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Research article

Stability analysis of delayed neural networks via compound-parameter -based integral inequality

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Abstract: This paper revisits the issue of stability analysis of neural networks subjected to time-varying delays. A novel approach, termed a compound-matrix-based integral inequality (CPBII), which accounts for delay derivatives using two adjustable parameters, is introduced. By appropriately adjusting these parameters, the CPBII efficiently incorporates coupling information along with delay derivatives within integral inequalities. By using CPBII, a novel stability criterion is established for neural networks with time-varying delays. The effectiveness of this approach is demonstrated through a numerical illustration.

Keywords: neural networks; time-varying delay; stability analysis; integral inequality; delay derivative
Mathematics Subject Classification: 37C75, 93C55, 92B20

1. Introduction

Over recent decades, neural networks (NNs) have garnered significant attention and demonstrated success across various engineering and research domains, thereby encompassing image processing, pattern recognition, optimization problems, and associative memory [1–3]. Stability properties, which are crucial for effective neural network deployment, include asymptotic and exponential stability. Time delays which are prevalent in numerous control systems [4], poses challenges by potentially destabilizing systems. Consequently, stability analysis, particularly regarding NNs with time delays, is imperative due to the substantial impact of equilibrium point dynamics on practical applications [5, 6].

The Lyapunov-Krasovskii functional (LKF) method is a powerful tool for examining the stability of a system. Its effectiveness lies in its ability to identify a positive definite function whose derivative is negative definite along the trajectory of the system [7–9]. The choice of an appropriate LKF is crucial to establish the stability criteria. In a notable study [10], a potent methodology known as the delay-product-type functional (DPTF) method was introduced. This method is distinguished by its inclusion of variables that are dependent on the amplitude of the delay. For example, a DPTF $V(t)$ is formulated as $V(t) = d(t)v_1^T(t)P_1v_1(t) + (h - d(t))v_2^T(t)P_2v_2(t)$, where $d(t)P_1 > 0$, $(h - d(t))P_2 > 0$ are delay amplitude-dependent matrices, and $v_1(t)$, $v_2(t)$ denote augmented terms related to the state. Importantly, the derivative of $V(t)$ unveils the interconnection among terms related to the state, delay amplitude, and delay derivative, which is subsequently integrated into the final linear matrix inequalities (LMIs). Afterwards, in [11], by using Wirtinger-based integral inequality, the nonintegral terms are connected to the integral terms. As shown in [12], it is efficient for reduction of the conservatism if some double integral terms are introduced in Lyapunov functionals. Nevertheless, there exists an intrinsic conservatism in the LKF due to incomplete state vectors $v_1(t)$ and $v_2(t)$. Various bounding methods have been developed for the stability analysis, including Jensen-based and Wirtinger-based integral inequalities [13, 14], as well as slack-matrix-based integral inequalities [15–19], such as the Bessel-Legendre inequality (BLI) [20, 21] and the Jacobi-Bessel inequality (JLI) [22]. While BLI offers analytical solutions for constant delay systems, its applicability to time-varying systems is limited due to a reliance on estimated boundaries [23–25]. In contrast, affine bessel-Legendre inequality (ABLI) addresses time-varying delay amplitude but suffers from conservatism due to incomplete vectors [26]. To overcome this, generalized free matrix-based integral inequality (GFMBII) was introduced to complement ABLI; however, it still lacked the full incorporation of delay amplitude-dependent slack variables [27]. Although the delay derivative-dependent integral inequality was first introduced in [28], further investigation is warranted as the decision matrices are fixed, thus limiting their flexibility and utilization. This highlights the need for continued research in this area to fully leverage the potential benefits of such integral inequalities.

With the above analysis, this paper focuses on investigating the stability of time-varying delayed NNs. Two kinds of slack matrices with two tunable parameters, which are dependent on both a delay amplitude and a delay derivative, are proposed. These advancements culminate in the formulation of a compound-matrix-based integral inequality (CPBII). By utilizing CPBII, a stability condition tailored for time-varying delayed NNs is developed. Compared to existing literature such as [6, 17, 19, 27], the most significant contribution of this paper is the successful incorporation of both delay amplitude and derivative information into the inequality with the help of a couple of convex parameters. This innovative approach enhances the robustness and accuracy of the analysis. The feasibility of the proposed criterion is demonstrated through a numerical example.

Notation: In this paper, $\mathbb{R}^n$ represents the n-dimensional Euclidean space; $\mathbb{N}$ represents the nature number; $He[X]$represents $X + X^T$; $Co\{\cdots\}$ represents a set of points; $col[X, Y]$ represents $[X^T, Y^T]^T$; diag{...} represents a block diagonal matrix; and X^T represents the transposition of X.

2. Problem formation and preliminaries

Let's take the NNs characterized by a time-dependent delays, as depicted by the following equation:

$$\dot{u}(t) = -Au(t) + F_0g(F_2u(t)) + F_1g(F_2u(t - \tau(t)), \tag{2.1}$$

where $u(t) = col[u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), ..., u_n(t)] \in \mathbb{R}^n$ represents the state vector, and $g(F_2u(t)) = col[g_1(F_{21}u(t)), g_2(F_{22}u(t)), ..., g_n(F_{2n}u(t))]$ is the activation function. $A = diag\{a_1, a_2, ..a_n\}$ is a positive definite diagonal matrix, and F_0, F_1, F_2 are the appropriately dimensional constant matrices. The delay amplitude $\tau(t)$ and the delay derivative $\dot{\tau}(t)$ are bounded by constants d and ρ, respectively, satisfying the following

$$0 \le \tau(t) \le d, \quad -\rho \le \dot{\tau}(t) \le \rho. \tag{2.2}$$

The activation functions g_i satisfy $g_i(0) = 0$ for all i. Define H^- and H^+ as diagonal matrices with known constants h_i^- and h_i^+, respectively, which can be either positive, negative, or zero. Similarly, let G^- and G^+ be positive definite diagonal matrices. Given these definitions and $m, m_1, m_2 \in \mathbb{R}$, we can deduce the following inequalities from the previously mentioned activation function properties:

$$h_i^- \le \frac{f_i(a) - f_i(b)}{a - b} \le h_i^+, a \ne b. \tag{2.3}$$

We can directly acheive the following inequalities from (2.3) with m, m_1, $m_2 \in \mathbb{R}$:

$$\ell_1(m, G^-) \ge 0, \ell_2(m_1, m_2, G^+) \ge 0, \tag{2.4}$$

where

$$\begin{aligned}
\ell_1(m, G^-) &= 2[g(F_2u(s)) - H^-F_2u(s)]^T G^- \\
&\times [H^+F_2x(s) - g(F_2u(s))] \\
\ell_2(m_1, m_2, G^+) &= 2[g(F_2u(m_1)) - g(F_2u(m_2)) \\
&-H^-F_2u(m_1) - u(m_2)]^T G^+[H^+F_2(x(s_1) \\
&\quad -u(m_2)) - g(F_2u(m_1)) + g(F_2u(m_2))].
\end{aligned}$$

For simplicity, we use the following notations for $S \in \mathbb{N}$:

$$\begin{aligned}
\tau &= \tau(t), d_\tau = d - \tau, \dot{\bar{\tau}} = 1 - \dot{\tau} \\
f_l(\alpha, \beta) &= \int_\alpha^\beta (\frac{s-\alpha}{\beta-\alpha})^l u(s)ds \\
v_{1l}(t) &= f_l(t, t-\tau), v_{2l}(t) = f_l(t-\tau, t-d) \\
\vartheta(\alpha,\beta) &= \begin{cases} col\begin{bmatrix} u(\alpha) & u(\beta) \end{bmatrix}, if\ h = 0 \\ col\begin{bmatrix} u(\alpha), u(\beta), \frac{1}{d}\Psi_0, \cdots, \frac{1}{d}\Psi_{h-1} \end{bmatrix}, if\ h > 0 \end{cases} \\
\Psi_k &= \int_{t-d}^t L_k(s)u(s)ds \\
L_m(s) &= (-1)^m \sum_{l=0}^m \left[(-1)^l \begin{pmatrix} k \\ l \end{pmatrix} \begin{pmatrix} m+l \\ l \end{pmatrix}\right]\left(\frac{s-t+d}{d}\right)^l \\
\pi_h(m) &= \begin{cases} \begin{bmatrix} I & -I \end{bmatrix}, if\ h = 0 \\ \begin{bmatrix} I, (-1)^{m+1}I, \varsigma_{hm}^0 I, \cdots, \varsigma_{hm}^{h-1} I \end{bmatrix}, if\ h > 0 \end{cases} \\
\varsigma_{hm}^l &= \begin{cases} -(2l+1)(1-(-1)^{m+l}), if\ l \le m \\ 0, if\ l \ge m+1 \end{cases}
\end{aligned}$$

$$
\begin{aligned}
\tilde{Q} &= diag\{Q^{-1}, 1/3Q^{-1}, ..., 1/(2h+1)Q^{-1}\} \\
\hat{Q} &= diag\{Q, 3Q, ..., (2h+1)Q\} \\
\Gamma_h &= col[\pi_h(0), \pi_h(1), ..., \pi_h(d)] \\
\vartheta_1 &= \vartheta(t-\tau, t), \vartheta_2 = \vartheta(t-\tau, t-d) \\
g_j^- &= g_j(s) - l_j^- s, g_j^+ = H^+ s - g_j(s) \\
o(t) &= col[u(t), u(t-\tau), u(t-d)] \\
\varrho_0(s) &= col[\dot{u}(s), u(s), g(F_2 u(s))] \\
\varrho_1(s) &= col[\int_s^t u(v)dv, \int_{t-d}^s u(v)dv] \\
\varrho_2(s) &= col[\int_s^{t-\tau} u(v)dv, \int_{t-d}^s u(v)dv] \\
\eta_{0h}(t) &= col[o(t), v_{10}(t), v_{20}(t), ..., v_{1h}(t), v_{2h}(t)] \\
\eta_{1h}(t) &= col[o(t), \frac{v_{10}(t)}{\tau}, \frac{v_{11}(t)}{\tau}, ..., \frac{v_{1h}(t)}{\tau}] \\
\eta_{2h}(t) &= col[o(t), \frac{v_{20}(t)}{\tau}, \frac{v_{21}(t)}{d_\tau}, ..., \frac{v_{2d}(t)}{d_\tau}] \\
\eta_{3h}(t,s) &= \begin{cases} \varrho(0)(s), h = 0 \\ \varrho_0(s), \varrho_1(s), h \geq 1 \end{cases} \\
\eta_{4h}(t,s) &= \begin{cases} \varrho(0)(s), h = 0 \\ \varrho_0(s), \varrho_5(s), h \geq 1 \end{cases} \\
\eta_5(t) &= col[g(F_2 u(t)), g(F_1 u(t-\tau)), g(F_2 u(t-d)) \\
&\quad \int_{t-\tau}^t g(F_2 u(s))ds, \int_{t-d}^{t-\tau} g(F_2 u(s))ds] \\
\eta_6(t) &= col[\dot{u}(t-\tau), \dot{u}(t-d)] \\
\eta_{7j}(t) &= col[\frac{v_{1j}(t)}{\tau}, \frac{v_{2j}(t)}{d_\tau}] \\
\eta_h(t) &= col[o(t), \eta_5(t), \eta_6(t), \eta_{70}(t), \eta_{71}(t), ..., \eta_{7h}(t)] \\
c_{j,N} &= [0_{n\times(j-1)n}, I_{n\times n}, 0_{n\times(N-j)n}], j \in 1, 2, ..., N.
\end{aligned}
$$

In the existing body of work, such as [18,26,27], the final LMIs often incorporate information about the delay derivative, which is typically derived from the derivatives of the LKFs. Despite this, there has been a noticeable absence of integral inequalities that directly pertain to the delay derivative in the context of time-varying delayed NNs. To address this deficiency, we introduce a novel approach in the form of a CPBII, which is outlined below.

Lemma 1. For any continuously differentiable function $u : [-d, 0] \to \mathbb{R}^n$, the subsequent inequality is valid for any given parameters γ_1 and γ_2, $R > 0$, any vector η, and slack variables M and N:

$$
\begin{aligned}
&- \int_{t-d}^{d} \dot{u}^T(s) R \dot{u}(s) ds \\
&\leq \frac{(\rho + \gamma_1 + \gamma_2 \dot{\tau})\rho^T}{\eta} \Big[\tau M^T \tilde{R} M + d_\tau N^T \tilde{R} N\Big]\eta
\end{aligned}
$$

$$+\Big(\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d^2}\Big)He[(\vartheta_1^T\Gamma_S^TM+\vartheta_2^T\Gamma_S^TN)\eta]$$
$$-\frac{(\gamma_1+\gamma_2\dot{\tau})}{\rho d^2}\Big\{d_\tau\vartheta_1^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_1+\tau\vartheta_2^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_2\Big\}. \tag{2.5}$$

Proof: For any parameters $\epsilon_1, \epsilon_2 \in [0,1]$ that satisfy $\epsilon_1+\epsilon_2=1$, the following relationship holds:

$$\begin{aligned}&\int_{t-d}^{t}\dot{u}^T(s)R\dot{u}(s)ds\\ =\ &\epsilon_1\int_{t-\tau}^{t}\dot{u}^T(s)R\dot{u}(s)ds+\epsilon_1\int_{t-d}^{t-\tau}\dot{u}^T(s)R\dot{u}(s)ds\\ &+\epsilon_2\int_{t-\tau}^{t}\dot{u}^T(s)R\dot{u}(s)ds+\epsilon_2\int_{t-d}^{t-\tau}\dot{u}^T(s)R\dot{u}(s)ds.\end{aligned}$$

By using the inequalities in [20, 27], for free matrices M and N, we have the following

$$\begin{aligned}&-\int_{t-d}^{t}\dot{u}^T(s)R\dot{u}(s)ds\\ \le\ &\epsilon_1\eta^T\Big[\tau M^T\tilde{R}M+d_\tau N^T\tilde{R}N\Big]\eta\\ &+\epsilon_1He[(\vartheta_1^T\Gamma_h^TM+\vartheta_2^T\Gamma_h^TN)\eta]\\ &-\epsilon_2\Big\{\frac{1}{\tau}\vartheta_1^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_1+\frac{1}{d_\tau}\vartheta_2^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_2\Big\}.\end{aligned} \tag{2.6}$$

From $\tau M^T\tilde{R}M+d_\tau N^T\tilde{R}N>0$, it yields

$$\begin{aligned}&-\int_{t-d}^{t}\dot{u}^T(s)R\dot{u}(s)ds\\ \le\ &\eta^T\Big[\tau M^T\tilde{R}M+d_\tau N^T\tilde{R}N\Big]\eta\\ &+\epsilon_1He[(\vartheta_1^T\Gamma_h^TM+\vartheta_2^T\Gamma_h^TN)\eta]\\ &-\epsilon_2\Big\{\frac{1}{d}\vartheta_1^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_1+\frac{1}{d_\tau}\vartheta_2^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_2\Big\}.\end{aligned} \tag{2.7}$$

From the fact

$$\epsilon_1=\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d^2},\epsilon_2=\frac{\tau d_\tau(\gamma_1+\gamma_2\dot{\tau})}{\rho d^2}, \tag{2.8}$$

one has $0\le\epsilon_1\le 1$, $0\le\epsilon_2\le 1$, $\epsilon_1+\epsilon_2=1$.

Substituting (2.8) into (2.6), we have

$$\begin{aligned}&-\int_{t-d}^{d}\dot{u}^T(s)R\dot{u}(s)ds\\ \le\ &\Big(\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d^2}\Big)\end{aligned}$$

$$\times \ \eta^T\Big[dM^T\tilde{R}M + d_\tau N^T\tilde{R}N\Big]\eta$$
$$+\Big(\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d^2}\Big)He[(\vartheta_1^T\Gamma_h^T M+\vartheta_2^T\Gamma_h^T N)\eta]$$
$$-\frac{(\gamma_1+\gamma_2\dot{\tau})}{\rho d^2}\Big\{d_\tau\vartheta_1^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_1+\tau\vartheta_2^T\Gamma_h^T\hat{R}\Gamma_h\vartheta_2\Big\} \tag{2.9}$$

Considering $\tau M^T\tilde{R}M + d_\tau N^T\tilde{R}N > 0$, $d \geq \tau \geq 0$, $0 \leq \frac{\tau d_\tau}{d^2} \leq 1$, and $0 \leq \frac{\gamma_1+\gamma_2\dot{\tau}}{\rho} \leq 1$, we obtain the following:

$$\begin{aligned}
&\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d^2}\\
\leq\ &\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho d}+\frac{(\gamma_1+\gamma_2\dot{\tau})d}{\rho d}\\
=\ &\frac{\rho d+(\gamma_1+\gamma_2\dot{\tau})(d-\tau)}{\rho d}\\
\leq\ &\frac{\rho d+(\gamma_1+\gamma_2\dot{\tau})d}{\rho d}\\
=\ &\frac{\rho+\gamma_1+\gamma_2\dot{\tau}}{\rho}.
\end{aligned} \tag{2.10}$$

By combining this inequality with the one from the previous lemma, (2.5) is derived. This completes the proof.

Remark 1. In Lemma 1, we present a unique integral inequality, termed as CPBII, which amalgamates slack matrices that are dependent on both the delay amplitude and the delay derivative. This is a pioneering approach in the literature where the delay derivative is factored in [28]. The advantages of CPBII are manifold:

- Through the integration of slack matrices that are influenced by both the delay amplitude and the derivative, the successfully forms a link between vectors related to the system states, the delay amplitude, and the delay derivative. This approach facilitates the retrieval of more interconnected data compared to DPTF, ABLI, and GFMBII, all without the need for extra decision variables.
- The inclusion of parameters γ_1 and γ_2 aid in circumventing certain incomplete terms. For example, when $\gamma_1 = 0$ and $\gamma_2 = 0$, the last term $d_\tau\vartheta^T 1\Gamma d^T\tilde{R}\Gamma_d\vartheta_1 + \tau\vartheta^T 2\Gamma h^T\tilde{R}\Gamma_h\vartheta_2$ is eliminated. Similarly, when $\gamma_1 = 0$, $\gamma_2 = 1$, and $\dot{d} = \rho$, the first term $\tau M^T\tilde{R}M + d_\tau N^T\tilde{R}N$ disappears. Furthermore, this parameter enhances the systems adaptability.

Remark 2. As noted from [10], there exist two strategies to mitigate the conservatism. The first involves striving to get as close to the left side of the inequality as possible, while the second entails introducing an adequate number of cross terms to ensure sufficient system information within the final conditions. Therefore, this paper opts for the second strategy, albeit at the expense of the first to a certain degree. The most significant challenge resides in the assignment of values to ϵ_1 and ϵ_2. If these values are not assigned appropriately, it becomes evidently impossible to counterbalance the discrepancy caused by the first strategy.

3. Main results

In the study [27], it is noted that the delay derivative $\dot{\tau}$ is present in $\Gamma_0(\tau, \dot{\tau})$, $\Gamma_1(\tau, \dot{\tau})$, and $\Gamma_3(\dot{\tau})$, where it is exclusively coupled with positive definite matrices such as P_{1h}, P_{2h}, Q_1, and Q_2. Interestingly, the slack matrices that are dependent on the delay derivative are not considered. To exploit the information offered by the delay derivative to its fullest extent, a stability condition for system (2.1) is formulated based on CPBII.

Theorem 1. Provided that there exist positive-definite symmetric matrices P, Q_1, Q_2, and R, and matrices H_1, H_2, M, and N, in conjunction with specific scalars γ_1, γ_2, d, and ρ, that meet the subsequent inequalities, it can be concluded that system (2.1) exhibits asymptotic stability:

$$\begin{bmatrix} \tilde{\Psi}(0,\dot{\tau}) & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dc_8^TH_2^T & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dN^T \\ * & -Z & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0 \tag{3.1}$$

$$\begin{bmatrix} \tilde{\Psi}(d,\dot{\tau}) & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dc_7^TH_1^T & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dM^T \\ * & -Z & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0 \tag{3.2}$$

$$\begin{bmatrix} \hat{\Psi}(0,\dot{\tau}) & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dc_8^TH_2^T & \sqrt{\frac{(\rho+\gamma_1+\gamma_2\dot{\tau})}{\rho}}dN^T \\ * & -Z & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0, \tag{3.3}$$

where

$$\begin{aligned}
\hat{\Psi}(0,\dot{\tau}) &= -d^2a_2(\dot{\tau}) + \tilde{\Psi}(0,\dot{\tau}) \\
a_2(\dot{\tau}) &= \bar{\gamma}_1^TQ_1\bar{\gamma}_1 - \dot{\bar{\tau}}\bar{\gamma}_2^TQ_1\bar{\gamma}_2 + He[\gamma_3^TQ_1\bar{\gamma}_4] \\
&\quad +\dot{\bar{\tau}}\bar{\gamma}_5^TQ_2\bar{\gamma}_5 - \bar{\gamma}_6^TQ_2\bar{\gamma}_6 + He[\gamma_7^TQ_2\bar{\gamma}_8] \\
&\quad +\frac{\gamma+\dot{\tau}}{\rho d}He[E_{1h}^T\Gamma_h^TM + E_{2h}^T\Gamma_h^TN \\
&\quad +c_7^TH_1c_7 + c_8^TH_2c_8] \\
\tilde{\Psi}(\tau,\dot{\tau}) &= \Psi_1(\tau,\dot{\tau}) + \Psi_2(\tau,\dot{\tau}) + \Psi_3(\tau,\dot{\tau}) \\
&\quad +\Psi_4(\dot{\tau}) + \Psi_5(\tau,\dot{\tau}) + \Psi_6 \\
\Psi_1(\tau,\dot{\tau}) &= He[\Pi_1^T(d)P_{0h}\Pi_2(\dot{\tau})] + \dot{\tau}\Pi_3^TP_{1h}\Pi_3 \\
&\quad +He[\Pi_3^TP_{1h}\Pi_4(\tau,\dot{\tau})] - \dot{\tau}\Pi_5^TP_{2h}\Pi_5 \\
\Psi_2(\tau,\dot{\tau}) &= \gamma_1^TQ_1\gamma_1 - \dot{\bar{\tau}}\gamma_2^TQ_1\gamma_2 + He[\gamma_3^TQ_1\gamma_4] \\
&\quad +\dot{\bar{\tau}}\gamma_5^TQ_2\gamma_5 - \gamma_6^TQ_2\gamma_6 + He[\gamma_7^TQ_2\gamma_8] \\
\Psi_3(\tau,\dot{\tau}) &= d^2c_a^TRc_a + \left(\frac{\rho d-(\gamma_1+\gamma_2\dot{\tau})\tau}{\rho} + \frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d}\right) \\
&\times \; He[E_{1h}^T\Gamma_h^TM + E_{2h}^T\Gamma_h^TN)] \\
&-\frac{(\gamma_1+\gamma_2\dot{\tau})}{\rho d}\left\{d_\tau E_{1h}^T\Gamma_h^T\hat{R}\Gamma_hC_{1h} + \tau E_{h2}^T\Gamma_h^T\hat{R}\Gamma_hC_{2h}\right\}
\end{aligned}$$

$$
\begin{aligned}
\Psi_4(\dot{\tau}) &= He[\rho_{31}^T F_2 c_a + \dot{\tau}\rho_{32}^T F_2 c_9 + \rho_{33}^T F_2 c_{10}] \\
\Psi_5(\tau,\dot{\tau}) &= d^2 c_4^T Z c_4 + \Big(\frac{\rho d - (\gamma_1+\gamma_2\dot{\tau})\tau}{\rho} + \frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d}\Big) \\
&\times He[c_7^T H_1 c_7 + c_8^T H_2 c_8] \\
&\quad -\frac{(\gamma_1+\gamma_2\dot{\tau})}{\rho d}(d_\tau c_7^T Z c_7 + \tau c_8^T Z c_8) \\
&\quad +\Big(\frac{\rho d - (\gamma_1+\gamma_2\dot{\tau})\tau}{\rho} + \frac{(\gamma_1+\gamma_2\dot{\tau})\tau^2}{\rho d}\Big) \\
&\times He[c_7^T H_1 c_7 + c_8^T H_2 c_8] \\
&\quad -\frac{(\gamma_1+\gamma_2\dot{\tau})}{\rho d}\Big\{d_\tau c_7^T Z c_7 + \tau c_8^T Z c_8\Big\}. \\
\Psi_6 &= \sum_{i=1}^{3} He\Big[(c_{3+i} - H^- F_2 c_i)^T U_i^- (H^+ F_2 c_i \\
&\quad -c_{3+i})\Big] + \sum_{i=1}^{2} He\Big[[c_{3+i} - c_{4+i} \\
&\quad -H^- F_2(c_i - c_{i+1})]^T U_i^+ \\
&\times [H^+ F_2(c_i - c_{i+1}) - c_{3+i} + c_{4+i}]\Big] \\
&\quad +He\Big[[c_4 - c_6 - H^- F_2(c_1 - c_3)]^T G_3^+ \\
&\times [H^+ F_2(c_1 - c_3) - c_4 + c_6]\Big] \\
\Lambda_1(\tau) &= col[c_o, c_{u_0}, c_{v_0}, \cdots, c_{u_h}, c_{v_h}] \\
\Lambda_2(\dot{\tau}) &= col[\dot{e}_0, \dot{e}_{u_0}, \dot{e}_{v_0}, \cdots, \dot{e}_{u_h}, \dot{e}_{v_h}] \\
\Lambda_3 &= col[c_0, c_{13}, \cdots, c_{2h+11}] \\
\Lambda_4(\tau,\dot{\tau}) &= col[\tau\dot{c}_0, \dot{c}_{u_0} - \dot{\tau}c_{11}, \dot{e}_{u_1} - \dot{\tau}c_{13}, \\
&\quad \cdots, \dot{e}_{u_h} - \dot{d}c_{2h+11}] \\
\Lambda_5 &= col[c_a, c_{12}, c_{14}, \cdots, c_{2h+14}] \\
\Lambda_6 &= col[d_\tau\dot{c}_0, \dot{e}_{v_0} + \dot{\tau}c_{12}, \dot{e}_{v_1} + \dot{\tau}c_{14}, \\
&\quad \cdots, \dot{e}_{v_h} + \dot{\tau}c_{2h+12}] \\
\ell_1 &= col[c_a, c_1, c_4, 0, \tau c_{11}] \\
\ell_2 &= col[c_9, c_2, c_5, \tau c_{11}, 0] \\
\ell_3 &= col[0, 0, 0, c_1, -\dot{\tau}c_2, 0] \\
\ell_4 &= col[c_1 - c_2, \tau c_{11}, c_7, \tau^2 c_{13}, \tau^2(c_{11} - c_{13})] \\
\ell_5 &= col[c_9, c_2, c_5, 0, d_\tau c_{12}] \\
\ell_6 &= col[c_{10}, c_3, c_6, d_\tau c_{12}, 0] \\
\ell_7 &= col[0, 0, 0, \dot{\tau}c_2, -c_3] \\
\ell_8 &= col[c_2 - c_3, d_\tau c_{12}, c_8, d_\tau^2 c_{14}, d_\tau^2(c_{12} - c_{14})]
\end{aligned}
$$

$$
\begin{aligned}
\bar{\ell}_1 &= col[0,0,0,0,c_{11}] \\
\bar{\ell}_2 &= col[0,0,0,c_{11},0] \\
\bar{\ell}_4 &= col[0,0,0,c_{13},(c_{11}-c_{13})] \\
\bar{\ell}_5 &= col[0,0,0,0,c_{12}] \\
\bar{\ell}_6 &= col[c_{10},0,0,c_{12},0] \\
\bar{\ell}_8 &= col[0,0,0,c_{14},(c_{12}-c_{14})] \\
\rho_{31} &= M_1(c_4 - H^- F_2 c_1) + M_2(H^+ F_2 c_1 - c_4) \\
\rho_{32} &= M_3(c_5 - H^- F_2 c_2) + M_4(H^+ F_2 c_2 - c_5) \\
\rho_{33} &= M_5(c_6 - H^- F_2 c_3) + M_6(H^+ F_2 c_3 - c_6) \\
\varepsilon_{h1} &= col[c_1, c_2, c_{11}, 2c_{13}, (h+1)c_{11+2h}] \\
\varepsilon_{h2} &= col[c_2, c_3, c_{12}, 2c_{14}, (h+1)c_{12+2h}] \\
c_{u_i} &= \tau c_{11+2i}, c_{v_i} = d_\tau c_{12+2i} \\
\dot{c}_{u_i} &= \begin{cases} c_1 - \dot{\tau} c_2, i = 1 \\ c_1 - i\dot{\tau} c_{11+2(i-1)} - i\dot{\tau} c_{11+2i}, i \geq 1 \end{cases} \\
c_a &= -Ac_1 + F_0 c_4 + F_1 c_5 \\
c_0 &= [c_1, c_2, c_2], \dot{c}_0 = col[c_a, \dot{\tau} c_9, c_{10}] \\
\dot{c}_{v_i} &= \begin{cases} \dot{\tau} c_2 - c_3, i = 1 \\ \dot{\tau} c_2 - i c_{12+2(i-1)} - i\dot{\tau} c_{12+2i}, i \geq 1 \end{cases} \\
c_i &= c_{i,10+2(h+1)}.
\end{aligned}
$$

Proof: An LKF candidate is formulated as follows:

$$V(t) = \sum_{i=1}^{5} (t) \tag{3.4}$$

$$
\begin{aligned}
V_1(t) &= \eta_{0h}^T(t) P_{0h} \eta_{0h}(t) + \tau \eta_{1h}^T(t) P_{1h} \eta_{1h}(t) \\
&\quad + d_\tau \eta_{2h}^T(t) P_{2h} \eta_{2h}(t) \\
V_2(t) &= \int_{t-\tau}^{t} \eta_{3h}^T(t,s) Q_{1h} \eta_{3h}(t,s) ds \\
&\quad + \int_{t-d}^{t-\tau} \eta_{4h}^T(t,s) Q_{2h} \eta_{4h}(t,s) \\
V_3(t) &= d \int_{t-d}^{t} \int_{s}^{t} \dot{u}^T(v) R \dot{u}(v) dv ds \\
V_4(t) &= 2 \sum_{l=1}^{n} \int_0^{F_{2l} u(t)} [m_{1l} g_l^-(v) + m_{2l} g_l^+(v)] dv \\
&\quad + 2 \sum_{l=1}^{n} \int_0^{F_{2l} u(t-d)} [m_{3l} g_l^-(v) + m_{4l} g_l^+(v)] dv \\
&\quad + 2 \sum_{l=1}^{n} \int_0^{F_{2l} u(t-h)} [m_{5l} g_l^-(v) + m_{6l} g_l^+(v)] dv
\end{aligned}
$$

$$V_5(t) = d\int_{t-d}^{t}\int_{s}^{t} g^T(F_2u(v))Zg(F_2u(v))dvds.$$

Setting

$$\begin{aligned} S_i &= \int_a^b\int_{s_1}^b\cdots\int_{s_i-1}^b ds_i\cdots ds_2ds_1 \\ \Psi_{0^i} &= \int_a^b\int_{s_1}^b\cdots\int_{s_i-1}^b u(s_i)ds_i\cdots ds_2ds_1, \end{aligned}$$

one has

$$\frac{1}{h_i}\Psi_{0^i} = \frac{i}{b-a}g^{i-1}_{(a,b)}.$$

Setting $a = t-\tau, b = t$, it yields the following:

$$\begin{aligned} \vartheta_1 &= col[u(t), u(t-\tau), \frac{s_0(t)}{d}, \frac{2s_1(t)}{d}, \cdots, \frac{(h+1)u_h(t)}{d}] \\ &= \varepsilon_{h1}\eta_h(t). \end{aligned}$$

If $a = t-d, b = t-\tau$, one has the following:

$$\vartheta_2 = \varepsilon_{h2}\eta_h(t).$$

Furthermore, from $\dot{V}(t)$, we have the following:

$$\begin{aligned} \dot{V}_1(t) &= 2\eta_{0h}^T(t)P_{0h}\dot{\eta}_{0h}(t) + \dot{\tau}\eta_{1h}^T(t)P_{1h}\eta_{1h}(t) \\ &\quad +2\tau\eta_{1h}^T(t)P_{1h}\dot{\eta}_{1h}(t) - \dot{d}\eta_{2h}^T(t)P_{2h}\eta_{2h}(t) \\ &\quad +2d_\tau\eta_{2h}^T(t)P_{2h}\dot{\eta}_{2h}(t) \\ &= \eta_h^T(t)\Psi_1(\tau,\dot{\tau})\eta_h(t) \qquad (3.5) \\ \dot{V}_2(t) &= \eta_{3h}^T(t,t)Q_1\eta_{3h}(t,t) \\ &\quad -\dot{\bar{\tau}}\eta_{3h}^T(t,t-\tau)Q_1\eta_{3h}(t,t-d) \\ &\quad +2\int_{t-\tau}^{t}\eta_{3h}^T(t,s)Q_1\frac{d\eta_{3h}(t,s)}{dt}ds \\ &\quad +\dot{\bar{\tau}}\eta_{4h}^T(t,t-\tau)Q_2\eta_{4h}(t,t-\tau) \\ &\quad -\eta_{4h}^T(t,t-d)Q_2\eta_{4h}(t,t-d) \\ &\quad +2\int_{t-d}^{t-\tau}\eta_{4h}^T(t,s)Q_2\frac{\tau\eta_{4h}(t,s)}{dt}ds \\ &= \eta_h^T(t)\Psi_2(\tau,\dot{\tau})\eta_h(t) \qquad (3.6) \\ \dot{V}_3(t) &= d^2\dot{u}^T(t)R\dot{u}(t) - d\int_{t-d}^{t}\dot{u}^T(s)R\dot{u}(s)ds \qquad (3.7) \\ \dot{V}_4(t) &= \eta_h^T(t)\Psi_4(\dot{d})\eta_h(t) \qquad (3.8) \\ \dot{V}_5(t) &= d^2g^T(F_2u(t))Zg(F_2u(t)) \end{aligned}$$

$$-d\int_{t-d}^{t} g^T(F_2u(s))Zg(F_2u(s))ds, \tag{3.9}$$

where the terms $\Psi_1(\tau,\dot{\tau})$, $\Psi_2(\tau,\dot{\tau})$, and $\Psi_4(\dot{\tau})$ remain consistent with those outlined in previous part. Upon differentiating $v_{1i}(t)$ and $v_{2i}(t)$ in $\eta_{1h}(t)$ and $\eta_{2h}(t)$, the following results are obtained:

$$\dot{f}_i(a,b) = \begin{cases} \dot{b}u(b) - \dot{a}u(a), i = 0 \\ \dot{b}u(b) - \frac{i\dot{a}}{b-a}f^{i-1} - \frac{i(\dot{b}-\dot{a})}{b-a}f_i, i \geq 1. \end{cases}$$

Utilizing Lemma 1, we obtain the following:

$$\dot{V}_3(t) \leq \eta_h^T(t)[\tilde{\Psi}_3(\tau,\dot{\tau}) + \Psi_3(\tau,\dot{\tau})]\eta_h(t). \tag{3.10}$$

where $\tilde{\Psi}_3(\tau,\dot{\tau}) = \frac{(\rho-\gamma_1-\gamma_2\dot{\tau})}{\rho}d[\tau M^T\tilde{R}M + d_\tau N^T\tilde{R}N\Big]$. Applying Lemma 1 with $h = 0$, one has

$$\dot{V}_5(t) \leq \eta_h^T(t)[\Psi_5(\tau,\dot{\tau}) + \hat{\Psi}_5(\tau,\dot{\tau})]\eta_h(t), \tag{3.11}$$

where $\hat{\Psi}_5(\tau,\dot{\tau}) = \frac{(\rho-\gamma_1-\gamma_2\dot{\tau})}{\rho}d\Big[\tau c_7^T H_1^T Z^{-1}H_1c_7 + d_\tau c_8^T H_2^T Z^{-1}H_2c_8\Big]$.

From the fact (2.4)

$$\begin{cases} \ell_1(t, G_1^-) \geq 0 \\ \ell_1(t-\tau, G_2^-) \geq 0 \\ \ell_1(t-d, G_3^-) \geq 0 \\ \ell_2(t, t-\tau, G_1^+) \geq 0 \\ \ell_2(t-\tau, t-d, G_2^+) \geq 0 \\ \ell_2(t, t-d, G_3^+) \geq 0, \end{cases}$$

one has

$$\eta_h^T(t)\Psi_6\eta_h(t) \geq 0. \tag{3.12}$$

Based on the discussions above, we can deduce

$$\dot{V}(t) \leq \eta_h^T(t)\bar{\Psi}(\tau,\dot{\tau})\eta_h(t), \tag{3.13}$$

where $\bar{\Psi}(\tau,\dot{\tau}) = \tilde{\Psi}(\tau,\dot{\tau}) + \hat{\Psi}_3(\tau) + \hat{\Psi}_5(\tau)$. Define

$$\bar{\Psi}(\tau,\dot{\tau}) = \tau^2 a_2(\dot{\tau}) + \tau a_1 + a_0$$

Here $a_2(\dot{\tau})$ has been defined in Theorem 1, and a_1, a_0 are the appropriate dimensional matrices. By the Schur complement lemma, the inequalities are equivalent to $\bar{\Psi}(0,\dot{\tau}) < 0$, $\bar{\Psi}(d,\dot{\tau}) < 0$, and $-d^2a_2(\dot{\tau}) + \bar{\Psi}(0,\dot{\tau}) < 0$. These correspond to the three conditions $f(0) < 0$, $f(d) < 0$, and $-d^2a_2 + f(0) < 0$ in Lemma 2 of Ref. [4]. Therefore, $\bar{\Psi}(\tau,\dot{\tau}) < 0$ is ensured for any $\tau \in [0,d]$.

Additionally, $\bar{\Psi}(\tau,\dot{\tau})$ is affine with respect to $\dot{\tau}$. Therefore, $\bar{\Psi}(\tau,\dot{\tau}) < 0$ is ensured for any $\dot{\tau} \in [\rho_1,\rho_2]$ by $\bar{\Psi}(\tau,\rho_1) < 0$ and $\bar{\Psi}(\tau,\rho_2) < 0$.

In conclusion, based on Theorem 1, for a small positive scalar ϵ, it follows that $\dot{V}(t) \leq -\epsilon\|u(t)\| < 0$ for $u(t) \neq 0$. This implies the asymptotical stability of NNs (2.1).

Remark 3. In earlier research such as [6, 27, 28], the inclusion of the delay derivative $\dot{\tau}$ has predominantly been dependent on the derivative of the LKFs. Yet, the full integration of the delay derivative $\dot{\tau}$ remains unaccomplished. In contrast to the LKFs, CPBII effectively incorporates the delay derivative $\dot{\tau}$. This methodology facilitates the concurrent introduction of $\dot{\tau}$, the delay amplitude ρ, slack matrices M, N, and the augmented vector $\eta_h(t)$ along with positive definite matrices. As a result, the system information can be efficiently interconnected. Additionally, the inclusion of parameters γ_1 and γ_2 assists in circumventing zero terms, thereby enabling the extraction of more coupling information. For example, consider $\rho d(\rho - \gamma_1 - \gamma_2\dot{\tau})$ in (3.1). If $\gamma_1 = 0$, $\gamma_2 = 1$, and $\dot{d} = \rho$ are set, the interrelation between $\dot{\tau}$ and $\eta_h(t)$ is connected via N would vanish. Furthermore, permitting any parameters where $\gamma_1 \leq 0$ and $\gamma_2 \leq 1$ (since $\rho - \gamma_1 - \gamma_2\dot{d} \geq 0$) enhances the adaptability of Theorem 1, in which demonstrates reduced conservatism.

4. Numerical example

In this section, we will demonstrate the effectiveness of the proposed stability condition.

We scrutinize NN that is characterized by the structure (2.1), where $F_2 = I$. The parameters employed in this analysis are derived from the study [27]:

$$
\begin{aligned}
A &= diag\{1.2769, 0.6231, 0.9230, 0.4480\} \\
F_0 &= \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.086 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix} \\
F_1 &= \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.022 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix} \\
H^- &= diag\{0, 0, 0, 0\} \\
H^+ &= diag\{0.1137, 0.1279, 0.7994, 0.2368\}.
\end{aligned}
$$

As depicted in Table 1, with the same LKF in [27], the maximum allowable upper bound of the delay amplitude is presented for a range of ρ values. A clear observation from the table is that the conservatism in the results derived in this study is less pronounced compared to previous studies [6, 17, 19, 27]. A comparative analysis between Theorem 1 of this paper and the results of [27] underscores the efficacy of CPBII in mitigating conservatism. Moreover, it is discerned that the parameters γ_1 and γ_2 augment the adaptability of the stability condition. It should be highlighted that the values of γ_1 and γ_2 in Table 1 are random under $\gamma_1 \leq 0$ and $\gamma_2 \leq 1$. Thus, the maximum allowable upper bounds of d may be larger by choosing more suitable values, which deserves a further study in the future.

Table 1. The maximum allowable upper bounds of d for different ρ.

ρ	0.1	0.5	0.9	NVs
Theorem 3, [19]	4.4167	3.5986	3.3755	$79n^2 + 15n$
Proposition1 [17]	4.5382	3.9313	3.4763	$60n^2 + 22n$
Proposition 3, [6]($N = 3$)	4.5468	4.0253	3.6246	$198n^2 + 26n$
Theorem 1, [27]($h = 1$)	4.5426	3.9438	3.4688	$83.5n^2 + 26.5n$
Theorem 1, [27]($h = 2$)	4.5470	3.9749	3.5052	$112.5n^2 + 28.5n$
Theorem 1($h = 2, \gamma_1 = -0.06, \gamma_2 = 0.7$)	4.8507	4.2714	3.8139	$112.5n^2 + 28.5n$
Theorem 1($h = 2, \gamma_1 = -0.02, \gamma_2 = 0.6$)	4.8601	4.2823	3.8244	$112.5n^2 + 28.5n$

On the other hand, by setting $u(t) = [0.50.3-0.3-0.5]^T$, $\tau(t) = 4.7601+0.1sin(t)$, $g(t) = 0.1tanh(u)$, it can be seen from Figure 1 that the state response is stable, which shows the effectiveness of proposed method.

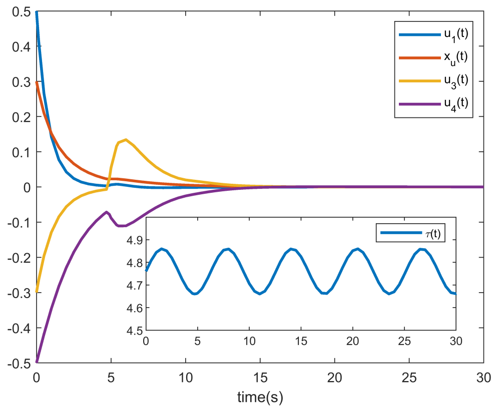

Figure 1. The state responses of system (2.1) under $\tau(t) = 4.7601 + 0.1sin(t)$.

5. Conclusions

This study addresseed stability analysis of neural networks with time-varying delays. We introduced to CPBII to incorporate delay derivatives into integral inequalities. Then, a novel stability criterion for such neural networks was derived using CPBII. Notably, CPBII encompassed all augmented vectors and their derivatives from the LKF, thus facilitating comprehensive coupling with the delay amplitude and the delayderivative via slack matrices and tunable parameters. This integration led to less conservative outcomes. The effectiveness of our approach was demonstrated through numerical examples.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Writing-original draft, Wenlong Xue; Validation, Zhenghong Jin; Writing-review & editing, Yufeng Tian.

Conflict of interest

The author declares that there is no conflict of interest.

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