AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 19557–19568.
DOI: 10.3934/math.2024954
Received: 31 January 2024
Revised: 10 April 2024
Accepted: 18 April 2024
Published: 13 June 2024

Research article

A reliable numerical algorithm for fractional Lienard equation arising in oscillating circuits

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Abstract: This work presents a numerical approach for handling a fractional Lienard equation (FLE) arising in an oscillating circuit. The scheme is based on the Vieta Lucas operational matrix of the fractional Liouville-Caputo derivative and the collocation method. This methodology involves a systematic approach wherein the operational matrix aids in expressing the fractional problem in terms of non-linear algebraic equations. The proposed numerical approach utilizing the operational matrix method offers a vital solution framework for efficiently tackling the fractional Lienard equation, addressing a key challenge in mathematical modeling. To analyze the fractional order system, we derive an approximate solution for the FLE. The solutions are explained graphically and in tabular form.

Keywords: fractional Lienard equation; operational matrix of differentiation; Vieta Lucas polynomials; collocation method; error analysis
Mathematics Subject Classification: 26A33, 33C45, 65L05

1. Introduction

In the advancement of radio and vacuum tube technology, the Lienard equation is used to explain an oscillating circuit, so it has received extensive attention from researchers. The Lienard equation is a nonlinear differential equation of second order and is expressed as [1]

$$z''(\eta) + \kappa_1(z)z'(\eta) + \kappa_2(z) = \kappa_3(\eta), \tag{1.1}$$

where, $\kappa_2(z)$ is restoring force, $\kappa_1(z)z'$ is a damping force and $\kappa_3(\eta)$ is an external force. The Lienard model exists in many physical phenomena for various options of $\kappa_1(z), \kappa_2(z)$ and $\kappa_3(\eta)$ [2, 3].

It is highly complicated to acquire exact solutions [4] of these nonlinear equations by conventional methods. Kong conducted an investigation into the particular structure of the Lienard model [5],

$$z''(\eta) + cz'(\eta) + dz^3(\eta) + ez^5(\eta) = 0, \tag{1.2}$$

where c, d and e are real constants.

Fractional derivatives allow for the precise identification of a physical phenomenon's perfect model, which depends on both the present and a prior time. In addition, there are several practical applications for fractional calculus in the fields of science and engineering [6–12]. Exploring textbooks, research papers, online courses and attending seminars or workshops can provide valuable insights into the practical applications of fractional calculus. Additionally, experimenting with software tools and numerical methods tailored for fractional calculus computations can offer hands-on experience and deepen comprehension of its real-world implementation [13–15]. The fractional Lienard equation is pivotal in analyzing oscillating circuits, representing the dynamics of voltage and current with fractional derivatives, crucial for understanding complex electrical systems and signal processing applications. The fractional operators are of non-local type, so they contain previous memory of the system. To study this system, we substitute the Liouville-Caputo derivative for the classical derivative in the Lienard equation. This substitution yields the FLE, which is formulated as follows:

$$z^{\alpha}(\eta) + cz'(\eta) + dz^3(\eta) + ez^5(\eta) = 0, 1 < \alpha \leq 2, \eta \in [0, 1], \tag{1.3}$$

with

$$z(0) = \sigma, \quad z'(0) = \delta, \tag{1.4}$$

initial conditions and $\sigma, \delta \in R$.

Kong [5] examined the classical Lienard equation and acquired the exact solution for specific values of constants c, d and e. Feng generalized Kong's outcome for the FLE [4]. For the approximate solutions of the Lienard equation, Matnifar et al. [16,17] suggested a variational homotopy perturbation technique and variational iteration method. To solve the FLE, Singh et al. [18] suggested a numerical approach by using the homotopy analysis transform technique.

The operational matrix method is an extremely efficient method for solving differential calculus problems. Advancements in the operational matrix method for arbitrary order differential equations include refined techniques like the Caputo and Riemann-Liouville fractional derivatives. These methods enhance accuracy and efficiency, especially in complex systems. Integration with other numerical approaches further extends its applicability, facilitating precise solutions for a wide range of fractional differential equations. Singh [19] obtained an approximate solution of the FLE by using Chebyshev operational matrix method. The FLE has been solved numerically by using an operational matrix of Legendre scaling polynomials and Jacobi polynomials [20, 21]. Singh et al. [22] investigated the FLE with exponential memory.

We provide a approximate technique to solve FLE in this work. For Vieta Lucas polynomials (VLPs), the proposed method merges the collocation technique with the operational matrix method. This combination produces a system of nonlinear algebraic equations (NLAEs) and solving these equations provides an approximate solution to the FLE. The solution's behavior is demonstrated for various fractional orders related to the FLE.

The organization of this article is outlined as follows: Section 2 offers a fundamental definition of fractional calculus and properties of Vieta Lucas polynomials. The computational procedure of the method is introduced in Section 3. Moving on to Section 4, we analyze the suggested technique for solving arbitrary order Lienard model. We present and discuss numerical results in Section 5. We give concluding remarks in Section 6.

2. Preliminaries

In this study, we employed the Liouville-Caputo type derivative of arbitrary order.

Definition 2.1. *The Liouville-Caputo derivative of fractional order $\alpha \geq 0$ is provided [14]:*

$$(D^{\alpha} g(\eta)) = \begin{cases} \frac{1}{\Gamma(l-\alpha)} \int_0^{\eta} (\eta - t)^{l-\alpha-1} \frac{d^l}{dt^l} g(t)dt, & l-1 < \alpha < l. \\ \frac{d^l}{d\eta^l} g(\eta), & \alpha = l \in N. \end{cases} \tag{2.1}$$

Definition 2.2. *For $\alpha > 0$ and $g(\eta) \in H_1(c, d)$ where $H_1(c, d)$ is the space of all integrable functions on (c, d), the Riemann-Liouville fractional integral of order α, indicated by I_0^{α}, is provided by*

$$I_0^{\alpha} g(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - t)^{\alpha-1} g(t)dt.$$

2.1. VLPs

For the shifted VLPs on $[0, 1]$, the analytical form is [23],

$$\Omega_n(\eta) = 2n \sum_{J=0}^{n} \frac{(-1)^J 4^{n-J} (2n - J - 1)!}{J!(2n - 2J)!} \eta^{n-J}; n \geq 1, \tag{2.2}$$

with $\Omega_0(\eta) = 2$.

It is feasible to extend the function g described in $L^2[0, 1]$ as an infinite sum of the shifted VLPs with $|g''(\eta)| \leq K$:

$$g(\eta) = \lim_{q\to\infty} \sum_{i=0}^{q} p_i \Omega_i(\eta), \tag{2.3}$$

where

$$p_i = \frac{1}{\theta_i \pi} \int_0^1 g(\eta)\Omega_i(\eta)\rho(\eta)d\eta; \quad i = 0, 1, 2, \ldots, \ \rho(\eta) = \frac{1}{\sqrt{\eta - \eta^2}}, \quad \theta_0 = 4 \ \text{and} \ \theta_i = 2(i \geq 1). \tag{2.4}$$

Using Eq (2.3)'s finite dimension approximations, we obtain

$$g(\eta) \cong \sum_{i=0}^{n} a_i \Omega_i(\eta) = P^T \Omega_n(\eta), \tag{2.5}$$

where the $(n+1)\times 1$ matrices $\Omega_n(\eta)$ and P are represented by

$$P = [p_0, p_1, \ldots, p_n]^T \ \ and\ \Omega_n(\eta) = [\Omega_0(\eta), \Omega_1(\eta), \ldots .\Omega_n(\eta)]^T . \tag{2.6}$$

2.2. Vieta Lucas operational matrix for fractional derivative

Theorem 2.1. *If $\Omega_n(\eta) = [\Omega_0(\eta), \Omega_1(\eta), \ldots\ldots, \Omega_n(\eta)]^T$ is the VLP vector and $\alpha > 0$, then*

$$D^{\alpha}\Omega_i(\eta) = D^{(\alpha)}\Omega_n(\eta), \tag{2.7}$$

where $D^{(\alpha)}$ is $(n+1)\times(n+1)$ operational matrix of Liouville-Caputo derivative of arbitrary order α and is explicitly formulated as follows [23]:

$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=0}^{i-\lceil\alpha\rceil}\xi_{i,0,k} & \sum_{k=0}^{i-\lceil\alpha\rceil}\xi_{i,1,k} & \cdots & \sum_{k=0}^{i-\lceil\alpha\rceil}\xi_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{m-\lceil\alpha\rceil}\xi_{m,0,k} & \sum_{k=0}^{m-\lceil\alpha\rceil}\xi_{m,1,k} & \cdots & \sum_{k=0}^{m-\lceil\alpha\rceil}\xi_{m,m,k} \end{pmatrix} \text{ and } \xi_{i,j,k} \text{ is given by}$$

$$\xi_{i,j,k} = \begin{cases} i\sum_{k=0}^{i-\lceil\alpha\rceil}(-1)^k \frac{4^{i-k}\Gamma(2i-k)\Gamma(i-k+1)\Gamma(i-k-\alpha+1/2)}{\sqrt{\pi}\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k-\alpha+1)^2}, & j=0, \\ 2i\sum_{k=0}^{i-\lceil\alpha\rceil}\sum_{r=0}^{j}\frac{(-1)^{k+r}}{\sqrt{\pi}}\frac{4^{i-k}\Gamma(2i-k)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k-\alpha+1)} & \\ \times\frac{4^{j-r}\Gamma(2j-r)\Gamma(i+j-k-r-\alpha+1/2)}{\Gamma(r+1)\Gamma(2j-2r+1)\Gamma(i+j+k-r-\alpha+1)}, & j=1,2,3,\ldots \end{cases}$$

Proof. Please see [23]. □

3. Computational procedure of the method

We suggest an approach for the approximate solution of the FLE in this section.

Step 1. For the unknown function in the FLE, the following approximation is created using Eq (2.5):

$$z(\eta) = \sum_{i=0}^{n} p_i\Omega_i(\eta) = P^T\Omega_n(\eta). \tag{3.1}$$

Step 2. Using Eq (3.1) in FLE (1.3), we obtain

$$P^T D^{(\alpha)}\Omega_n(\eta) + cP^T D^{(1)}\Omega_n(\eta) + d\left(P^T\Omega_n(\eta)\right)^3 + e\left(P^T\Omega_n(\eta)\right)^5 = 0, \tag{3.2}$$

where operational matrix of differentiations of order α and 1 are represented by $D^{(\alpha)}$ and $D^{(1)}$ accordingly and can obtained by Eq (2.7).

Step 3. The residual for Eq (3.2) is

$$R_n(\eta) = P^T D^{(\alpha)}\Omega_n(\eta) + cP^T D^{(1)}\Omega_n(\eta) + d\left(P^T\Omega_n(\eta)\right)^3 + e\left(P^T\Omega_n(\eta)\right)^5 . \tag{3.3}$$

Step 4. Now collocate $n-1$ points in Eq (3.3) given by $\eta_i = i/n, i = 0, 1, 2, 3, \dots, n-2$.
By Eqs (1.4), (3.1) and (3.3), we find

$$R_n(\eta_i) = P^T D^{(\alpha)}\Omega_n(\eta_i) + cP^T D^{(1)}\Omega_n(\eta_i) + d\left(P^T\Omega_n(\eta_i)\right)^3 + e\left(P^T\Omega_n(\eta_i)\right)^5 = 0, \tag{3.4}$$

$$P^T\Omega_n(0) = \sigma, \quad P^T D^{(1)}\Omega_n(0) = \delta. \tag{3.5}$$

Step 5. We obtain a system of $(n+1)$ NLAEs. The solution of these equations provides the approximation's unknowns by using collocation points in Eqs (3.4) and (3.5). The approximate solution for the FLE is obtained utilizing these unknowns in Eq (3.1).

4. Error analysis of the scheme

Theorem 4.1. *Let the function $z : [0,1] \to R$, $z \in C^{(n+1)}[0,1]$ and $z_n(t)$ represents the n^{th} approximation obtained by employing VLP. Then*

$$E^h_{z,n} = \|z - z_n\|_{L^2_\delta[0,1]}, \tag{4.1}$$

and as $n \to \infty$, $E^h_{z,n}$ approaches 0.

Proof. See [23]. □

Theorem 4.2. *Consider H as a Hilbert space, with X being a closed subspace of H s.t.* $\dim X < \infty$ *and $\{x_1, x_2, \cdots, x_M\}$ is any basis for X. Let z be an arbitrary element in H and x_0 be the unique best approximation to z out of X. Then,*

$$\|z - x_0\|_2^2 = \frac{G(z; x_1, x_2, \dots, x_M)}{G(x_1, x_2, \dots, x_M)},$$

where

$$G(z; x_1, x_2, \dots, x_M) = \begin{vmatrix} \langle z, z\rangle & \langle z, x_1\rangle & \cdots & \langle z, x_M\rangle \\ \langle x_1, z\rangle & \langle x_1, x_1\rangle & \cdots & \langle x_1, x_M\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_M, z\rangle & \langle x_M, x_1\rangle & \cdots & \langle x_M, x_M\rangle \end{vmatrix}.$$

Proof. Please see [24, 25]. □

Theorem 4.3. *Consider a function $g \in L^2[0,1]$, is approximated as in Eq (2.5) by $f_M(\eta)$ as*

$$f_M(\eta) = \sum_{k=0}^{M} P_k\Omega_k(\eta).$$

Consider $S_M(g) = \int_0^1 [g(\eta) - f_N(\eta)]^2\, d\eta$. Then, we have $\lim_{M\to\infty} S_M(g) = 0$.

Proof. Please see [24–26]. □

Theorem 4.4. *If $E^{\alpha,h}_{D,n}$ represents the error vector for operational matrix of differentiation of α order, obtained by utilizing $(n+1)$ VLPs, then in this scenario, we have*

$$E^{\alpha,h}_{D,n} = D^{(\alpha)}\Omega_n(\eta) - D^\alpha\Omega_n(\eta), \tag{4.2}$$

tending to 0 *as $n \to \infty$.*

Proof. Consider $E_{D,n}^{\alpha,h}$ is the error of an operational matrix of the Liouville-Caputo operator of fractional order. Then,

$$E_{D,n}^{\alpha,h} = D^{(\alpha)}\Omega_n(\eta) - D^{\alpha}\Omega_n(\eta),$$

and

$$E_{D,n}^{\alpha,h} = \left[E_{D,0}^{\alpha,h}, E_{D,1}^{\alpha,h}, \cdots, E_{D,n}^{\alpha,h}\right]^T.$$

By approximating $(\eta)^{i-k-\alpha}$ as in Eq (2.5) and from Theorem 4.2,

$$\left\|(\eta)^{i-k-\alpha} - \sum_{j=0}^{m} P_j\Omega_n(\eta)\right\|_2 = \left(\frac{G\left((\eta)^{i-k-\alpha}; \Omega_0(\eta), \cdots, \Omega_n(\eta)\right)}{G\left(\Omega_o(\eta), \Omega_1(\eta)\cdots, \Omega_n(\eta)\right)}\right)^{\frac{1}{2}}. \tag{4.3}$$

In virtue of Eqs (2.7), (4.2), and (4.3), we have

$$\begin{aligned}\left\|E_{D,n}^{\alpha,h}\right\|_2 =& \left\|D^{\alpha}\Omega_n(\eta) - \sum_{j=0}^{n} \xi_{i,j,k}\Omega_n(\eta)\right\|_2 \\ \leq& 2i\sum_{k=0}^{i-\lceil\alpha\rceil}\left|\frac{(-1)^k 4^{i-k}\Gamma(2i+1)\Gamma(i-k+1)}{\Gamma(2i-2k+1)\Gamma(k+1)\Gamma(i-k-\alpha)}\right| \\ &\times\left(\frac{G\left((\eta)^{i-k-\alpha}; \Omega_0(\eta), \cdots, \Omega_n(\eta)\right)}{G\left(\Omega_0(\eta), \Omega_1(\eta), \cdots, \Omega_n(\eta)\right)}\right)^{\frac{1}{2}}, \quad 0 \leq i \leq n.\end{aligned} \tag{4.4}$$

By Eq (4.4) with use of results (4.2) and (4.3), it is easily interpreted that as n increases, $E_{D,n}^{\alpha,h}$ tends to zero.

Suppose that U_n is the n-dimensional subspace generated by $(\Omega_i)_{0\leq i\leq n}$ for $L_h^2[0,1]$. Let ω_n be the infimum of the functional on the space U_n. Then, it can be expressed as:

$$U_n \subset U_{n+1} \text{ and } \omega_{n+1} \geq \omega_n.$$

□

Theorem 4.5. *Let L denote the functional. Then,*

$$\lim_{n\to\infty} \omega_n(\eta) = \omega(\eta) = \inf_{\eta\in[0,1]} L(\eta).$$

Proof. See [27, 28]. □

For the FLE, the functional is

$$N(\eta) = D^{\alpha}z(\eta) + cD'z + dz^3 + ez^5 = 0. \tag{4.5}$$

Using Eqs (3.1)–(3.4), we get

$$\begin{aligned}N^{(E)}(\eta) =& P^T D^{(\alpha)}\Omega_n(\eta) + E_{D,n}^{\alpha,h} + cP^T D^{(1)}\Omega_n(\eta) + cE_{D,n}^{1,h} \\ &+ d\left(P^T\Omega_n(\eta + E_{z,n}^h\right)^3 + e\left(P^T\Omega_n(\eta) + E_{z,n}^h\right)^5,\end{aligned} \tag{4.6}$$

where

$$E_{z,n}^{h} = P^{T}\Omega(\eta) - P^{T}\Omega_{n}(\eta), \tag{4.7}$$

$$E_{D,n}^{\alpha,h} = D^{(\alpha)}\Omega_{n}(\eta) - D^{\alpha}\Omega_{n}(\eta), \tag{4.8}$$

$$E_{D,n}^{1,h} = D^{(1)}\Omega_{n}(\eta) - D^{1}\Omega_{n}(\eta). \tag{4.9}$$

We have the residual for Eq (4.6),

$$\begin{aligned} R_{n}^{(E)}(\eta) =& P^{T}D^{(\alpha)}\Omega_{n}(\eta) + E_{D,n}^{\alpha,h} + cP^{T}D^{(1)}\Omega_{n}(\eta) + cE_{D,n}^{1,h} \\ &+ d\left(P^{T}\Omega_{n}(\eta) + E_{z,n}^{h}\right)^{3} + e\left(P^{T}\Omega_{n}(\eta) + E_{z,n}^{h}\right)^{5}. \end{aligned} \tag{4.10}$$

Now, collocating $n-1$ points in Eq (4.10) by $\eta_i = \frac{i}{n}, \quad i = 0, 1, 2, \ldots, n-2$, we determine

$$R_{n}^{(E)}(\eta_i) = 0. \tag{4.11}$$

A system of NLAEs is obtained by combining Eq (3.5) with the collocation points in Eq (4.10). The solution for this system yields the result for the FLE, represented by $\omega_n^*(\eta)$. By utilizing results (4.1) and (4.4) and letting $n \to \infty$,

$$\omega_n^*(\eta) \to \omega_n(\eta). \tag{4.12}$$

From result (4.5) and Eq (4.12), we obtain

$$\lim_{n\to\infty} \omega_n^*(\eta) = \omega_(\eta).$$

5. Numerical results and discussion

Using various initial estimations, we apply our proposed approach to the Lienard equation in this section. The Lienard equation's constants are selected for comparison, providing that exact solutions are known for these particular constant values.

Case 1. With the following initial conditions, we generate the approximate result for the FLE given by Eq (1.3) in this case [16, 17]:

$$z(0) = \sigma = \sqrt{\frac{-2c}{d}} \text{ and } z'(0) = \delta = -\frac{c\sqrt{-c}}{d\sqrt{\frac{-2c}{d}}}. \tag{5.1}$$

We take $e = -3, d = 4$ and $c = -1$. For Case 1, the exact solution for the classical Lienard equation is

$$z(\eta) = \sqrt{\frac{-2c(1 + \tanh\sqrt{-c\eta})}{d}}. \tag{5.2}$$

Figure 1 illustrates the approximate solutions for different values of α specifically 1.8, 1.9 and 2. The results clearly show a smooth transition from fractional to integer order. Furthermore, Table 1 provides a comparison between the exact solutions and the approximate solutions derived using our proposed method.

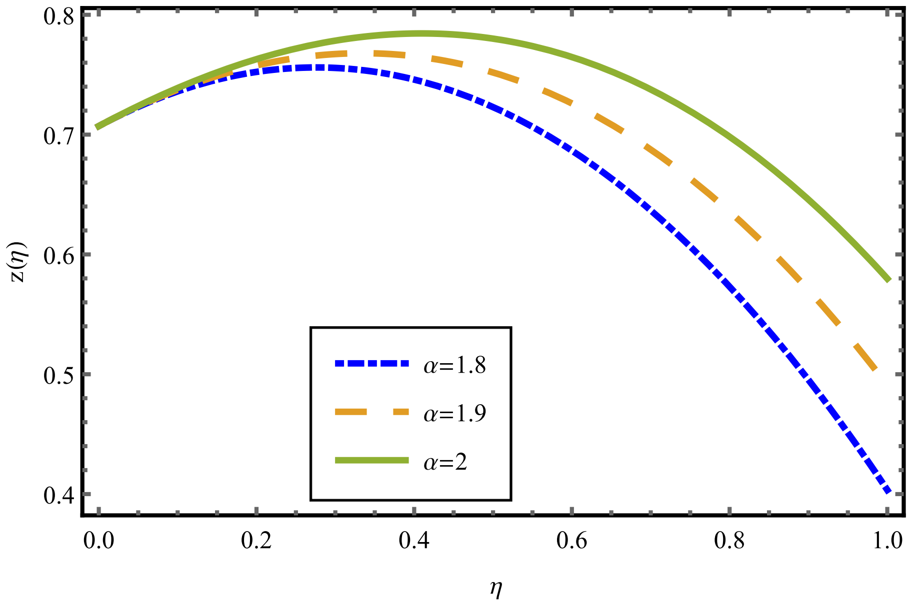

Figure 1. Behavior of $z(\eta)$ at various α for Case 1.

Table 1. Analysis of the obtained and exact solution for case 1 when $\alpha = 2$ and $n = 4$.

η	Exact solution	Present method
0.00	0.7071067	0.7071067
0.01	0.7106334	0.7106081
0.02	0.7141419	0.7140406
0.03	0.7176318	0.7174032
0.04	0.7211028	0.7206950
0.05	0.7245544	0.7239150
0.06	0.7279862	0.7270623
0.07	0.7313979	0.7301360
0.08	0.7347890	0.7331350
0.09	0.7381591	0.7360586
0.1	0.7415079	0.7389057

Table 1 demonstrates that the solutions obtained using the suggested method are reliable for real world implementations of the FLE.

Case 2. Here, we solved the FLE with the subsequent initial guess [16, 17]:

$$z(0) = \sigma = \sqrt{\frac{\phi}{2+\zeta}} \text{ and } z'(0) = \delta = 0, \tag{5.3}$$

where

$$\phi = 4\sqrt{\frac{3c^2}{(3d^2 - 16ce)}} \quad \text{ and } \zeta = -1 + \frac{\sqrt{3}d}{\sqrt{(3d^2 - 16ce)}}. \tag{5.4}$$

Here, we put $c = -1, d = 4$ and $e = 3$. The exact solution for the classical Lienard equation with initial conditions in Eq (1.4), is

$$z(\eta) = \sqrt{\frac{\phi \operatorname{sech}^2 \sqrt{-c}\eta}{2 + \zeta \operatorname{sech}^2 \sqrt{-c}\eta}},$$

where ϕ and ζ are as given in Eq (5.4).

In Figure 2, we present the responses for various values of $\alpha = 1.8, 1.9$ and 2, respectively. The results clearly show a smooth transition from fractional to integer order. Table 2 presents both the exact solutions and the approximate solutions obtained using our proposed method.

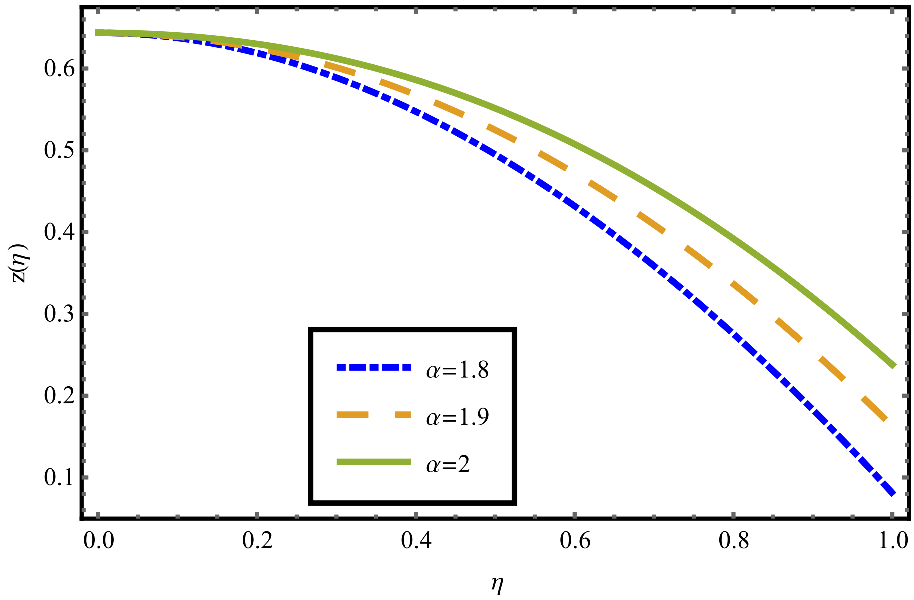

Figure 2. Behavior of $z(\eta)$ at various α for Case 2.

Table 2. Analysis of the obtained and exact solution for Case 2 when $\alpha = 2$ and $n = 4$.

η	Exact solution	Present method
0.00	0.6435942	0.6435942
0.01	0.6435565	0.6435619
0.02	0.6434434	0.6434668
0.03	0.6432551	0.6433074
0.04	0.6429915	0.6430830
0.05	0.6426530	0.6427927
0.06	0.6422396	0.6424360
0.07	0.6417518	0.6420119
0.08	0.6411897	0.6415199
0.09	0.6405539	0.6409593
0.1	0.6398446	0.6403293

6. Conclusions

In this paper, we proposed a computational method for the FLE involving the Liouville-Caputo operator. This method stands out for its simplicity and user-friendliness, making it easier to implement than other techniques. The ease primarily arises from the straightforward construction of the operational matrix for the differential equation. We have developed the operational matrix for Liouville-Caputo differentiation in connection with VLPs. We have observed that at $\alpha = 2$, the solution of fractional Lienard equation by applying suggested techniques is in great agreement with the exact solution of the FLE. The results suggest that the proposed technique is highly suitable and accurate for analyzing fractional-order models involving the Liouville-Caputo operator. The operational matrix method for the FLE is pivotal in engineering and physics. It enables precise modeling of diverse systems like electrical circuits and particle dynamics, aiding in control system design, vibration analysis and understanding nonlinear phenomena in various fields.

Author contributions

Jagdev Singh: Conceptualization, Methodology, Software, Validation, Writing—original draft, Project administration; Jitendra Kumar: Conceptualization, Methodology, Software, Validation, Formal analysis, Writing—original draft, Writing—review and editing; Devendra Kumar: Conceptualization, Methodology, Software, Validation, Formal analysis, Writing—original draft, Writing—review and editing, Writing—review and editing; Dumitru Baleanu: Conceptualization, Methodology, Validation, Writing—original draft. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this manuscript.

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