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Research article

Boundedness of the product of some operators from the natural Bloch space into weighted-type space

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Abstract: Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X, which may be infinite dimensional. The authors characterize the boundedness of the product of the radial derivative operator and the weighted composition operator from the natural Bloch space (or the little Bloch-type space) into the weighted-type space (or the little weighted-type space) on $\mathbb{B}_X$.

Keywords: weighted composition operator; radial derivative operator; natural Bloch space; weighted-type space; boundedness
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1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$, and let X be a complex Banach space. Let $\mathbb{B}_X := \{z \in X : \|z\| < 1\}$ be the unit ball of X, and $H(\mathbb{B}_X)$ be the algebra of holomorphic functions from $\mathbb{B}_X$ into $\mathbb{C}$. Let $H(\mathbb{B}_X, \mathbb{B}_X)$ denote the class of holomorphic mappings from $\mathbb{B}_X$ into $\mathbb{B}_X$.

We shall consider one class of Banach spaces, $\mathcal{B}_{nat}(\mathbb{B}_X)$, defined in [29], which is defined as follows:

The natural Bloch space $\mathcal{B}_{nat}(\mathbb{B}_X)$ is defined to be the space of all $f \in H(\mathbb{B}_X)$ for which

$$\|f\|_{nat} := \sup\{(1 - \|z\|^2)\|f'(z)\| : z \in \mathbb{B}_X\} < \infty,$$

where $f'(z) = Df(z) \in X^*$ (the dual space of X) denotes the Fréchet derivative of f at the point z. Endowed with the norm $\|f\|_{nat-Bloch} = |f(0)| + \|f\|_{nat}$, the natural Bloch space $\mathcal{B}_{nat}(\mathbb{B}_X)$ becomes a Banach space. When $X = \mathbb{C}$ and $\mathbb{B}_X = \mathbb{D}$, $\mathcal{B}_{nat}(\mathbb{B}_X)$ is the classical Bloch space $\mathcal{B}(\mathbb{D})$ defined in [1, 32]. If X is a Hilbert space H, we have that the spaces $\mathcal{B}_{nat}(\mathbb{B}_H)$, $\mathcal{B}(\mathbb{B}_H)$, $\mathcal{B}_{\mathcal{R}}(\mathbb{B}_H)$, $\mathcal{B}_{weak}(\mathbb{B}_H)$, and $\mathcal{B}_{inv}(\mathbb{B}_H)$ defined in [4] coincide. One has studied the Bloch space on some homogeneous domains of $\mathbb{C}^n$ in [3, 4, 46, 51]. The definition of a complex-valued Bloch function on the infinite-dimensional bounded symmetric domains was later given in [5, 9].

The weighted-type space $H^\infty_\omega(\mathbb{B}_X)$ [47] consists of all functions $f \in H(\mathbb{B}_X)$ such that $\|f\|_{\omega,\infty} = \sup\{\omega(z)|f(z)| : z \in \mathbb{B}_X\}$ is finite, and the little weighted-type space $H^\infty_{\omega,0}(\mathbb{B}_X)$ (the closed subspace of $H^\infty_\omega(\mathbb{B}_X)$) consists of all functions $f \in H^\infty_\omega(\mathbb{B}_X)$ with $\lim_{\|z\|\to 1} \omega(z)|f(z)| = 0$. It is clear that both $H^\infty_\omega(\mathbb{B}_X)$ and $H^\infty_{\omega,0}(\mathbb{B}_X)$ are Banach spaces with the norm $\|\cdot\|_{\omega,\infty}$, where ω denotes a normal function on $[0,1)$ [35], and ω can be extended to a function on $\mathbb{B}_X$ by $\omega(z) = \omega(\|z\|)$.

Given $\psi \in H(\mathbb{B}_X)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. The weighted composition operator with symbols ψ and φ is defined, for $f \in H(\mathbb{B}_X)$, by $W_{\psi,\varphi} f = \psi(f \circ \varphi)$ (see [47]). The operator with symbol ψ is defined, $f \in H(\mathbb{B}_X)$, by $M_\psi f = \psi f$, usually called the multiplication operator, and the operator with symbol φ is defined, $f \in H(\mathbb{B}_X)$, by $C_\varphi f = f \circ \varphi$, usually called the composition operator. $W_{\psi,\varphi}$ is a product-type operator, as $W_{\psi,\varphi} = M_\psi C_\varphi$.

An extensive study concerning the theory of weighted composition operators on Banach and Hilbert spaces of holomorphic functions has been established during the past four decades. It plays a central role in the study of the isometries on several spaces of holomorphic functions. The study of the weighted composition operators on $\mathcal{B}(\mathbb{D})$ began with the work of Ohno and Zhao in [31]. They characterized the weighted composition operators between spaces $\mathcal{B}(\mathbb{D})$. More results on weighted composition operators in various settings can be found in [6, 8, 10, 20, 24, 27, 28, 30] and the references therein. Product-type operators on some spaces of analytic functions on $\mathbb{D}$ or $\mathbb{B}^n$ have become a subject of increasing interest in the last twenty years (see, e.g., [15, 16, 21–23, 34, 36–38, 42–45, 49, 50, 52] and the related references therein). On a sum of more complex product-type operators from Bloch-type spaces to the weighted-type spaces, Huang and Jiang in [18] completely characterized the boundedness and compactness of the sum operator from Bloch-type spaces to weighted-type spaces on $\mathbb{B}^n$ (also see [17, 19, 39–41, 48] and the related references therein).

F. Colonna and M. Tjani [7] characterized the bounded weighted composition operators from a large class of Banach spaces of analytic functions on $\mathbb{D}$ into weighted Banach spaces. H. Hamada [13] characterized the bounded weighted composition operator $W_{\psi,\varphi}$ from the space of bounded holomorphic functions $H^\infty(\mathbb{B}_X)$ into the Bloch space $\mathcal{B}(\mathbb{B}_X)$ of infinite-dimensional bounded symmetric domains. Some studies have also been devoted to the situation when $\mathbb{B}_X$ is the open unit ball of a Banach space X (see, e.g., [2, 11, 12, 14, 26, 33, 47]). There has been a huge interest in the operators on subspaces of $H(\mathbb{B}_X)$. The radial derivative operator on $H(\mathbb{B}_X)$ is defined as follows:

$$\mathcal{R}f(z) = Df(z)z, \ z \in \mathbb{B}_X.$$

Motivated by some of the above-mentioned investigations, and using some modifications of the methods and ideas therein, the primary purpose of this paper is to bring the current results on the boundedness of the product of $\mathcal{R}$ and $W_{\psi,\varphi}$ from the natural Bloch spaces $\mathcal{B}_{nat}(\mathbb{B}_X)$ (or the little Bloch-type spaces $\mathcal{B}_{nat,0}(\mathbb{B}_X)$) into the weighted-type space $H^\infty_\omega(\mathbb{B}_X)$ (or the little weighted-type space $H^\infty_{\omega,0}(\mathbb{B}_X)$). There are still many open questions on this topic. Thus, our hope is that our study will inspire more work in this area. Before we formulate the main theorem, we need the following auxiliary result [25].

Lemma 1.1. *Let $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$. Then*

$$|f(z)| \le C \log \frac{2}{1 - \|z\|^2} \|f\|_{nat-Bloch}, \ for \ z \in \mathbb{B}_X,$$

where C is a positive constant independent of $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$.

2. The boundedness of the product-type operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H_\omega^\infty(\mathbb{B}_X)$

In this section, we consider the boundedness of the product of the radial derivative operator and the weighted composition operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H_\omega^\infty(\mathbb{B}_X)$ and obtain a necessary and sufficient condition. For $x \in \mathbb{B}_X \backslash \{0\}$, let the set

$$T(x) = \{\ell_x \in X^* : \ell_x(x) = \|x\|, \|\ell_x\| = 1\},$$

then $T(x)$ is nonempty by the Hahn–Banach theorem.

Theorem 2.1. *Suppose $\psi \in H(\mathbb{B}_X)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$.*

(1) *If*

$$\sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1 - \|\varphi(z)\|^2} < \infty, \tag{2.1}$$

and

$$\sup_{z\in\mathbb{B}_X} \omega(z)\, |\psi'(z)z| \log \frac{2}{1 - \|\varphi(z)\|^2} < \infty, \tag{2.2}$$

then $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H_\omega^\infty(\mathbb{B}_X)$ is bounded.

(2) *If $\sup_{z\in\overline{\mathbb{B}_X}} |\ell_{\varphi(a)}(\varphi'(a)z)| = \|\varphi'(a)\|$, for $a \in \mathbb{B}_X$ with $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$ and $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H_\omega^\infty(\mathbb{B}_X)$ is bounded, then* (2.2) *holds and*

$$\sup_{\{z\in\mathbb{B}_X : r<\|\varphi(z)\|<1\}} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1 - \|\varphi(z)\|^2} < \infty, \tag{2.3}$$

for $r \in (0, 1)$.

Proof. (1) First, assume (2.1) and (2.2). Using the chain rule, it is easy to see that for $f \in H(\mathbb{B}_X)$, $z \in \mathbb{B}_X$

$$\begin{aligned}
&\omega(z)|\mathcal{R}W_{\psi,\varphi}f(z)| = \omega(z)|D(W_{\psi,\varphi}f)(z)z| = \omega(z)|(\psi f \circ \varphi)'(z)z| \\
&\le \omega(z)|\psi(z)|\|(f \circ \varphi)'(z)z\| + \omega(z)|f(\varphi(z))||\psi'(z)z| \\
&= \omega(z)|\psi(z)|\|f'(\varphi(z))\varphi'(z)z\| + \omega(z)|f(\varphi(z))||\psi'(z)z| \\
&\le \omega(z)|\psi(z)|\|f'(\varphi(z))\|\|\varphi'(z)\| + \omega(z)|f(\varphi(z))||\psi'(z)z|.
\end{aligned}$$

Let $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$. Then, by Lemma 1.1, we get

$$\begin{aligned}
&\left\|\mathcal{R}W_{\psi,\varphi}f\right\|_{\omega,\infty} = \sup_{z\in\mathbb{B}_X} \omega(z)|\mathcal{R}W_{\psi,\varphi}f(z)| \\
&\le \sup_{z\in\mathbb{B}_X} \left(\omega(z)|\psi(z)|\|f'(\varphi(z))\|\|\varphi'(z)\| + \omega(z)|f(\varphi(z))||\psi'(z)z|\right) \\
&\le \sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1 - \|\varphi(z)\|^2}\|f\|_{nat} \\
&+C \sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1 - \|\varphi(z)\|^2}\|f\|_{nat-Bloch}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2}\|f\|_{nat-Bloch}\\
&+C\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|\log\frac{2}{1-\|\varphi(z)\|^2}\|f\|_{nat-Bloch}.
\end{aligned}\tag{2.4}
$$

By conditions (2.1), (2.2), and (2.4), it follows that the operator $\mathcal{R}W_{\psi,\varphi}:\mathcal{B}_{nat}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)$ is bounded.

(2) Now suppose that $\mathcal{R}W_{\psi,\varphi}:\mathcal{B}_{nat}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)$ is bounded. Then there is a positive constant C, for which

$$\|\mathcal{R}W_{\psi,\varphi}f\|_{\omega,\infty}\leq C\|f\|_{nat-Bloch},$$

for all $f\in\mathcal{B}_{nat}(\mathbb{B}_X)$. Choose $f(z)=1$, then $f\in\mathcal{B}_{nat}(\mathbb{B}_X)$ and $\|f\|_{nat-Bloch}=1$, from which we get

$$\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|=\|\mathcal{R}W_{\psi,\varphi}f\|_{\omega,\infty}\leq C\|f\|_{nat-Bloch}=C<\infty.\tag{2.5}$$

To prove (2.3) holds, fix $a\in\mathbb{B}_X$; if $\varphi(a)\in\mathbb{B}_X\backslash\{0\}$, let $w=\varphi(a)$ and $\ell_w\in T(w)$ be fixed. Set

$$f_a(z)=\frac{1}{1-\|\varphi(a)\|\ell_w(z)},\ z\in\mathbb{B}_X.\tag{2.6}$$

Then

$$f_a'(z)=\frac{\|\varphi(a)\|}{(1-\|\varphi(a)\|\ell_w(z))^2}\ell_w'(z),$$

and

$$f_a'(\varphi(a))=\frac{\|\varphi(a)\|\ell_w'(\varphi(a))}{(1-\|\varphi(a)\|^2)^2}=\frac{\|\varphi(a)\|\ell_w}{(1-\|\varphi(a)\|^2)^2}.\tag{2.7}$$

Thus,

$$
\begin{aligned}
\|f_a'(z)\| &\leq \frac{\|\varphi(a)\|}{|1-\|\varphi(a)\|\ell_w(z)|^2}\leq\frac{1}{(1-\|\varphi(a)\|\|z\|)^2}\\
&\leq\frac{1}{(1-\|\varphi(a)\|)}\frac{1}{(1-\|z\|)},
\end{aligned}
$$

which implies that $f_a\in\mathcal{B}_{nat}(\mathbb{B}_X)$ with $\|f_a\|_{nat}\leq\frac{2}{1-\|\varphi(a)\|}\leq\frac{4}{1-\|\varphi(a)\|^2}$. Hence, using the triangle inequality, (2.6) and (2.7), we have that

$$
\begin{aligned}
&\left(1+\frac{4}{1-\|\varphi(a)\|^2}\right)\left\|\mathcal{R}W_{\psi,\varphi}\right\|\geq\|f_a\|_{nat-Bloch}\left\|\mathcal{R}W_{\psi,\varphi}\right\|\\
&\geq\left\|\mathcal{R}W_{\psi,\varphi}f_a\right\|_{\omega,\infty}=\sup_{z\in\mathbb{B}_X}\omega(z)\left|\left(W_{\psi,\varphi}f_a\right)'(z)z\right|\\
&=\sup_{z\in\mathbb{B}_X}\omega(z)|\psi(z)(f_a\circ\varphi)'(z)z+(f_a\circ\varphi)(z)\psi'(z)z|\\
&\geq\omega(a)|\psi(a)(f_a\circ\varphi)'(a)a+\omega(a)(f_a\circ\varphi)(a)\psi'(a)a|\\
&\geq\omega(a)|\psi(a)(f_a\circ\varphi)'(a)a|-\omega(a)|(f_a\circ\varphi)(a)\psi'(a)a|
\end{aligned}
$$

$$
\begin{aligned}
&= \omega(a)|\psi(a)||f_a'(\varphi(a))\varphi'(a)a| - \omega(a)|f_a(\varphi(a))||\psi'(a)a| \\
&= \frac{\omega(a)|\psi(a)|\|\varphi(a)\|\|\ell_w\varphi'(a)a\|}{(1-\|\varphi(a)\|^2)^2} - \frac{\omega(a)|\psi'(a)a|}{1-\|\varphi(a)\|^2} \\
&= \frac{\omega(a)|\psi(a)|\|\varphi(a)\|}{(1-\|\varphi(a)\|^2)^2}\sup_{z\in\overline{\mathbb{B}_X}}\left|\ell_w\left(\varphi'(a)z\right)\right| - \frac{\omega(a)|\psi'(a)a|}{1-\|\varphi(a)\|^2} \\
&= \frac{\omega(a)|\psi(a)|\|\varphi(a)\|\|\varphi'(a)\|}{(1-\|\varphi(a)\|^2)^2} - \frac{\omega(a)|\psi'(a)a|}{1-\|\varphi(a)\|^2}.
\end{aligned} \tag{2.8}
$$

From (2.5) and (2.8), we easily get for $r \in (0,1)$

$$
\begin{aligned}
&\sup_{\{z\in\mathbb{B}_X: r<\|\varphi(z)\|<1\}} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2} \\
&\le \frac{1}{r}\sup_{\{z\in\mathbb{B}_X: r<\|\varphi(z)\|<1\}} \frac{\omega(z)\|\varphi(z)\||\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2} \\
&\le \frac{1}{r}\left((5-r^2)\left\|\mathcal{R}W_{\psi,\varphi}\right\| + \sup_{\{z\in\mathbb{B}_X: r<\|\varphi(z)\|<1\}} \omega(z)|\psi'(z)z|\right) \\
&\le \frac{1}{r}\left((5-r^2)\left\|\mathcal{R}W_{\psi,\varphi}\right\| + \sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z|\right) \\
&< \infty,
\end{aligned}
$$

that is, (2.3) holds.

Next, we will prove (2.2). For given $a \in \mathbb{B}_X$ if $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$, consider the function

$$
\begin{aligned}
g_a(z) &= 2\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)} \\
&\quad -\left(\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}\right)^2 \Bigg/ \log\frac{2}{1-\|\varphi(a)\|^2}\ , \text{ for } z \in \mathbb{B}_X.
\end{aligned} \tag{2.9}
$$

Then

$$
\begin{aligned}
&g_a'(z) \\
&= \frac{2\|\varphi(a)\|\ell_{\varphi(a)}'(z)}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)} - \frac{2\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}}{\log\frac{2}{1-\|\varphi(a)\|^2}}\frac{\|\varphi(a)\|\ell_{\varphi(a)}'(z)}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)} \\
&= \frac{2\|\varphi(a)\|\ell_{\varphi(a)}}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)} - \frac{2\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}}{\log\frac{2}{1-\|\varphi(a)\|^2}}\frac{\|\varphi(a)\|\ell_{\varphi(a)}}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}, \text{ for } z \in \mathbb{B}_X,
\end{aligned} \tag{2.10}
$$

so that

$$|g_a(0)| \le 3\log 2, \tag{2.11}$$

$$g_a'(\varphi(a)) = \theta \ \text{(null operator) and } g_a(\varphi(a)) = \log\frac{2}{1-\|\varphi(a)\|^2}. \tag{2.12}$$

On the other hand, we have

$$
\begin{aligned}
\|g_a'(z)\| &\le \frac{2\|\ell_{\varphi(a)}\|}{1-|\ell_{\varphi(a)}(z)|} + \left|\frac{2\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}}{\log\frac{2}{1-\|\varphi(a)\|^2}}\right| \frac{2\|\ell_{\varphi(a)}\|}{1-|\ell_{\varphi(a)}(z)|} \\
&\le \frac{2}{1-\|z\|} + \frac{4\left(\log\left|\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}\right| + \frac{\pi}{2}\right)}{(1-\|z\|)\log\frac{2}{1-\|\varphi(a)\|^2}} \\
&\le \frac{2}{1-\|z\|} + \frac{4\left(\log\frac{4}{1-\|\varphi(a)\|^2} + \frac{\pi}{2}\right)}{(1-\|z\|)\log\frac{2}{1-\|\varphi(a)\|^2}} \\
&\le \frac{2}{1-\|z\|} + \frac{4\left(2\log\frac{2}{1-\|\varphi(a)\|^2} + \frac{\pi}{2}\right)}{(1-\|z\|)\log\frac{2}{1-\|\varphi(a)\|^2}} \\
&\le \frac{2}{1-\|z\|} + \frac{4}{1-\|z\|}\left(2 + \frac{\pi}{2\log 2}\right) \\
&\le \frac{C}{1-\|z\|^2}, \text{ for } z \in \mathbb{B}_X,
\end{aligned}
$$

hence $g_a \in \mathcal{B}_{nat}(\mathbb{B}_X)$ for $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$ and $\sup\limits_{\varphi(a)\in\mathbb{B}_X\backslash\{0\}} \|g_a\|_{nat} \le C$. By (2.11) and (2.12), we obtain

$$
\begin{aligned}
&\omega(z)|\psi'(z)z| \log\frac{2}{1-\|\varphi(z)\|^2} \\
&\le \omega(z)|\psi(z)g_z'(\varphi(z))\varphi'(z)z + g_z(\varphi(z))\psi'(z)z| \\
&\le \sup_{z\in\mathbb{B}_X} \omega(z)\left|\left(W_{\psi,\varphi}g_z\right)'(z)z\right| \\
&\le \sup_{z\in\mathbb{B}_X} \omega(z)\left|\mathcal{R}W_{\psi,\varphi}g_z(z)\right| \\
&= \|\mathcal{R}W_{\psi,\varphi}g_z\|_{\omega,\infty} \le \|\mathcal{R}W_{\psi,\varphi}\|\|g_z\|_{nat-Bloch} \\
&= \|\mathcal{R}W_{\psi,\varphi}\|\left(\|g_z\|_{nat} + |g_z(0)|\right) \\
&\le \|\mathcal{R}W_{\psi,\varphi}\|\left(C + 3\log 2\right) < \infty
\end{aligned} \tag{2.13}
$$

for all $\|\varphi(z)\| > r > 0$. If $\|\varphi(z)\| \le r < 1$, using (2.5), we have

$$
\begin{aligned}
&\omega(z)|\psi'(z)z| \log\frac{2}{1-\|\varphi(z)\|^2} \\
&\le \log\frac{2}{1-r^2}\omega(z)|\psi'(z)z| \\
&\le \log\frac{2}{1-r^2}\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z| < \infty,
\end{aligned}
$$

which, together with (2.13), proves that the condition in (2.2) is necessary. □

Remark 2.2. For $\lambda \in \partial\mathbb{B}_X$, set $\varphi(z) = \frac{1}{2}(z-\lambda)$, $\forall z \in \mathbb{B}_X$. Then $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$, $\varphi'(z) = \frac{1}{2}I_d$, and

$$
\sup_{z\in\overline{\mathbb{B}_X}} |\ell_{\varphi(a)}(\varphi'(a)z)| = \frac{1}{2}\sup_{z\in\overline{\mathbb{B}_X}} |\ell_{\varphi(a)}(z)| = \frac{1}{2}\|\ell_{\varphi(a)}\| = \frac{1}{2} = \|\varphi'(a)\|,
$$

for $a \in \mathbb{B}_X$.

Corollary 2.3. *Suppose $\psi \in H(\mathbb{B}_X)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. If $\sup_{z\in\overline{\mathbb{B}_X}} |\ell_{\varphi(a)}(\varphi'(a)z)| = \|\varphi'(a)\|$, for $a \in \mathbb{B}_X$ with $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$ and for $r \in (0,1)$,*

$$\sup_{|\varphi(z)|\le r} \omega(z)|\psi(z)|\|\varphi'(z)\| \le C < \infty,$$

then $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H^\infty_\omega(\mathbb{B}_X)$ is bounded if and only if

$$\sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1-\|\varphi(z)\|^2} < \infty$$

and

$$\sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2} < \infty.$$

Since taking $\varphi(z) = z$, $\sup_{z\in\overline{\mathbb{B}_X}} |\ell_w(\varphi'(a)z)| = 1 = \|\varphi'(a)\|$, for $w = \varphi(a) \in \mathbb{B}_X\backslash\{0\}$, we have the following result:

Corollary 2.4. *Suppose $\psi \in H(\mathbb{B}_X)$. Then the operator $\mathcal{R}M_\psi : \mathcal{B}_{nat}(\mathbb{B}_X) \to H^\infty_\omega(\mathbb{B}_X)$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1-\|z\|^2} < \infty \tag{2.14}$$

and

$$\sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|}{1-\|z\|^2} < \infty. \tag{2.15}$$

Proof. Necessity. Assume that $\mathcal{R}M_\psi : \mathcal{B}_{nat}(\mathbb{B}_X) \to H^\infty_\omega(\mathbb{B}_X)$ is bounded. By Theorem 2.1, we have

$$\sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1-\|z\|^2} < \infty$$

and for $r \in (0,1)$

$$\sup_{\{z\in\mathbb{B}_X : r<\|z\|<1\}} \frac{\omega(z)|\psi(z)|}{1-\|z\|^2} \le C < \infty.$$

So

$$\begin{aligned}
\sup_{z\in\mathbb{B}_X} \tfrac{\omega(z)|\psi(z)|}{1-\|z\|^2} &\le \sup_{\{z\in\mathbb{B}_X : \|z\|\le r\}} \frac{\omega(z)|\psi(z)|}{1-\|z\|^2} + \sup_{\{z\in\mathbb{B}_X : r<\|z\|<1\}} \frac{\omega(z)|\psi(z)|}{1-\|z\|^2} \\
&\le \frac{\max_{r\in[0,r]} \omega(r)}{1-r^2} \sup_{\{z\in\mathbb{B}_X : \|z\|\le r\}} |\psi(z)| + C \\
&< \infty,
\end{aligned}$$

that is, (2.15) holds.

Sufficiency. It is clear. □

When $X = \mathbb{C}$ and $\mathbb{B}_X = \mathbb{D}$, we find

$$\sup_{z\in\overline{\mathbb{D}}} |\ell_{\varphi(a)}(\varphi'(a)z)| = |\varphi'(a)|\|\ell_{\varphi(a)}\| = |\varphi'(a)|,$$

for $a \in \mathbb{D}$ with $\varphi(a) \in \mathbb{D}\backslash\{0\}$. Let $D_r = \{z \in \mathbb{D} : |\varphi(z)| \leq r\}$ $(0 < r < 1)$ and $h(z) = \omega(z)|\psi(z)||\varphi'(z)|$. By the fact that the derivative of an analytic function is itself analytic, we have h is a continuous function on the compact subset D_r of $\mathbb{D}$. Thus

$$\sup_{z\in D_r} \omega(z)|\psi(z)||\varphi'(z)| \leq C < \infty.$$

So we have the following corollary:

Corollary 2.5. *Let $\psi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Then $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}) \to H^\infty_\omega(\mathbb{D})$ is a bounded operator if and only if*

$$\sup_{z\in\mathbb{D}} \omega(z)|\psi'(z)z| \log \frac{2}{1 - |\varphi(z)|^2} < \infty$$

and

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi(z)||\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty.$$

Example. Suppose $\omega(z) = 1 - \|z\|$. We will give an example of a holomorphic mapping ψ from $\mathbb{B}_X$ into $\mathbb{C}$ and a holomorphic mapping φ from $\mathbb{B}_X$ into $\mathbb{B}_X$ such that $\mathcal{R}W_{\psi,\varphi}$ is bounded from $\mathcal{B}_{nat}(\mathbb{B}_X)$ into $H^\infty_\omega(\mathbb{B}_X)$, but the operator $\mathcal{R}M_\psi$ is not bounded from $\mathcal{B}_{nat}(\mathbb{B}_X)$ into $H^\infty_\omega(\mathbb{B}_X)$. Let $a \in \mathbb{B}_X\backslash\{0\}$ and define the function $\psi(z) = \log \frac{1}{1-\ell_a(z)}$ and $\varphi(z) = \frac{1}{2}(a - z)$ for $z \in \mathbb{B}_X$. Then $\psi \in H(\mathbb{B}_X)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. The operator $\mathcal{R}M_\psi$ is not bounded from $\mathcal{B}_{nat}(\mathbb{B}_X)$ into $H^\infty_\omega(\mathbb{B}_X)$ by Corollary 2.4 since $\psi \notin H^\infty(\mathbb{B}_X)$. On the other hand, it is straightforward to verify that

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1 - \|\varphi(z)\|^2} \\
&= \sup_{z\in\mathbb{B}_X} \omega(z) \left\| \frac{\ell_a'(z)}{1 - \ell_a(z)} \right\| \log \frac{2}{1 - \frac{\|a-z\|^2}{4}} \\
&\leq \sup_{z\in\mathbb{B}_X} \frac{1}{1 - \|z\|} \log \frac{8}{4 - \|a - z\|^2} \\
&\leq 2 \log \frac{4}{1 - \|a\|}.
\end{aligned}$$

Additionally, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1 - \|\varphi(z)\|^2} \\
&\leq \sup_{z\in\mathbb{B}_X} \frac{\frac{1}{2}\left|\log \frac{1}{1-\ell_a(z)}\right|}{1 - \frac{\|a-z\|^2}{4}}
\end{aligned}$$

$$\leq \sup_{z\in\mathbb{B}_X} \frac{\left(\log|\frac{1}{1-\ell_a(z)}| + \frac{\pi}{2}\right)}{1-\|a\|}$$
$$\leq \sup_{z\in\mathbb{B}_X} \frac{\left(\log\frac{1}{1-\|z\|} + \frac{\pi}{2}\right)}{1-\|a\|}$$
$$\leq \frac{C}{1-\|a\|}.$$

Therefore, the operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H^\infty_\omega(\mathbb{B}_X)$ is bounded by Theorem 2.1.

3. The boundedness of the product-type operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$

The classical Bloch space $\mathcal{B}(\mathbb{D})$ plays an important role in geometric function theory, and it has been studied by many authors. In this section, we will also be interested in the generalization of the little Bloch space $\mathcal{B}_0(\mathbb{D})$ consisting of functions f in $\mathcal{B}(\mathbb{D})$ with $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. Thus, we first introduce the little Bloch-type space $\mathcal{B}_{nat,0}(\mathbb{B}_X)$. Then we study the boundedness of the product-type operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$.

Definition 3.1. The little Bloch-type space $\mathcal{B}_{nat,0}(\mathbb{B}_X)$ (the subspace of $\mathcal{B}_{nat}(\mathbb{B}_X)$) consists of all functions $f \in \mathcal{B}_{nat}(\mathbb{B}_X)$ for which

$$\lim_{\|z\|\to 1}(1-\|z\|^2)\|f'(z)\| = 0.$$

Next, we formulate and prove several auxiliary results.

Lemma 3.2. *If $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$, then*

$$\lim_{\|z\|\to 1} \frac{|f(z)|}{\log\frac{2}{1-\|z\|^2}} = 0.$$

Proof. If $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$, then $\forall\epsilon > 0$, there is a $\delta > 0$ such that

$$\|f'(z)\| < \frac{\epsilon}{3(1-\|z\|^2)} \tag{3.1}$$

and

$$\frac{|f(0)| + \log 2\|f\|_{nat}}{\log\frac{2}{1-\|z\|^2}} < \frac{\epsilon}{3}, \tag{3.2}$$

for all z with $\delta < \|z\| < 1$. Using the following limit again:

$$\lim_{\|z\|\to 1} \frac{1}{\log\frac{2}{1-\|z\|^2}} = 0,$$

there is a $\tau \in (\delta, 1)$ such that

$$\frac{1}{\log\frac{2}{1-\|z\|^2}} < \frac{\epsilon}{3\log\frac{2}{1-\delta}\|f\|_{nat}}, \tag{3.3}$$

for all z with $\tau < \|z\| < 1$. By (3.1–3.3), we have

$$\begin{aligned}
|f(z)| &\le |f(0)| + \int_0^1 \|f'(tz)\|\|z\|dt \\
&= |f(0)| + \int_0^{\frac{\delta}{\|z\|}} \|f'(tz)\|\|z\|dt + \int_{\frac{\delta}{\|z\|}}^1 \|f'(tz)\|\|z\|dt \\
&\le |f(0)| + \|z\| \int_0^{\frac{\delta}{\|z\|}} \frac{\|f\|_{nat}}{1-\|tz\|^2}dt + \int_{\frac{\delta}{\|z\|}}^1 \frac{\epsilon\|z\|}{3(1-t^2\|z\|^2)}dt \\
&\le |f(0)| + \frac{1}{2}\log\frac{1+\delta}{1-\delta}\|f\|_{nat} + \frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}\frac{\epsilon}{3} \\
&\le |f(0)| + \log\frac{2}{1-\delta}\|f\|_{nat} + \log\frac{2}{1-\|z\|^2}\frac{\epsilon}{3} \\
&< \log\frac{2}{1-\|z\|^2}\frac{\epsilon}{3} + \log\frac{2}{1-\|z\|^2}\frac{\epsilon}{3} + \log\frac{2}{1-\|z\|^2}\frac{\epsilon}{3} \\
&= \log\frac{2}{1-\|z\|^2}\epsilon,
\end{aligned}$$

for all z with $\tau < \|z\| < 1$, that is

$$\lim_{\|z\|\to 1} \frac{|f(z)|}{\log\frac{2}{1-\|z\|^2}} = 0.$$

□

Proposition 3.3. *The little Bloch-type space $\mathcal{B}_{nat,0}(\mathbb{B}_X)$ is a closed subspace of $\mathcal{B}_{nat}(\mathbb{B}_X)$.*

Proof. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{nat,0}(\mathbb{B}_X)$ with

$$\lim_{j\to\infty} \|f_j - f\|_{nat-Bloch} = 0, \text{for } f \in \mathcal{B}_{nat}(\mathbb{B}_X).$$

$\forall\epsilon > 0$, there is a $j_0 \in \mathbb{N}$ such that

$$\|f_j - f\|_{nat-Bloch} < \frac{\epsilon}{2},$$

for all j with $j \ge j_0$. Since $f_{j_0} \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$, there exists a $\delta > 0$ such that

$$(1-\|z\|^2)\|f'_{j_0}(z)\| < \frac{\epsilon}{2}, \text{ for } \delta < \|z\| < 1.$$

Thus, we get

$$\begin{aligned}
&(1-\|z\|^2)\|f'(z)\| \\
&\le (1-\|z\|^2)\|f'(z) - f'_{j_0}(z)\| + (1-\|z\|^2)\|f'_{j_0}(z)\| \\
&\le \sup_{z\in\mathbb{B}_X}(1-\|z\|^2)\|f'(z) - f'_{j_0}(z)\| + (1-\|z\|^2)\|f'_{j_0}(z)\| \\
&\le \|f_{j_0} - f\|_{nat-Bloch} + (1-\|z\|^2)\|f'_{j_0}(z)\| \\
&< \epsilon, \text{ for } \delta < \|z\| < 1,
\end{aligned}$$

that is $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$. □

The following theorem describes the boundedness of the product-type operator $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$.

Theorem 3.4. *Suppose $\psi \in H(\mathbb{B}_X)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$.*

(1) *If*

$$\lim_{\|z\|\to 1} \omega(z)|\psi'(z)z| = 0, \tag{3.4}$$

$$\lim_{\|z\|\to 1} \omega(z)|\psi(z)|\|\varphi'(z)\| = 0, \tag{3.5}$$

$$M := \sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| \log \frac{2}{1-\|\varphi(z)\|^2} < \infty \tag{3.6}$$

and

$$L := \sup_{z\in\mathbb{B}_X} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2} < \infty, \tag{3.7}$$

then $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded.

(2) *If $\sup_{z\in\overline{\mathbb{B}_X}} |\ell_{\varphi(a)}(\varphi'(a)z)| = \|\varphi'(a)\|$, for $a \in \mathbb{B}_X$ with $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$ and $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded, then* (3.4), (3.6) *holds and*

$$\sup_{\{z\in\mathbb{B}_X: r<\|\varphi(z)\|<1\}} \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|}{1-\|\varphi(z)\|^2} < \infty, \tag{3.8}$$

for $r \in (0,1)$.

Proof. (1) First, assume that (3.4)–(3.7) holds. By Theorem 2.1, we know that $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat}(\mathbb{B}_X) \to H^\infty_{\omega}(\mathbb{B}_X)$ is bounded. Since $\mathcal{B}_{nat,0}(\mathbb{B}_X) \subset \mathcal{B}_{nat}(\mathbb{B}_X)$, $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega}(\mathbb{B}_X)$ is bounded. Therefore, from the closed graph theorem, we only need to prove that $\mathcal{R}W_{\psi,\varphi}f \in H^\infty_{\omega,0}(\mathbb{B}_X)$ for all $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$. For an arbitrarily small positive number ϵ, if $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$, by Lemma 3.2, there exists a $0 < \delta_1 < 1$ such that

$$(1-\|z\|^2)\|f'(z)\| < \frac{\epsilon}{2L},$$

$$\left(\log \frac{2}{1-\|z\|^2}\right)^{-1} |f(z)| < \frac{\epsilon}{2M},$$

for all z with $\delta_1 < \|z\| < 1$. If $\|\varphi(z)\| > \delta_1$, it from (3.6) and (3.7) follows that

$$\begin{aligned}
&\omega(z)\left|\mathcal{R}(W_{\psi,\varphi}f)(z)\right| = \omega(z)\left|(W_{\psi,\varphi}f)'(z)z\right| \\
&\le \omega(z)|\psi(z)|\|f'(\varphi(z))\|\|\varphi'(z)z\| + \omega(z)|f(\varphi(z))||\psi'(z)z| \\
&\le \frac{\omega(z)|\psi(z)|\|\varphi'(z)\|\epsilon}{2L(1-\|\varphi(z)\|^2)} + \frac{\epsilon\omega(z)|\psi'(z)z|\log\frac{2}{1-\|\varphi(z)\|^2}}{2M} \\
&< \epsilon,
\end{aligned} \tag{3.9}$$

for all z with $\delta_1 < \|z\| < 1$. Choose a constant K_1 such that $|f(z)| \le K_1$ for all $\|z\| \le \delta_1$. Since for $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X) \subset \mathcal{B}_{nat}(\mathbb{B}_X)$, we have $\|f\|_{nat} = \sup\{(1-\|z\|^2)\|f'(z)\| : z \in \mathbb{B}_X\} < \infty$, therefore

$$\|f'(z)\| \le \frac{\|f\|_{nat}}{1-\|z\|^2} \le \frac{\|f\|_{nat}}{1-\delta_1^2},$$

for all $\|z\| \le \delta_1$. Choose a constant $K = \max\left\{K_1, \frac{\|f\|_{nat}}{1-\delta_1^2}\right\}$, then $|f(z)| \le K$ and $\|f'(z)\| \le K$ for all $\|z\| \le \delta_1$. Using (3.4) and (3.5), there is a $\delta_2 \in (\delta_1, 1)$ such that

$$\omega(z)|\psi'(z)z| < \frac{\epsilon}{2K}, \tag{3.10}$$

and

$$\omega(z)|\psi(z)|\|\varphi'(z)\| < \frac{\epsilon}{2K}, \tag{3.11}$$

for all z with $\delta_2 < \|z\| < 1$. If $\|\varphi(z)\| \le \delta_1$, using (3.10) and (3.11), we have

$$\begin{aligned}
&\omega(z)\left|\mathcal{R}(W_{\psi,\varphi}f)(z)\right| \\
&\le \omega(z)|\psi(z)|\|f'(\varphi(z))\|\|\varphi'(z)z\| + \omega(z)|f(\varphi(z))||\psi'(z)z| \\
&\le K\omega(z)|\psi(z)|\|\varphi'(z)\| + K\omega(z)|\psi'(z)z|, \\
&< \epsilon,
\end{aligned} \tag{3.12}$$

for all z with $\delta_2 < \|z\| < 1$. From (3.9) and (3.12), we conclude that

$$\lim_{\|z\|\to 1} \omega(z)\left|\mathcal{R}(W_{\psi,\varphi}f)(z)\right| = 0.$$

Hence $\mathcal{R}W_{\psi,\varphi}f \in H^\infty_{\omega,0}(\mathbb{B}_X)$ for all $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$. So $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded.

(2) Suppose $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded. That means that $\mathcal{R}W_{\psi,\varphi}f \in H^\infty_{\omega,0}(\mathbb{B}_X)$ for all $f \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$. If we choose $f(z) = 1 \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$, we have

$$\sup_{z\in\mathbb{B}_X} \omega(z)|\psi'(z)z| = \|\mathcal{R}W_{\psi,\varphi}f\|_{\omega,\infty} \le C\|f\|_{nat-Bloch} = C < \infty. \tag{3.13}$$

and

$$\lim_{\|z\|\to 1} \omega(z)|\psi'(z)z| = \lim_{\|z\|\to 1} \omega(z)\left|\mathcal{R}W_{\psi,\varphi}f(z)\right| = 0,$$

that is, (3.4) holds.

To prove (3.8) holds, fix $a \in \mathbb{B}_X$; if $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$, let $w = \varphi(a)$ and $\ell_w \in T(w)$ be fixed. We take $f_a : \mathbb{B}_X \to \mathbb{C}$ (see (2.6)). Then

$$f_a'(z) = \frac{\|\varphi(a)\|}{(1-\|\varphi(a)\|\ell_w(z))^2}\ell_w'(z),$$

therefore,

$$\|f_a'(z)\| \le \frac{\|\varphi(a)\|}{|1-\|\varphi(a)\|\ell_w(z)|^2} \le \frac{1}{(1-\|\varphi(a)\|)^2},$$

so $f_a \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$ with $\|f_a\|_{nat} \leq \frac{2}{1-\|\varphi(a)\|} \leq \frac{4}{1-\|\varphi(a)\|^2}$. Hence, using the triangle inequality and (2.6)–(2.8), we get for $0 < r < \|\varphi(a)\| < 1$

$$\begin{aligned}
&\left(1+\frac{4}{1-\|\varphi(a)\|^2}\right)\left\|\mathcal{R}W_{\psi,\varphi}\right\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)}\\
&\geq \|f_a\|_{nat-Bloch}\left\|\mathcal{R}W_{\psi,\varphi}\right\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)} \geq \left\|\mathcal{R}W_{\psi,\varphi}f_a\right\|_{\omega,\infty}\\
&= \sup_{z\in\mathbb{B}_X}\omega(z)\left|\left(W_{\psi,\varphi}f_a\right)'(z)z\right|\\
&= \sup_{z\in\mathbb{B}_X}\omega(z)|\psi(z)(f_a\circ\varphi)'(z)z+(f_a\circ\varphi)(z)\psi'(z)z|\\
&\geq \frac{\omega(a)|\psi(a)|\|\varphi(a)\|\|\varphi'(a)a\|}{\left(1-\|\varphi(a)\|^2\right)^2}-\frac{\omega(a)|\psi'(a)a|}{1-\|\varphi(a)\|^2}.
\end{aligned}\tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned}
&\sup_{r<\|\varphi(z)\|<1}\frac{\omega(z)|\psi(z)|\|\varphi'(z)z\|}{1-\|\varphi(z)\|^2}\\
&\leq \frac{1}{r}\sup_{r<\|\varphi(z)\|<1}\frac{\omega(z)\|\varphi(z)\||\psi(z)|\|\varphi'(z)z\|}{1-\|\varphi(z)\|^2}\\
&\leq \frac{1}{r}\left((5-r^2)\left\|\mathcal{R}W_{\psi,\varphi}\right\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)}+\sup_{r<\|\varphi(z)\|<1}\omega(z)|\psi'(z)z|\right)\\
&\leq \frac{1}{r}\left((5-r^2)\left\|\mathcal{R}W_{\psi,\varphi}\right\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)}+\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|\right),
\end{aligned}$$

that is, (3.8) holds.

Next, we will prove (3.6). For given $a \in \mathbb{B}_X$ if $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$, consider the function g_a given by (2.9). Using (2.10), we have for $z \in \mathbb{B}_X$

$$\begin{aligned}
\|g_a'(z)\| &\leq \frac{2}{1-\|\varphi(a)\|}+\left|\frac{2\log\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}}{\log\frac{2}{1-\|\varphi(a)\|^2}}\right|\frac{2}{1-\|\varphi(a)\|}\\
&\leq \frac{2}{1-\|\varphi(a)\|}+\frac{4\left(\log\left|\frac{2}{1-\|\varphi(a)\|\ell_{\varphi(a)}(z)}\right|+\frac{\pi}{2}\right)}{(1-\|\varphi(a)\|)\log\frac{2}{1-\|\varphi(a)\|^2}}\\
&\leq \frac{2}{1-\|\varphi(a)\|}+\frac{4\left(\log\frac{4}{1-\|\varphi(a)\|^2}+\frac{\pi}{2}\right)}{(1-\|\varphi(a)\|)\log\frac{2}{1-\|\varphi(a)\|^2}},
\end{aligned}$$

therefore, $g_a \in \mathcal{B}_{nat,0}(\mathbb{B}_X)$ for $\varphi(a) \in \mathbb{B}_X\backslash\{0\}$ and $\sup\limits_{\varphi(a)\in\mathbb{B}_X\backslash\{0\}}\|g_a\|_{nat} \leq C$. By (2.11) and (2.12), we conclude that

$$\begin{aligned}
&\omega(z)|\psi'(z)z|\log\frac{2}{1-\|\varphi(z)\|^2}\\
&\leq \omega(z)|\psi(z)g_z'(\varphi(z))\varphi'(z)z+g_z(\varphi(z))\psi'(z)z|\\
&\leq \sup_{z\in\mathbb{B}_X}\omega(z)\left|\left(\mathcal{R}W_{\psi,\varphi}g_z\right)(z)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \|\mathcal{R}W_{\psi,\varphi}g_z\|_{\omega,\infty} \le \|\mathcal{R}W_{\psi,\varphi}\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)}\|g_z\|_{nat-Bloch} \\
&\le \|\mathcal{R}W_{\psi,\varphi}\|_{\mathcal{B}_{nat,0}(\mathbb{B}_X)\to H^\infty_\omega(\mathbb{B}_X)}(C+3\log 2)<\infty
\end{aligned} \tag{3.15}
$$

for all $\varphi(z) \in \mathbb{B}_X\backslash\{0\}$. If $\varphi(z)=0$, using (3.13) and $\mathcal{B}_{nat,0}(\mathbb{B}_X) \subset \mathcal{B}_{nat}(\mathbb{B}_X)$, we have

$$
\begin{aligned}
&\omega(z)|\psi'(z)z|\log\frac{2}{1-\|\varphi(z)\|^2} \\
&= \log 2\omega(z)|\psi'(z)z| \le \log 2\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|<\infty,
\end{aligned}
$$

which and (3.15) prove that the condition in (3.6) is necessary, finishing the proof of the theorem. □

Corollary 3.5. *Suppose $\psi \in H(\mathbb{B}_X)$. Then $\mathcal{R}M_\psi : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded if and only if $\psi \in H^\infty_{\omega,0}(\mathbb{B}_X)$,*

$$
\lim_{\|z\|\to 1}\omega(z)|\psi'(z)z| = 0, \tag{3.16}
$$

$$
J := \sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|\log\frac{2}{1-\|z\|^2}<\infty, \tag{3.17}
$$

and

$$
\sup_{z\in\mathbb{B}_X}\frac{\omega(z)|\psi(z)|}{1-\|z\|^2}<\infty. \tag{3.18}
$$

Proof. Sufficiency. It is clear.

Necessity. Assume that $\mathcal{R}M_\psi : \mathcal{B}_{nat,0}(\mathbb{B}_X) \to H^\infty_{\omega,0}(\mathbb{B}_X)$ is bounded. Then $\psi \in H^\infty_{\omega,0}(\mathbb{B}_X)$, (3.17) holds and

$$
\sup_{\{z\in\mathbb{B}_X:r<\|z\|<1\}}\frac{\omega(z)|\psi(z)|}{1-\|z\|^2}<\infty,
$$

for $r \in (0,1)$. From which we get

$$
\begin{aligned}
\sup_{z\in\mathbb{B}_X}\tfrac{\omega(z)|\psi(z)|}{1-\|z\|^2} &\le \sup_{\{z\in\mathbb{B}_X:\|z\|\le r\}}\frac{\omega(z)|\psi(z)|}{1-\|z\|^2}+\sup_{\{z\in\mathbb{B}_X:r<\|z\|<1\}}\frac{\omega(z)|\psi(z)|}{1-\|z\|^2} \\
&\le \sup_{\{z\in\mathbb{B}_X:\|z\|\le r\}}\frac{\omega(z)|\psi(z)|}{1-\|z\|^2}+C<\infty,
\end{aligned}
$$

that is condition (3.18) holds. Moreover,

$$
\begin{aligned}
\omega(z)|\psi'(z)z| &\le \frac{1}{\log\frac{2}{1-\|z\|^2}}\sup_{z\in\mathbb{B}_X}\omega(z)|\psi'(z)z|\log\frac{2}{1-\|z\|^2} \\
&= \frac{J}{\log\frac{2}{1-\|z\|^2}} \to 0,\ (\|z\|\to 1),
\end{aligned}
$$

that is, (3.16) holds. □

When $X = \mathbb{C}$ and $\mathbb{B}_X = \mathbb{D}$, we have the following corollary:

Corollary 3.6. *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{D}) \to H^\infty_{\omega,0}(\mathbb{D})$ is bounded if and only if $\psi' \in H^\infty_{\omega,0}(\mathbb{D})$*

$$\lim_{|z|\to 1} \omega(z)|\psi(z)||\varphi'(z)| = 0, \tag{3.19}$$

$$\sup_{z\in\mathbb{D}} \omega(z)|\psi'(z)z| \log \frac{2}{1-|\varphi(z)|^2} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)|\psi(z)||\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

Proof. By Theorem 3.4, we only need to prove that if $\mathcal{R}W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{D}) \to H^\infty_{\omega,0}(\mathbb{D})$ is bounded, (3.19) holds. Taking $f(z) = z$, we have $f \in \mathcal{B}_0(\mathbb{D})$, so $\mathcal{R}W_{\psi,\varphi}f \in H^\infty_{\omega,0}(\mathbb{D})$. In addition, we have for every $z \in \mathbb{D}$

$$\begin{aligned} & \omega(z)\left|\mathcal{R}W_{\psi,\varphi}f(z)\right| = \omega(z)\left|(W_{\psi,\varphi}f)'(z)z\right| \\ = \;& \omega(z)\,|\psi'(z)\varphi(z)z + \psi(z)\varphi'(z)z| \\ \geq \;& \omega(z)|\psi(z)\varphi'(z)z| - \omega(z)|\psi'(z)\varphi(z)z|. \end{aligned}$$

From $\psi' \in H^\infty_{\omega,0}(\mathbb{D})$ it follows that

$$\begin{aligned} & \omega(z)\,|\psi(z)|\,|\varphi'(z)z| \\ \leq \;& \omega(z)\left|\left(\mathcal{R}W_{\psi,\varphi}f\right)(z)\right| + \omega(z)|\psi'(z)||\varphi(z)z| \\ \leq \;& \omega(z)\left|\left(\mathcal{R}W_{\psi,\varphi}f\right)(z)\right| + \omega(z)|\psi'(z)| \\ \to \;& 0\ (|z| \to 1), \end{aligned}$$

that is, (3.19) holds. □

4. Conclusions

There has been huge interest in the operators on subspaces of $H(\mathbb{B}_X)$. Up to now, there have been fewer results on the product of the weighted composition operator and the radial derivative operator on subspaces of $H(\mathbb{B}_X)$. Thus, our hope is that this exposition will inspire more work in this area. In this study, our aim is to investigate the boundedness of the product of the radial derivative operator and the weighted composition operator from the natural Bloch spaces $\mathcal{B}_{nat}(\mathbb{B}_X)$ (or the little Bloch spaces $\mathcal{B}_{nat,0}(\mathbb{B}_X)$) into the weighted-type spaces $H^\infty_\omega(\mathbb{B}_X)$ (or the little weighted-type spaces $H^\infty_{\omega,0}(\mathbb{B}_X)$). This provides a good starting point for discussion and further research. Of course, working with operators on the unit ball of Banach spaces X has some difficulties compared to the product of operators on the subspace of all holomorphic functions on the open unit disc or the unit ball. Mainly because the test function in the natural Bloch space $\mathcal{B}_{nat}(\mathbb{B}_X)$ or $\mathcal{B}_{nat,0}(\mathbb{B}_X)$ is not easy to obtain. Just because of this, this is an interesting topic for future work.

Author contributions

Xiaoman Liu: Investigation; validation; supervision; writing-review and editing. **Yongmin Liu**: Proposed the investigation in the paper; project administration; writing-review and editing. All authors contributed equally to the writing of this paper. They also read and approved the final manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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