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Research article

Oscillation criterion of Kneser type for half-linear second-order dynamic equations with deviating arguments

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Abstract: This paper employed the well-known Riccati transformation method to deduce a Kneser-type oscillation criterion for second-order dynamic equations. These results are considered an extension and improvement of the known Kneser results for second-order differential equations and are new for other time scales. We have included examples to highlight the significance of the results we achieved.

Keywords: Kneser-type; oscillation; second-order; half-linear; dynamic equation; time scales
Mathematics Subject Classification: 39A10, 39A99, 34N05, 34K11

1. Introduction

Stefan Hilger [1] proposed the concept of dynamic equations on time scales to unite continuous and discrete analysis. Dynamic equation theory includes classical theories for differential and difference equations and instances in between. The q-difference equations, with significant consequences in quantum theory (refer to [2]), can be analyzed across several time scales. The time scales include $\mathbb{T}=q^{\mathbb{N}_0} := \{q^\lambda : \lambda \in \mathbb{N}_0 \text{ for } q > 1\}$, $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=\mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ denotes harmonic numbers. See the sources [3, 4] for more information on time scale calculus.

Researchers from a wide variety of applied fields have demonstrated a substantial amount of interest in the phenomena of oscillation. This is primarily owing to the fact that oscillation has its roots in mechanical vibrations and has a wide range of applications in the fields of science and engineering. It is possible for oscillation models to integrate advanced terms or delays in order to take into account the influence that temporal contexts have on different solutions. There has been a substantial amount of investigation carried out on the topic of oscillation in delay equations, as evidenced by the contributions made by [5–11]. Compared to other areas of research, the extant literature on advanced oscillation is rather limited, consisting of only a few publications that specifically explore this topic [12–17].

In order to examine and gain an understanding of the phenomenon of oscillation, which is present in a wide variety of practical applications, a wide variety of models are applied. Through the integration of cross-diffusion factors, particular models within the field of mathematical biology have been improved in order to take into consideration the effects of delay and/or oscillation. It is recommended that interested parties consult the scholarly publications with the titles [18, 19] in order to have a more in-depth understanding of this subject matter. Since differential equations are of such critical importance in understanding and analyzing a wide variety of events that occur in the real world, the focus of the inquiry that is now being conducted is on the examination of these equations. In this study, differential equations are utilized to investigate the turbulent flow of a polytrophic gas through a porous material and non-Newtonian fluid theory. The non-Newtonian fluid theory is also taken into consideration. A comprehensive understanding of the mathematical principles that support these fields is required, since they have significant practical implications and require a comprehensive understanding of those principles. Individuals who are interested in further information might consult the papers [20–25] that were previously mentioned. Therefore, this aims to study the oscillatory behavior of a specific class of second-order half-linear dynamic equations with deviating arguments of the form

$$\left[r\left|z^{\Delta}\right|^{\gamma-1}z^{\Delta}\right]^{\Delta}(\tau)+q(\tau)\left|z(\varphi(\tau))\right|^{\gamma-1}z(\varphi(\tau))=0 \tag{1.1}$$

on an unbounded above arbitrary time scale $\mathbb{T}$, where $\tau\in[\tau_0,\infty)_{\mathbb{T}}$, $\tau_0\geq 0$, $\tau_0\in\mathbb{T}$, $\gamma>0$, $r,q\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{+})$ such that

$$R(\tau):=\int_{\tau_0}^{\tau}\frac{\Delta\omega}{r^{\frac{1}{\gamma}}(\omega)}\to\infty \quad \text{as} \quad \tau\to\infty, \tag{1.2}$$

and $\varphi\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ satisfying $\lim_{\tau\to\infty}\varphi(\tau)=\infty$. By a solution of Eq (1.1), we mean a nontrivial real-valued function $z\in C^{1}_{\mathrm{rd}}[T_z,\infty)_{\mathbb{T}}$, $T_z\in[\tau_0,\infty)_{\mathbb{T}}$ such that $r\left|z^{\Delta}\right|^{\gamma-1}z^{\Delta}\in C^{1}_{\mathrm{rd}}[T_z,\infty)_{\mathbb{T}}$ and z satisfies (1.1) on $[T_z,\infty)_{\mathbb{T}}$, where C_{rd} is the set of rd-continuous functions. A solution z of (1.1) is considered oscillatory if it does not eventually become positive or negative. Otherwise, we refer to it as nonoscillatory. We will exclude solutions that vanish in the vicinity of infinity. Note that if $\mathbb{T}=\mathbb{R}$, then (1.1) becomes the second-order half-linear differential equation

$$\left[r\left|z'\right|^{\gamma-1}z'\right]'(\tau)+q(\tau)\left|z(\varphi(\tau))\right|^{\gamma-1}z(\varphi(\tau))=0.$$

If $\mathbb{T}=\mathbb{Z}$, then (1.1) gets the second-order half-linear difference equation

$$\Delta\left[r\left|\Delta z\right|^{\gamma-1}\Delta z\right](\tau)+q(\tau)\left|z(\varphi(\tau))\right|^{\gamma-1}z(\varphi(\tau))=0,$$

where

$$\sigma(\tau)=\tau+1 \quad \text{and} \quad \Delta z(\tau):=z(\tau+1)-z(\tau).$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, thus (1.1) converts the second order half-linear difference equation

$$\Delta_h\left[r\left|\Delta_h z\right|^{\gamma-1}\Delta_h z\right](\tau)+q(\tau)\left|z(\varphi(\tau))\right|^{\gamma-1}z(\varphi(\tau))=0,$$

where
$$\sigma(\tau) = \tau + h \quad \text{and} \quad \Delta_h z(\tau) := \frac{z(\tau+h) - z(\tau)}{h}.$$
If $\mathbb{T} = \{\tau : \tau = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then (1.1) becomes the second-order half-linear q-difference equation
$$\Delta_q \left[r \left| \Delta_q z \right|^{\gamma-1} \Delta_q z \right](\tau) + q(\tau) \left| z(\varphi(\tau)) \right|^{\gamma-1} z(\varphi(\tau)) = 0,$$
where
$$\sigma(\tau) = q\tau \quad \text{and} \quad \Delta_q z(\tau) = \frac{z(q\tau) - z(\tau)}{(q-1)\tau}.$$
If $\mathbb{T} = \mathbb{N}_0^2 := \{\tau^2 : \tau \in \mathbb{N}_0\}$, then (1.1) gets the second-order half-linear difference equation
$$\Delta_N \left[r \left| \Delta_N z \right|^{\gamma-1} \Delta_N z \right](\tau) + q(\tau) \left| z(\varphi(\tau)) \right|^{\gamma-1} z(\varphi(\tau)) = 0,$$
where
$$\sigma(\tau) = (\sqrt{\tau} + 1)^2 \quad \text{and} \quad \Delta_N z(\tau) = \frac{z((\sqrt{\tau}+1)^2) - z(\tau)}{1 + 2\sqrt{\tau}}.$$
If $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$, then (1.1) converts the second-order half-linear harmonic difference equation
$$\Delta_{H_n} \left[r \left| \Delta_{H_n} z \right|^{\gamma-1} \Delta_{H_n} z \right](H_n) + q(H_n) \left| z(\varphi(H_n)) \right|^{\gamma-1} z(\varphi(H_n)) = 0,$$
where
$$\sigma(H_n) = H_{n+1} \quad \text{and} \quad \Delta_{H_n} z(H_n) = (n+1)\Delta z(H_n).$$
The oscillation results for differentials that are linked to the oscillation results for (1.1) on time scales are presented in the following. In addition, it offers a comprehensive summary of the significant contributions that this paper has made. Oscillation theory has consistently relied heavily on Euler differential equations and their numerous generalizations ever since Sturm's significant contribution to the literature. One of the most well-known and widely used is the second-order Euler equation
$$z''(\tau) + \frac{q_0}{\tau^2} z(\tau) = 0,\ q_0 > 0, \tag{1.3}$$
which is oscillatory if and only if
$$q_0 > \frac{1}{4}.$$
Among the most important oscillation criteria of second-order differential equations are Kneser-type (see [26]), which used Sturmian comparison methods, and the oscillatory behavior of the Euler equation (1.3) to show that the linear differential equation
$$z''(\tau) + q(\tau) z(\tau) = 0, \tag{1.4}$$
is oscillatory if
$$\liminf_{\tau \to \infty} \tau^2 q(\tau) > \frac{1}{4}. \tag{1.5}$$
Since then, in the same way, many works have appeared that deduce Kneser-type criteria for different types of differential equations. Some of these works follow; see [27–29]:

(I) The linear differential equation

$$[rz']'(\tau) + q(\tau)z(\tau) = 0 \tag{1.6}$$

is oscillatory if

$$\liminf_{\tau\to\infty} r(\tau)R^2(\tau)q(\tau) > \frac{1}{4}. \tag{1.7}$$

(II) The half-linear differential equation

$$\left[|z'|^{\gamma-1} z'\right]'(\tau) + q(\tau)\,|z(\tau)|^{\gamma-1} z(\tau) = 0 \tag{1.8}$$

is oscillatory if

$$\liminf_{\tau\to\infty} \tau^{\gamma+1} q(\tau) > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}. \tag{1.9}$$

(III) The half-linear differential equation

$$\left[r\,|z'|^{\gamma-1} z'\right]'(\tau) + q(\tau)\,|z(\tau)|^{\gamma-1} z(\tau) = 0 \tag{1.10}$$

is oscillatory if

$$\liminf_{\tau\to\infty} r^{\frac{1}{\gamma}}(\tau)R^{\gamma+1}(\tau)q(\tau) > \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}. \tag{1.11}$$

We note that the Euler equation

$$\left[r\,|z'|^{\gamma-1} z'\right]'(\tau) + \frac{q_0}{r^{\frac{1}{\gamma}}(\tau)R^{\gamma+1}(\tau)}\,|z(\tau)|^{\gamma-1} z(\tau) = 0,\ q_0 > 0 \tag{1.12}$$

has a nonoscillatory solution $z(\tau) = R^{\frac{\gamma}{\gamma+1}}(\tau)$ if $q_0 = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}$. That is to say, the constant $\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}$ serves as the lower bound of oscillation for all solutions of the Eq (1.12).

In consideration of the aforementioned comments, we establish the Kneser-type oscillation criterion for the dynamic equation (1.1) on time scales with deviating arguments by employing the Riccati transformation technique:

(i) Include the oscillation criterion (1.5) that has been given by Kneser [26] for the Eq (1.4).
(ii) Include the oscillation criterion (1.7). (1.9), and (1.11) for the differential equations (1.6), (1.8), and (1.10), respectively.
(iii) Obtained results are applicable to all time scales, whether continuous or discrete.

2. Main results

We begin this section with the following lemma, which we need to substantiate the main results.

Lemma 2.1 (see [30, Theorem 1])**.** *Assume z is a positive solution of* (1.1) *on* $[\tau_0,\infty)_{\mathbb{T}}$. *Then*

$$z^{\Delta}(\tau) > 0,\ \left[\frac{z}{R}\right]^{\Delta}(\tau) < 0,\ z(\tau) \geq \left[r^{\frac{1}{\gamma}} z^{\Delta} R\right](\tau),\ \textit{and}\ \left[r\left|z^{\Delta}\right|^{\gamma-1} z^{\Delta}\right]^{\Delta}(\tau) < 0 \tag{2.1}$$

eventually.

The following main theorem is the Kneser-type oscillation criterion in Eq (1.1).

Theorem 2.1. *If* $l := \liminf_{\tau\to\infty} \dfrac{R(\tau)}{R(\sigma(\tau))} > 0$ *and*

$$A := \liminf_{\tau\to\infty} r^{\frac{1}{\gamma}}(\tau) R(\tau) R^{\gamma}(\eta(\tau)) q(\tau) > \frac{1}{l^{\gamma(\gamma+1)}}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}, \tag{2.2}$$

where the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ *is given by*

$$\sigma(\tau) := \inf\{\omega \in \mathbb{T} : \omega > \tau\}, \tag{2.3}$$

and $\eta(\tau) := \min\{\tau, \varphi(\tau)\}$, *then all solutions of Eq* (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution z on $[\tau_0,\infty)_{\mathbb{T}}$. Without loss of generality, we let $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ for $\tau \in [\tau_0,\infty)_{\mathbb{T}}$. By using Lemma 2.1, there exists $\tau_1 \in (\tau_0,\infty)_{\mathbb{T}}$ such that for $\tau \geq \tau_1$,

$$z^{\Delta}(\tau) > 0,\ \left[\frac{z}{R}\right]^{\Delta}(\tau) < 0,\ z(\tau) \geq \left[r^{\frac{1}{\gamma}} z^{\Delta} R\right](\tau),\ \text{and}\ \left[r\left|z^{\Delta}\right|^{\gamma-1} z^{\Delta}\right]^{\Delta}(\tau) < 0. \tag{2.4}$$

Let

$$w(\tau) := \frac{r(\tau)\left(z^{\Delta}(\tau)\right)^{\gamma}}{z^{\gamma}(\tau)}. \tag{2.5}$$

It follows that

$$\begin{aligned} w^{\Delta}(\tau) &= \frac{1}{z^{\gamma}(\tau)}\left[r\left(z^{\Delta}\right)^{\gamma}\right]^{\Delta}(\tau) - \frac{(z^{\gamma})^{\Delta}(\tau)}{z^{\gamma}(\tau) z^{\gamma}(\sigma(\tau))}\left[r\left(z^{\Delta}\right)^{\gamma}\right]^{\sigma}(\tau) \\ &\overset{(1.1)}{=} -q(\tau)\left(\frac{z(\varphi(\tau))}{z(\tau)}\right)^{\gamma} - \frac{(z^{\gamma})^{\Delta}(\tau)}{z^{\gamma}(\tau)} w^{\sigma}(\tau). \end{aligned} \tag{2.6}$$

Pötzsche chain rule application yields

$$\begin{aligned} \frac{(z^{\gamma})^{\Delta}(\tau)}{z^{\gamma}(\tau)} &\geq \begin{cases} \gamma\left(\dfrac{z^{\sigma}(\tau)}{z(\tau)}\right)^{\gamma} \dfrac{z^{\Delta}(\tau)}{z^{\sigma}(\tau)}, & 0 < \gamma \leq 1 \\[2ex] \gamma \dfrac{z^{\sigma}(\tau)}{z(\tau)} \dfrac{z^{\Delta}(\tau)}{z^{\sigma}(\tau)}, & \gamma \geq 1 \end{cases} \\ &\overset{(2.4)}{\geq} \gamma r^{-\frac{1}{\gamma}}(\tau)\left[\frac{r^{-\frac{1}{\gamma}} z^{\Delta}}{z}\right]^{\sigma}(\tau) \\ &= \gamma r^{-\frac{1}{\gamma}}(\tau)\left(w^{\sigma}(\tau)\right)^{\frac{1}{\gamma}}. \end{aligned}$$

Hence,

$$w^{\Delta}(\tau) \leq -q(\tau)\left(\frac{z(\varphi(\tau))}{z(\tau)}\right)^{\gamma} - \gamma r^{-\frac{1}{\gamma}}(\tau)\left(w^{\sigma}(\tau)\right)^{1+\frac{1}{\gamma}}.$$

By using the fact that $\left[\dfrac{z}{R}\right]^{\Delta}(\tau) < 0$, we get for $\tau \geq \tau_1$,

$$w^{\Delta}(\tau) \leq -q(\tau)\left(\frac{R(\eta(\tau))}{R(\tau)}\right)^{\gamma} - \gamma r^{-\frac{1}{\gamma}}(\tau)\left(w^{\sigma}(\tau)\right)^{1+\frac{1}{\gamma}}. \tag{2.7}$$

From the definitions of l and A, we obtain that, for any $\varepsilon \in (0,1)$, there exists a $\tau_2 \in [\tau_1, \infty)_{\mathbb{T}}$ such that, for $\tau \in [\tau_2, \infty)_{\mathbb{T}}$,

$$\frac{R(\tau)}{R^{\sigma}(\tau)} \geq \varepsilon l \quad \text{and} \quad r^{\frac{1}{\gamma}}(\tau) R(\tau) R^{\gamma}(\eta(\tau)) q(\tau) \geq \varepsilon A, \tag{2.8}$$

and

$$R^{\gamma}(\tau) w^{\sigma}(\tau) \geq \varepsilon W_* \tag{2.9}$$

where

$$W_* := \liminf_{\tau \to \infty} R^{\gamma}(\tau) w^{\sigma}(\tau), \quad 0 \leq W_* \leq 1 \tag{2.10}$$

due to (2.4) and (2.5). Therefore, (2.7) becomes

$$\begin{aligned} w^{\Delta}(\tau) &\leq -\frac{\varepsilon A r^{-\frac{1}{\gamma}}(\tau)}{R^{\gamma+1}(\tau)} - \frac{\gamma r^{-\frac{1}{\gamma}}(\tau)}{R^{\gamma+1}(\tau)} (\varepsilon W_*)^{1+\frac{1}{\gamma}} \\ &\leq -\left[\frac{\varepsilon A}{\gamma} + (\varepsilon W_*)^{1+\frac{1}{\gamma}}\right] \frac{\gamma r^{-\frac{1}{\gamma}}(\tau)}{R^{\gamma+1}(\tau)}. \end{aligned} \tag{2.11}$$

Applying the Pötzsche chain rule, we obtain

$$\left[\frac{-1}{R^{\gamma}}\right]^{\Delta}(\tau) \leq \frac{\gamma r^{-\frac{1}{\gamma}}(\tau)}{R^{\gamma+1}(\tau)}. \tag{2.12}$$

Substituting (2.12) into (2.11), we have

$$w^{\Delta}(\tau) \leq -\left[\frac{\varepsilon A}{\gamma} + (\varepsilon W_*)^{1+\frac{1}{\gamma}}\right] \left(\frac{-1}{R^{\gamma}(\tau)}\right)^{\Delta}.$$

Integrating from $\sigma(\tau)$ to v, we get

$$w(v) - w^{\sigma}(\tau) \leq -\left[\frac{\varepsilon A}{\gamma} + (\varepsilon W_*)^{1+\frac{1}{\gamma}}\right] \left(\frac{1}{R^{\gamma\sigma}(\tau)} - \frac{1}{R^{\gamma}(v)}\right).$$

Due to $w > 0$ and letting $v \to \infty$, we see

$$-w^{\sigma}(\tau) \leq -\left[\frac{\varepsilon A}{\gamma} + (\varepsilon W_*)^{1+\frac{1}{\gamma}}\right] \left(\frac{1}{R^{\gamma\sigma}(\tau)}\right).$$

Therefore,

$$\varepsilon A \leq \gamma R^{\gamma\sigma}(\tau) w^{\sigma}(\tau) - \gamma (\varepsilon W_*)^{1+\frac{1}{\gamma}}.$$

By using (2.8), we achieve

$$\varepsilon A \leq \frac{\gamma}{(\varepsilon l)^{\gamma}} R^{\gamma}(\tau) w^{\sigma}(\tau) - \gamma (\varepsilon W_*)^{1+\frac{1}{\gamma}}.$$

Taking the lim inf of both sides as $\tau \to \infty$, we obtain

$$\varepsilon A \leq \frac{\gamma}{(\varepsilon l)^{\gamma}} W_* - \gamma (\varepsilon W_*)^{1+\frac{1}{\gamma}}.$$

Since ε is arbitrary, we arrive at

$$A \leq \frac{\gamma}{l^{\gamma}} W_* - \gamma W_*^{1+\frac{1}{\gamma}}.$$

Let

$$Y = \gamma, \quad X = \frac{\gamma}{l^{\gamma}}, \quad \text{and} \quad U = W_*.$$

By the inequality

$$XU - YU^{1+\frac{1}{\gamma}} \leq \frac{X^{\gamma+1}}{Y^{\gamma}} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \quad X, Y > 0. \tag{2.13}$$

Hence,

$$A \leq \frac{1}{l^{\gamma(\gamma+1)}} \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1},$$

which gives us the contradiction in (2.2). This completes the proof. □

Motivated by Theorem 2.1, we can prove the following result, which is the Kneser-type oscillation criterion for Eq (1.1) in the case when $r^{\Delta} \geq 0$ on $[\tau_0, \infty)_{\mathbb{T}}$.

Corollary 2.1. *Let* $r^{\Delta} \geq 0$ *on* $[\tau_0, \infty)_{\mathbb{T}}$. *If* $l_* := \liminf_{\tau\to\infty} \dfrac{\tau}{\sigma(\tau)} > 0$ *and*

$$B := \liminf_{\tau\to\infty} \frac{\tau \eta^{\gamma}(\tau) q(\tau)}{r(\tau)} > \frac{1}{l_*^{\gamma(\gamma+1)}} \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}, \tag{2.14}$$

where $\eta(\tau) := \min\{\tau, \varphi(\tau)\}$, *then all solutions of Eq.* (1.1) *oscillate.*

Proof. Assume to the contrary that Eq (1.1) has a nonoscillatory solution z on $[\tau_0, \infty)_{\mathbb{T}}$. Without loss of generality, we let $z(\tau) > 0$ and $z(\varphi(\tau)) > 0$ for $\tau \in [\tau_0, \infty)_{\mathbb{T}}$. As shown in the proof of Theorem 2.1, we have

$$w^{\Delta}(\tau) \leq -q(\tau) \left(\frac{z(\varphi(\tau))}{z(\tau)} \right)^{\gamma} - \gamma r^{-\frac{1}{\gamma}}(\tau) (w^{\sigma}(\tau))^{1+\frac{1}{\gamma}},$$

where w is defined as in (2.5). By using [21, Lemma 2.2], there exists $\tau_1 \in (\tau_0, \infty)_{\mathbb{T}}$ such that

$$z^{\Delta}(\tau) > 0, \ \left[\frac{z(\tau)}{\tau - \tau_0} \right]^{\Delta} < 0, \text{ and } \left[r \left| z^{\Delta} \right|^{\gamma-1} z^{\Delta} \right]^{\Delta}(\tau) < 0 \quad \text{for } \tau \geq \tau_1. \tag{2.15}$$

Assume $\kappa \in (0, 1)$ is arbitrary. We have from (2.15) that there is a $\tau_{\kappa} \in [\tau_1, \infty)_{\mathbb{T}}$ such that for $\tau \in [\tau_{\kappa}, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\tau) \leq -\kappa q(\tau) \left(\frac{\varphi(\tau)}{\tau} \right)^{\gamma} - \gamma r^{-\frac{1}{\gamma}}(\tau) (w^{\sigma}(\tau))^{1+\frac{1}{\gamma}}. \tag{2.16}$$

Now, for any $\varepsilon \in (0, 1)$, there exists a $\tau_2 \in [\tau_1, \infty)_{\mathbb{T}}$ such that, for $\tau \in [\tau_2, \infty)_{\mathbb{T}}$,

$$\frac{\tau^{\gamma} w^{\sigma}(\tau)}{r(\tau)} \geq \varepsilon W^*, \ \frac{\tau}{\sigma(\tau)} \geq \varepsilon l_*, \text{ and } \frac{\tau \varphi^{\gamma}(\tau) c(\tau)}{r(\tau)} \geq \varepsilon B, \tag{2.17}$$

where

$$W^* := \liminf_{\tau\to\infty} \frac{\tau^{\gamma} w^{\sigma}(\tau)}{r(\tau)}, \quad 0 \leq W^* \leq 1. \tag{2.18}$$

Hence,

$$\begin{aligned} w^{\Delta}(\tau) &\leq -\varepsilon\kappa B\frac{r(\tau)}{\tau^{\gamma+1}} - \frac{\gamma r(\tau)}{\tau^{\gamma+1}}(\varepsilon W^{*})^{1+\frac{1}{\gamma}} \\ &= -\frac{\gamma r(\tau)}{\tau^{\gamma+1}}\left[\frac{\varepsilon\kappa B}{\gamma} + (\varepsilon W^{*})^{1+\frac{1}{\gamma}}\right] \\ &\leq -\frac{\gamma r(\tau)}{\tau^{\gamma+1}}\left[\frac{\varepsilon\kappa B}{\gamma} + (\varepsilon W^{*})^{1+\frac{1}{\gamma}}\right]. \end{aligned} \tag{2.19}$$

Applying the Pötzsche chain rule, we obtain

$$\left[\frac{-1}{\tau^{\gamma}}\right]^{\Delta} \leq \frac{\gamma}{\tau^{\gamma+1}}. \tag{2.20}$$

Substituting (2.20) into (2.19), we have

$$w^{\Delta}(\tau) \leq -r(\tau)\left[\frac{\varepsilon\kappa B}{\gamma} + (\varepsilon W^{*})^{1+\frac{1}{\gamma}}\right]\left[\frac{-1}{\tau^{\gamma}}\right]^{\Delta}.$$

Integrating from $\sigma(\tau)$ to v, we get

$$w(v) - w^{\sigma}(\tau) \leq -\left[\frac{\varepsilon\kappa B}{\gamma} + (\varepsilon W^{*})^{1+\frac{1}{\gamma}}\right]\int_{\sigma(\tau)}^{v} r(\omega)\left[\frac{-1}{\omega^{\gamma}}\right]^{\Delta}\Delta\omega.$$

Due to $r^{\Delta} \geq 0$ and $w > 0$, and letting $v \to \infty$, we see

$$-w^{\sigma}(\tau) \leq -\frac{r(\tau)}{\sigma^{\gamma}(\tau)}\left[\frac{\varepsilon\kappa B}{\gamma} + (\varepsilon W^{*})^{1+\frac{1}{\gamma}}\right].$$

Therefore,

$$\varepsilon\kappa B \leq \gamma\frac{\sigma^{\gamma}(\tau)\,w^{\sigma}(\tau)}{r(\tau)} - \gamma(\varepsilon W^{*})^{1+\frac{1}{\gamma}}.$$

By using (2.17), we obtain

$$\varepsilon\kappa B \leq \frac{\gamma}{(\varepsilon l_{*})^{\gamma}}\frac{\tau^{\gamma}w^{\sigma}(\tau)}{r(\tau)} - \gamma(\varepsilon W^{*})^{1+\frac{1}{\gamma}}.$$

Taking the lim inf of both sides as $\tau \to \infty$, we obtain

$$\varepsilon\kappa B \leq \frac{\gamma}{(\varepsilon l_{*})^{\gamma}}W^{*} - \gamma(\varepsilon W^{*})^{1+\frac{1}{\gamma}}.$$

Since ε and κ are arbitrary, we arrive at

$$B \leq \frac{\gamma}{l_{*}^{\gamma}}W^{*} - \gamma(W^{*})^{1+\frac{1}{\gamma}}.$$

Let

$$Y = \gamma, \quad X = \frac{\gamma}{l_{*}^{\gamma}}, \quad \text{and} \quad U = W^{*}.$$

By the inequality (2.13), we have

$$B \leq \frac{1}{l_{*}^{\gamma(\gamma+1)}}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1},$$

which gives us the contradiction in (2.14). This completes the proof. □

3. Examples

The applications of the theoretical results presented are illustrated through the following examples:
Example 3.1. The Euler dynamic equations

$$\left[r\left|z^{\Delta}\right|^{\gamma-1}z^{\Delta}\right]^{\Delta}(\tau)+\frac{q_0}{r^{\frac{1}{\gamma}}(\tau)R^{\gamma+1}(\tau)}\left|z(\tau)\right|^{\gamma-1}z(\tau)=0$$

and

$$\left[r\left|z^{\Delta}\right|^{\gamma-1}z^{\Delta}\right]^{\Delta}(\tau)+\frac{q_0}{r^{\frac{1}{\gamma}}(\tau)R^{\gamma+1}(\tau)}\left|z(\sigma(\tau))\right|^{\gamma-1}z(\sigma(\tau))=0,$$

are oscillatory if $q_0>\frac{1}{l^{\gamma(\gamma+1)}}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}$ by using Theorem 2.1. This condition is known to be the optimal one for the second-order Euler differential equation

$$\left[r\left|z'\right|^{\gamma-1}z'\right]'(\tau)+\frac{q_0}{r^{\frac{1}{\gamma}}(\tau)R^{\gamma+1}(\tau)}\left|z(\tau)\right|^{\gamma-1}z(\tau)=0.$$

Example 3.2. Consider a second-order half-linear delay dynamic equation for $\tau\in[\tau_0,\infty)_{\mathbb{T}}$,

$$\left[\tau^2\left(z^{\Delta}(\tau)\right)^3\right]^{\Delta}+\frac{q_0}{64l^{12}\tau\varphi(\tau)}z^3(\varphi(\tau))=0, \tag{3.1}$$

where $q_0>0$ is a constant and $\varphi(\tau)\leq\tau$ for $\tau\in[\tau_0,\infty)_{\mathbb{T}}$. It is evident that (1.2) holds since

$$\int_{\tau_0}^{\infty}\frac{\Delta\omega}{r^{\frac{1}{\gamma}}(\omega)}=\int_{\tau_0}^{\infty}\frac{\Delta\omega}{\sqrt[3]{\omega^2}}=\infty,$$

by [4, Example 5.60]. Also, by the Pötzsche chain rule, we have

$$R(\tau)=\int_{\tau_0}^{\tau}\frac{\Delta\omega}{\sqrt[3]{\omega^2}}\geq 3\int_{\tau_0}^{\tau}\left(\sqrt[3]{\omega}\right)^{\Delta}\Delta\omega=3\left(\sqrt[3]{\tau}-\sqrt[3]{\tau_0}\right),$$

and so,

$$\begin{aligned}\liminf_{\tau\to\infty}r^{\frac{1}{\gamma}}(\tau)R(\tau)R^{\gamma}(\eta(\tau))q(\tau) &= \frac{81q_0}{64l^{12}}\liminf_{\tau\to\infty}\left(1-\sqrt[3]{\frac{\tau_0}{\tau}}\right)\left(1-\sqrt[3]{\frac{\tau_0}{\varphi(\tau)}}\right)\\ &= \frac{81q_0}{64l^{12}}.\end{aligned}$$

Then, by applying Theorem 2.1, all solutions of Eq (3.1) oscillate if $q_0>\frac{1}{4}$.
Example 3.3. Consider a second-order half-linear functional advanced dynamic equation for $\tau\in[\tau_0,\infty)_{\mathbb{T}}$,

$$\left[\sqrt[3]{\frac{\left(z^{\Delta}(\tau)\right)^5}{\sigma(\tau)}}\right]^{\Delta}+q_0\frac{\sigma(\tau)}{4\sqrt[3]{\tau^8}}\sqrt[3]{z(\varphi(\tau))}=0, \tag{3.2}$$

where $q_0 > 0$ is a constant and $\varphi(\tau) \geq \tau$ for $\tau \in [\tau_0, \infty)_{\mathbb{T}}$. It is evident that (1.2) holds since, by the Pötzsche chain rule, we have

$$R(\tau) = \int_{\tau_0}^{\tau} \sigma(\omega)\,\Delta\omega \geq \frac{1}{2}\int_{\tau_0}^{\tau} \left(\omega^2\right)^{\Delta} \Delta\omega = \frac{1}{2}\left(\tau^2 - \tau_0^2\right).$$

Also,

$$\liminf_{\tau\to\infty} r^{\frac{1}{\gamma}}(\tau)\, R(\tau)\, R^{\gamma}(\eta(\tau))\, q(\tau) = \frac{q_0}{4}\liminf_{\tau\to\infty}\left(1 - \left(\frac{\tau_0}{\tau}\right)^2\right)\sqrt[3]{1 - \left(\frac{\tau_0}{\tau}\right)^2} = \frac{q_0}{4}.$$

Then, by applying Theorem 2.1, all solutions of Eq (3.2) oscillate if $q_0 > \dfrac{1}{\sqrt[9]{64 l^4}}$.

Example 3.4. Consider a second-order half-linear functional dynamic equation for $\tau \in [\tau_0, \infty)_{\mathbb{T}}$,

$$\left[\sqrt[4]{\tau}\frac{z^{\Delta}(\tau)}{\sqrt{\left|z^{\Delta}(\tau)\right|}}\right]^{\Delta} + \frac{q_0}{3\sqrt[4]{\tau^3}\sqrt{3\eta(\tau)}}\frac{z(\varphi(\tau))}{\sqrt{|z(\varphi(\tau))|}} = 0 \tag{3.3}$$

where $q_0 > 0$ is a constant. It is evident that (1.2) holds since

$$\int_{\tau_0}^{\infty} \frac{\Delta\omega}{r^{\frac{1}{\gamma}}(\omega)} = \int_{\tau_0}^{\infty} \frac{\Delta\omega}{\sqrt{\omega}} = \infty,$$

by [4, Example 5.60]. Also,

$$\liminf_{\tau\to\infty} \frac{\tau\eta^{\gamma}(\tau)\, q(\tau)}{r(\tau)} = \frac{q_0}{3\sqrt{3}}.$$

Hence, by Corollary 2.1, all solutions of Eq (3.3) oscillate if $q_0 > \dfrac{1}{\sqrt[4]{l_*^3}}$.

4. Discussion and conclusions

This research paper introduces a criterion for Kneser-type oscillations that can be applied to (1.1) in both cases, $\varphi(\tau) \leq \tau$ and $\varphi(\tau) \geq \tau$, and on any arbitrary time scale. Also, our results expand related contributions to second-order differential equations; see the details below:

(1) Criterion (2.14) becomes (1.4) in the case when $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, $r(\tau) = \tau$, and $\varphi(\tau) = \tau$;
(2) Criterion (2.14) becomes (1.9) in the case where $\mathbb{T} = \mathbb{R}$, $r(\tau) = \tau$, and $\varphi(\tau) = \tau$;
(3) Criterion (2.2) becomes (1.7) supposing that $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, and $\varphi(\tau) = \tau$;
(4) Criterion (2.2) becomes (1.11) in the case when $\mathbb{T} = \mathbb{R}$ and $\varphi(\tau) = \tau$.

Remark 4.1. It would be valuable to propose a methodology for examining the Kneser-type oscillation criterion (1.1) under the assumption that

$$\int_{\tau_0}^{\infty} \frac{\Delta\omega}{r^{\frac{1}{\gamma}}(\omega)} < \infty.$$

Author contributions

Hassan oversaw the study and helped with the inspection. All authors carried out the main results of this article, drafted the manuscript, and read and approved the final manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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