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Research article

Characterization of solitons in a pseudo-quasi-conformally flat and pseudo-W_8 flat Lorentzian Kähler space-time manifolds

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Abstract: The present paper dealt with the study of solitons of Lorentzian Kähler space-time manifolds. In this paper, we have discussed different conditions for solitons to be steady, expanding, or shrinking in terms of isotropic pressure, the cosmological constant, energy density, nonlinear equations, and gravitational constant in pseudo-quasi-conformally flat and pseudo-W_8 flat Lorentzian Kähler space-time manifolds.

Keywords: Lorentzian Kähler space-time manifolds; solitons; differential equations; partial differential equations; pseudo-quasi-conformal curvature tensor; pseudo-W_8 curvature tensor; nonlinear equations
Mathematics Subject Classification: 35C08, 53C50, 53C55

1. Introduction

The relativistic fluid models being involved in the study of different branches of physics, like astrophysics and plasma physics. According to general relativity, a perfect fluid space-time is a four-dimensional Riemannian manifold with Lorentzian metric g. For the study of idealized distribution of matter, such as the interior of a star or an isotropic pressure, we use perfect fluid in cosmology. The behavior of a perfect fluid inside a spherical object is described by Einstein's field equation. A perfect fluid is said to be a radiation fluid if its mass density ψ is three times that of the isotropic pressure p. O'Neill in 1983 studied the properties of semi-Riemannian geometry in relativity theory [23]. Also, Kaigorodov [19] studied the structure of space-time curvature. Later on, different differential geometers [1,6,9,12,30] extended their study to the curvature structure of space-time. In 2020, Panday and Chaturvedi [24] studied Lorentzian complex space form. In the fields of soliton theory, theory of relativity, and related topics, numerous geometers have investigated geometric and topological characteristics concerning symmetry. Their works from references [2, 14, 17, 18] are a great place to start when looking for ideas and a desire to learn more about symmetry. The Ricci solitons and Einstein solitons generate self-similar solutions to partial differential equations. Ricci flow [16] and Einstein flow [5] are defined as

$$\frac{\partial g}{\partial t} = -2S \ and \ \frac{\partial g}{\partial t} = -2\Big(S - \frac{r}{2}g\Big), \tag{1.1}$$

respectively, where S is the Ricci tensor, g is the Riemannian metric, and r is the scalar curvature. Solitons are waves that travel across space with little energy loss and maintain their shape and speed even after colliding with other waves of the same type. The trajectory of wave transmission is characterized by nonlinear partial differential equations. The concept of η-Ricci soliton was given by Cho and Kimura [11] . An η-Ricci soliton equation is given by

$$L_\xi g + 2S + 2ug + 2v\eta \otimes \eta = 0, \tag{1.2}$$

for real constants u and v, where g, S, and ξ denote pseudo-Riemannian metric, Ricci curvature, and vector field, respectively, and η is a 1-form. The data (g, ξ, u, v) that satisfies Eq (1.2) is called η-Ricci solitons. If $v = 0$, then the data (g, ξ, u) is called Ricci soliton [16]. It is referred to as shrinking, steady, or expanding according to whether u is zero, positive, or negative, accordingly [10].

The Lie derivative of $g(\pi_1, \pi_2)$ with respect to ξ is given by

$$(L_\xi g)(\pi_1, \pi_2) = g(\nabla_{\pi_1}\xi, \pi_2) + g(\pi_1, \nabla_{\pi_2}\xi). \tag{1.3}$$

With the help of differential Eq (1.3), Eq (1.2) can be written as

$$S(\pi_1, \pi_2) = -ug(\pi_1, \pi_2) - v\eta(\pi_1)\eta(\pi_2) - \frac{1}{2}[g(\nabla_{\pi_1}\xi, \pi_2) + g(\pi_1, \nabla_{\pi_2}\xi)], \tag{1.4}$$

for any $\pi_1, \pi_2 \in \chi(M)$.

The equation of the η-Einstein soliton is introduced by Blaga [3] in 2018 as follows:

$$L_\xi g + 2S + (2u - r)g + 2v\eta \otimes \eta = 0, \tag{1.5}$$

where g, S, r, ξ, η, u, and v are stated as above. The data (g, ξ, u, v) that satisfies differential Eq (1.5) is called an η-Einstein soliton in M; for $v = 0$, the data (g, ξ, u) is called an Einstein soliton [5].
Using (1.3) in (1.5), we have

$$S(\pi_1, \pi_2) = -\Big(u - \frac{r}{2}\Big)g(\pi_1, \pi_2) - v\eta(\pi_1)\eta(\pi_2) - \frac{1}{2}[g(\nabla_{\pi_1}\xi, \pi_2) + g(\pi_1, \nabla_{\pi_2}\xi)], \tag{1.6}$$

for any $\pi_1, \pi_2 \in \chi(M)$.

Ricci soliton has been studied by Praveena and Bagewadi in [28,29], in which they obtained results in almost pseudo-symmetric Kähler manifolds. De et al. [13] in 2012 studied conformally flat almost pseudo-Ricci symmetric space-times. Ricci soliton associated with perfect fluid space-time has been discussed by Venkatesha et al. in [20, 34]. In [32], Siddiqi and Siddiqi have discussed conformal Ricci soliton and geometrical structure in a perfect fluid space-time. Pratyay Debnath and Arabinada Konar studied [15] quasi-Einstein manifolds and quasi-Einstein space-times. In 2020, Blaga [4] studied solitons and geometrical structures in perfect fluid space-time. Recently, in 2023, Catuevedi et al. [8] studied the concept of solitons in Bochner Flat Lorentzian Kähler space-time manifold. These ideas motivated us to study the η-Ricci soliton and η-Einstein soliton in pseudo-quasi-conformally flat and pseudo-W_8 flat Lorentzian Kähler space-time manifolds.

2. Basics of Lorentzian Kähler space-time manifold

A semi-Riemannian manifold (dim=n, even) (M^n, g) endowed with a Lorentzian metric g is referred to as a Lorentzian Kähler manifold if it satisfies the following conditions [24]:

$$F^2 = -I, \quad g(F\pi_1, F\pi_2) = g(\pi_1, \pi_2), \textit{and} \quad (\nabla_{\pi_1} F)\pi_2 = 0, \tag{2.1}$$

where F is a (1,1) tensor field, I is an identity matrix, π_1 and π_2 are arbitrary differentiable vector fields, and ∇ is a Riemannian connection. We know that in a Lorentzian Kähler manifold, the following relation holds:

$$R(\pi_1, \pi_2, \pi_3, \pi_4) = R(F\pi_1, F\pi_2, \pi_3, \pi_4) = R(\pi_1, \pi_2, F\pi_3, F\pi_4). \tag{2.2}$$

$$\begin{cases} S(\pi_1, \pi_2) = S(F\pi_1, F\pi_2), \\ S(F\pi_1, \pi_2) = -S(\pi_1, F\pi_2), \\ g(F\pi_1, \pi_2) = -g(\pi_1, F\pi_2). \end{cases} \tag{2.3}$$

We refer to a four-dimensional Lorentzian Kähler manifold as a Lorentzian Kähler space-time manifold. This assumption is taken into consideration throughout the study.

The Einstein equation with cosmological constant for the perfect fluid space-time is as follows:

$$S(\pi_1, \pi_2) = -\Big(\lambda - \frac{r}{2} - Kp\Big)g(\pi_1, \pi_2) + K(\psi + p)\eta(\pi_1)\eta(\pi_2), \tag{2.4}$$

for any $\pi_1, \pi_2 \in \chi(M)$, where p, ψ, λ, K, S, and r are the isotropic pressure, energy density, cosmological constant, gravitational constant, Ricci tensor, and scalar curvature, respectively. η is an associated 1-form such that $\eta(\xi) = -1$, g is the metric tensor of minkowski space-time [22], and ξ is the velocity vector of the fluid. Here, the Ricci tensor S is the functional combination of g and $\eta \otimes \eta$ called quasi-Einstein [7].

Consider an orthonormal frame field $\{E_i\}_{1 \le i \le 4}$, that is, $g(E_i, E_j) = \epsilon_{ij}\delta_{ij}$, $i, j \in \{1, 2, 3, 4\}$ with $\epsilon_{11} = -1$, $\epsilon_{ii} = -1$, $i \in \{2, 3, 4\}$, $\epsilon_{ij} = 0$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

Let $\xi = \sum_{i=1}^{n} \xi^i E_i$, then we can write

$$-1 = g(\xi, \xi) = \sum_{1 \le i, j \le 4} \xi^i \xi^j g(E_i, E_j) = \sum_{i=1}^{4} \epsilon_{ii}\{\xi^i\}^2, \tag{2.5}$$

and

$$\eta(E_i) = g(E_i, \xi) = \sum_{j=1}^{4} \xi^i g(E_i, E_j) = \epsilon_{ii}\xi^i. \tag{2.6}$$

Contracting Eq (2.4), we get

$$r = 4\lambda + K(\psi - 3p). \tag{2.7}$$

The Einstein equation without cosmological constant for perfect fluid space-time is as follows:

$$S(\pi_1, \pi_2) = \Big(\frac{r}{2} + Kp\Big)g(\pi_1, \pi_2) + K(\psi + p)\eta(\pi_1)\eta(\pi_2). \tag{2.8}$$

Now, contracting Eq (2.8), we have

$$r = K(\psi - 3p). \tag{2.9}$$

3. Pseudo-quasi-conformal curvature tensor

In this section, we studied pseudo-quasi-conformal curvature tensor in a Lorentzian Kähler space-time manifold. In 2005, Shaikh and Jana [31] introduced and studied pseudo-quasi-conformal curvature tensor on Riemannian manifolds. Kundu and Prakash et al. [21, 26] studied pseudo-quasi-conformal curvature tensor on P-sasakian manifolds. In 2021, Suh et al. [33], studied pseudo-quasi-conformal curvature tensor in space-times of general relativity.

The pseudo-quasi-conformal curvature tensor is defined by:

$$\begin{aligned}\tilde{V}(\pi_1,\pi_2,\pi_3,\pi_4) = (\alpha_1 + d)R(\pi_1,\pi_2,\pi_3,\pi_4) + \Big(\alpha_2 - \frac{d}{3}\Big)[S(\pi_2,\pi_3)g(\pi_1,\pi_4)\\ - S(\pi_1,\pi_3)g(\pi_2,\pi_4)] + \alpha_2[g(\pi_2,\pi_3)S(\pi_1,\pi_4) - g(\pi_1,\pi_3)S(\pi_2,\pi_4)]\\ - \frac{r}{12}\{\alpha_1 + 6\alpha_2\}[g(\pi_2,\pi_3)g(\pi_1,\pi_4) - g(\pi_1,\pi_3)g(\pi_2,\pi_4)],\end{aligned} \tag{3.1}$$

where R is the curvature tensor, S is the Ricci tensor, $\tilde{V}(\pi_1,\pi_2,\pi_3,\pi_4) = g(\tilde{V}(\pi_1,\pi_2),\pi_3,\pi_4)$, and $R(\pi_1,\pi_2,\pi_3,\pi_4) = g(R(\pi_1,\pi_2)\pi_3,\pi_4)$.

If the manifold is pseudo-quasi-conformally flat, then from Eq (3.1), we can write

$$\begin{aligned}&(\alpha_1 + d)R(\pi_1,\pi_2,\pi_3,\pi_4) + \Big(\alpha_2 - \frac{d}{3}\Big)[S(\pi_2,\pi_3)g(\pi_1,\pi_4) - S(\pi_1,\pi_3)g(\pi_2,\pi_4)]\\ &+ \alpha_2[g(\pi_2,\pi_3)S(\pi_1,\pi_4) - g(\pi_1,\pi_3)S(\pi_2,\pi_4)]\\ &- \frac{r}{12}\{\alpha_1 + 6\alpha_2\}[g(\pi_2,\pi_3)g(\pi_1,\pi_4) - g(\pi_1,\pi_3)g(\pi_2,\pi_4)] = 0.\end{aligned} \tag{3.2}$$

Now, replacing π_1 by $F\pi_1$ and π_2 by $F\pi_2$ in Eq (3.2), we get

$$\begin{aligned}&(\alpha_1 + d)R(F\pi_1,F\pi_2,\pi_3,\pi_4) + \Big(\alpha_2 - \frac{d}{3}\Big)[S(F\pi_2,\pi_3)g(F\pi_1,\pi_4) - S(F\pi_1,\pi_3)g(F\pi_2,\pi_4)]\\ &+ \alpha_2[g(F\pi_2,\pi_3)S(F\pi_1,\pi_4) - g(F\pi_1,\pi_3)S(F\pi_2,\pi_4)]\\ &- \frac{r}{12}\{\alpha_1 + 6\alpha_2\}[g(F\pi_2,\pi_3)g(F\pi_1,\pi_4) - g(F\pi_1,\pi_3)g(F\pi_2,\pi_4)] = 0.\end{aligned} \tag{3.3}$$

Subtracting Eq (3.3) from Eq (3.2), we have

$$\begin{aligned}&(\alpha_1 + d)[R(\pi_1,\pi_2,\pi_3,\pi_4) - R(F\pi_1,F\pi_2,\pi_3,\pi_4)] + \Big(\alpha_2 - \frac{d}{3}\Big)\\ &[S(\pi_2,\pi_3)g(\pi_1,\pi_4) - S(\pi_1,\pi_3)g(\pi_2,\pi_4) - S(F\pi_2,\pi_3)g(F\pi_1,\pi_4)\\ &+ S(F\pi_1,\pi_3)g(F\pi_2,\pi_4)] + \alpha_2[g(\pi_2,\pi_3)S(\pi_1,\pi_4) - g(\pi_1,\pi_3)S(\pi_2,\pi_4)\\ &- g(F\pi_2,\pi_3)S(F\pi_1,\pi_4) + g(F\pi_1,\pi_3)S(F\pi_2,\pi_4)] - \frac{r}{12}\{\alpha_1 + 6\alpha_2\}[g(\pi_2,\pi_3)g(\pi_1,\pi_4)\\ &- g(\pi_1,\pi_3)g(\pi_2,\pi_4) - g(F\pi_2,\pi_3)g(F\pi_1,\pi_4) + g(F\pi_1,\pi_3)g(F\pi_2,\pi_4)] = 0.\end{aligned} \tag{3.4}$$

Now, using Eq (2.2) in Eq (3.4), we get

$$\begin{aligned}&\Big(\alpha_2 - \frac{d}{3}\Big)[S(\pi_2,\pi_3)g(\pi_1,\pi_4) - S(\pi_1,\pi_3)g(\pi_2,\pi_4) - S(F\pi_2,\pi_3)g(F\pi_1,\pi_4)\\ &+ S(F\pi_1,\pi_3)g(F\pi_2,\pi_4)] + \alpha_2[g(\pi_2,\pi_3)S(\pi_1,\pi_4) - g(\pi_1,\pi_3)S(\pi_2,\pi_4)\\ &- g(F\pi_2,\pi_3)S(F\pi_1,\pi_4) + g(F\pi_1,\pi_3)S(F\pi_2,\pi_4)] - \frac{r}{12}\{\alpha_1 + 6\alpha_2\}[g(\pi_2,\pi_3)g(\pi_1,\pi_4)\\ &- g(\pi_1,\pi_3)g(\pi_2,\pi_4) - g(F\pi_2,\pi_3)g(F\pi_1,\pi_4) + g(F\pi_1,\pi_3)g(F\pi_2,\pi_4)] = 0.\end{aligned} \tag{3.5}$$

Taking a frame field over π_1 and π_4 in Eq (3.5) and using Eqs (2.1) and (2.3), we get

$$S(\pi_2, \pi_3) = -\frac{r\alpha_1}{4d} g(\pi_2, \pi_3). \tag{3.6}$$

From Eqs (3.6) and (1.4), we get

$$-\frac{r\alpha_1}{4d} g(\pi_2, \pi_3) = -ug(\pi_2, \pi_3) - v\eta(\pi_2)\eta(\pi_3) - \frac{1}{2}[g(\nabla_{\pi_2}\xi, \pi_3) + g(\pi_2, \nabla_{\pi_3}\xi)]. \tag{3.7}$$

Multiplying (3.7) by ϵ_{ii}, taking $\pi_2 = \pi_3 = E_i$, and using Eqs (2.5) and (2.6), we get

$$4u - v = r\frac{\alpha_1}{d} + 4div\xi. \tag{3.8}$$

Using Eq (2.7) in Eq (3.8), we get

$$4u - v = [4\lambda + K(\psi - 3p)]\frac{\alpha_1}{d} + 4div\xi. \tag{3.9}$$

Now, putting $\pi_2 = \pi_3 = \xi$ in (3.7), we get

$$u - v = \frac{r}{4}\frac{\alpha_1}{d}. \tag{3.10}$$

Again, using Eq (2.7) in Eq (3.10), we get

$$u - v = \left[\lambda + \frac{K}{4}(\psi - 3p)\right]\frac{\alpha_1}{d}. \tag{3.11}$$

From Eqs (3.9) and (3.11), we have

$$u = \left[\lambda + \frac{K}{4}(\psi - 3p)\right]\frac{\alpha_1}{d} + \frac{4}{3}div\xi \text{ and } v = \frac{4div\xi}{3}. \tag{3.12}$$

If $v = 0$, then we get the Ricci soliton with $u = \left[\lambda + \frac{K}{4}(\psi - 3p)\right]\frac{\alpha_1}{d}$.
This will be steady if $u = 0$; therefore, from Eq (3.12), we get $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, or $\frac{\alpha_1}{d} = 0$.
Shrinking if $u < 0$, therefore, from Eq (3.12), we get $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} < 0$.
Expanding if $u > 0$, therefore, from Eq (3.12), we get $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} < 0$.

Thus, we conclude:

Theorem 3.1. *Ricci soliton (g, ξ, u) in a pseudo-quasi-conformally flat Lorentzian Kähler space-time manifold with cosmological constant is:*

(i) steady: if $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\frac{\alpha_1}{d} = 0$,
(ii) shrinking: if $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} < 0$,
(iii) or expanding: if $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\frac{\alpha_1}{d} < 0$.
Using Eqs (2.9) and (3.8), we get

$$4u - v = [K(\psi - 3p)]\frac{\alpha_1}{d} + 4div\xi. \tag{3.13}$$

Using Eqs (2.9) and (3.10), we get

$$u - v = \Big[\frac{K(\psi - 3p)}{4}\Big]\frac{\alpha_1}{d}. \tag{3.14}$$

After calculating (3.13) and (3.14), we get

$$u = \Big[\frac{K(\psi - 3p)}{4}\Big]\frac{\alpha_1}{d} + \frac{4}{3}div\xi \ \text{ and } \ v = \frac{4}{3}div\xi, \tag{3.15}$$

and if $v = 0$, then we get the Ricci soliton with $u = \Big[\frac{K(\psi-3p)}{4}\Big]\frac{\alpha_1}{d}$.
This will be steady if $u = 0$, therefore, from Eq (3.15), we get $p = \frac{K\psi}{3}$ or $\frac{\alpha_1}{d} = 0$.
Shrinking if $u < 0$, therefore, from Eq (3.15), we get $p < \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p > \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} < 0$.
Expanding if $u > 0$, therefore, from Eq (3.15), we get $p < \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} < 0$ or $p > \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} > 0$.

Thus, we conclude:

Theorem 3.2. *A Ricci soliton (g, ξ, u) in a pseudo-quasi-conformally flat Lorentzian Kähler space-time manifold without cosmological constant is:*

(i) steady: if $p = \frac{K\psi}{3}$ or $\frac{\alpha_1}{d} = 0$,
(ii) shrinking: if $p < \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} > 0$ or $p > \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} < 0$,
(iii) or expanding: if $p < \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} < 0$ or $p > \frac{K\psi}{3}$, and $\frac{\alpha_1}{d} > 0$.

From Eqs (1.6) and (3.6), we obtain

$$-\frac{r\alpha_1}{4d}g(\pi_2, \pi_3) = -\Big(u - \frac{r}{2}\Big)g(\pi_2, \pi_3) - v\eta(\pi_2)\eta(\pi_3) - \frac{1}{2}[g(\nabla_{\pi_2}\xi, \pi_3) + g(\pi_2, \nabla_{\pi_3}\xi)]. \tag{3.16}$$

Taking $\pi_2 = \pi_3 = E_i$, multiplying Eq (3.16) by ϵ_{ii}, and using Eqs (2.5) and (2.6), we get

$$4u - v = r\Big(\frac{\alpha_1}{d} + 2\Big) + 4div\xi. \tag{3.17}$$

Using Eq (2.7) in Eq (3.17), we get

$$4u - v = [4\lambda + K(\psi - 3p)]\Big(\frac{\alpha_1}{d} + 2\Big) + 4div\xi, \tag{3.18}$$

and taking $\pi_2 = \pi_3 = \xi$ in Eq (3.16), we have

$$u - v = \frac{r}{4}\Big(\frac{\alpha_1}{d} + 2\Big). \tag{3.19}$$

Again, using Eq (2.7) in Eq (3.19), we get

$$u - v = \Big[\lambda + \frac{K}{4}(\psi - 3p)\Big]\Big(\frac{\alpha_1}{d} + 2\Big). \tag{3.20}$$

From Eqs (3.18) and (3.20), we have

$$u = \Big[\lambda + \frac{K}{4}(\psi - 3p)\Big]\Big(\frac{\alpha_1}{d} + 2\Big) + \frac{4}{3}div\xi \ \text{ and } \ b = \frac{4}{3}div\xi, \tag{3.21}$$

if $v = 0$, then we get the Einstein soliton with $u = \left[\lambda + \frac{K}{4}(\psi - 3p)\right]\left(\frac{\alpha_1}{d} + 2\right)$.
This will be steady if $u = 0$, therefore, from Eq (3.21), we get $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\left(\frac{\alpha_1}{d} + 2\right) = 0$.
Shrinking if $u < 0$, therefore, from Eq (3.21), we get $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.
Expanding if $u > 0$, therefore, from Eq (3.21), we get $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.

Thus, we conclude:

Theorem 3.3. *An Einstein soliton (g, ξ, u) in a pseudo-quasi-conformally flat Lorentzian Kähler space-time manifold with cosmological constant is:*

(i) steady: if $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\left(\frac{\alpha_1}{d} + 2\right) = 0$,
(ii) shrinking: if $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$,
(iii) or expanding: if $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.

Using Eqs (2.9) and (3.17), we get

$$4u - v = K(\psi - 3p)\left(\frac{\alpha_1}{d} + 2\right) + 4div\xi. \tag{3.22}$$

Using Eqs (2.9) and (3.19), we get

$$u - v = \frac{K(\psi - 3p)}{4}\left(\frac{\alpha_1}{d} + 2\right). \tag{3.23}$$

After calculating Eqs (3.22) and (3.23), we get

$$u = \frac{K(\psi - 3p)}{4}\left(\frac{\alpha_1}{d} + 2\right) + \frac{4}{3}div\xi \quad \text{and} \quad v = \frac{4}{3}div\xi, \tag{3.24}$$

if $v = 0$, then we obtain the Einstein soliton with $u = \frac{K(\psi-3p)}{4}\left(\frac{\alpha_1}{d} + 2\right)$.
This will be steady if $u = 0$, therefore, from Eq (3.24), we get $p = \frac{K\psi}{3}$ or $\left(\frac{\alpha_1}{d} + 2\right) = 0$.
Shrinking if $u < 0$, therefore, from Eq (3.24), we get $p < \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p > \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.
Expanding if $u > 0$, therefore, from Eq (3.24), we get $p > \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p < \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.

Thus, we conclude:

Theorem 3.4. *An Einstein soliton (g, ξ, u) in a pseudo-quasi-conformally flat Lorentzian Kähler space-time manifold without cosmological constant is:*

(i) steady: if $p = \frac{K\psi}{3}$ or $\left(\frac{\alpha_1}{d} + 2\right) = 0$,
(ii) shrinking: if $p < \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p > \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$,
(iii) or expanding: if $p > \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) > 0$ or $p < \frac{K\psi}{3}$, and $\left(\frac{\alpha_1}{d} + 2\right) < 0$.

4. Pseudo-W_8 curvature tensor

In this section, we studied the pseudo-W_8 curvature tensor in Lorentzian Kähler space-time manifold. In 1982, Pokhariyal and Mishra [25] defined W_8 curvature tensor. Later in 2018, Pandey et al. [27] gave the concept of the pseudo W_8 curvature tensor on a Riemannian manifold. The pseudo-W_8 curvature tensor is defined by:

$$\begin{aligned}\tilde{W}_8(\pi_1,\pi_2,\pi_3,\pi_5) &= \alpha_1\tilde{R}(\pi_1,\pi_2,\pi_3,\pi_5) + \alpha_2[S(\pi_1,\pi_2)g(\pi_3,\pi_5) - S(\pi_2,\pi_3)g(\pi_1,\pi_5)]\\ &- \frac{r}{n}\Big[\frac{\alpha_1}{n-1} - \alpha_2\Big][g(\pi_1,\pi_2)g(\pi_3,\pi_5) - g(\pi_2,\pi_3)g(\pi_1,\pi_5)].\end{aligned} \tag{4.1}$$

If the manifold is pseudo-W_8 flat, then from Eq (4.1), we can write

$$\begin{aligned}&\alpha_1\tilde{R}(\pi_1,\pi_2,\pi_3,\pi_5) + \alpha_2[S(\pi_1,\pi_2)g(\pi_3,\pi_5) - S(\pi_2,\pi_3)g(\pi_1,\pi_5)]\\ &- \frac{r}{4}\Big[\frac{\alpha_1}{3} - \alpha_2\Big][g(\pi_1,\pi_2)g(\pi_3,\pi_5) - g(\pi_2,\pi_3)g(\pi_1,\pi_5)] = 0.\end{aligned} \tag{4.2}$$

Now, replacing π_1 by $F\pi_1$ and π_2 by $F\pi_2$ in Eq (4.2), we get

$$\begin{aligned}&\alpha_1\tilde{R}(F\pi_1,F\pi_2,\pi_3,\pi_5) + \alpha_2[S(F\pi_1,F\pi_2)g(\pi_3,\pi_5) - S(F\pi_2,\pi_3)g(F\pi_1,\pi_5)]\\ &- \frac{r}{4}\Big[\frac{\alpha_1}{3} - \alpha_2\Big][g(F\pi_1,F\pi_2)g(\pi_3,\pi_5) - g(F\pi_2,\pi_3)g(F\pi_1,\pi_5)] = 0.\end{aligned} \tag{4.3}$$

Subtracting Eq (4.3) from Eq (4.2), we have

$$\begin{aligned}&\alpha_1[\tilde{R}(\pi_1,\pi_2,\pi_3,\pi_5) - \tilde{R}(F\pi_1,F\pi_2,\pi_3,\pi_5)] + \alpha_2[S(\pi_1,\pi_2)g(\pi_3,\pi_5) - S(\pi_2,\pi_3)g(\pi_1,\pi_5)\\ &- S(F\pi_1,F\pi_2)g(\pi_3,\pi_5) + S(F\pi_2,\pi_3)g(F\pi_1,\pi_5)] - \frac{r}{4}\Big[\frac{\alpha_1}{3} - \alpha_2\Big][g(\pi_1,\pi_2)g(\pi_3,\pi_5)\\ &- g(\pi_2,\pi_3)g(\pi_1,\pi_5) - g(F\pi_1,F\pi_2)g(\pi_3,\pi_5) + g(F\pi_2,\pi_3)g(F\pi_1,\pi_5)] = 0.\end{aligned} \tag{4.4}$$

Using Eq (2.2) in Eq (4.4), we get

$$\begin{aligned}&\alpha_2[S(\pi_1,\pi_2)g(\pi_3,\pi_5) - S(\pi_2,\pi_3)g(\pi_1,\pi_5) - S(F\pi_1,F\pi_2)g(\pi_3,\pi_5) + S(F\pi_2,\pi_3)g(F\pi_1,\pi_5)]\\ &- \frac{r}{4}\Big[\frac{\alpha_1}{3} - \alpha_2\Big][g(\pi_1,\pi_2)g(\pi_3,\pi_5) - g(\pi_2,\pi_3)g(\pi_1,\pi_5) - g(F\pi_1,F\pi_2)g(\pi_3,\pi_5)\\ &+ g(F\pi_2,\pi_3)g(F\pi_1,\pi_5)] = 0.\end{aligned} \tag{4.5}$$

Taking a frame field over π_1 and π_5 in Eq (4.5) and using Eqs (2.1) and (2.3), we get

$$S(\pi_2,\pi_3) = \frac{r}{4}\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big)g(\pi_2,\pi_3). \tag{4.6}$$

From Eqs (4.6) and (1.4), we get

$$\frac{r}{4}\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big)g(\pi_2,\pi_3) = -ug(\pi_2,\pi_3) - v\eta(\pi_2)\eta(\pi_3) - \frac{1}{2}[g(\nabla_{\pi_2}\xi,\pi_3) + g(\pi_2,\nabla_{\pi_3}\xi)]. \tag{4.7}$$

Taking $\pi_2 = \pi_3 = E_i$, multiplying (4.7) by ϵ_{ii}, and using Eqs (2.5) and (2.6), we get

$$4u - v = -r\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) + 4div\xi. \tag{4.8}$$

Using Eq (2.7) in Eq (4.8), we get

$$4u - v = [-4\lambda - K(\psi - 3p)]\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) + 4div\xi. \tag{4.9}$$

Now, putting $\pi_2 = \pi_3 = \xi$ in (4.7), we have

$$u - v = -\frac{r}{4}\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big). \tag{4.10}$$

Using Eq (2.7) in Eq (4.10), we get

$$u - v = \Big[-\lambda - \frac{K}{4}(\psi - 3p)\Big]\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big). \tag{4.11}$$

From Eqs (4.9) and (4.11), we have

$$u = \Big[-\lambda - \frac{K}{4}(\psi - 3p)\Big]\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) + \frac{4}{3}div\xi \ \text{ and } \ v = \frac{4}{3}div\xi, \tag{4.12}$$

if $v = 0$, then we get the Ricci soliton with $u = \Big[-\lambda - \frac{K}{4}(\psi - 3p)\Big]\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big)$.
This will be steady if $u = 0$, therefore, from Eq (4.12), we get $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) = 0$.
Shrinking if $u < 0$, therefore, from Eq (4.12), we get $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.
Expanding if $u > 0$, therefore, from Eq (4.12), we get $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.

Thus, we conclude:

Theorem 4.1. *A Ricci soliton (g, ξ, u) in a pseudo-W_8 flat Lorentzian Kähler space-time manifold with cosmological constant is:*

(i) steady: if $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) = 0$,
(ii) shrinking: $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$,
(iii) or expanding: if $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.

Using Eqs (2.9) and (4.8), we get

$$4u - v = -K(\psi - 3p)\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) + 4div\xi. \tag{4.13}$$

Using Eqs (2.9) and (4.10), we get

$$u - v = -\frac{K}{4}(\psi - 3p)\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big). \tag{4.14}$$

After calculating (4.13) and (4.14), we have

$$u = -\frac{K}{4}(\psi - 3p)\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) + \frac{4}{3}div\xi \ \text{ and } \ v = \frac{4}{3}div\xi, \tag{4.15}$$

if $v = 0$, then we get the Ricci soliton with $u = -\frac{K}{4}(\psi - 3p)\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big)$.
This will be steady if $u = 0$, therefore, from Eq (4.15), we get $p = \frac{\psi}{3}$ or $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) = 0$.
Shrinking if $u < 0$, therefore, from Eq (4.15), we get $p < \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p > \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.
Expanding if $u > 0$, therefore, from Eq (4.15), we get $p > \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p < \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.

Thus, we conclude:

Theorem 4.2. *A Ricci soliton (g, ξ, u) in a pseudo-W_8 flat Lorentzian Kähler space-time manifold with cosmological constant is:*

(i) steady: if $p = \frac{\psi}{3}$ or $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) = 0$,
(ii) shrinking: $p < \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p > \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$,
(iii) or expanding: if $p > \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) > 0$ or $p < \frac{\psi}{3}$, and $\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big) < 0$.

From Eq (1.6) and (4.6), we get

$$\frac{r}{4}\Big(\frac{\alpha_1}{3\alpha_2} - 1\Big)g(\pi_2, \pi_3) = -(u - \frac{r}{2})g(\pi_2, \pi_3) - v\eta(\pi_2)\eta(\pi_3) - \frac{1}{2}[g(\nabla_{\pi_2}\xi, \pi_3) + g(\pi_2, \nabla_{\pi_3}\xi)]. \tag{4.16}$$

Taking $\pi_2 = \pi_3 = E_i$, multiplying Eq (4.16) by ϵ_{ii}, and using Eqs (2.5) and (2.6), we get

$$4u - v = r\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big) + 4div\xi. \tag{4.17}$$

Using Eq (2.7) in Eq (4.17), we have

$$4u - v = [4\lambda + K(\psi - 3p)]\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big) + 4div\xi. \tag{4.18}$$

Now, putting $\pi_2 = \pi_3 = \xi$ in (4.16), we get

$$u - v = \frac{r}{4}\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big). \tag{4.19}$$

Using Eq (2.7) in Eq (4.19), we get

$$u - v = \Big[\lambda + \frac{K}{4}(\psi - 3p)\Big]\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big). \tag{4.20}$$

From Eqs (4.18) and (4.20), we have

$$u = \Big[\lambda + \frac{K}{4}(\psi - 3p)\Big]\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big) + \frac{4}{3}div\xi \text{ and } v = \frac{4}{3}div\xi, \tag{4.21}$$

if $v = 0$, then we get the Einstein soliton with $u = \Big[\lambda + \frac{K}{4}(\psi - 3p)\Big]\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big)$.
This will be steady if $u = 0$, therefore, from Eq (4.21), we get $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$ or $\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big) = 0$.
Shrinking if $u < 0$, therefore, from Eq (4.21), we get $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and $\Big(3 - \frac{\alpha_1}{3\alpha_2}\Big) > 0$ or $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\psi}{3}$, and

$\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.
Expanding if $u>0$, therefore, from Eq (4.21), we get $p>\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p<\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.

Thus, we conclude:

Theorem 4.3. *An Einstein soliton (g,ξ,u) in a pseudo-W_8 flat Lorentzian Kähler space-time manifold with cosmological constant is:*

(i) steady: if $p=\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$ or $\left(3-\frac{\alpha_1}{3\alpha_2}\right)=0$,
(ii) shrinking: if $p<\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p>\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$,
(iii) or expanding: if $p>\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p<\frac{4}{3}\frac{\lambda}{K}+\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.
Using Eqs (2.9) and (4.17), we get

$$4u-v=K(\psi-3p)\Big(3-\frac{\alpha_1}{3\alpha_2}\Big)+4div\xi. \tag{4.22}$$

Using Eqs (2.9) and (4.19), we get

$$u-v=\frac{K(\psi-3p)}{4}\Big(3-\frac{\alpha_1}{3\alpha_2}\Big). \tag{4.23}$$

After calculating Eqs (4.22) and (4.23), we get

$$u=\frac{K(\psi-3p)}{4}\Big(3-\frac{\alpha_1}{3\alpha_2}\Big)+\frac{4}{3}div\xi \ \text{ and } \ v=\frac{4}{3}div\xi, \tag{4.24}$$

if $v=0$, then we get the Einstein soliton with $u=\frac{K(\psi-3p)}{4}\left(3-\frac{\alpha_1}{3\alpha_2}\right)$.
This will be steady if $u=0$, therefore, from Eq (4.24), we get $p=\frac{\psi}{3}$ or $\left(3-\frac{\alpha_1}{3\alpha_2}\right)=0$.
Shrinking if $u<0$, therefore, from Eq (4.24), we get $p<\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p>\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.
Expanding if $u>0$, therefore, from Eq (4.24), we get $p>\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p<\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.

Thus, we conclude:

Theorem 4.4. *An Einstein soliton (g,ξ,u) in a pseudo-W_8 flat Lorentzian Kähler space-time manifold without cosmological constant is:*

(i) steady: if $p=\frac{\psi}{3}$ or $\left(3-\frac{\alpha_1}{3\alpha_2}\right)=0$,
(ii) shrinking: if $p<\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$ or $p>\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$,
(iii) or expanding: if $p>\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)>0$, and or $p<\frac{\psi}{3}$, and $\left(3-\frac{\alpha_1}{3\alpha_2}\right)<0$.

Author contributions

B. B. Chaturvedi, Kunj Bihari Kaushik, Prabhawati Bhagat, Mohammad Nazrul Islam Khan: conceptualization, methodology, investigation, writing-original draft, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. A. Awane, A. Chkiriba, M. Goze, E. Azizi, M. B. Bah, Vectorial polarized manifolds, *Africain Journal of Mathematical Physics*, **4** (2007), 33–43.
2. M. B. Ayed, K. E. Mehdi, M. O. Ahmedou, F. Pacella, Energy and Morse index of solutions of Yamabe type problems on thin annuli, *J. Eur. Math. Soc.*, **7** (2005), 283–304. https://doi.org/10.4171/JEMS/29
3. A. M. Blaga, On gradient η-Einstein solitons, *Kragujev. J. Math.*, **42** (2018), 229–237.
4. A. M. Blaga, Solitons and geometrical structures in a perfect fluid spacetime, *Rocky Mountain J. Math.*, **50** (2020), 41–53. https://doi.org/10.1216/rmj.2020.50.41
5. G. Catino, L. Mazzieri, Gradient Einstein solitons, *Nonlinear Analysis*, **132** (2016), 66–94. https://doi.org/10.1016/j.na.2015.10.021
6. M. C. Chaki, S. Ray, Space-times with covariant-constant energy-momentum tensor, *Int. J. Theor. Phys.*, **35** (1996), 1027–1032. https://doi.org/10.1007/BF02302387
7. M. C. Chaki, R. K. Maity, On quasi Einstein manifolds, *Publ. Math. Debrecen*, **57** (2000), 297–306. https://doi.org/10.5486/PMD.2000.2169
8. B. B. Chaturvedi, P. Bhagat, M. N. I. Khan, Novel theorems for a Bochner flat Lorentzian Kähler space-time manifold with η-Ricci-Yamabe solitons, *Chaos Soliton. Fract.*, **11** (2023), 100097. https://doi.org/10.1016/j.csfx.2023.100097
9. H. Chtioui, K. E. Mehdi, N. Gamara, The Webster scalar curvature problem on the three dimensional CR manifolds, *B. Sci. Math.*, **131** (2007), 361–374. https://doi.org/10.1016/j.bulsci.2006.05.003
10. B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Rhode Island: AMS and Science Press, 2006.
11. J. T. Cho, M. Kimura, Ricci solitons and real hyper surfaces in a complex space form, *Tohoku Math. J.*, **61** (2009), 205–212. https://doi.org/10.2748/tmj/1245849443
12. U. C. De, G. C. Ghosh, On weakly Ricci symmetric spacetime manifolds, *Radovi Matematicki*, **13** (2004), 93–101.

13. A. De, C. Özgür, U. C. De, On conformally flat almost pseudo-Ricci symmetric spacetimes, *Int. J. Theor. Phys.*, **51** (2012), 2878–2887. https://doi.org/10.1007/s10773-012-1164-0

14. U. C. De, Y. J. Suh, S. K. Chaubey, Semi-symmetric curvature properties of Robertson-Walker spacetimes, *J. Math. Phys. Anal. Geo.*, **18** (2022), 368–381. https://doi.org/10.15407/mag18.03.368

15. P. Debnath, A. Konar, On quasi-Einstein manifolds and quasi-Einstein spacetimes, *Differ. Geom. Dyn. Syst.*, **12** (2010), 73–82.

16. R. S. Hamilton, The Ricci flow on surfaces, *Contemp. Math.*, **71** (1988), 237–262.

17. A. Haseeb, M. Bilal, S. K. Chaubey, A. A. H. Ahmadini, ξ-Conformally flat LP-kenmotsu manifolds and Ricci-Yamabe solitons, *Mathematics*, **11** (2022), 212. https://doi.org/10.3390/math11010212

18. A. Haseeb, S. K. Chaubey, F. Mofarreh, A. A. H. Ahmadini, A solitonic study of riemannian manifolds equipped with a semi-symmetric metric ξ-connection, *Axioms*, **12** (2023), 809. https://doi.org/10.3390/axioms12090809

19. V. R. Kaigorodov, Structure of space-time curvature, *J. Math. Sci.*, **28** (1985), 256–273. https://doi.org/10.1007/BF02105213

20. Venkatesha, H. A. Kumara, Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field, *Afr. Mat.*, **30** (2019), 725–736. https://doi.org/10.1007/s13370-019-00679-y

21. S. Kundu, On P-Sasakian manifolds, *Math. Rep.*, **15** (2013), 221–232.

22. H. Mohajan, Minkowski geometry and space-time manifold in relativity, *Journal of Environmental Treatment Techniques*, **1** (2013), 101–109.

23. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, New York: Academic Press, 1983.

24. P. Pandey, B. B. Chaturvedi, On a Lorentzian complex space form, *Natl. Acad. Sci. Lett.*, **43** (2020), 351–353. https://doi.org/10.1007/s40009-020-00874-7

25. G. P. Pokhariyal, Relativistic significance of curvature tensors, *International Journal of Mathematics and Mathematical Sciences*, **5** (1982), 133–139. https://doi.org/10.1155/S0161171282000131

26. D. G. Prakasha, S. R. Talawar, K. K. Mitji, On the pseudo-quasi-conformal curvature tensor of P-Sasakian manifolds, *Electronic Journal of Mathematical Analysis and Applications*, **5** (2017), 147–155.

27. B. Prasad, R. P. S. Yadav, S. N. Pandey, Pseudo $W8$ curvature tensor $\tilde{W8}$ on a Riemannian manifold, *Journal of Progressive Science*, **9** (2018), 35–43.

28. M. M. Praveena, C. S. Bagewadi, On almost pseudo Bochner symmetric generalized complex space forms, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, **32** (2016), 149–159.

29. M. M. Praveena, C. S. Bagewadi, On almost pseudo symmetric Kähler manifolds, *Palestine Journal of Mathematics*, **6** (2017), 272–278.

30. A. K. Raychaudhuri, B. Sriranjan, A. Banerjee, *General relativity, astrophysics, and cosmology*, New York: Springer, 1992.

31. A. A. Shaikh, S. K. Jana, A pseudo quasi-conformal curvature tensor on a Riemannian manifold, *South East Asian J of Mathematics and Mathematical Sciences*, **4** (2005), 15–20.

32. M. D. Siddiqi, S. A. Siddiqi, Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime, *Int. J. Geom. Methods M.*, **17** (2020), 2050083. https://doi.org/10.1142/S0219887820500838

33. Y. J. Suh, V. Chavan, N. A. Pundeer, Pseudo-quasi-conformal curvature tensor and spacetimes of general relativity, *Filomat*, **35** (2021), 657–666. https://doi.org/10.2298/FIL2102657S

34. Venkatesha, S. Chidananda, η-Ricci soliton and almost η-Ricci soliton on almost coKähler manifolds, *Acta Math. Univ. Comenianae*, **2** (2021), 217–230.

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