem.

For the convenience of the reader, we outline the proof of (2.21), keeping a part of the notation used in [1], Thm. 3.2, so the changes are slightly more evident. For any fixed $x \in M$, an n-dimensional Hadamard manifold, we have a unique chart given by the exponential map $\exp_x(r\xi)$. The volume element in geodesic polar coordinate is given by $d\mu = r^{n-1}\sqrt{|g|}\,dr\,d\xi$, where $g = (g_{ij})$ is the metric tensor, evaluated at $\exp_x(r\xi)$, $|g| = \det(g_{ij})$, and where $d\xi$ is the measure on the unit sphere S^{n-1}.

Letting $\Phi(r) = -\frac{1}{2\pi}\log r$ and $r = d(x,y)$, viewed as a function of y for fixed x, we have, for all $r > 0$,

$$\Delta\Phi(r) = \Phi''(r)|\nabla r|^2 + \Phi'(r)\Delta r = -\frac{1}{2\pi r}\partial_r \log\sqrt{|g|} \tag{2.22}$$

where we used $|\nabla r| = 1$ and $\Delta r = \partial_r \log(r^{n-1}\sqrt{|g|})$ in any dimensions ([7], Lemma 11.13). Letting

$$H_x(y) = \frac{1}{2\pi r}\partial_r \log\sqrt{|g|}, \qquad y = \exp_x(r\xi) \tag{2.23}$$

it is then easy to check that, in the sense of distributions,

$$\Delta\Phi(d(x,\cdot)) = -\delta_x - H_x. \tag{2.24}$$

From the Laplacian comparison ([7], Thm. 11.5), we have, in any dimension n,

$$(n-1)a\coth(ar) \le \Delta r = \frac{n-1}{r} + \partial_r \log\sqrt{|g|} \le (n-1)b\coth(br) \tag{2.25}$$

from which we deduce, when $n = 2$,

$$\left|\partial_r \log\sqrt{|g|}\right| \le Cr, \qquad r \le 4 \tag{2.26}$$

where C is independent of x. Hence, we get

$$|H_x(y)| \le C, \qquad d(x,y) \le 4. \tag{2.27}$$

Define now

$$U_x(y) = \int_M G(y,z)\psi(d(x,z))H_x(z)d\mu(z) \tag{2.28}$$

where $\psi : [0,\infty) \to \mathbb{R}$, is smooth and such that $\psi = 1$ on $[0,2]$ and $\psi = 0$ on $[4,\infty)$. One then has that the function $h^x(y) := G(x,y) - \Phi(d(x,y)) - U_x(y)$ is harmonic in the ball $B(x,2)$, as a function of y, and also uniformly bounded on $\partial B(0,2)$. This last fact follows from a uniform estimate on U_x, which can be verified as in [1] with some small modifications. First, we have the Green function comparison

$$G(x,y) \le G_a(d(x,y)) = -\frac{1}{2\pi}\log\tanh\Big(\frac{ad(x,y)}{2}\Big), \qquad x,y \in M \tag{2.29}$$

where $G_a(d_a(\tilde{x},\tilde{y}))$ gives the Green function on the two-dimensional space form of constant curvature $-a^2 < 0$. Since for $r > 0$ we have $|G_a(r)| \le \max\{\log\frac{2}{r}, 1\} + C_a$ for some $C_a > 0$, then using the volume and the Rauch comparison theorems, we get for $y = \exp_x(\overline{y}), z = \exp_x(\overline{z})$, with $\overline{y}, \overline{z} \in \mathbb{R}^2$,

$$|U_x(y)| \le C\int_{B(x,4)} G(y,z)d\mu(z) \le C\int_{|\overline{y}-\overline{z}|\le 2} \log\frac{2}{|\overline{y}-\overline{z}|}d\overline{z} + C\int_{|\overline{z}|\le 4}(1+C_a)d\overline{z} \le C. \tag{2.30}$$

(We note that in [1], the authors used the bound $G(x,y) \le G_0(d(x,y)) = c_n d(x,y)^{2-n}$, but this cannot be done when $n = 2$.)

Using the gradient bound for harmonic functions, we can conclude that $|\nabla h^x|$ is bounded on $B(x,1)$, uniformly w.r. to x. Moreover, using (2.11), the fact that G is symmetric, and arguing as in (2.30), if $d(x,y) \le 1$, we obtain

$$|\nabla_y U_x(y)| \le C\int_{B(x,4)} |\nabla_y G(y,z)|d\mu(z) \le C\int_{d(x,z)\le 4} d(y,z)^{-1}d\mu(z) \le C. \tag{2.31}$$

Finally, we have

$$|\nabla_y G(x,y) - \nabla_y \Phi(d(x,y))| \le |\nabla_y U_x(y) + \nabla_y h^x(y)| \le C, \qquad d(x,y) \le 1, \tag{2.32}$$

which gives (2.21).

Next, note that estimate (3.25) in [1] still holds when $n = 2$, i.e.,

$$\mu(\{y : G(x,y) > s\}) \le \frac{4}{a^2 s}, \quad s > 0. \tag{2.33}$$

Using (2.33) and the gradient estimate (2.12), for $s < 1$ we get

$$\begin{aligned}
&\mu(\{y : |\nabla_y G| > s\}) \le \mu(\{y : G(b + Cd^{-1}) > s\}) \\
&\le \mu(\{y : d \le 1\}) + \mu(\{y : d > 1,\ G(b + Cd^{-1}) > s\}) \le C + \mu(\{y : (C + b)G > s\}) \\
&\le C + Cs^{-1} \le Cs^{-1}.
\end{aligned} \tag{2.34}$$

Hence,

$$|\nabla_y G(x, \cdot)|^*(t) \le Ct^{-1}, \quad t > 1 \tag{2.35}$$

and (2.5) holds. Note also that $|\nabla_y G| \le C$ for large distances therefore, by Remark 17 in [14], (2.7) follows. Furthermore, (2.35) and (2.3) imply (2.8).

By (2.33), (2.12), and the symmetry of $G(x, y)$ in x, y, we have

$$|\nabla_y G(\cdot, y)|^*(t) \le Ct^{-1}, \quad t > 1, \tag{2.36}$$

which combined with (2.21) gives (2.4). To prove (2.10), we follow the same argument as in [14], which still works when $n = 2$, given (2.11).

The sharpness of the exponential constant 4π in (1.1) and (1.2) is proved in the usual way, by considering the "Moser family" of functions $v_\epsilon \in W_0^{1,n}(B(x_0, 1))$, and some fixed $x_0 \in M$, defined as

$$v_\epsilon(y) = \begin{cases} \log \dfrac{1}{\epsilon} & \text{if } d(x_0, y) \le \epsilon \\ \log \dfrac{1}{d(x_0, y)} & \text{if } \epsilon < d(x_0, y) \le 1 \\ 0 & \text{if } d(x_0, y) > 1. \end{cases} \tag{2.37}$$

3. Conclusions

We derived two types of sharp Moser-Trudinger inequalities on complete, simply connected, two-dimensional Riemannian manifolds whose sectional curvatures K satisfy the bounds $-b^2 \le K \le -a^2 < 0$. The results fill gaps that were left in [1] and [14], where such inequalities were proved in dimensions $n \ge 3$.

Author contributions

Carlo Morpurgo and Liuyu Qin: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing-original draft, Writing-review & editing, Visualization, Supervision. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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