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Research article

Unilateral global interval bifurcation and one-sign solutions for Kirchhoff type problems

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Abstract: In this paper, we study the following Kirchhoff type problems:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u^3 + g(u, \lambda), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where λ is a parameter. Under some natural hypotheses on g and Ω, we establish a unilateral global bifurcation result from interval for the above problem. By applying the above result, under some suitable assumptions on nonlinearity, we shall investigate the existence of one-sign solutions for a class of Kirchhoff type problems.

Keywords: unilateral global interval bifurcation; one-sign solutions; Kirchhoff type problems
Mathematics Subject Classification: 35B32, 35P05

1. Introduction

Consider the following Kirchhoff type problem:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u^3 + g(u, \lambda), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a parameter. The problem (1.1) is nonlocal as the appearance of the term $\int_\Omega |\nabla u|^2 dx$ which implies that it is not a pointwise identity. This causes some mathematical difficulties which make the study for the problem (1.1) particularly interesting. The main difficulties when dealing with this problem lie in the presence of the nonlocal terms which arises in nonlinear vibrations and the analogous to the stationary case of equations that arise in the study of string or membrane vibrations.

The problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 [1]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx)\frac{\partial^2 u}{\partial x^2}=f(x,u),$$

where ρ, ρ_0, h, E and L are constants, f is the external force, by considering the effect of the changing in the length of the string during the vibration. After the famous article by Lions [2], Eq (1.1) received much attention, and some important and interesting results have been obtained, for example, see [3–5]. Recently, by using mountain pass theorem, and so on, the authors [6–8] have studied the existence of weak solution, etc. for the Kirchhoff equations. Meanwhile, by applying the bifurcation techniques, there are some papers to study Kirchhoff type problems, see for example [9–13].

Assume Ω and $g\in C(\mathbb{R}^2,\mathbb{R})$ satisfy the following conditions:

(H0) Ω satisfy one of the following conditions:

(i) Ω is an open ball in $\mathbb{R}^N$ for $N=1,2,3$;

(ii) $\Omega\subset\mathbb{R}^2$ is symmetric in x and y, and convex in the x and y directions or

(iii) $\Omega\subset\mathbb{R}^2$ is convex.

(H1) There exist $c>0$ and $p\in(1,2^*)$ such that $|g(s,\lambda)|\le c(1+|s|^{(p-1)})$, where

$$2^*=\begin{cases}\frac{2N}{N-2}, & N>2,\\ +\infty, & N\le 2.\end{cases}$$

$$\lim_{s\to 0}\frac{g(s,\lambda))}{s^3}=0 \tag{1.2}$$

uniformly for $\lambda\in\mathbb{R}$.

Let $E:=H_0^1(\Omega)$ with the norm $\|u\|=(\int_\Omega|\nabla u|^2dx)^{1/2}$. Let $Q^+=\{u\in E|u>0 \text{ in } \Omega\}$ and set $Q^-=-Q^+$ and $Q=Q^+\cup Q^-$. Let $\mathscr{S}^\pm$ denote the closure in $\mathbb{R}\times Q^\pm$ of the set of nontrivial solutions of (1.1).

From [13], obviously, the problem (1.1) can be equivalently written as $u=F(\lambda u^3+H(\lambda,u))):=G_\lambda(u)$, where $H(\lambda,\cdot)$ denotes the usual Nemytskii operator associated with g. From condition (H1) and noting $4<2^*$, we can see that $G_\lambda: E\to E$ is completely continuous and $G_\lambda(0)=0,\forall\lambda\in\mathbb{R}$.

By Rabinowitz [14], Dai [13] obtained the following global bifurcation lemma.

Lemma 1.1. (See [13]) Let (H0), (H1) and (1.2) hold. Then $(\lambda_1,0)$ is a bifurcation point of problem (1.1) and the associated bifurcation continuum $\mathscr{C}$ in $\mathbb{R}\times H_0^1(\Omega)$, whose closure contains $(\lambda_1,0)$, is either unbounded or contains a pair $(\mu,0)$, where μ is another eigenvalue of problem (1.3), and λ_1 is the principal eigenvalue of the following problem:

$$\begin{cases}-(\int_\Omega|\nabla u|^2dx)\Delta u=\lambda u^3, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases} \tag{1.3}$$

By [12, Theorem 1.2], λ_1 is simple, isolated and is the unique principal eigenvalue of the problem, and φ_1 is the principal eigenfunction corresponding to eigenvalue λ_1.

By Dancer [15, Theorem 2.1], there are two continua $\mathscr{C}^+$ and $\mathscr{C}^-$ consisting of the bifurcation branch $\mathscr{C}$ emanating from $(\lambda_1,0)$, which satisfy:

Lemma 1.2. Both $\mathscr{C}^+$ and $\mathscr{C}^-$ are unbounded and

$$\mathscr{C}^\nu \subset (\mathbb{R} \times P^\nu \cup \{(\lambda_1, 0)\}),$$

where $\nu \in \{+, -\}$.

However, among the above papers, the nonlinearities are differentiable at the origin. In [16], Berestycki established an important global bifurcation theorem from intervals for a class of second-order problems involving non-differentiable nonlinearity. Recently, Dai and Ma [17] also considered a class of high-dimensional p-Laplacian problems involving non-differentiable nonlinearity, and the author [18,19] studied interval bifurcation for the Monge-Ampere equations and Kirchhoff type problems involving non-differentiable nonlinearity, respectively. In 2018, Dai [20] established interval bifurcation theorem from the trivial solutions axis by a new method.

Motivated by above papers, we shall consider the Kirchhoff type problem (1.1). We assume that $g \in C(\mathbb{R}^2, \mathbb{R})$ satisfies the following conditions:

(H2) $g(s, \mu)s > 0$ for any $s \neq 0$ and $\mu \in \mathbb{R}$.

(H3) There exist $\underline{g}_0, \overline{g}_0 \in \mathbb{R}$ and $\underline{g}_0 \neq \overline{g}_0$ such that

$$\underline{g}_0 = \liminf_{|s| \to 0} \frac{g(s, \mu)}{s^3}, \ \overline{g}_0 = \limsup_{|s| \to 0} \frac{g(s, \mu)}{s^3}$$

uniformly for $\mu \in \mathbb{R}$.

(H4) There exist $\underline{g}_\infty, \overline{g}_\infty \in \mathbb{R}$ and $\underline{g}_\infty \neq \overline{g}_\infty$such that

$$\underline{g}_\infty = \liminf_{|s| \to +\infty} \frac{g(s, \mu)}{s^3}, \ \overline{g}_\infty = \limsup_{|s| \to +\infty} \frac{g(s, \mu)}{s^3}$$

uniformly for $\mu \in \mathbb{R}$.

Under (H0)–(H4), we shall establish the Theorems 2.1 and 2.2 for the problem (1.1), which bifurcates from the trivial solutions axis or from infinity, respectively.

Furthermore, by applying the above results (Theorems 2.1 and 2.2), we shall investigate the existence of one-sign solutions for the following Kirchhoff type problems:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = rf(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where r is a parameter. Let f satisfies:

(H5) $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(s)s > 0$ for any $s \neq 0$;

(H6) There exist $\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty \in (0, +\infty)$ with $\overline{f}_\infty < \underline{f}_0$ or $\overline{f}_0 < \underline{f}_\infty$ such that

$$\underline{f}_0 = \liminf_{|s| \to 0} \frac{f(s)}{s^3}, \ \overline{f}_0 = \limsup_{|s| \to 0} \frac{f(s)}{s^3},$$

$$\underline{f}_\infty = \liminf_{|s| \to +\infty} \frac{f(s)}{s^3}, \ \overline{f}_\infty = \limsup_{|s| \to +\infty} \frac{f(s)}{s^3}.$$

The rest of this paper is arranged as follows. In Section 2, we establish the unilateral global bifurcation results for the problem (1.1) from the trivial solutions axis or from infinity, respectively. In Section 3, on the basis of the unilateral global interval bifurcation results, we shall investigate the existence of one-sign solutions for the Kirchhoff type problems (1.4).

2. Unilateral global bifurcation from an interval at 0 and ∞

We have our first result on (1.1):

Theorem 2.1. Assume that (H0)–(H3) hold. Let $I_0 = [\lambda_1 - \overline{g}_0, \lambda_1 - \underline{g}_0]$. The component $\mathscr{C}^\nu$ of $\mathscr{S}^\nu \cup (I_0 \times \{0\})$, containing $I_0 \times \{0\}$ is unbounded and lies in $(\mathbb{R} \times P^\nu) \cup (I_0 \times \{0\})$.

To prove Theorem 2.1, we introduce the following auxiliary approximate problem:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda u^3 + g(u|u|^\epsilon, \lambda), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The next lemma provides the a priori bounds for the solutions of problem (2.1) near the trivial solution.

Lemma 2.1. Let ϵ_n, $0 < \epsilon_n < 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times P^\nu$ such that (λ_n, u_n) is a nontrivial solution of problem (2.1) corresponding to $\epsilon = \epsilon_n$, and (λ_n, u_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times X$, then $\lambda \in I_0$.

Proof. Let $v_n = u_n/\|u_n\|$, then v_n is a solution of the following problem:

$$\begin{cases} -(\int_\Omega |\nabla v_n|^2 dx)\Delta v_n = \lambda_n v_n^3 + \frac{g(u_n|u_n|^{\epsilon_n}, \lambda_n)}{\|u_n\|^3}, & \text{in } \Omega, \\ v_n = 0, & \text{on } \partial\Omega. \end{cases}$$

It is not difficult to see that

$$\underline{g}_0 \leq \frac{g(u_n|u_n|^{\epsilon_n}, \lambda_n)}{\|u_n\|^3} \leq \overline{g}_0. \tag{2.2}$$

So, up to a subsequence, there exists $\alpha \in [\underline{g}_0, \overline{g}_0]$ such that

$$\lim_{n\to\infty} \frac{g(u_n|u_n|^\epsilon, \lambda_n)}{u_n^3} = \alpha$$

uniformly for n large enough.

Using this fact with (2.2), we have that $\lambda_n v_n^3 + \frac{g(u_n|u_n|^{\epsilon_n}, \lambda_n)}{\|u_n\|^3}$ is bounded for n large enough. By the completely continuous of G_λ implies that v_n is strong convergence in $C^1(\overline{\Omega})$. Without loss of generality, we may assume that $v_n \to v$ with $\|v\| = 1$. Clearly, we have $v \in \overline{P}$.

So, up to a subsequence, we obtain that

$$v = \lambda v^3 + \alpha v^3.$$

It follows that $\lambda + \alpha = \lambda_1$, which implies $\lambda \in I_0$.

Therefore, we have that $\lambda \in I_0$.

Proof of Theorem 2.1. We only prove the case of $\mathscr{C}^+$ since the case of $\mathscr{C}^-$ is similar.

We divide the rest of proofs into two steps.

Step 1. We show that $\mathscr{C}^+ \subset \mathbb{R} \times P^+ \cup (I_0 \times \{0\})$.

For any $(\lambda, u) \in \mathscr{C}^+$, there are two possibilities: (i) $u \geq 0$ but $u \not\equiv 0$, or (ii) $u \equiv 0$. If the case (ii) occurs, there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times P^+$ such that (λ_n, u_n) is a solution of problem (1.1), and (λ_n, u_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times X$. By Lemma 2.1, we have that $\lambda \in I_0$, i.e., $(\lambda, u) \in I_0 \times \{0\}$. Hence, $\mathscr{C}^+ \subset \mathbb{R} \times P^+ \cup (I_0 \times \{0\})$ in the case of (ii). If the case (i) occurs, by the strong maximum principle [21, Theorem 8.19], we know that $u > 0$ in Ω. Hence, $\mathscr{C}^+ \subset \mathbb{R} \times P^+ \cup (I_0 \times \{0\})$.

Step 2. Similar to the argument of Theorem 3.1 in [18], we prove that $\mathscr{C}^+$ is unbounded.

We add the points $\{(\lambda, \infty)|\lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$. Let S_N denote the spectral set of problem (1.3). Let $\overline{I}_\infty = [\overline{\lambda} - \overline{g}_\infty, \overline{\lambda} - \underline{g}_\infty]$, where $\overline{\lambda} \in S_N \setminus \{\lambda_1\}$. By Rabinowitz [22], our second main result for (1.1) is the following theorem.

Theorem 2.2. Let (H0), (H1), (H2) and (H4) hold. Let $I_\infty = [\lambda_1 - \overline{g}_\infty, \lambda_1 - \underline{g}_\infty]$. There exists a connected component $\mathscr{D}^\nu$ of $\mathscr{S}^\nu \cup (I_\infty \times \{\infty\})$, containing $I_\infty \times \{\infty\}$. Moreover, if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap (\cup_{\overline{\lambda} \in S_N \setminus \{\lambda_1\}} (\overline{I}_\infty \cup I_\infty)) = I_\infty$ and $\mathscr{M}$ is a neighborhood of $I_\infty \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either

(1) $\mathscr{D}^\nu - \mathscr{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathscr{D}^\nu - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0)|\lambda \in \mathbb{R}\}$ or

(2) $\mathscr{D}^\nu - \mathscr{M}$ is unbounded.

If (2) occurs and $\mathscr{D}^\nu - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}^\nu - \mathscr{M}$ meets $\overline{I}_\infty$.

Proof. The idea is similar to the proof of Theorem 1.6 of [22], but we give a rough sketch of the proof for readers convenience. If $(\lambda, u) \in \mathscr{S}^\nu$ with $\|u\| \neq 0$, dividing (2.1) by $\|u\|^6$ and setting $v = u/\|u\|^2$ yield

$$\begin{cases} -(\int_\Omega |\nabla v|^2 dx)\Delta v = \lambda v^3 + \frac{g(u|u|^\epsilon, \lambda)}{\|u\|^6}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Define

$$\widetilde{g}(v, \lambda) = \begin{cases} \|v\|^6 g(\frac{v}{\|v\|^2}, \lambda), & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

Clearly, (2.3) is equivalent to

$$\begin{cases} -\Delta v = \lambda v^3 + \widetilde{g}(v|v|^\epsilon, \lambda)\ \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

It is obvious that $(\lambda, 0)$ is always the solution of (2.4). Now, applying Theorem 2.1 to the problem (2.4), we have a connected component $\mathscr{C}^\nu$ of $\mathscr{S}^\nu \cup (I_0 \times \{0\})$. Under the inversion $v \to \frac{v}{\|v\|^2} = u$, $\mathscr{C}^\nu \to \mathscr{D}^\nu$ satisfying the problem (1.1). Clearly, $\mathscr{D}^\nu$ satisfy the conclusions of this theorem.

3. One-sign solutions for Kirchhoff type problems

Let $\Sigma^\pm$ denote the closure in $K^\pm$ of the set of nontrivial solutions of (1.4).

In order to prove Theorem 3.1, we need the following Sturm-type comparison result.

Lemma 3.1. Let (H0) hold. Assume that $b_i(x)$ are two weight functions such that $b_i(x) \in C(\Omega), i = 1, 2$. Also let $u_i (i = 1, 2)$ be weak solution of the following differential equations:

$$\begin{cases} -\|u\|^2 \Delta u = b_i(x) u^3, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

respectively, $u_1 \neq 0$ for almost every $x \in \Omega$. If $b_1(x)u_1^2 \leq b_2(x)u_2^2$ for $x \in \Omega$, then u_2 must change sign.

Proof. Without loss of generality, we may assume that $\|u\| = 1$ and $u_1 > 0, u_2 > 0$. Then, by [23, (1.1)] the following Picones identity:

$$0 \leq |\nabla u_1|^2 + |\frac{u_1}{u_2}\nabla u_2|^2 - 2\nabla u_1 (\frac{u_1}{u_2}\nabla u_2) = |\nabla u_1|^2 - \nabla(\frac{u_1^2}{u_2})\nabla u_2.$$

Furthermore, we have

$$\int_\Omega \left[|\nabla u_1|^2 - \nabla(\frac{u_1^2}{u_2})\nabla u_2\right] dx = \int_\Omega \left[\frac{u_1}{u_2}(u_2 L[u_1] - u_1 L[u_2]\right] = \int_\Omega \left[b_1(x)u_1^2 - b_2(x)u_2^2]u_1^2\right] \le 0,$$

where $L[u] = -\Delta u$.

Moreover, we have

$$|\nabla u_2|^2 + |\frac{u_2}{u_1}\nabla u_1|^2 - 2\nabla u_2(\frac{u_2}{u_1}\nabla u_1) \equiv 0,$$

so $\nabla(\frac{u_2}{u_1}) = 0$, a.e. Ω, i.e., $u_1 = ku_2$ for some constant k in each component of Ω. But this is impossible since $b_2(x) \not\equiv b_1(x)$ almost everywhere in Ω. This accomplishes the proof.

The main results of this section are the following theorems.

Theorem 3.1. Let (H0), (H5) and (H6) hold. For all

$$r \in (\frac{\lambda_1}{\underline{f}_0}, \frac{\lambda_1}{\overline{f}_\infty}) \cup (\frac{\lambda_1}{r\underline{f}_\infty}, \frac{\lambda_1}{r\overline{f}_0}), \tag{3.1}$$

problem (1.4) possesses at least two one-sign solutions u^+ and u^- such that νu^ν is positive for $\nu = +, -$.

Theorem 3.2. Let (H0) and (H5) hold. Suppose that $\underline{f}_0 = +\infty, \underline{f}_\infty, \overline{f}_\infty \in (0, +\infty)$. For all

$$r \in (0, \frac{\lambda_1}{\overline{f}_\infty}),$$

problem (1.4) possesses at least two one-sign solutions u^+ and u^- such that νu^ν is positive for $\nu = +, -$.

Theorem 3.3. Let (H0) and (H5) hold. Suppose that $\underline{f}_\infty = +\infty, \underline{f}_0, \overline{f}_0 \in (0, +\infty)$. For all

$$r \in (0, \frac{\lambda_1}{\overline{f}_0}),$$

problem (1.4) possesses at least two one-sign solutions u^+ and u^- such that νu^ν is positive for $\nu = +, -$.

Theorem 3.4. Let (H0) and (H5) hold. Suppose that $\overline{f}_0 = 0, \underline{f}_\infty, \overline{f}_\infty \in (0, +\infty)$. For all

$$r \in (\frac{\lambda_1}{\underline{f}_\infty}, +\infty),$$

problem (1.4) possesses at least two one-sign solutions u^+ and u^- such that νu^ν is positive for $\nu = +, -$.

Theorem 3.5. Let (H0) and (H5) hold. Suppose that $\overline{f}_\infty = 0, \underline{f}_0, \overline{f}_0 \in (0, +\infty)$. For all

$$r \in (\frac{\lambda_1}{\underline{f}_0}, +\infty),$$

problem (1.4) possesses at least two one-sign solutions u^+ and u^- such that νu^ν is positive for $\nu = +, -$.

We construct the following problem:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = r(u^3 + g(u)), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $g = f - u^3$. Clearly, problem (3.2) is equivalent to problem (1.4). Furthermore, we have that $\underline{g}_0 = \underline{f}_0 - 1$, $\overline{g}_0 = \overline{f}_0 - 1$, $\underline{g}_\infty = \underline{f}_\infty - 1$, $\overline{g}_\infty = \overline{f}_\infty - 1$.

Proof of Theorem 3.1. We consider the following problem:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = \mu r u^3 + r g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Let $I_0 = [\frac{\lambda_1}{r} - \overline{f}_0 + 1, \frac{\lambda_1}{r} - \underline{f}_0 + 1]$. By Theorem 2.1, There is a distinct unbounded component $\mathscr{C}^\nu$ of $\mathscr{S}^\nu \cup (I_0 \times \{0\})$, containing $I_0 \times \{0\}$ and lying in $(K^\nu \cup (I_0 \times \{0\}))$.

Set $I_\infty = [\frac{\lambda_1}{r} - \overline{f}_\infty + 1, \frac{\lambda_1}{r} - \underline{f}_\infty + 1]$. By Theorem 2.2, There is a distinct unbounded component $\mathscr{D}^\nu$ of $\mathscr{S}^\nu \cup (I_\infty \times \{\infty\})$, which satisfy the alternates of Theorem 2.2.

We claim that $\mathscr{C}^\nu = \mathscr{D}^\nu$.

Firstly, we shall show that 2^o of Theorem 2.2 does not occur.

On the contrary, we suppose that $(\mu_n, v_n) \in \mathscr{D}^\nu - \mathscr{M}$ such that

$$\lim_{n\to\infty} \mu_n = \pm\infty.$$

It follows that $\lim_{n\to\infty} r(\mu_n + \frac{g(u)}{u^3})) = \pm\infty$.

If $\lim_{n\to\infty} r(\mu_n + \frac{g(u)}{u^3})) = -\infty$, applying Lemma 3.1 to v_n and φ_1, we can get that φ_1 must change its sign for n large enough. While, this is impossible. So we have that $\lim_{n\to\infty} r(\mu_n + \frac{g(u)}{u^3})) = +\infty$. Applying Lemma 3.1 to φ_1 and v_n, we get that v_n must change its sign for n large enough, and this contradicts the fact that $v_n \in P^\nu$.

So case (1) of Theorem 2.2 must happen, i.e., there exist some point $(\mu_*, 0) \in \mathscr{D}^\nu$. By Lemma 2.1, one obtain that $\mu_* \in I_0$, and it follows that $\mathscr{C}^\nu = \mathscr{D}^\nu$.

Obviously, for any solution $(1, v)$ of (3.3), it yields a solution v of (3.2). By (H6) and (3.1), we have that $I_0 \cap I_\infty = \emptyset$ and $\frac{\lambda_1}{r} - \underline{f}_0 + 1 < 1 < \frac{\lambda_1}{r} - \overline{f}_\infty + 1$ or $\frac{\lambda_1}{r} - \underline{f}_\infty + 1 < 1 < \frac{\lambda_1}{r} - \overline{f}_0 + 1$. It follows that the subsets $I_0 \times E$ and $I_\infty \times E$ of $\mathbb{R} \times E$ can be separated by the hyperplane $\{1\} \times E$. Furthermore, we have $\mathscr{C}^\nu$ cross the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Proof of Theorem 3.2. By [24], define f as the following:

$$f^{[n]}(s) := \begin{cases} ns, & s \in [-\frac{1}{n}, \frac{1}{n}], \\ \left[f(\frac{2}{n}) - 1\right](ns - 2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -\left[f(-\frac{2}{n}) + 1\right](ns + 2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

We consider the following problem:

$$\begin{cases} -(\int_\Omega |\nabla u|^2 dx)\Delta u = \mu r f^{[n]}(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, $\underline{f}^{[n]}_\infty = \underline{f}_\infty$, $\overline{f}^{[n]}_\infty = \overline{f}_\infty$.

By Theorem 2.1, there exists an unbounded continuum $\mathscr{D}^{\nu[n]}$ of solutions of the problem (3.4) emanating from $(\frac{\lambda_1}{rn}, 0)$ such that $\mathscr{D}^{\nu[n]} \subset ((\mathbb{R} \times \mathscr{S}^\nu) \cup \{(\frac{\lambda_1}{rn}, 0)\})$.

Taking $z^* = (0,0)$, we easily obtain that $z^* \in \liminf_{n\to+\infty} \mathscr{D}^{\nu[n]}$. We obtain that $\left(\cup_{n=1}^{\infty} \mathscr{D}^{\nu[n]}\right) \cap \overline{B}_R$ is s pre-compact.

Therefore, by [25, Lemma 2.1], $\mathscr{D}^{\nu} = \limsup_{n\to\infty} \mathscr{D}^{\nu[n]}$ is unbounded closed connected such that $z^* \in \mathscr{D}^{\nu}$.

For any $(\mu, u) \in \mathscr{D}^{\nu}$, by [26, P. 7], it follows that $\mathscr{D}^{\nu} \subseteq \cup_{n=1}^{\infty} \mathscr{D}^{\nu[n]}$. So we have that $\mathscr{D}^{\nu} \subset ((\mathbb{R}\times\mathscr{S}^{\nu}) \cup \{(0,0)\})$.

For any sequence $(\mu_n, u_n) \in \mathscr{D}^{\nu}$ with $\lim_{n\to+\infty} \|u_n\| = +\infty$. By $\overline{f}_{\infty}, \underline{f}_{\infty} \in (0,+\infty)$, for n large enough, one can obtain that $\frac{f(u_n)}{u_n^3}$ is bounded. In the following, we have $\alpha \in [\overline{f}_{\infty}, \underline{f}_{\infty}]$ such that

$$\frac{f(u_n)}{u_n^3} = \alpha$$

for n large enough. Similar the proof of Theorem 3.1, Lemma 3.1 implies that μ_n is bounded. So, up to a sequence, we have that $\mu_n \to \mu \in \mathbb{R}$. Then reasoning as that of Lemma 2.1, we obtain that

$$\mu \in [\frac{\mu_1}{r\overline{f}_{\infty}}, \frac{\mu_1}{r\underline{f}_{\infty}}].$$

So, $\mathscr{D}^{\nu}$ joins $(0,0)$ to $[\frac{\mu_1}{r\overline{f}_{\infty}}, \frac{\mu_1}{r\underline{f}_{\infty}}] \times \{\infty\}$. By the structure of $\mathscr{D}^{\nu}$, there exist two one-sign solutions u^+ and u^-.

Proof of Theorem 3.3. We define the cut-off function of f as the following:

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2n^2+f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \frac{2n^2-f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, n]. \end{cases}$$

We consider the following problem:

$$\begin{cases} -(\int_{\Omega} |\nabla u|^2 dx)\Delta u = \mu r f^{[n]}(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_{\infty} = n$, $\underline{f}_0^{[n]} = \underline{f}_0$, $\overline{f}_0^{[n]} = \overline{f}_0$.

By Theorem 2.2, there exists an unbounded continuum $\mathscr{D}^{\nu[n]}$ of solutions of the problem (3.5) emanating from $(\frac{\lambda_1}{rn}, \infty)$, such that $\mathscr{D}^{\nu[n]} \subset ((\mathbb{R}\times P^{\nu}) \cup \{(\frac{\lambda_1}{rn}, \infty)\})$, $\nu \in \{+,-\}$.

Taking $z_n = (\frac{\lambda_1}{rn}, \infty)$ and $z_n \to z^* = (0, \infty)$, we easily obtain that $z^* \in \liminf_{n\to+\infty} \mathscr{D}^{\nu[n]}$ with $\|z^*\| = +\infty$.

Therefore, by [25, Theorem 2.2], there exists an unbounded component $\mathscr{D}^{\nu}$ in $E := \liminf_{n\to\infty} \mathscr{D}^{\nu[n]}$ and $z^* \in \mathscr{D}^{\nu}$, $\nu \in \{+,-\}$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (3.5) can be converted to the equivalent equation (1.4). Since $\mathscr{D}^{\nu[n]} \subset (\mathbb{R}\times P^{\nu})$, we conclude $\mathscr{D}^{\nu} \subset (\mathbb{R}\times P^{\nu})$. Moreover, $\mathscr{D}^{\nu} \subset \sum^{\nu}$ by (1.4).

For any sequence $(\mu_n, u_n) \in \mathscr{D}^{\nu}$ with $\lim_{n\to+\infty} \|u_n\| = 0$. In view of $\overline{f}_0, \underline{f}_0 \in (0,+\infty)$, we can see that $\frac{f(u_n)}{u_n^3}$ is bounded for n large enough.

For n large enough, one can obtain that $\frac{f(u_n)}{u_n^3}$ is bounded. In the following, we have $\alpha \in [\overline{f}_0, \underline{f}_0]$ such that

$$\frac{f(u_n)}{u_n^3} = \alpha$$

for n large enough. If $\mu_n \to \mu \in \mathbb{R}$ as $n \to +\infty$, by Lemma 2.1, it follows that

$$\mu \in [\frac{\mu_1}{r\overline{f}_0}, \frac{\mu_1}{r\underline{f}_0}].$$

So, $\mathscr{D}^\nu$ joins $[\frac{\mu_1}{r\overline{f}_0}, \frac{\mu_1}{r\underline{f}_0}] \times \{0\}$ to $(0, \infty)$.

By the structure of $\mathscr{D}^\nu$, there exist two one-sign solutions u^+ and u^-.

Proof of Theorems 3.4 and 3.5. Similar the proof of Theorems 3.2 and 3.3, we can obtain Theorems 3.4 and 3.5.

4. Conclusions

In this study, I establish a unilateral global bifurcation result for the Kirchhoff type problem (1.1) from an interval lying on trivial solution axis or an interval at infinity,respectively. By applying the above results, under some suitable assumptions on nonlinearity, I shall investigate the existence of one-sign solutions for a class of Kirchhoff type problem (1.4).

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this article.

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