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Research article

The continuity of biased random walk's spectral radius on free product graphs

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Abstract: R. Lyons, R. Pemantle and Y. Peres (Ann. Probab. 24 (4), 1996, 1993–2006) conjectured that for a Cayley graph G with a growth rate $\mathrm{gr}(G) > 1$, the speed of a biased random walk exists and is positive for the biased parameter $\lambda \in (1, \mathrm{gr}(G))$. And Gábor Pete (Probability and geometry on groups, Chaper 9, 2024) sheds light on the intricate relationship between the spectral radius of the graph and the speed of the biased random walk. Here, we focus on an example of a Cayley graph, a free product of complete graphs. In this paper, we establish the continuity of the spectral radius of biased random walks with respect to the bias parameter in this class of Cayley graphs. Our method relies on the Kesten-Cheeger-Dodziuk-Mohar theorem and the analysis of generating functions.

Keywords: biased random walk; spectral radius; continuous; Cayley graph; free product
Mathematics Subject Classification: Primary 60J10, 60G50, 05C81, Secondary 60C05, 05C63, 05C80

1. Introduction

Consider an infinite, locally finite, and connected graph $G = (V(G), E(G), o)$, where $V(G)$ denotes the vertex set, $E(G)$ the edge set, and o is a designated root. Consider Markov chain. There is a stationary measure $\pi(\cdot)$ such that for any two adjacent vertices x and y, $\pi(x)p(x, y) = \pi(y)p(y, x)$, where $p(x, y)$ is the transition probability. For the edge joining vertices x and y, assign a weight

$$c(x, y) = \pi(x)p(x, y).$$

Now we call the weights of the edges conductance and their reciprocals resistance. In this paper, we study the spectral radius of irreducible Markov chains on weighted graphs. More precisely, we focus on the spectral radius of a biased random walk, which is defined as follows:

Fix a root o in a graph G. Let $|x|$ be the graph distance between x and o for any vertex x of G. Write $\mathbb{N}$ for the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{Z}_+$, we define two subsets of vertices:

The ball of radius n centered at o is denoted by $B_G(n) = \{x \in V(G) : \ |x| \le n\}$.

The boundary of this ball is $\partial B_G(n) = \{x \in V(G) : \ |x| = n\}$. Fix any $\lambda \in [0, \infty)$. If edge e is at distance n from o, then let its conductance be λ^{-n}. Denote by RW_λ the random walk associated the above conductances, and we call RW_λ a biased random walk. Recall that a random walk on an infinite connected network is transient iff the effective conductance from any of its vertices to infinity is positive [19].

The motivation for introducing RW_λ is to design a Monte-Carlo algorithm for self-avoiding walks by Berretti and Sokal [5]. See [13, 20, 21] for refinements of this idea. Due to interesting phenomenology and similarities to concrete physical systems ([7, 9, 10, 12, 23–25]), biased random walks and biased diffusions in disordered media have attracted much attention in the mathematical and physics communities since the 1980s.

In the following, we assume G is transitive. Let $M_n = |\partial B_G(n)|$ be the cardinality of $\partial B_G(n)$ for any $n \in \mathbb{Z}_+$. Define the growth rate of G as

$$\mathrm{gr}(G) = \liminf_{n\to\infty} \sqrt[n]{M_n}.$$

Since the sequence $\{M_n\}_{n=0}^{\infty}$ is submultiplicative, the limit $\mathrm{gr}(G) = \lim_{n\to\infty} \sqrt[n]{M_n}$ exists indeed. R. Lyons [15] showed that the critical parameter for RW_λ on a general tree is exactly the exponent of the Hausdorff dimension of the tree boundary. Moreover, R. Lyons [16] proved that for Cayley graphs and degree-bounded transitive graphs, the growth rate is exactly the critical parameter of the RW_λ.

Let d be the vertex degree of G. For any vertex v of G except o, denote by d_v^- the number of edges connecting v to $\partial B_G(|v| - 1)$. For the definition of RW_λ $(X_n)_{n=0}^{\infty}$ starting at o, the transition probability from v to an adjacent vertex u is

$$p(v,u) = \begin{cases} 1/d & \text{if } v = o, \\ \frac{\lambda}{d+(\lambda-1)d_v^-} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o, \\ \frac{1}{d+(\lambda-1)d_v^-} & \text{otherwise.} \end{cases}$$

Note that RW_1 is just a simple random walk on G.

We write

$$p^{(n)}(x,y) = p^{(n)}(x,y,\lambda) = \mathbb{P}_x(X_n = y).$$

The Green function is defined by

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n, \ x, \ y \in V(G), \ z \in \mathbb{C}.$$

Define

$$\tau_y = \tau_y(\lambda) = \inf\{n \ge 1 |\ X_n = y\}, \ f^{(n)}(x,y) = \mathbb{P}_x(\tau_y = n),$$

the associated generating function

$$U(x,y|z)=\sum_{n=1}^{\infty}f^{(n)}(x,y)z^n,\ x,\ y\in V(G),\ z\in\mathbb{C}.$$

Given any function $g(z)$, let us denote the radius of convergence by R_g. By the Cauchy-Hadamard criterion,

$$R_G=R_G(\lambda)=\frac{1}{\limsup_{n\to\infty}\sqrt[n]{p^n(x,y)}}.$$

Recall [19, *Exercise 1.2*] that R_G is independent of x and y due to its irreducibility. Define

$$\rho=\rho_G(\lambda)=\rho(\lambda)=\frac{1}{R_G}=\limsup_{n\to\infty}\sqrt[n]{p^n(x,x)}.$$

ρ is called the spectral radius of the biased random walk. The reason for this name we can refer to [14, 19] for more information on the spectral radius. Moreover,

$$\rho=\limsup_{n\to\infty}\sqrt[n]{p^n(o,o)}.$$

Define the speed of RW_λ, $(X_n)_{n=0}^{\infty}$, as the limit of $\frac{|X_n|}{n}$ as $n\to\infty$, if it exists almost surely (or in probability).

There are many deep and important questions related to how the spectral radius and the speed depend on the bias parameter λ (see [14], Chapter 9). In [17, p.2005 *Questions*], R. Lyons, R. Pemantle, and Y. Peres raised the Lyons-Pemantle-Peres monotonicity problem. *For a Cayley graph G with* $\mathrm{gr}(G)>1$, *does the speed of RW_λ exist and be positive for all $\lambda\in(1,\mathrm{gr}(G))$*? For more information, readers can refer to [1, 4].

Moreover, R. Lyons, R. Pemantle and Y. Peres [18] conjectured that the speed of RW_λ on the supercritical Galton-Watson tree without leaves is strictly decreasing. The conjecture has been confirmed for λ lying in some regions (see [2, 4, 26]). For Galton-Watson trees without leaves, the Lyons-Pemantle-Peres monotonicity problem was answered positively for $\lambda\le\frac{m_1}{1160}$ by Ben Arous, Fribergh and Sidoravicius [4], where m_1 is the minimal degree of the Galton-Watson tree. And Aïdékon [1] improved the just-mentioned result to $\lambda\le\frac{1}{2}$ by a completely different approach. In [3], Ben Arous, Hu, Olla, and Zeitouni obtained the Einstein relation for RW_λ on Galton-Watson trees, which implies the conjecture holds in a neighborhood of m.

We are interested in the continuity of the spectral radius of RW_λ. For $\lambda>\mathrm{gr}(G)$, RW_λ is recurrent, $\rho=1$. When $\lambda=0$ RW_λ is not irreducible.

Problem 1.1. *For a Cayley graph G, is the spectral radius of RW_λ continuous for all $\lambda\in(0,\mathrm{gr}(G)]$?*

In this paper, our focus is centered on addressing Problem 1.1 within the realm of a distinctive class of block-free product groups. To ensure a comprehensive understanding, the precise definition and pertinent details regarding free products will be meticulously introduced in Section 2.

Theorem 1.2. *Let graph G be a free product of complete graphs. Then the spectral radius $\rho(\lambda)$ of RW_λ on G is continuous in $\lambda\in(0,\mathrm{gr}(G)]$.*

To prove Theorem 1.2, the primary technical obstacle lies in establishing the generating function and subsequently demonstrating that the Cheeger constant is indeed positive.

Theorem 1.2 affirmatively resolves Problem 1.1, specifically for the scenario involving the free product of complete graphs. A key observation here is that the d-regular tree $\mathbb{T}_d$, represents a particular instance of such free products of complete graphs. Consequently, it is deduced from the theorem that the spectral radius, $\rho(\lambda)$ of RW_λ defined on $\mathbb{T}_d$, exhibits a characteristic of continuity over the interval $(0,\mathrm{gr}(\mathbb{T}_d)]$, where $\mathrm{gr}(\mathbb{T}_d)$ signifies the growth rate of the d-regular tree. This finding underscores the robustness of the spectral property with respect to variations in the parameter λ.

2. Proof of Theorem 1.2

2.1. Free product of graphs and growth rate

Intuitively, a "free product" of finite Cayley graphs G_i is a rule to construct a new Cayley graph $\mathcal{G}$ by gluing these m cells at "common vertices" without edge intersection, step by step. Concretely, we construct the Cayley graph $\mathcal{G}$ of $H_1 * H_2 * \cdots * H_r$ by the following steps: Here, $*$ denotes a free product.

Step 1. Glue each i-cell $(1 \le i \le r)$ at a common vertex o such that any two of the r cells only have one common vertex o. View o as the birth root of these m cells. Usually o is chosen to be the identity element 1. Denote the obtained graph as $G^{(1)}$, and mark any vertex x in the j-cell as $[j]$.

Step 2. For any $x \in G^{(1)} \backslash \{o\}$, it must be in some cell, say an i-cell, then we glue each j-cell $(1 \le j \ne i \le r)$ at x such that any of the $r-1$ cells has only one common vertex x with $G^{(1)}$ and any two of the $r-1$ cells only have one common vertex x. View x as the birth root of just added $r-1$ cells. For any distinctive two vertices x and y of $G^{(1)} \backslash \{o\}$, we require that any cell glued at x is disjoint from any cell glued at y. Let $G^{(2)}$ be the resulting graph. And for any $1 \le j \le r$, also mark a vertex x of $G^{(2)} \backslash G^{(1)}$ in the j-cell as $[j]$.

Step 3. For all $x \in G^{(2)} \backslash G^{(1)}$, according to $\langle x \rangle$, we can determine which type of cell x belongs to, and glue other type cells as Step 2. And then mark all new added vertices y and define $\langle y \rangle$ and $[y]$ as Step 2. Denote by $G^{(3)}$ the obtained graph. And so on, we obtain $\mathcal{G}$ with a type function $[\cdot]$ in its vertices, where for convenience we let $[o] = 0$.

2.2. Spectral radius: Free product of complete graphs

Now let G be the Cayley graph of $H_1 * H_2 * \cdots * H_r$ with root o, where each H_i $(1 \le i \le r)$ is a finite group whose Cayley graph is the complete graph K_{m_i+1} on $m_i + 1$ vertices. Let $\sum_{i=1}^{r} m_i = m$. Thus, the transition probability from v to an adjacent vertex u is

$$p(v,u) = \begin{cases} 1/m & \text{if } v = o, \\ \frac{\lambda}{m+\lambda-1} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o, \\ \frac{1}{m+\lambda-1} & \text{otherwise.} \end{cases}$$

Let $f_i^{(n)}(o,o)$ $(i = 1, 2, \cdots, r)$ be the probability of the biased random walk on G starting at o and $\tau_o = n$, which does visit a vertex of K_{m_i+1}. Define

$$U_i(o,o|z) = \sum_{n=1}^{\infty} f_i^{(n)}(x,y) z^n.$$

Hence,

$$U(o,o|z)=\sum_{i=1}^{r}U_i(o,o|z). \tag{2.1}$$

Note the tree-like structure of G. To compute $f_i^{(n)}(o,o)$, a biased random walk must reach an edge in K_{m_i+1} from o and return o by an edge in K_{m_i+1} at last step. Each vertex of K_{m_i+1} glues a copy of K_{m_j+1} ($j\neq i$). By the symmetry of K_{m_i+1}, we can regard K_{m_i+1} as an edge with a cycle and glue the same structure as in G. Thus

$$U_i(o,o|z)=\frac{m_i}{m}z\frac{\lambda}{m+\lambda-1}z\sum_{n=0}^{\infty}(\sum_{j=0}^{i-1}M_j(z)+\widetilde{M}_i(z)+\sum_{k=i+1}^{r}M_k(z))^n.$$

Here $M_j(z)$ denotes the generating function associated with the hitting probability of K_{m_i+1}, which starts at a vertex of K_{m_j+1} ($j\neq i$). Note that the probability from o to verteices of K_{m_j+1} is $\frac{m_i}{m}$ and from a vertex of K_{m_i+1} to the vertices of K_{m_j+1} is $\frac{m_i}{m+\lambda-1}$. And the other steps obey the same law as $U_j(o,o|z)$. Thus

$$M_j(z)=\frac{m}{m+\lambda-1}U_j(o,o|z),\quad \widetilde{M}_i(z)=\frac{m_i-1}{m+\lambda-1}z.$$

Therefore, in the domain $\{z\in\mathbb{C}|\ |z|<R_U\}$,

$$\begin{aligned}U_i(o,o|z)&=\frac{\lambda m_i}{(m+\lambda-1)^2}z^2\frac{1}{1-\left(\frac{m}{m+\lambda-1}\sum_{i=1}^{r}U_j(o,o|z)-\frac{m}{m+\lambda-1}U_i(o,o|z)+\widetilde{M}_i(z)\right)}\\&=\frac{\lambda m_i}{(m+\lambda-1)^2}z^2\frac{1}{1-\left(\frac{m}{m+\lambda-1}U(o,o|z)-\frac{m}{m+\lambda-1}U_i(o,o|z)+\widetilde{M}_i(z)\right)}.\end{aligned}$$

Hence,

$$\begin{aligned}&U_i(o,o|z)\\&=\frac{-\left((m+\lambda-1)-mU(o,o|z)-(m_i-1)\,z\right)}{2m}\\&+\frac{\sqrt{((m+\lambda-1)-mU(o,o|z)-(m_i-1)\,z)^2+4\lambda m_i z^2}}{2m}.\end{aligned} \tag{2.2}$$

Drawing upon the results established in Eqs 2.1 and 2.2, we are able to deduce that

$$\begin{aligned}&U(o,o|z)\\&=\sum_{i=1}^{r}U_i(o,o|z)\\&=\sum_{i=1}^{r}\frac{-\left((m+\lambda-1)-mU(o,o|z)-(m_i-1)\,z\right)}{2m}\\&+\sum_{i=1}^{r}\frac{\sqrt{((m+\lambda-1)-mU(o,o|z)-(m_i-1)\,z)^2+4\lambda m_i z^2}}{2m}.\end{aligned}$$

Let

$$\partial B_{G_i}(n) = \{x \in V(G_i) : \ |x| = n\}.$$

Define

$$S_i(z) = \sum_{n \geq 1} |\partial B_{G_i}(n)| z^n.$$

Lemma 2.1. [8]

$$\mathrm{gr}(G) = \frac{1}{z_S},$$

where z_S is the unique real number with

$$\sum_{i=1}^{r} \frac{S_i(z_S)}{1 + S_i(z_S)} = 1. \tag{2.3}$$

In the work of E. Candellero, L. A. Gilch, and S. Müller [8], they derive an upper bound for the upper box-counting dimension and a complementary lower bound for the Hausdorff dimension of the geometric endpoint boundary of the trace. This analysis is instrumental in establishing Lemma 2.1. Our approach subsequently leverages Lemma 2.1 as a pivotal step to demonstrate that the growth rate of the free product of complete graphs is indeed positive.

Since the Cayley graph of H_i $(1 \leq i \leq r)$ is a complete graph on $m_i + 1$ vertices,

$$S_i(z) = m_i z.$$

Recall Lemma 2.1,

$$\mathrm{gr}(G) = \frac{1}{z_S},$$

where z_S is the unique real positive number with

$$\sum_{i=1}^{r} \frac{m_i z_S}{1 + m_i z_S} = 1.$$

Let $\Gamma_i = (V_i, \ E_i)$ generated by the vertex set

$$V_i = \{x \in V(G), \text{ the geodesic between } o \text{ and } x \text{ starting at } o \text{ to visit } K_{m_i+1}\}.$$

Clearly, for any $n \in \mathbb{Z}_+$, $1 \leq i \leq r$,

$$|B_{\Gamma_i}(n)| = m_i \sum_{j \neq i} |B_{\Gamma_j}(n-1)|.$$

Hence, for n large enough and $1 \leq i \neq j \leq r$,

$$|\partial B_{\Gamma_i}(n)| \sim |\partial B_{\Gamma_j}(n)| \sim \mathrm{gr}(G)^n.$$

To continue, we introduce some preliminaries about the Cheeger constant. For a weighted graph H (with weight $c(x, y)$ for the edge joining vertices x and y), we say that H satisfies the isoperimetric inequality, briefly IP_∞, if there exists a $\kappa > 0$ such that $C(\partial_E(S)) \geq \kappa C(S)$ for any finite connected subset S. Here $C(S) = \sum_{x \in S} C_x$ and $C(\partial_E(S)) = \sum_{x \in S, \ y \notin S} c(x, y)$, where $C_x = \sum_{y \sim x} c(x, y)$. The largest possible κ is the: Kesten-Cheeger-Dodziuk-Mohar theorem, see [14] Chapter 7, Theorem 7.3. A weighted graph can also be regarded as a network. Readers can refer to [19] for details.

Theorem 2.2. *For any connected infinite network H, the following are equivalent:*
(1) H satisfies IP_∞ *with* $\kappa > 0$.
(2) $0 < \rho < 1$.
In fact, $\kappa^2/2 \le 1 - \rho \le \kappa$.

In the subsequent discussion, we will undertake the task of proving that the Cheeger constant of G is indeed positive. This assertion is facilitated by invoking the Kesten-Cheeger-Dodziuk-Mohar theorem (Theorem 2.2), which leads us directly to the derivation of the following lemma.

Lemma 2.3. *For any* $\lambda \in (0, \mathrm{gr}(G))$, $0 < \rho(\lambda) < 1$.

Proof. To return o, a path can hit a vertex x of type $[i]$ with probability $\frac{1}{m}$, then move to another neighbour of x with type $[j]$ with probability $\frac{m_j-1}{m+\lambda-1}$. Until the $n-th$ step, the random walk runs away from o. From the $(n+1)-th$ until the $2n-th$ the random walk returns o with probability $\frac{\lambda}{m+\lambda-1}$ for every step. Hence, for any $n \ge 2$

$$p^{(2n)}(o,o) \ge \sum_{i=1}^{r} \frac{1}{m} (\min_{1\le i\le r} \frac{m_i}{m+\lambda-1})^{n-1} (\frac{\lambda}{m+\lambda-1})^n.$$

Thus

$$\rho(\lambda) \ge \min_{1\le i\le r} \frac{\sqrt{m_i\lambda}}{m+\lambda-1} > 0.$$

Let S be any connected, finite subgraph of G. Assume that S is between $\partial B_G(n)$ and $\partial B_G(n+m)$. Note that

$$S = \cup_{j=n}^{n+m}(S \cap \partial B_G(j)) = \cup_{j=n}^{n+m} S_j, \ |S| = |\partial S| + |S^o|,$$

where S_j denotes $S \cap \partial B_G(j)$ and S^o denotes the interior points of S.

For any $x \in S_n$, denote the subgraph of x as $S_x = (V(S_x), E(G))$. Here

$$V(S_x) = \{y \in V(G), \text{ the geodesic between } y \text{ and } o \text{ visits } x\},$$

$$E(S_x) = \{xy \in E(G), \ x, y \in V(S_x)\},$$

$$S_x(n) = \{y \in V(S_x) : \text{the graph distance of } x \text{ and } y \le n\}.$$

Given any fixed $\epsilon > 0$, from the definition of $\mathrm{gr}(G)$, there exists n_0 such that, for any $k > n_0$,

$$|\partial B_G(k)| \ge (gr(G) - \epsilon)^k.$$

Define

$$n_1 = \inf\{k > n, \text{ there exists } y \in \partial S \cap S_x(k-n)\},$$

which means that at $\partial S_x(k-n)$, we can find at least one vertex belonging to ∂S. Hence,

$$\frac{C(\partial S_x \cap S_x(n_1-n))}{C(S_x(n_1-n))} \ge \frac{\lambda^{-n_1}}{m^{n_1-n}(m+\lambda-1)\lambda^{-n}} = \frac{\lambda^{n-n_1}}{m^{n_1-n}(m+\lambda-1)}.$$

If $n_1 - n \le n_0$, then

$$\frac{C(\partial S_x \cap S_x(n_1-n))}{C(S_x(n_1-n))} \ge \frac{\lambda^{-n_0}}{m^{n_0}(m+\lambda-1)} > 0.$$

Now assume that, $n_1 - n > n_0$. Notice that every vertex $x \neq o$ has $m - m_i$ neighbors in $B_G(|x| + 1)$ and only one neighbour in $B_G(|x| - 1)$ if $x \in [i]$. Let V_j^i be the set of all vertices of type $[i]$ in $S^o \cap S_j$. It is easy to see that

$$\sum_{i=1}^{r}(m - m_i)|V_j^i| \leq |S_{j+1}|.$$

Notice that $\sum_{i=1}^{r}(m - m_i)|V_j^i|$ denotes the growth way of $S^o \cap S_j$. Since $n_1 - n > n_0$, for $n_0 \leq j \leq n_1 - n$,

$$|S^o \cap \partial S_x(j)|(\mathrm{gr}(G) - \epsilon) \leq \sum_{i=1}^{r}(m - m_i)|V_j^i| \leq |\partial S_x(j + 1)|.$$

Write $c = \mathrm{gr}(G) - \epsilon$. Hence, for any $n_0 \leq j \leq n_1 - n$,

$$\begin{aligned}
&|S^o \cap \partial S_x(j)| \\
&\leq \frac{1}{c}|\partial S_x(j + 1)| = \frac{1}{c}|S^o \cap \partial S_x(j + 1)| + \frac{1}{c}|\partial S \cap \partial S_x(j + 1)| \\
&\leq \frac{1}{c}\left(\frac{1}{c}|S^o \cap \partial S_x(j + 2)| + \frac{1}{c}|\partial S \cap \partial S_x(j + 2)|\right) + \frac{1}{c}|\partial S \cap \partial S_x(j + 1)| \\
&= \left(\frac{1}{c}\right)^2 |S^o \cap \partial S_x(j + 2)| + \left(\frac{1}{c}\right)^2 |\partial S \cap \partial S_x(j + 2)| \\
&\leq \left(\frac{1}{c}\right)^{n_1-n-j} |S^o \cap \partial S_x(n_1 - n)| + \left(\frac{1}{c}\right)^{n_1-n-j} |\partial S \cap \partial S_x(n_1 - n)|.
\end{aligned}$$

In the case when $|\partial S \cap \partial S_x(n_1 - n)| \geq \epsilon|\partial S_x(n_1 - n)|$,

$$|S^o \cap \partial S_x(n_1 - n)| \leq (1 - \epsilon)|\partial S_x(n_1 - n)| \leq \frac{1 - \epsilon}{\epsilon}|\partial S \cap \partial S_x(n_1 - n)|.$$

Therefore, for any $n_0 \leq j \leq n_1 - n$,

$$|S^o \cap S_x(j)| \leq \frac{1}{\epsilon}\left(\frac{1}{c}\right)^{n_1-n-j} |\partial S \cap \partial S_x(n_1 - n)|.$$

Notice that every vertex in $S^o \cap S_x(j)$ has weight $\lambda^{-j}(m + \lambda - 1)$. Therefore,

$$\begin{aligned}
C(S^o \cap \partial S_x(n_0 + 1)) &\leq \frac{m + \lambda - 1}{\lambda^{n+n_0+1}}\frac{1}{\epsilon}\left(\frac{1}{c}\right)^{n_1-n-n_0-1} |\partial S \cap \partial S_x(n_1 - n)|, \\
C(S^o \cap \partial S_x(n_0 + 2)) &\leq \frac{m + \lambda - 1}{\lambda^{n+n_0+2}}\frac{1}{\epsilon}\left(\frac{1}{c}\right)^{n_1-n-n_0-2} |\partial S \cap \partial S_x(n_1 - n)|, \\
C(S^o \cap \partial S_x(n_1 - n - 1)) &\leq \frac{m + \lambda - 1}{\lambda^{n_1-1}}\frac{1}{\epsilon}\frac{1}{c}|\partial S \cap \partial S_x(n_1 - n)|.
\end{aligned}$$

Combining with that

$$C(S^o \cap S_x(n_1 - n)) = \sum_{i=1}^{n_1-n} C(S^o \cap \partial S_x(i)),$$

$$C(\partial S \cap S_x(n_1 - n)) = \sum_{i=0}^{n_1-n} C(\partial S \cap \partial S_x(i)),$$

we have that

$$C(S^o \cap S_x(n_1 - n)) \leq C(S^0 \cap S_x(n_0)) + \sum_{i=1}^{n_1-n-n_0-1} \frac{m+\lambda-1}{\epsilon} \left(\frac{\lambda}{c}\right)^i \lambda^{-n_1} |\partial S \cap \partial S_x(n_1 - n)|.$$

Since $\lambda < \mathrm{gr}(G) - \epsilon$,

$$\sum_{i=1}^{n_1-n-n_0-1} \left(\frac{\lambda}{c}\right)^i \leq \frac{\frac{\lambda}{c}}{1-\frac{\lambda}{c}}.$$

Hence, we obtain that,

$$C(S^o \cap S_x(n_1 - n)) \leq C(S^0 \cap S_x(n_0)) + \frac{m+\lambda-1}{\epsilon} \frac{\frac{\lambda}{c}}{1-\frac{\lambda}{c}} \lambda^{-n_1} |\partial S \cap \partial S_x(n_1 - n)|.$$

Notice that

$$\lambda^{-n_1} |\partial S \cap \partial S_x(n_1 - n)| \leq C(\partial S_x \cap (S_x(n_1 - n) \setminus S_x(n_0)).$$

Thus,

$$\begin{aligned} &\frac{C(\partial S_x \cap S_x(n_1 - n))}{C(S_x(n_1 - n))} \\ &\frac{C(S^o \cap S_x(n_0)) + C(S^o \cap (S_x(n_1 - n) \setminus S_x(n_0))}{C(\partial S \cap S_x(n_0)) + C(\partial S \cap (S_x(n_1 - n) \setminus S_x(n_0))} \\ &\geq \min\{\frac{\lambda^{-n_0}}{m^{n_0}(m+\lambda-1)}, \frac{1}{\frac{m+\lambda-1}{\epsilon}\frac{\frac{\lambda}{c}}{1-\frac{\lambda}{c}} + 1}\} > 0. \end{aligned}$$

Notice that the left case is $|\partial S \cap \partial S_x(n_1 - n)| < \epsilon |\partial S_x(n_1 - n)|$. However, we will discuss the case in the following ways. If $n_1 - n \leq n_0$, for $x_1 \in \partial S_x(n_1)$, define

$$n_2(x_1) = \inf\{k > n_1, \text{ there exists } y \in \partial S \cap S_{x_1}(k - n_1)\}.$$

Then, similar to the proof of $S_x(n_1 - n)$, we can discuss the case of $S_{x_1}(n_2 - n_1)$. If $n_1 - n > n_0$, define

$$n_2 = \inf\{n_2(y),\ y \in \partial S_x(n_1) \cap S^o\}.$$

Moreover, if $|\partial S \cap \partial S_x(n_2 - n)| \geq \epsilon |\partial S_x(n_2 - n)|$, then for $n_1 \leq j \leq n_2$,

$$|S^o \cap \partial S_x(j)|(\mathrm{gr}(G) - 2\epsilon) \leq \sum_{i=1}^{r} (m - m_i)|V_j^i| \leq |\partial S_x(j+1)|.$$

Similarly, as above, we have

$$\frac{C(\partial S \cap S_x(n_2 - n))}{C(S_x(n_2 - n))} \geq \min\{\frac{\lambda^{-n_0}}{m^{n_0}(m+\lambda-1)}, \frac{1}{\frac{m+\lambda-1}{\epsilon}\frac{\frac{\lambda}{c_1}}{1-\frac{\lambda}{c_1}} + 1}\} > 0.$$

Here $c_1 = \mathrm{gr}(G) - 2\epsilon$. The left case is $|\partial S \cap \partial S_x(n_1 - n)| < \epsilon|\partial S_x(n_1 - n)|$ and $|\partial S \cap \partial S_x(n_2 - n_1)| < \epsilon|\partial S_x(n_2 - n_1)|$. Notice that S is a finite graph; then there exists $K > 0$ such that $n_K = n + m$. And

$$\partial S \cap \partial S_x(n_K - n) = \partial S \cap \partial S_x(m) = \partial S_x(n_K - n).$$

Hence,

$$|\partial S \cap \partial S_x(n_K - n)| > \epsilon|\partial S_x(n_K - n)|.$$

Whatever, from the above discussion, we can divide S_x into several parts. S_x^1, S_x^2, $\cdots$ such that every part satisfies $\frac{C(\partial S \cap S_x^i)}{C(S_x^i)} > 0$ ($i \in \mathbb{N}$). Therefore,

$$\frac{C(\partial S \cap \partial S_x)}{C(\partial S_x)} \geq \min\{\frac{\lambda^{-n_0}}{m^{n_0}(m + \lambda - 1)}, \frac{1}{\frac{m+\lambda-1}{\epsilon}\frac{\frac{\lambda}{c_{K-1}}}{1-\frac{\lambda}{c_{K-1}}} + 1}\} > 0.$$

Here $c_{K-1} = \mathrm{gr}(G) - K\epsilon$. Hence, for $\lambda \in (0, \mathrm{gr}(G) - K\epsilon)$,

$$\frac{C(\partial S)}{C(S)} > 0.$$

Note that K is decreasing as $\epsilon \downarrow 0$. Thus, we prove that G is a positive Cheeger constant for $\lambda \in (0, \mathrm{gr}(G))$, which completes the proof by Theorem 2.2.

Lemma 2.4. *$G(o, o|R_G) < \infty$, $U(o, o|R_G) < 1$, and $R_G = R_U$.*

Proof. Note that for any $z > 0$

$$G(o, o|z) = 1 + \sum_{i=1}^{\infty}(U(o, o|z))^i.$$

It is easy to see that $R_G \leq R_U$, and in $|z| < R_G$.

$$G(o, o|z) = \frac{1}{1 - U(o, o|z)}.$$

So $U(o, o|R_G) \leq 1$.

Recall the following: Pringsheim's Theorem: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$, then the radius of the convergence is the smallest positive singularity of $f(z)$.

Hence, the smallest positive singularity R_G of $G(o, o|z)$ is either one of the radius of convergence R_U of $U(o, o|z)$ or the smallest positive number z_1 with $U(o, o|z_1) = 1$. Whatever, notice that $U(o, o|z)$ is strictly increasing for z. Therefore, z_1 is the unique positive number satisfying $U(o, o|z) = 1$. Thus, to prove the theorem, we only need to prove $U(o, o|R_G) < 1$.

Assume that $U(o, o|R_G) = 1$. We exclude the trivial case where $m_i = 1$ for $1 \leq i \leq r$. Notice that $R_G \geq 1$. If $R_G = 1$, then $U(o, o|1) < 1$ by transience. So the left-case is $R_G > 1$. Firstly, let us consider the case of $0 < \lambda \leq 1$. Recall that

$$\begin{aligned} U(o, o|z) &= \sum_{i=1}^{r} U_i(o, o|z) \\ &= \sum_{i=1}^{r} \frac{-((m + \lambda - 1) - mU(o, o|z) - (m_i - 1)\, z)}{2m} \end{aligned}$$

$$+\frac{\sqrt{((m+\lambda-1)-mU(o,o|z)-(m_i-1)\,z)^2+4\lambda m_i z^2}}{2m}.$$

Thus

$$\begin{aligned}1&=\sum_{i=1}^{r}\frac{-((\lambda-1)-(m_i-1)\,R_G)+\sqrt{((\lambda-1)-(m_i-1)\,R_G)^2+4\lambda m_i R_G^2}}{2m}\\&=\sum_{i=1}^{r}\frac{((1-\lambda)+(m_i-1)\,R_G)+\sqrt{((1-\lambda)+(m_i-1)\,R_G)^2+4\lambda m_i R_G^2}}{2m}.\end{aligned}$$

Notice $(1-\lambda)+(m_i-1)R_G \geq (1-\lambda)+(m_i-1)$. Since there is at least one $m_i > 1$, which implies $(1-\lambda)+(m_i-1)R_G > (1-\lambda)+(m_i-1)$, we have

$$\begin{aligned}1&>\sum_{i=1}^{r}\frac{((1-\lambda)+(m_i-1))+\sqrt{((1-\lambda)+(m_i-1))^2+4\lambda m_i}}{2m}\\&=\sum_{i=1}^{r}\frac{m_i-\lambda+m_i+\lambda}{2m}=1.\end{aligned}$$

It is a contradiction. That is for $0<\lambda\leq 1$, $U(o,o|R_G)<1$.

Consider the case when $\lambda>1$. Notice that

$$[\lambda-1-(m_i-1)z]^2+4\lambda m_i z^2=[\lambda-1+(m_i+1)z]^2+4\lambda m_i z^2-4m_i z^2-4(\lambda-1)m_i z.$$

For $\lambda>1$,

$$\begin{aligned}&4\lambda m_i R_G^2-4m_i R_G^2-4(\lambda-1)m_i R_G\\=&4R_G(\lambda m_i R_G-m_i R_G-(\lambda-1)m_i)\\=&4R_G((\lambda-1)m_i R_G-(\lambda-1)m_i)\\=&4R_G(\lambda-1)m_i(R_G-1)\geq 0.\end{aligned}$$

Since $R_G>1$, we have

$$4\lambda m_i R_G^2-4m_i R_G^2-4(\lambda-1)m_i R_G>0.$$

Thus, we get the following contradiction:

$$\begin{aligned}1&=\sum_{i=1}^{r}\frac{-((\lambda-1)-(m_i-1)\,R_G)+\sqrt{((\lambda-1)-(m_i-1)\,R_G)^2+4\lambda m_i R_G^2}}{2m}\\&=\sum_{i=1}^{r}\frac{-((\lambda-1)-(m_i-1)\,R_G)}{2m}\\&+\sum_{i=1}^{r}\frac{\sqrt{((\lambda-1)-(m_i+1)\,R_G)^2+4\lambda m_i R_G^2-4R_G^2+4(\lambda-1)m_i R_G}}{2m}\end{aligned}$$

$$> \sum_{i=1}^{r} \frac{-((\lambda-1)-(m_i-1)R_G)+((\lambda-1)+(m_i+1)R_G)}{2m}$$
$$= R_G > 1.$$

Hence, for $\lambda \geq 1$, $U(o,o|R_G) < 1$.

Suppose that $R_G < R_U$. There exists $\widetilde{z}$ with $R_G < \widetilde{z} < R_U$ such that $U(o,o|\widetilde{z}) < 1$ since $U(o,o|R_G) < 1$. Therefore, $G(o,o|\widetilde{z}) < \infty$. It is in contradiction with R_G, which is the convergence radius of $G(o,o|z)$.

Therefore, we complete the proof.

Lemma 2.5. *For any $\lambda_0 \in (0, \mathrm{gr}(G))$, $\lim_{\lambda\to\lambda_0} R_G(\lambda) = R_G(\lambda_0)$.*

Proof. We will employ proof by contradiction to establish the aforementioned theorem.

Fix a sequence $\{\lambda_k\}_{k\geq 1}$ with $\lambda_k \uparrow \lambda_0$. Suppose $\limsup_{\lambda\to\lambda_0} R_G(\lambda) > R_G(\lambda_0) = z_*$. Then we can find a subsequence n_k with $\lim_{k\to\infty} R_G(\lambda_{n_k}) > z_*$. Without loss of generality, we can assume $z' = \lim_{k\to\infty} R_G(\lambda_k) > z_*$. For a large enough k,

$$1 > U(R_G(\lambda_k), \lambda_k) = \sum_{n=0}^{\infty} f^{(n)}(o,o,\lambda_k) R_G(\lambda_k)^n.$$

Applying Fatou's lemma,

$$1 > \liminf_{k\to\infty} U(R_G(\lambda_k),\lambda_k) = \liminf_{k\to\infty} \sum_{n=0}^{\infty} f^{(n)}(o,o,\lambda_k) R_G(\lambda_k)^n = \sum_{n=0}^{\infty} f^{(n)}(o,o,\lambda_0) z'^n = \infty.$$

It is a contradiction. Hence, $\limsup_{\lambda\to\lambda_0} R_G(\lambda) \leq R_G(\lambda_0)$, and especially $\liminf_{\lambda\to\lambda_0} \rho(\lambda) \leq \rho(\lambda_0)$.

Specially $\limsup_{\lambda\to gr(G)} R_G(\lambda) \leq R_G(gr(G)) = 1$. Notice that for any λ, $R_G(\lambda) \geq 1$. So $\lim_{\lambda\to gr(G)} R_G(\lambda) = R_G(gr(G)) = 1$.

Here we use $f^{(n)}(\lambda)$ to denote $f^{(n)}(o,o)$ and $U(o,o,\lambda)$ to $U(o,o|z)$. Let

$$\Pi_n = \{ \text{ all the paths with length } n \text{ and } \tau_o = n\}.$$

Thus

$$f^{(n)}(\lambda) = \sum_{\gamma\in\Pi_n} \mathbb{P}(\gamma,\lambda).$$

Here $\mathbb{P}(\gamma,\lambda) = \prod_{i=0}^{n} p(w_i, w_{i+1})$ for $\gamma = w_0 w_1 \cdots w_n$. Note that

$$p(v,u) = \begin{cases} 1/m & \text{if } v = o, \\ \frac{\lambda}{m+\lambda-1} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o, \\ \frac{1}{m+\lambda-1} & \text{otherwise.} \end{cases}$$

Given $\lambda_0 \in (0, \mathrm{gr}(G))$ and z_0. For any $\epsilon > 0$, $R_G(\lambda), R_G(\lambda_0)$, $z < z_0$ with $|z-z_0|^2 + |\lambda-\lambda_0|^2 < \epsilon$, there exists a $0 < \delta < \sqrt{\epsilon}$ such that $\mathbb{P}(\gamma,\lambda) \leq (1+\delta)^n \mathbb{P}(\gamma,\lambda_0)$. Hence, $f^{(n)}(\lambda) \leq (1+\delta)^n f^{(n)}(\lambda_0)$. And there exists a $\delta_1 > 0$ such that

$$(1+\delta)\frac{z}{z_0} \leq (1+\delta_1) < \frac{R_{\lambda_0}}{z_0}.$$

Therefore,

$$U(o,o,\lambda)=\Sigma_{n=0}^{\infty}f^{(n)}(\lambda)z^n\le\Sigma_{n=0}^{\infty}(1+\delta_1)^n f^{(n)}(\lambda_0)z_0^n<\infty.$$

If $\limsup_{\lambda\to\lambda_0} R_G(\lambda)<R_G(\lambda_0)$. We can find a subsequence n_k such that $\lim_{\lambda_{n_k}\to\lambda_0} R_G(\lambda)<R_G(\lambda_0)$. Thus for any $\epsilon>0$, let $z=R_{\lambda_0}-\frac{\epsilon}{2}$. Since $\lim_{\lambda\to\lambda_0} R_G(\lambda)<R_G(\lambda_0)$,

$$U(o,o,z)=\infty.$$

It is impossible. It means that $\lim_{\lambda\to\lambda_0} R_G(\lambda)=R_G(\lambda_0)$.

3. Concluding of some open problems

For any vertex set A and Z, let $\tau_A=\inf\{n\ge 0|\ X_n\in A\}$. If RW_λ starts at a vertex in A, then $\tau_A=0$. Write $\tau_A^+=\inf\{n>0|\ X_n\in A\}$. τ_A^+ is different from τ_A only when RW_λ starts in A. Consider the probability of a RW_λ starting at a vertex x that visits A before its visit Z :

$$\mathbb{P}_x(A\to Z,\lambda)=\mathbb{P}_x(\tau_Z<\tau_A^+).$$

For $\lambda\in[0,\lambda_c]$, define $\theta(\lambda)=\mathbb{P}_o(\tau_o^+(\lambda)<\infty)$. Clearly, $\theta(\lambda)=U(o,o|1)=\sum_{n=1}^{\infty}f^{(n)}(o,o,\lambda)$, and $\theta(0)=0$. Suppose $A=\{o\}$ and $(G_n)_{n\ge1}$ be any sequence of finite subgraphs of G that exhaust G. That is $G_n\subseteq G_{n+1}$ and $G=\bigcup G_n$. And let Z_n be the set of vertices in $G\setminus G_n$. So $\lim_{n\to\infty}\mathbb{P}_o(o\to Z_n,\lambda)$ is the probability of never returning to o. And $\lim_{n\to\infty}\mathbb{P}_o(o\to Z_n,\lambda)$ is independent of $(Z_n)_{n\ge1}$ see [19] Exercise 2.4. We may regard the entire circuit between o and Z_n as a single conductor of effective conductance $\mathscr{C}_c(o\leftrightarrow Z_n)$. Recall [19] Chapter 2.2, $\mathscr{C}_c(o\leftrightarrow Z_n)=\pi(o)\mathbb{P}_o(o\to Z_n,\lambda)$. Hence,

$$\theta(\lambda)=1-\lim_{n\to\infty}\mathbb{P}_o(o\to Z_n,\lambda).$$

Recall the following: Rayleigh's monotonicity principle.

Theorem 3.1. *Let H be a finite graph and A and Z be two disjoint subsets of its vertices. If c and c' are two assignments of conductances on H with $c\le c'$, then $\mathscr{C}_c(A\leftrightarrow Z)\le\mathscr{C}_{c'}(A\leftrightarrow Z)$.*

Notice that $c(x,y)=c(x,y,\lambda)$ is the edge weight of edge xy. And recall the definition of RW_λ, $c(x,y)$ is decreasing of λ. By Theorem 3.1, for $\lambda_1\le\lambda_2$,

$$\mathscr{C}_{c(\lambda_1)}(o\leftrightarrow Z_n)\ge\mathscr{C}_{c(\lambda_2)}(o\leftrightarrow Z_n).$$

Whatever, for o, $\pi(o)$ can be any positive constant that is independent of λ. In fact, for any vertex $x\in G$, we can choose $\pi(x)=\sum_{y\sim x}c(x,y)$. Therefore, for $\lambda_1\le\lambda_2$,

$$\lim_{n\to\infty}\mathbb{P}_o(o\to Z_n,\lambda_1)\ge\lim_{n\to\infty}\mathbb{P}_o(o\to Z_n,\lambda_2).$$

And

$$\theta(\lambda_1)\le\theta(\lambda_2).$$

The number of visits to vertex o prior to escape is modeled by a geometric random variable, which has an expected value or mean $(1-\theta(\lambda))^{-1}$. Note that the mean of $\tau_o^+(\lambda)$,

$$\mathbb{E}_o(\tau_o^+(\lambda)) = \sum_{n=1}^{\infty} n f^{(n)}(o,o,\lambda).$$

Recall the Varopoulos-Carne bound ([19] Chapter 13.2, Theorem 13.4) of n-step transition probability. For any x, y,

$$p^{(n)}(x,y,\lambda) \le 2\sqrt{\pi(y)/\pi(x)}\rho^n.$$

Clearly, if $\rho < 1$, then

$$\mathbb{E}_o(\tau_o^+(\lambda)) = \sum_{n=1}^{\infty} n f^{(n)}(o,o,\lambda) < \infty.$$

Hence, for large enough n, by the Strong Law of Large Numbers,

$$\sharp\{\text{the number of visits o before time } n\} \sim \frac{n}{\mathbb{E}_o(\tau_o^+(\lambda))}.$$

Problem 3.2. *Is $\mathbb{E}_o(\tau_o^+(\lambda))$ increasing for $\lambda \in [0, \mathrm{gr}(G)]$?*

$$\limsup_{n\to\infty} p^{(n)}(o,o,\lambda) = \mathbb{P}_o(\tau_o^+(\lambda) < \infty)^{\frac{1}{\mathbb{E}_o(\tau_o^+(\lambda))}} ?$$

If both problems have a positive answer, then the spectral radius ρ is increasing with λ. And for $\lambda \in [0, \mathrm{gr}(G))$, if $\mathbb{E}_o(\tau_o^+(\lambda)) < \infty$, then $\rho < 1$.

Moreover,

Problem 3.3. *For RW_λ on the free product of complete graphs with $\lambda \in [0, \lambda_c]$, is $\theta(\lambda)$ continuous and strictly increasing?*

Recall the following result: [19] Chapter 6, Proposition 6.6.

Proposition 3.4. *Consider a graph H with an upper exponent growth rate $b > 1$. For a reversible Markov chain $(X_n)_{n\ge 0}$ starting at o on its vertex set with reversible measure $\pi(\cdot)$ is bounded and $\pi(o) > 0$ and $\rho < 1$. Then, in the graph metric,*

$$\liminf_{n\to\infty} \frac{|X_n|}{n} > -\frac{\ln\rho}{\ln b}.$$

Hence, for RW_λ on G with $\rho < 1$, $\liminf_{n\to\infty} \frac{|X_n|}{n} > 0$. So the key, to answer Lyons-Pemantle-Peres monotonicity problem, is to prove the existence of the speed. Furthermore, R. Lyons, R. Pemantle, and Y. Peres [17] proved that on the lamplighter group $\mathbb{Z} \ltimes \sum_{x\in\mathbb{Z}} \mathbb{Z}_2$, which has a growth rate of $(1+\sqrt{5})/2$, the speed of RW_λ is 0 at $\lambda = 1$, and is strictly positive when $1 < \lambda < (1+\sqrt{5})/2$. We wonder whether the spectral radius ρ has a similar property or not. That is

Problem 3.5. *For $\lambda \in (0,1]$, $\rho(\lambda) = 1$? And for $1 < \lambda < (1+\sqrt{5})/2$, $\rho(\lambda) < 1$?*
Moreover, does a similar result hold for amenable groups with exponential growth?

Citing Chapter 9 of [14], it is demonstrated that the n-step transition probability is governed by the spectral radius $\rho(\lambda)$, with the following upper bound holding:

$$p^{(n)}(x, y) \leq 2\sqrt{\frac{\pi(y)}{\pi(x)}}\rho^n.$$

This inequality underscores the influence of the spectral radius; as the spectral radius decreases, the probability of transition between any two states x and y after n steps becomes more restricted. Given that the speed, a measure of how rapidly the random walk explores the graph, is inherently tied to the transition probabilities, a positive speed necessitates that $p^{(n)}(x, y)$ decreases over time, implicating a requirement for $\rho(\lambda)$ to be sufficiently small. Based on these considerations, we posit that the existence of a positive speed aligns with the premise that the spectral radius $\rho(\lambda)$ must be adequately restrained.

4. Conclusions

In the paper, we establish the continuity of the spectral radius $\rho(\lambda)$ of RW_λ on the free product of complete graphs, as a function of the parameter λ within the interval $(0, \mathrm{gr}(G)]$.

Author contributions

He Song, Longmin Wang, Kainan Xiang contributed to the conception of the study; He Song, Longmin Wang, Kainan Xiang, Qingpei Zang contributed significantly to analysis and manuscript preparation; He Song, Longmin Wang, Kainan Xiang, Qingpei Zang contributed to the writing of the manuscript. He Song, Longmin Wang, Kainan Xiang, Qingpei Zang helped perform the analysis with constructive discussions. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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