Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(7): 19299–19331.
DOI: 10.3934/math.2024940
Received: 26 March 2023
Revised: 23 May 2024
Accepted: 27 May 2024
Published: 11 June 2024

Research article

Generalized $(\alpha_s, \xi, \hbar, \tau)$-Geraghty contractive mappings and common fixed point results in partial b-metric spaces

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Abstract: In this paper, we introduce two new classes of mixed $(\mathcal{S}, \mathcal{T})$-$\alpha$-admissible mappings and interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha$-admissible mappings and study the sufficient conditions for the existence and uniqueness of a common fixed point of generalized $(\alpha_s, \xi, \hbar, \tau)$-Geraghty contractive mapping in the framework of partial b-metric spaces. We also provide two examples to show the applicability and validity of our results. Moreover, we present an application to the existence of solutions to an integral equation by means of one of our results.

Keywords: common fixed point; weakly contractive mapping; partial b-metric; generalized $(\alpha_s, \xi, \hbar, \tau)$-Geraghty contractive mapping
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

The Banach fixed point theorem [1], popularly known as the Banach contraction mapping principle, is a rewarding result in fixed point theory. It has widespread applications in both pure and applied mathematics and has been extended in many different directions. One of the most popular and interesting topics among them is the study of new classes of spaces and their fundamental properties.

In 2014, Shukla [2] introduced the concept of partial b-metric space and proved some fixed point theorems of contractive mappings in partial b-metric space. After that, some authors have researched the fixed point theorems of new contractive conditions in partial b-metric space. Mustafa et al. [3] proved some fixed point and common fixed point results for (ψ, φ)-weakly contractive mappings in ordered partial b-metric spaces. After that, authors in [4] studied the sufficient conditions for the existence of a unique common fixed point of generalized α_s-ψ-Geraghty contractions in an α_s-complete partial b-metric space. Mukheimer [5] and Vujaković et al. [6] introduced the concept of α-ψ-φ-contractive self mapping, and Latif et al. [7] considered α-admissible mappings in the setup of partial

b-metric spaces and established some fixed and common fixed point results for ordered cyclic weakly (ψ, φ, L, A, B)-contractive mappings. In 2021, Gautam et al. [8] established coincidence and common fixed point theorems for weakly compatible pairs of mapping in quasi-partial b-metric spaces.

A partial b-metric space means that it is a generalization of b-metric spaces and partial metric spaces, and researchers have researched many fixed point theorems of new contractive conditions in b-metric spaces and partial metric spaces. In [9–11], Dolatabad et al. introduced the concept of φ-contractive self mapping in ordered rectangular b-metric spaces. In 2012, Aydi et al. [12] proved a general common fixed point theorem for two pairs of weakly compatible self-mappings of a partial metric space satisfying a generalized Meir-Keeler type contractive condition. Roshan et al. [13] introduced the notion of almost generalized $(\psi, \varphi)_s$-contractive mappings, and Dinarvand [14] presented a new class of almost contractive mappings called almost generalized $(\psi, \varphi, \theta)_s$-contractive mappings in partially ordered b-metric spaces. Ameer et al. [15] introduced the notion of generalized α^*-ψ-Geraghty contraction type for multi-valued mappings in b-metric spaces. Tiwari and Heeramani [16] introduced the notion of generalized α-β-ψ contractive mappings involving rational expressions and established existence and uniqueness of fixed points of Berinde type generalized α-β-ψ contractive mappings in the context of partial metric spaces. Zada et al. [17] introduced the notion of cyclic (α,β)-$(\psi, \varphi)_s$-rational-type contraction in b-metric spaces. In [18, 19], the authors presented the notion of almost generalized $(\alpha, \beta, \psi, \varphi)$-Geraghty type contractive mappings in partial metric spaces. Debnath [20, 21] introduced separately some new set-valued Meir-Keeler, Geraghty, and Edelstein type multivalued contractive mappings in a b-metric space and multivalued Geraghty type contractive mappings in a complete metric space. For recent development on fixed point theory, we refer to [22–31].

Motivated and inspired by Theorems 2.7 in [22], Theorem 3.2 in [23], and Theorem 12 in [24], in this paper, our purpose is to introduce two new classes of mixed $(\mathcal{S}, \mathcal{T})$-$\alpha$-admissible mappings and interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha$-admissible mappings and obtain a few common fixed point results involving generalized contractive conditions in the framework of partial b-metric spaces. Furthermore, we provide examples that elaborate the useability of our results. Moreover, we present an application to the existence of solutions to an integral equation by means of one of our results.

2. Preliminaries

First, we recall some definitions and lemmas in partial b-metric spaces.

Definition 2.1. [2] Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $\partial_b : \mathcal{X}\times\mathcal{X} \to [0, +\infty)$ is said to be a partial $b-$metric, if for all $\mathfrak{u}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$, the following conditions are satisfied:

(i) $\partial_b(\mathfrak{u}, \mathfrak{y}) \geq 0$,$\mathfrak{u} = \mathfrak{y}$ if and only if $\partial_b(\mathfrak{u}, \mathfrak{y}) = \partial_b(\mathfrak{u}, \mathfrak{u}) = \partial_b(\mathfrak{y}, \mathfrak{y})$;

(ii) $\partial_b(\mathfrak{u}, \mathfrak{u}) \leq \partial_b(\mathfrak{u}, \mathfrak{y})$;

(iii) $\partial_b(\mathfrak{u}, \mathfrak{y}) = \partial_b(\mathfrak{y}, \mathfrak{u})$;

(iv) $\partial_b(\mathfrak{u}, \mathfrak{y}) \leq s(\partial_b(\mathfrak{u}, \mathfrak{z}) + \partial_b(\mathfrak{y}, \mathfrak{z})) - \partial_b(\mathfrak{z}, \mathfrak{z})$.

The pair $(\mathcal{X}, \partial_b)$ is called a partial b-metric space. It is clear that, the class of partial b-metric spaces is larger than that of partial metric spaces.

On the other hand, Mustafa et al. [3] modified (iv) in the above definition and got the following result:

Definition 2.2. [3] Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ be a given real number. If the mapping

$\partial_b : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ satisfies the conditions (i)–(iii) in the above definition, and

(iv*) $\partial_b(\mathfrak{u}, \mathfrak{y}) \leq s(\partial_b(\mathfrak{u}, \mathfrak{z}) + \partial_b(\mathfrak{y}, \mathfrak{z}) - \partial_b(\mathfrak{z}, \mathfrak{z})) + \frac{1-s}{2}(\partial_b(\mathfrak{u}, \mathfrak{u}) + \partial_b(\mathfrak{y}, \mathfrak{y}))$, for all $\mathfrak{u}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$,

then ∂_b is a partial b-metric with a coefficient $s \geq 1$. In this case, we also call the pair $(\mathcal{X}, \partial_b)$ is a partial b-metric space.

In this paper, we consider fixed point results in the setting of partial b-metric spaces which are defined by Definition 2.2.

Proposition 2.3. Every partial b-metric ∂_b defines b-metric ∂_b^w and ∂_b^s, where $\partial_b^w(\mathfrak{u}, \mathfrak{y}) = \partial_b(\mathfrak{u}, \mathfrak{y}) - \min\{\partial_b(\mathfrak{u}, \mathfrak{u}), \partial_b(\mathfrak{y}, \mathfrak{y})\}$ and $\partial_b^s(\mathfrak{u}, \mathfrak{y}) = 2\partial_b(\mathfrak{u}, \mathfrak{y}) - \partial_b(\mathfrak{u}, \mathfrak{u}) - \partial_b(\mathfrak{y}, \mathfrak{y})$.

Proof. Without loss of generality, assume that $\min\{\partial_b(\mathfrak{u}, \mathfrak{u}), \partial_b(\mathfrak{y}, \mathfrak{y})\} = \partial_b(\mathfrak{u}, \mathfrak{u})$, then

$$\begin{aligned}
\partial_b^w(\mathfrak{u}, \mathfrak{y}) &= \partial_b(\mathfrak{u}, \mathfrak{y}) - \min\{\partial_b(\mathfrak{u}, \mathfrak{y}), \partial_b(\mathfrak{y}, \mathfrak{y})\} \\
&= \partial_b(\mathfrak{u}, \mathfrak{y}) - \partial_b(\mathfrak{u}, \mathfrak{u}) \\
&\leq s(\partial_b(\mathfrak{u}, \mathfrak{z}) + \partial_b(\mathfrak{y}, \mathfrak{z}) - \partial_b(\mathfrak{z}, \mathfrak{z})) + \frac{1-s}{2}(\partial_b(\mathfrak{u}, \mathfrak{u}) + \partial_b(\mathfrak{y}, \mathfrak{y})) - \partial_b(\mathfrak{u}, \mathfrak{u}) \\
&\leq s\partial_b(\mathfrak{u}, \mathfrak{z}) + s\partial_b(\mathfrak{z}, \mathfrak{y}) - s\partial_b(\mathfrak{z}, \mathfrak{z}) - s\partial_b(\mathfrak{u}, \mathfrak{u}) \\
&\leq s(\partial_b^w(\mathfrak{u}, \mathfrak{z}) + \partial_b^w(\mathfrak{z}, \mathfrak{y})).
\end{aligned}$$

So, ∂_b^w is a b-metric. Similarly, ∂_b^s is also a b-metric. □

Definition 2.4. [3] Let $(\mathcal{X}, \partial_b)$ be a partial b−metric space with parameter $s \geq 1$. Then a sequence $\{\mathfrak{u}_n\}$ in $\mathcal{X}$ is said to be:

(i) p_b-convergent if and only if there exists $\mathfrak{u} \in \mathcal{X}$ such that $\lim_{n\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}) = \partial_b(\mathfrak{u}, \mathfrak{u})$;

(ii) a p_b-Cauchy sequence if and only if $\lim_{n,m\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}_m)$ exists (and is finite).

A partial b-metric space is called p_b-complete if each p_b-Cauchy sequence in this space is p_b-convergent to a point $\mathfrak{u} \in \mathcal{X}$ such that

$$\lim_{n,m\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}_m) = \lim_{n\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}) = \partial_b(\mathfrak{u}, \mathfrak{u}).$$

Lemma 2.5. [3] A sequence $\{\mathfrak{u}_n\}$ is a p_b-Cauchy sequence in a partial b-metric space $(\mathcal{X}, \partial_b)$ iff it is a b-Cauchy sequence in the b-metric space $(\mathcal{X}, \partial_b^s)$.

Lemma 2.6. [3] A partial b-metric space $(\mathcal{X}, \partial_b)$ is p_b-complete if and only if the b-metric space $(\mathcal{X}, \partial_b^s)$ is b-complete. Moreover, $\lim_{n,m\to\infty} \partial_b^s(\mathfrak{u}_n, \mathfrak{u}_m) = 0$ if and only if

$$\lim_{n,m\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}_m) = \lim_{n\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}) = \partial_b(\mathfrak{u}, \mathfrak{u}).$$

Definition 2.7. Let $\mathcal{P}$ and $\mathcal{Q}$ be two self-mappings on a nonempty set $\mathcal{X}$. If $\mathfrak{w} = \mathcal{P}\mathfrak{u} = \mathcal{Q}\mathfrak{u}$, for some $\mathfrak{u} \in \mathcal{X}$, then $\mathfrak{u}$ is said to be the coincidence point of $\mathcal{P}$ and $\mathcal{Q}$, where $\mathfrak{w}$ is called the point of coincidence of $\mathcal{P}$ and $\mathcal{Q}$. Let $C(\mathcal{P}, \mathcal{Q})$ denote the set of all coincidence points of $\mathcal{P}$ and $\mathcal{Q}$.

Definition 2.8. [26] Let $\mathcal{P}$ and $\mathcal{Q}$ be two self-mappings defined on a nonempty set $\mathcal{X}$. Then $\mathcal{P}$ and $\mathcal{Q}$ is said to be weakly compatible if they commute at every coincidence point, that is, $\mathcal{P}\mathfrak{u} = \mathcal{Q}\mathfrak{u} \Rightarrow \mathcal{P}\mathcal{Q}\mathfrak{u} = \mathcal{Q}\mathcal{P}\mathfrak{u}$ for every $\mathfrak{u} \in C(\mathcal{P}, \mathcal{Q})$.

We cite the following lemma to obtain our main results:

Lemma 2.9. [3] Let $(\mathcal{X}, \partial_b)$ be a partial b-metric space with parameter $s \geq 1$. Assume that $\{\mathfrak{u}_n\}$ and $\{\mathfrak{y}_n\}$ are $b-$convergent to $\mathfrak{u}$ and $\mathfrak{y}$, respectively. Then,

$$\frac{1}{s^2}\partial_b(\mathfrak{u}, \mathfrak{y})-\frac{1}{s}\partial_b(\mathfrak{u}, \mathfrak{u})-\partial_b(\mathfrak{y}, \mathfrak{y}) \leq \liminf_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{y}_n) \leq \limsup_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{y}_n) \leq s\partial_b(\mathfrak{u}, \mathfrak{u})+s^2\partial_b(\mathfrak{y}, \mathfrak{y})+s^2\partial_b(\mathfrak{u}, \mathfrak{y}).$$

In particular, if $\partial_b(\mathfrak{u}, \mathfrak{y}) = 0$, then $\lim_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{y}_n) = 0$. Moreover, for each $\mathfrak{z} \in \mathcal{X}$,

$$\frac{1}{s}\partial_b(\mathfrak{u}, \mathfrak{z}) - \partial_b(\mathfrak{u}, \mathfrak{u}) \leq \liminf_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{z}) \leq \limsup_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{z}) \leq s\partial_b(\mathfrak{u}, \mathfrak{z}) + s\partial_b(\mathfrak{u}, \mathfrak{u}).$$

Furthermore, if $\partial_b(\mathfrak{u}, \mathfrak{u}) = 0$, then

$$\frac{1}{s}\partial_b(\mathfrak{u}, \mathfrak{z}) \leq \liminf_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{z}) \leq \limsup_{n\to+\infty} \partial_b(\mathfrak{u}_n, \mathfrak{z}) \leq s\partial_b(\mathfrak{u}, \mathfrak{z}).$$

Lemma 2.10. Let $(\mathcal{X}, \partial_b)$ be a partial $b-$mertic space and let $\{\mathfrak{u}_n\}$ be a sequence in $\mathcal{X}$ such that $\{\partial_b(\mathfrak{u}_n, \mathfrak{u}_{n+1})\}$ is non-increasing and that

$$\lim_{n\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0.$$

If $\{\mathfrak{u}_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and the following four sequences

$$\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}), \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}), \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k+1}),$$

satisfy

$$\epsilon \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) \leq s\epsilon,$$
$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) \leq s^2\epsilon,$$
$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) \leq s\epsilon,$$
$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k+1}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k+1}) \leq s^2\epsilon.$$

Proof. Let $(\mathcal{X}, \partial_b)$ be a partial b-mertic space and $\{\mathfrak{u}_n\} \subset \mathcal{X}$ be a sequence satisfying $\lim_{n\to\infty} \partial_b(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0$. If $\{\mathfrak{u}_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$m_k > n_k > k, \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2m_k-2}) < \epsilon, \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2m_k}) \geq \epsilon,$$

for all positive integers k.

Step 1. By the triangle inequality, we have

$$\begin{aligned}
\epsilon &\leq \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k})\\
&\leq s[\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-2}) + \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k})] - \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2m_k-2})\\
&\leq s\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k}) + s\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-2})\\
&\leq s\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k}) + s[s(\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-1}) + \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2})) - \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-1})]\\
&\leq s\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k}) + s^2[\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-1}) + \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2})]\\
&\leq s\epsilon + s^2\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-1}) + s^2\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}).
\end{aligned} \tag{2.1}$$

Taking the upper and lower limits as $k \to \infty$ in (2.1), we obtain that

$$\epsilon \le \liminf_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) \le \limsup_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) \le s\epsilon.$$

Step 2. Again, by the triangle inequality, one can deduce

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) &\le s[\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) + \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k+1})] - \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) + s\partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k+1}).\end{aligned} \tag{2.2}$$

Letting $k \to \infty$ in (2.2), we arrive at

$$\limsup_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) \le s^2\epsilon.$$

Also,

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) &\le s[\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{u}_{2n_k+1}, \mathfrak{u}_{2n_k})] - \partial_b(\mathfrak{u}_{2n_k+1}, \mathfrak{u}_{2n_k+1}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) + s\partial_b(\mathfrak{u}_{2n_k+1}, \mathfrak{u}_{2n_k}),\end{aligned}$$

we get that

$$\epsilon \le \liminf_{k\to\infty} s\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}).$$

Thus,

$$\frac{\epsilon}{s} \le \liminf_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) \le \limsup_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) \le s^2\epsilon.$$

Step 3. Using a similar method in the Step 2, we get

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) &\le s[\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) + \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k})] - \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2m_k-2}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k}) + s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) \\ &\le s\epsilon + s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}),\end{aligned}$$

$$\limsup_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) \le s\epsilon,$$

and

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k}) &\le s[\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-1}) + \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k})] - \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-1}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2m_k-1}) + s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}),\end{aligned}$$

$$\epsilon \le \liminf_{k\to\infty} s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}).$$

Thus,

$$\frac{\epsilon}{s} \le \liminf_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) \le \limsup_{k\to\infty} \partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k}) \le s\epsilon.$$

Step 4. From $\partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2m_k-2}) < \epsilon$, and by using the triangular inequality again, one can deduce

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2n_k+1}) &\le s[\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) + \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k+1})] - \partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2m_k-2}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) + s\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k+1}) \\ &\le s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) + s[s(\partial_b(\mathfrak{u}_{2m_k-2}, \mathfrak{u}_{2n_k}) + \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k+1})) - \partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k})] \\ &\le s\partial_b(\mathfrak{u}_{2m_k-1}, \mathfrak{u}_{2m_k-2}) + s^2\epsilon + s^2\partial_b(\mathfrak{u}_{2n_k}, \mathfrak{u}_{2n_k+1}),\end{aligned}$$

and

$$\begin{aligned}\partial_b(\mathfrak{u}_{2m_k},\mathfrak{u}_{2n_k}) &\le s[\partial_b(\mathfrak{u}_{2m_k},\mathfrak{u}_{2m_k-1})+\partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k})]-\partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2m_k-1})\\ &\le s\partial_b(\mathfrak{u}_{2m_k},\mathfrak{u}_{2m_k-1})+s\partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k})\\ &\le s\partial_b(\mathfrak{u}_{2m_k},\mathfrak{u}_{2m_k-1})+s[s(\partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k+1})+\partial_b(\mathfrak{u}_{2n_k+1},\mathfrak{u}_{2n_k}))-\partial_b(\mathfrak{u}_{2n_k+1},\mathfrak{u}_{2n_k+1})]\\ &\le s\partial_b(\mathfrak{u}_{2m_k},\mathfrak{u}_{2m_k-1})+s^2\partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k+1})+s^2\partial_b(\mathfrak{u}_{2n_k+1},\mathfrak{u}_{2n_k}).\end{aligned}$$

Taking the upper and lower limits as $k \to \infty$, we have

$$\frac{\epsilon}{s^2} \le \liminf_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k+1}) \le \limsup_{k\to\infty}\ \partial_b(\mathfrak{u}_{2m_k-1},\mathfrak{u}_{2n_k+1}) \le s^2\epsilon.$$

□

3. Main results

In this section, we will establish some results for the existence of a common fixed point of generalized weakly contractive mappings in the setup of p_b-complete partial b-metric spaces.

Let Ξ denote the class of all mappings $\xi : \mathbb{R}_0{}^+ \to [0,\zeta)$, $\zeta < 1$ and $\mathcal{H}$ denote the class of the functions $\hbar : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

(1) $\hbar$ is non-decreasing,

(2) $\hbar$ is continuous,

(3) $\hbar(\kappa) = 0$ if and only if $\kappa = 0$.

Let $\mathfrak{I}$ denote the class of the functions $\tau : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

(1) τ is upper semi-continuous,

(2) $\tau(\kappa) = 0$ if and only if $\kappa = 0$.

Definition 3.1. Let $(\mathcal{X},\partial_b)$ be a partial b-metric space with parameter $s \ge 1$, and let $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\alpha_s : \mathcal{X}\times\mathcal{X} \to [0,+\infty)$ be given mappings. The mapping pair $(\mathfrak{f},\mathfrak{g})$ is said to be $(\mathcal{S},\mathcal{T})$-$\alpha_s$-admissible, if for all $\mathfrak{u},\mathfrak{y} \in \mathcal{X}$, $\alpha_s(\mathcal{T}\mathfrak{y},\mathcal{S}\mathfrak{u}) \ge s$ implies $\alpha_s(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y}) \ge s$.

Definition 3.2. Let $(\mathcal{X},d)$ be a partial b-metric space with parameter $s \ge 1$, and let $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be four mappings. Assume that $\alpha_s : \mathcal{X}\times\mathcal{X} \to [0,+\infty)$. The mapping pair $(\mathfrak{f},\mathfrak{g})$ is called a mixed $(\mathcal{S},\mathcal{T})$-$\alpha_s$-admissible, if $(\mathfrak{f},\mathfrak{g})$ satisfies the following conditions:

(1) $(\mathfrak{f},\mathfrak{g})$ is $(\mathcal{S},\mathcal{T})$-$\alpha_s$-admissible;

(2) $\alpha_s(\mathfrak{u},\mathfrak{y}) = \alpha_s(\mathfrak{y},\mathfrak{u})$;

(3) $\alpha_s(\mathfrak{u},\mathfrak{y}) \ge s$ and $\alpha_s(\mathfrak{y},\mathfrak{z}) \ge s$ imply $\alpha_s(\mathfrak{u},\mathfrak{z}) \ge s$.

Remark 3.3. For $s = 1$, the Definition 3.2 reduces to the definition of a mixed $(\mathcal{S},\mathcal{T})$-$\alpha$-admissible mapping in a partial metric space.

Let $(\mathcal{X},\partial_b)$ be a p_b-complete partial b-metric space with parameter $s \ge 1$ and let $\alpha_s : \mathcal{X}\times\mathcal{X} \to [0,+\infty)$. Then

(a) If $\{\mathcal{S}\mathfrak{u}_{2n}\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{S}\mathfrak{u}_{2n} \to \varpi$ as $n \to \infty$, then there exists a subsequence $\{\mathcal{S}\mathfrak{u}_{2n_i}\}$ of $\{\mathcal{S}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathcal{S}\mathfrak{u}_{2n_i},\varpi) \ge s$ for all $i \in \mathbb{N}$.

(b) If $\{\mathcal{S}\mathfrak{u}_{2n}\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{S}\mathfrak{u}_{2n} \to \varpi$ as $n \to \infty$, then there exists a subsequence $\{\mathcal{S}\mathfrak{u}_{2n_k}\}$ of $\{\mathcal{S}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathcal{S}\mathfrak{u}_{2n_k},\mathcal{T}\varpi) \ge s$ for all $k \in \mathbb{N}$.

(c) For all $\varpi,\varsigma \in C(\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T})$, we have the condition of $\alpha_s(\varpi,\varsigma) \ge s$.

Lemma 3.4. Let $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S}$ be four self-mappings in a partial b-metric space $(\mathcal{X}, \partial_b)$, such that the pair $(\mathfrak{f}, \mathfrak{g})$ is mixed $(\mathcal{S}, \mathcal{T})$-$\alpha_s$-admissible. Assume that there exists $\mathfrak{u}_0 \in \mathcal{X}$ satisfying $\mathfrak{y}_0 = \mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_1$, $\mathfrak{y}_1 = \mathfrak{g}\mathfrak{u}_1 = \mathcal{S}\mathfrak{u}_2$ such that $\alpha_s(\mathfrak{y}_0, \mathfrak{y}_1) = \alpha_s(\mathcal{T}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$. Define two sequences $\{\mathfrak{u}_n\}$, $\{\mathfrak{y}_n\} \subseteq \mathcal{X}$ by $\mathfrak{y}_{2n} = \mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{u}_{2n+1}$ and $\mathfrak{y}_{2n+1} = \mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}$, where $n = 1, 2, 3, \ldots$. Then for $n, m \in \mathbb{N} \cup \{0\}$, we have $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$.

Proof. Since $\alpha_s(\mathcal{T}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$ and $(\mathfrak{f}, \mathfrak{g})$ is mixed $(\mathcal{S}, \mathcal{T})$-$\alpha_s$-admissible, $\alpha_s(\mathcal{T}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) = \alpha_s(\mathfrak{y}_0, \mathfrak{Y}_1) \geq s$ implies $\alpha_s(\mathfrak{f}\mathfrak{u}_2, \mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_2) = \alpha_s(\mathfrak{y}_1, \mathfrak{y}_2) \geq s$, $\alpha_s(\mathcal{T}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_2) \geq s$ implies $\alpha_s(\mathfrak{f}\mathfrak{u}_2, \mathfrak{g}\mathfrak{u}_3) = \alpha_s(\mathcal{T}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_4) = \alpha_s(\mathfrak{y}_2, \mathfrak{y}_3) \geq s$, $\alpha_s(\mathcal{T}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_4) \geq s$ implies $\alpha_s(\mathfrak{f}\mathfrak{u}_4, \mathfrak{g}\mathfrak{u}_3) = \alpha_s(\mathcal{T}\mathfrak{u}_5, \mathcal{S}\mathfrak{u}_4) = \alpha_s(\mathfrak{y}_3, \mathfrak{y}_4) \geq s$. Applying the above argument repeatedly, one can obtain $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \geq s$ for any $n \in \mathbb{N}$. Since $\alpha_s(\mathfrak{u}, \mathfrak{y}) \geq s$ and $\alpha_s(\mathfrak{y}, \mathfrak{z}) \geq s$ imply $\alpha_s(\mathfrak{u}, \mathfrak{z}) \geq s$, and $\alpha_s(\mathfrak{u}, \mathfrak{y}) = \alpha_s(\mathfrak{y}, \mathfrak{u})$. So, $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$ for all $n, m \in \mathbb{N} \cup \{0\}$. □

Theorem 3.5. Let $(\mathcal{X}, \partial_b)$ be a p_b-complete partial b-metric space with parameter $s \geq 1$ and let $\alpha_s : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$, $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ be given mappings and $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. Suppose $\hbar \in \mathcal{H}$ and $\tau \in \mathfrak{I}$. If the following conditions are satisfied:

(i) $(\mathfrak{f}, \mathfrak{g})$ is mixed $(\mathcal{T}, \mathcal{S})$-$\alpha_s$-admissible,
(ii) there is $\mathfrak{u}_0 \in \mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0, \mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$,
(iii) properties (a), (b) and (c) are satisfied,
(iv) $(\mathfrak{f}, \mathcal{S})$ and $(\mathfrak{g}, \mathcal{T})$ are weakly compatible,
(v) one of $\mathfrak{f}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is p_b-complete,
(vi) for any $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$ and $\mathrm{L} \geq 0$,

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})) \leq \xi(\hbar(M(\mathfrak{u}, \mathfrak{y})))\hbar(M(\mathfrak{u}, \mathfrak{y})) + \mathrm{L}\tau(N(\mathfrak{u}, \mathfrak{y})), \tag{3.1}$$

where

$$M(\mathfrak{u}, \mathfrak{y}) = \max\Big\{\partial_b(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}), \frac{\partial_b(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})}{2s}, \frac{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})\partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})}, \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}) + \partial_b(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}) + s\partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})}\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})\Big\},$$

and

$$N(\mathfrak{u}, \mathfrak{y}) = \min\left\{\partial_b^w(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b^w(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}), \partial_b^w(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}), \partial_b^w(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})\right\},$$

then $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S}$ have a unique common fixed point.

Proof. Let $\mathfrak{u}_0$ be an arbitrary point in $\mathcal{X}$ and meet condition (ii) , since $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, we can find $\mathfrak{u}_1 \in \mathcal{X}$ such that $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_1$, as $\mathfrak{g}\mathfrak{u}_1 \in \mathfrak{g}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, there exists $\mathfrak{u}_2 \in \mathcal{X}$ such that $\mathfrak{g}\mathfrak{u}_1 = \mathcal{S}\mathfrak{u}_2$. In general , $\{\mathfrak{u}_{2n+1}\} \subseteq \mathcal{X}$ is chosen such that $\mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{u}_{2n+1}$ and $\{\mathfrak{u}_{2n+2}\} \subseteq \mathcal{X}$ such that $\mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}$, we obtain a sequences $\{\mathfrak{y}_n\} \subseteq \mathcal{X}$ such that

$$\mathfrak{y}_{2n} = \mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{u}_{2n+1}, \mathfrak{y}_{2n+1} = \mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}.$$

According to conditions (i) and (ii), we get $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$.

Step 1. Suppose $\partial_b(\mathfrak{y}_{2m}, \mathfrak{y}_{2m+1}) = 0$ for some m, that is, $\mathfrak{y}_{2m} = \mathfrak{y}_{2m+1}$, then $\mathfrak{g}, \mathcal{T}$ have a coincidence point. From Lemma 3.4, we get $\alpha_s(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1}) = \alpha_s(\mathfrak{y}_{2m}, \mathfrak{y}_{2m+1}) \geq s$. If $\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}) > 0$, applying (3.1), we arrive at

$$\begin{aligned}\hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})) &= \hbar(\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1}))\\ &\leq \alpha_s(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1})\hbar(\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1}))\\ &\leq \alpha_s(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1}))\\ &\leq \xi(\hbar(M(\mathfrak{u}_{2m+2}, \mathfrak{u}_{2m+1})))\hbar(M(\mathfrak{u}_{2m+2}, \mathfrak{u}_{2m+1})) + \mathrm{L}\tau(N(\mathfrak{u}_{2m+2}, \mathfrak{u}_{2m+1})),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M(\mathfrak{u}_{2m+2}, \mathfrak{u}_{2m+1}) =& \max\{\partial_b(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{S}\mathfrak{u}_{2m+2}), \partial_b(\mathfrak{g}\mathfrak{u}_{2m+1}, \mathcal{T}\mathfrak{u}_{2m+1}),\\ &\frac{\partial_b(\mathcal{S}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1}) + \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1})}{2s}, \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{S}\mathfrak{u}_{2m+2})\partial_b(\mathfrak{g}\mathfrak{u}_{2m+1}, \mathcal{T}\mathfrak{u}_{2m+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1})},\\ &\frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1}) + \partial_b(\mathcal{S}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{S}\mathfrak{u}_{2m+2}) + s\partial_b(\mathfrak{g}\mathfrak{u}_{2m+1}, \mathcal{T}\mathfrak{u}_{2m+1})}\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{S}\mathfrak{u}_{2m+2})\}\\ =& \max\{\partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m}), \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m}), \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}), \partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m})\\ &\frac{\partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m+1}) + \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m})}{2s}, \frac{\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})\partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m})}{1 + \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})},\\ &\frac{1 + \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m}) + \partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m+1})}{1 + s\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}) + s\partial_b(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m})}\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})\}\\ =& \partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}),\end{aligned}$$

and

$$\begin{aligned}N(\mathfrak{u}_{2m+2}, \mathfrak{u}_{2m+1}) =& \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{S}\mathfrak{u}_{2m+2}), \partial_b^w(\mathfrak{g}\mathfrak{u}_{2m+1}, \mathcal{T}\mathfrak{u}_{2m+1}), \partial_b^w(\mathcal{S}\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1}), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{u}_{2m+1})\}\\ =& \min\{\partial_b^w(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}), \partial_b^w(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m}), \partial_b^w(\mathfrak{y}_{2m+1}, \mathfrak{y}_{2m+1}), \partial_b^w(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m})\}\\ =& 0.\end{aligned}$$

So, from (3.2) and $\xi(\varepsilon) < \zeta < 1$, we have

$$\begin{aligned}\hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})) &\leq \alpha_s(\mathfrak{y}_{2m}, \mathfrak{y}_{2m+1})\hbar(s^2\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}))\\ &\leq \xi(\hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})))\hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}))\\ &< \hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})),\end{aligned}$$

which is a contradiction, then $\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}) = 0$, that is, $\mathfrak{y}_{2m+2} = \mathfrak{y}_{2m+1}$. So, $\mathfrak{f}$ and $\mathcal{S}$ have a coincidence point. Similarly, when $\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}) = 0$, we get $\partial_b(\mathfrak{y}_{2m+3}, \mathfrak{y}_{2m+2}) = 0$, that is, $\mathfrak{y}_{2m+1} = \mathfrak{y}_{2m+2} = \mathfrak{y}_{2m+3}$. In this case, $\{\mathfrak{y}_n\}$ is a Cauchy sequence in X.

Step 2. Suppose $\partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) > 0$ for all $n \geq 0$. First, it is easy to show that $\alpha_s(\mathcal{S}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{u}_{2n+1}) = \alpha_s(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n}) \geq s$. From (3.1), we have

$$\begin{aligned}\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})) &= \hbar(\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \alpha_s(\mathcal{S}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{u}_{2n+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \xi(\hbar(M(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1})))\hbar(M(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1})) + \mathrm{L}\tau(N(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1}))\\ &\leq \zeta\hbar(M(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1})) + \mathrm{L}\tau(N(\mathfrak{u}_{2n}, \mathfrak{u}_{2n+1})),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}M(\mathfrak{u}_{2n},\mathfrak{u}_{2n+1}) =& \max\{\partial_b(\mathcal{S}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1}),\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1}),\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{S}\mathfrak{u}_{2n}),\partial_b(\mathfrak{g}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}),\\ & \frac{\partial_b(\mathcal{S}\mathfrak{u}_{2n},\mathfrak{g}\mathfrak{u}_{2n+1})+\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1})}{2s},\frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{S}\mathfrak{u}_{2n})\partial_b(\mathfrak{g}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1})}{1+\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathfrak{g}\mathfrak{u}_{2n+1})},\\ & \frac{1+\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1})+\partial_b(\mathcal{S}\mathfrak{u}_{2n},\mathfrak{g}\mathfrak{u}_{2n+1})}{1+s\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{S}\mathfrak{u}_{2n})+s\partial_b(\mathfrak{g}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1})}\partial_b(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{S}\mathfrak{u}_{2n})\}\\ =& \max\{\partial_b(\mathfrak{y}_{2n-1},\mathfrak{y}_{2n}),\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n}),\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1}),\partial_b(\mathfrak{y}_{2n+1},\mathfrak{y}_{2n}),\\ & \frac{\partial_b(\mathfrak{y}_{2n-1},\mathfrak{y}_{2n+1})+\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n})}{2s},\frac{\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})\partial_b(\mathfrak{y}_{2n+1},\mathfrak{y}_{2n})}{1+\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})},\\ & \frac{1+\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n})+\partial_b(\mathfrak{y}_{2n-1},\mathfrak{y}_{2n+1})}{1+s\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})+s\partial_b(\mathfrak{y}_{2n+1},\mathfrak{y}_{2n})}\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})\}\\ =& \max\{\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1}),\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})\},\end{aligned}$$

and

$$\begin{aligned}N(\mathfrak{u}_{2n},\mathfrak{u}_{2n+1}) =& \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{S}\mathfrak{u}_{2n}),\partial_b^w(\mathfrak{g}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}),\partial_b^w(\mathcal{S}\mathfrak{u}_{2n},\mathfrak{g}\mathfrak{u}_{2n+1}),\partial_b^w(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1})\}\\ =& \min\{\partial_b^w(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1}),\partial_b^w(\mathfrak{y}_{2n+1},\mathfrak{y}_{2n}),\partial_b^w(\mathfrak{y}_{2n-1},\mathfrak{y}_{2n+1}),\partial_b^w(\mathfrak{y}_{2n},\mathfrak{y}_{2n})\}\\ =& 0.\end{aligned}$$

Hence, $M(\mathfrak{u}_{2n},\mathfrak{u}_{2n+1}) = \max\{\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1}),\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})\}$. If $M(\mathfrak{u}_{2n},\mathfrak{u}_{2n+1}) = \partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})$, then by (3.3), one can deduce that

$$\hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})) \le \xi(\hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})))\hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})) < \hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})),$$

it gives a contradiction. It follows that $M(\mathfrak{u}_{2n},\mathfrak{u}_{2n+1}) = \partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})$, then by (3.3) again, we have

$$\hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})) \le \zeta\hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})) < \hbar(\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})). \tag{3.4}$$

Since $\hbar$ is non-decreasing, then $\partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1}) < \partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n-1})$. In the same way, we can also get $\partial_b(\mathfrak{y}_{2n+1},\mathfrak{y}_{2n+2}) < \partial_b(\mathfrak{y}_{2n},\mathfrak{y}_{2n+1})$. So, $\partial_b(\mathfrak{y}_n,\mathfrak{y}_{n+1}) < \partial_b(\mathfrak{y}_{n-1},\mathfrak{y}_n)$ for all n. Therefore, $\{\partial_b(\mathfrak{y}_n,\mathfrak{y}_{n+1})\}$ is a decreasing sequence, and $\lim_{n\to\infty}\partial_b(\mathfrak{y}_n,\mathfrak{y}_{n+1}) = \varrho \ge 0$. Let $\varrho > 0$. Passing to the limit in (3.4) as $n \to \infty$, we obtain that $\hbar(\varrho) \le \zeta\hbar(\varrho)$ and $\varrho = 0$ by the properties of function $\hbar \in \mathcal{H}$. Hence, $\lim_{n\to\infty}\partial_b(\mathfrak{y}_n,\mathfrak{y}_{n+1}) = 0$.

We next prove that $\{\mathfrak{y}_n\}$ is a Cauchy sequence in the partial b-metric space $(\mathcal{X},\partial_b)$. Suppose that is not the case. Using Lemma 2.10, there exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and the following four sequences

$$\partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k}),\partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k+1}),\partial_b(\mathfrak{y}_{2m_k-1},\mathfrak{y}_{2n_k}),\partial_b(\mathfrak{y}_{2m_k-1},\mathfrak{y}_{2n_k+1})$$

satisfy

$$\epsilon \le \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k}) \le \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k}) \le s\epsilon,$$

$$\frac{\epsilon}{s} \le \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k+1}) \le \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k},\mathfrak{y}_{2n_k+1}) \le s^2\epsilon,$$

$$\frac{\epsilon}{s} \leq \liminf_{k\to\infty} \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \leq \limsup_{k\to\infty} \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \leq s\epsilon,$$

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}) \leq \limsup_{k\to\infty} \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}) \leq s^2\epsilon.$$

Applying condition (3.1) to elements $\mathfrak{u} = \mathfrak{u}_{2m_k}$ and $\mathfrak{y} = \mathfrak{u}_{2n_k+1}$, we have

$$\begin{aligned}\alpha_s(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) &= \alpha_s(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k})\hbar(s^2\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1}))\\ &\leq \xi(\hbar(M(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})))\hbar(M(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})) + \mathrm{L}\tau(N(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})),\end{aligned}$$

where

$$\begin{aligned}M(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) = &\max\{\partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}), \partial_b(\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{u}_{2n_k+1}),\\ &\frac{\partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1})}{2s}, \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k})\partial_b(\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{u}_{2n_k+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})},\\ &\frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}) + s\partial_b(\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{u}_{2n_k+1})}\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k})\}\\ = &\max\{\partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}), \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k}),\\ &\frac{\partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}) + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k})}{2s}, \frac{\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1})\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k})}{1 + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1})},\\ &\frac{1 + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}) + \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1})}{1 + s\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}) + s\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k})}\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1})\},\end{aligned}$$

and

$$\begin{aligned}N(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1}) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}), \partial_b^w(\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{u}_{2n_k+1}), \partial_b^w(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1})\}\\ &= \min\{\partial_b^w(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}), \partial_b^w(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k}), \partial_b^w(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}), \partial_b^w(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k})\}.\end{aligned}$$

Taking the upper limit as $k \to \infty$, and according the properties of functions $\hbar \in \mathcal{H}$ and $\tau \in \mathfrak{I}$, we obtain

$$\hbar(s\epsilon) \leq \limsup_{k\to\infty} \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})),$$

$$\limsup_{k\to\infty} \hbar(M(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})) = \hbar(\limsup_{k\to\infty} M(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})) \leq \hbar(s\epsilon).$$

Since $\partial_b^w(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = \partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) - \min\{\partial_b(\mathfrak{y}_n, \mathfrak{y}_n), \partial_b(\mathfrak{y}_{n+1}, \mathfrak{y}_{n+1})\}$, then $\partial_b^w(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \to 0$ as $n \to \infty$, so

$$\limsup_{k\to\infty} \tau(N(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})) \leq \tau(\limsup_{k\to\infty} N(\mathfrak{u}_{2m_k}, \mathfrak{u}_{2n_k+1})) = 0.$$

In view of condition (i) and Lemma 3.4, we get

$$\alpha(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{u}_{2n_k+1}) = \alpha(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \geq s.$$

Thus, we obtain

$$s\hbar(s\epsilon) \leq \zeta\hbar(s\epsilon) < \hbar(s\epsilon),$$

which is a contradiction. This shows that $\{\mathfrak{y}_{2n}\}$ is a Cauchy sequence and hence $\{\mathfrak{y}_n\}$ is a Cauchy sequence both in $(\mathcal{X}, \partial_b)$ and in $(\mathcal{X}, \partial_b^s)$.

Step 3. Since $(\mathcal{X}, \partial_b)$ is p_b-complete if and only if $(\mathcal{X}, \partial_b^s)$ is b-complete. Therefore, there exists $\varpi \in \mathcal{X}$ such that $\lim_{n,m\to\infty} \partial_b^s(\mathfrak{y}_n, \mathfrak{y}_m) = 0$, or equivalently

$$\lim_{n,m\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_m) = \lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \varpi) = d(\varpi, \varpi).$$

Moreover, since $\partial_b^s(\mathfrak{y}_n, \mathfrak{y}_m) = 2\partial_b(\mathfrak{y}_n, \mathfrak{y}_m) - \partial_b(\mathfrak{y}_n, \mathfrak{y}_n) - \partial_b(\mathfrak{y}_m, \mathfrak{y}_m)$ and $\lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = 0$, we have $\lim_{n,m\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_m) = 0$. Then we obtain $\lim_{n,m\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_m) = \lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \varpi) = \partial_b(\varpi, \varpi) = 0$.
It follows from $\mathfrak{y}_n \to \varpi$ that $\{\mathfrak{f}\mathfrak{u}_{2n}\}$, $\{\mathcal{T}\mathfrak{u}_{2n+1}\}$, $\{\mathfrak{g}\mathfrak{u}_{2n+1}\}$, $\{\mathcal{S}\mathfrak{u}_{2n+2}\}$ converge to ϖ respectively.

Step 4. Assume that $\mathcal{T}(\mathcal{X})$ is p_b-complete. Hence, there exists $\mathfrak{p} \in \mathcal{X}$ such that $\varpi = \mathcal{T}\mathfrak{p}$. We will show that $\varpi = \mathfrak{g}\mathfrak{p}$. On the contrary, assume that $\partial_b(\varpi, \mathfrak{g}\mathfrak{p}) > 0$. First of all, from condition (a), we can choose a subsequence $\{\mathfrak{u}_{2n_i}\}$ of $\{\mathfrak{u}_{2n}\}$ such that $\{\mathcal{S}\mathfrak{u}_{2n_i}\} \to \varpi$ of $\{\mathcal{S}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathcal{S}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p}) = \alpha_s(\mathcal{S}\mathfrak{u}_{2n_i}, \varpi) \geq s$. From (3.1), we get

$$\begin{aligned}\alpha_s(\mathcal{S}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p})) &\leq \xi(\hbar(M(\mathfrak{u}_{2n_i}, \mathfrak{p})))\hbar(M(\mathfrak{u}_{2n_i}, \mathfrak{p})) + \mathrm{L}\tau(N(\mathfrak{u}_{2n_i}, \mathfrak{p})) \\ &< \zeta\hbar(M(\mathfrak{u}_{2n_i}, \mathfrak{p})) + \mathrm{L}\tau(N(\mathfrak{u}_{2n_i}, \mathfrak{p})),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}M(\mathfrak{u}_{2n_i}, \mathfrak{p}) &= \max\{\partial_b(\mathcal{S}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p}), \partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p}), \partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{S}\mathfrak{u}_{2n_i}), \partial_b(\mathfrak{g}\mathfrak{p}, \mathcal{T}\mathfrak{p}), \frac{\partial_b(\mathcal{S}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p}) + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p})}{2s}, \\ &\quad \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{S}\mathfrak{u}_{2n_i})\partial_b(\mathfrak{g}\mathfrak{p}, \mathcal{T}\mathfrak{p})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p})}, \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p}) + \partial_b(\mathcal{S}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{S}\mathfrak{u}_{2n_i}) + s\partial_b(\mathfrak{g}\mathfrak{p}, \mathcal{T}\mathfrak{p})}\partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{S}\mathfrak{u}_{2n_i})\} \\ &= \max\{\partial_b(\mathfrak{y}_{2n_i-1}, \varpi), \partial_b(\mathfrak{y}_{2n_i}, \varpi), \partial_b(\mathfrak{y}_{2n_i}, \mathfrak{y}_{2n_i-1}), \partial_b(\mathfrak{g}\mathfrak{p}, \varpi), \frac{\partial_b(\mathfrak{y}_{2n_i-1}, \mathfrak{g}\mathfrak{p}) + \partial_b(\mathfrak{y}_{2n_i}, \varpi)}{2s}, \\ &\quad \frac{\partial_b(\mathfrak{y}_{2n_i}, \mathfrak{y}_{2n_i-1})\partial_b(\mathfrak{g}\mathfrak{p}, \varpi)}{1 + \partial_b(\mathfrak{y}_{2n_i}, \mathfrak{g}\mathfrak{p})}, \frac{1 + \partial_b(\mathfrak{y}_{2n_i}, \varpi) + \partial_b(\mathfrak{y}_{2n_i-1}, \mathfrak{g}\mathfrak{p})}{1 + s\partial_b(\mathfrak{y}_{2n_i}, \mathfrak{y}_{2n_i-1}) + s\partial_b(\mathfrak{g}\mathfrak{p}, \varpi)}\partial_b(\mathfrak{y}_{2n_i}, \mathfrak{y}_{2n_i-1})\},\end{aligned}$$

and

$$\begin{aligned}N(\mathfrak{u}_{2n_i}, \mathfrak{p}) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{S}\mathfrak{u}_{2n_i}), \partial_b^w(\mathfrak{g}\mathfrak{p}, \mathcal{T}\mathfrak{p}), \partial_b^w(\mathcal{S}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p}), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathcal{T}\mathfrak{p})\} \\ &= \min\{\partial_b^w(\mathfrak{y}_{2n_i}, \mathfrak{y}_{2n_i-1}), \partial_b^w(\mathfrak{g}\mathfrak{p}, \varpi), \partial_b^w(\mathfrak{y}_{2n_i-1}, \mathfrak{g}\mathfrak{p}), \partial_b^w(\mathfrak{y}_{2n_i}, \varpi)\}.\end{aligned}$$

Letting $i \to \infty$, we obtain

$$\limsup_{i\to\infty} \hbar(M(\mathfrak{u}_{2n_i}, \mathfrak{p})) = \hbar(\limsup_{i\to\infty} M(\mathfrak{u}_{2n_i}, \mathfrak{p})) \leq \hbar(\partial_b(\varpi, \mathfrak{g}\mathfrak{p})),$$

$$\limsup_{i\to\infty} \tau(N(\mathfrak{u}_{2n_i}, \mathfrak{p})) \leq \tau(\limsup_{i\to\infty} N(\mathfrak{u}_{2n_i}, \mathfrak{p})) = 0,$$

$$\hbar(s\partial_b(\varpi, \mathfrak{g}\mathfrak{p})) \leq \limsup_{i\to\infty} \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathfrak{g}\mathfrak{p})).$$

Thus, we deduce

$$s\hbar(s\partial_b(\varpi, \mathfrak{g}\mathfrak{p})) \leq \zeta\hbar(\partial_b(\varpi, \mathfrak{g}\mathfrak{p})) < \hbar(\partial_b(\varpi, \mathfrak{g}\mathfrak{p})),$$

which is a contradiction, so $\varpi = \mathfrak{g}\mathfrak{p} = \mathcal{T}\mathfrak{p}$. Since $(\mathfrak{g}, \mathcal{T})$ is weakly compatible, $\mathfrak{g}\varpi = \mathfrak{g}\mathcal{T}\mathfrak{p} = \mathcal{T}\mathfrak{g}\mathfrak{p} = \mathcal{T}\varpi$.

Next, we will show that $\mathfrak{g}\varpi = \varpi$. Similarly, by condition (b), we can also get a subsequence $\{\mathfrak{u}_{2n_k}\}$ of $\{\mathfrak{u}_{2n}\}$ such that $\{\mathcal{S}\mathfrak{u}_{2n_k}\} \to \varpi$ of $\{\mathcal{S}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathcal{S}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi) \geq s$. In view of (3.1), we have

$$\begin{aligned}\alpha_s(\mathcal{S}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi)\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathfrak{g}\varpi)) &\leq \xi(\hbar(M(\mathfrak{u}_{2n_k}, \varpi)))\hbar(M(\mathfrak{u}_{2n_k}, \varpi)) + \mathrm{L}\tau(N(\mathfrak{u}_{2n_k}, \varpi)) \\ &< \zeta\hbar(M(\mathfrak{u}_{2n_k}, \varpi)) + \mathrm{L}\tau(N(\mathfrak{u}_{2n_k}, \varpi)),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}&M(\mathfrak{u}_{2n_k}, \varpi)\\ &= \max\Big\{\partial_b(\mathcal{S}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi), \partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi), \partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{u}_{2n_k}), \partial_b(\mathfrak{g}\varpi, \mathcal{T}\varpi), \frac{\partial_b(\mathcal{S}\mathfrak{u}_{2n_k}, \mathfrak{g}\varpi) + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi)}{2s},\\ &\quad \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{u}_{2n_k})\partial_b(\mathfrak{g}\varpi, \mathcal{T}\varpi)}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathfrak{g}\varpi)}, \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi) + \partial_b(\mathcal{S}\mathfrak{u}_{2n_k}, \mathfrak{g}\varpi)}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{u}_{2n_k}) + s\partial_b(\mathfrak{g}\varpi, \mathcal{T}\varpi)}\partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{u}_{2n_k})\Big\}\\ &= \max\Big\{\partial_b(\mathfrak{y}_{2n_k-1}, \mathfrak{g}\varpi), \partial_b(\mathfrak{y}_{2n_k}, \mathfrak{g}\varpi), \partial_b(\mathfrak{y}_{2n_k}, \mathfrak{y}_{2n_k-1}), \partial_b(\mathfrak{g}\varpi, \mathfrak{g}\varpi), \frac{\partial_b(\mathfrak{y}_{2n_k-1}, \mathfrak{g}\varpi) + \partial_b(\mathfrak{y}_{2n_k}, \mathfrak{g}\varpi)}{2s},\\ &\quad \frac{\partial_b(\mathfrak{y}_{2n_k}, \mathfrak{y}_{2n_k-1})\partial_b(\mathfrak{g}\varpi, \mathfrak{g}\varpi)}{1 + \partial_b(\mathfrak{y}_{2n_k}, \mathfrak{g}\varpi)}, \frac{1 + \partial_b(\mathfrak{y}_{2n_k}, \mathfrak{g}\varpi) + \partial_b(\mathfrak{y}_{2n_k-1}, \mathfrak{g}\varpi)}{1 + s\partial_b(\mathfrak{y}_{2n_k}, \mathfrak{y}_{2n_k-1}) + s\partial_b(\mathfrak{g}\varpi, \mathfrak{g}\varpi)}\partial_b(\mathfrak{y}_{2n_k}, \mathfrak{y}_{2n_k-1})\Big\},\end{aligned}$$

and

$$\begin{aligned}N(\mathfrak{u}_{2n_k}, \varpi) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{u}_{2n_k}), \partial_b^w(\mathfrak{g}\varpi, \mathcal{T}\varpi), \partial_b^w(\mathcal{S}\mathfrak{u}_{2n_k}, \mathfrak{g}\varpi), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{T}\varpi)\}\\ &= \min\{\partial_b^w(\mathfrak{y}_{2n_k}, \mathfrak{y}_{2n_k-1}), \partial_b^w(\mathfrak{g}\varpi, \mathfrak{g}\varpi), \partial_b^w(\mathfrak{y}_{2n_k-1}, \mathfrak{g}\varpi), \partial_b^w(\mathfrak{y}_{2n_k}, \mathfrak{g}\varpi)\}\\ &= 0.\end{aligned}$$

Taking the upper limit as $k \to \infty$, we have

$$\limsup_{k\to\infty}\hbar(M(\mathfrak{u}_{2n_k}, \varpi)) = \hbar(\limsup_{k\to\infty} M(\mathfrak{u}_{2n_k}, \varpi)) \leq \hbar(s\partial_b(\varpi, \mathfrak{g}\varpi)),$$

$$\hbar(s\partial_b(\varpi, \mathfrak{g}\varpi)) \leq \liminf_{k\to\infty}\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n_k}, \varpi)).$$

Thus, we obtain

$$s\hbar(s\partial_b(\varpi, \mathfrak{g}\varpi)) \leq \zeta\hbar(s\partial_b(\varpi, \mathfrak{g}\varpi)) < \hbar(s\partial_b(\varpi, \mathfrak{g}\varpi)),$$

so $\varpi = \mathfrak{g}\varpi = \mathcal{T}\varpi$. Since $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$ and $\mathfrak{g}\varpi = \varpi$, there exists $\mathfrak{q} \in \mathcal{X}$ such that $\varpi = \mathcal{S}\mathfrak{q}$.

Similarly, we show that ϖ is also fixed point of $\mathfrak{f}$ and $\mathcal{S}$. Hence, $\mathfrak{f}\varpi = \mathfrak{g}\varpi = \mathcal{T}\varpi = \mathcal{S}\varpi = \varpi$. The proofs for the cases in which $\mathcal{S}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X})$, or $\mathfrak{f}(\mathcal{X})$ is p_b-complete are similar.

Step 5. We prove that the uniqueness of common fixed point. On the contrary, there is another common fixed point ς of $\mathfrak{f}$, $\mathfrak{g}$, $\mathcal{S}$ and $\mathcal{T}$. By condition (c), we can also get $\alpha_s(\varpi, \varsigma) = \alpha_s(\varsigma, \varpi) \geq s$. Then,

$$\alpha_s(\varsigma, \varpi)\varphi(s^2\partial_b(\mathfrak{f}\varsigma, \mathfrak{g}\varpi)) \leq \xi(\hbar(M(\varsigma, \varpi)))\hbar(M(\varsigma, \varpi)) + \mathrm{L}\tau(N(\varsigma, \varpi)),$$

where $M(\varsigma, \varpi) = \partial_b(\varsigma, \varpi)$ and $N(\varsigma, \varpi) = 0$. Therefore, we obtain

$$s\hbar(s^2\partial_b(\varsigma, \varpi)) \leq \xi(\hbar(M(\varsigma, \varpi)))\hbar(M(\varsigma, \varpi)) < \hbar(\partial_b(\varsigma, \varpi)),$$

a contradiction, so $\varpi = \varsigma$. □

Example 3.6. Let $\mathcal{X} = [0, +\infty)$, $\partial_b(\mathfrak{u}, \mathfrak{y}) = \max\{\mathfrak{u}^2, \mathfrak{y}^2\} + (\mathfrak{u} - \mathfrak{y})^2$, $\partial_b^w(\mathfrak{u}, \mathfrak{y}) = \max\{\mathfrak{u}^2, \mathfrak{y}^2\} - \min\{\mathfrak{u}^2, \mathfrak{y}^2\} + (\mathfrak{u} - \mathfrak{y})^2 = |\mathfrak{u}^2 - \mathfrak{y}^2| + (\mathfrak{u} - \mathfrak{y})^2$, $\mathrm{L} = 2$ and $s = 2$. Define mappings $\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T}$ by

$$\mathfrak{f}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{64}, & \mathfrak{u} \in [0, 1] \\ e^{\mathfrak{u}} - e + \frac{1}{2}, & \mathfrak{u} > 1 \end{cases}, \mathcal{T}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{3}, & \mathfrak{u} \in [0, 1] \\ e^{2\mathfrak{u}} - e^2 + \frac{1}{2}, & \mathfrak{u} > 1 \end{cases},$$

$$\mathfrak{g}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{16}, & \mathfrak{u} \in [0, 1] \\ \frac{1}{4}, & \mathfrak{u} > 1 \end{cases}, \mathcal{S}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{2}, & \mathfrak{u} \in [0, 1] \\ \frac{7\mathfrak{u}}{4}, & \mathfrak{u} > 1 \end{cases}.$$

Since $\mathfrak{f}\mathcal{S}0 = \mathcal{S}\mathfrak{f}0 = 0$ and $\mathfrak{g}\mathcal{T}0 = \mathcal{T}\mathfrak{g}0 = 0$, $(\mathfrak{f}, \mathcal{S})$ and $(\mathfrak{g}, \mathcal{T})$ are weakly compatible.

Define mappings $\alpha_s : \mathcal{S}(\mathcal{X}) \times \mathcal{T}(\mathcal{X}) \to [0, +\infty)$ and $\alpha_s(\mathfrak{u}, \mathfrak{y}) = \alpha_s(\mathfrak{y}, \mathfrak{u})$ by

$$\alpha_s(\mathfrak{u}, \mathfrak{y}) = \begin{cases} s, & \mathfrak{u}, \mathfrak{y} \in [0, \frac{1}{2}] \\ 0, & \text{otherwise} \end{cases},$$

and

$$\hbar(\kappa) = \kappa, \tau(\kappa) = \begin{cases} 0, & \kappa = 0 \\ 128\kappa + 1, & \kappa \in (0, +\infty) \end{cases},$$

$$\xi(\iota) = \begin{cases} \frac{81}{224} + \frac{15}{112}\iota, & \iota \in [0, 1] \\ 0, & \iota > 1 \end{cases}.$$

It is clear that $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X})$ is p_b-complete. For $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$ such that $\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y}) \geq s$, we know that $\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y} \in [0, \frac{1}{2}]$, and which implies that $\mathfrak{u}, \mathfrak{y} \in [0, 1]$. It follows that $\alpha_s(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) \geq s$ and $(\mathfrak{f}, \mathfrak{g})$ is mixed $(\mathcal{S}, \mathcal{T})$-$\alpha_s$-admissible.

For $\mathfrak{u}, \mathfrak{y} \in [0, 1]$ and $\mathfrak{u} = \mathfrak{y}$, we get

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})) = 2\hbar(4d(\frac{\mathfrak{u}}{64}, \frac{\mathfrak{u}}{16})) = 8(\max\{\frac{\mathfrak{u}^2}{64^2}, \frac{\mathfrak{u}^2}{16^2}\} + (\frac{\mathfrak{u}}{64} - \frac{\mathfrak{u}}{16})^2) = \frac{200}{4096}\mathfrak{u}^2,$$

and

$$\xi(\hbar(M(\mathfrak{u}, \mathfrak{u})))\hbar(M(\mathfrak{u}, \mathfrak{u})) \geq \xi(\hbar(\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})))\hbar(\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})) \geq \frac{265}{4096}\mathfrak{u}^2.$$

So, $\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{u})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{u})) \leq \zeta(\hbar(M(\mathfrak{u}, \mathfrak{u})))\hbar(M(\mathfrak{u}, \mathfrak{u})) + \mathrm{L}\tau(N(\mathfrak{u}, \mathfrak{u}))$.

If $\mathfrak{u} \neq \mathfrak{y}$, we have

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})) = 2\hbar(4d(\frac{\mathfrak{u}}{64}, \frac{\mathfrak{y}}{16})) = 8(\max\{\frac{\mathfrak{u}^2}{64^2}, \frac{\mathfrak{y}^2}{16^2}\} + (\frac{\mathfrak{u}}{64} - \frac{\mathfrak{y}}{16})^2) \leq \frac{3}{32}\max\{\mathfrak{u}^2, \mathfrak{y}^2\},$$

$$\hbar(M(\mathfrak{u}, \mathfrak{y})) \geq \hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})\}) = \max\{\frac{1985}{4096}\mathfrak{u}^2, \frac{3825}{20736}\mathfrak{y}^2\} \geq \frac{7}{81}\max\{\mathfrak{u}^2, \mathfrak{y}^2\}.$$

It follows that

$$\xi(\hbar(M(\mathfrak{u}, \mathfrak{y})))\hbar(M(\mathfrak{u}, \mathfrak{y})) \geq \xi(\hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})\}))\hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})\})$$

$$\geq \frac{1}{32}\max\{\mathfrak{u}^2, \mathfrak{y}^2\},$$

and

$$\begin{aligned}
N(\mathfrak{u},\mathfrak{y}) &= \min\{\partial_b^w(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}), \partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}), \partial_b^w(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y}), \partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})\}\\
&= \min\{\frac{416}{2304}\mathfrak{y}^2, \frac{1984}{4096}\mathfrak{u}^2, |\frac{\mathfrak{u}^2}{4}-\frac{\mathfrak{y}^2}{256}|+(\frac{\mathfrak{u}}{2}-\frac{\mathfrak{y}}{16})^2, |\frac{\mathfrak{u}^2}{4096}-\frac{\mathfrak{y}^2}{9}|+(\frac{\mathfrak{u}}{64}-\frac{\mathfrak{y}}{3})^2\}\\
&\geq \min\{\frac{416}{2304}\mathfrak{y}^2, \frac{1984}{4096}\mathfrak{u}^2, |\frac{\mathfrak{u}^2}{4}-\frac{\mathfrak{y}^2}{256}|, |\frac{\mathfrak{u}^2}{4096}-\frac{\mathfrak{y}^2}{9}|\}\\
&\geq \min\{\frac{416}{2304}\mathfrak{y}^2, \frac{1984}{4096}\mathfrak{u}^2, \frac{1}{256}|\mathfrak{u}^2-\mathfrak{y}^2|, \frac{1}{4096}|\mathfrak{u}^2-\mathfrak{y}^2|\}\\
&= \frac{1}{4096}|\mathfrak{u}^2-\mathfrak{y}^2| = \frac{1}{4096}(\max\{\mathfrak{u}^2,\mathfrak{y}^2\}-\min\{\mathfrak{u}^2,\mathfrak{y}^2\}).
\end{aligned}$$

So,

$$\mathrm{L}\tau(N(\mathfrak{u},\mathfrak{y})) \geq 2\tau(\frac{1}{4096}(\max\{\mathfrak{u}^2,\mathfrak{y}^2\}-\min\{\mathfrak{u}^2,\mathfrak{y}^2\})) \geq 2\frac{1}{32}\max\{\mathfrak{u}^2,\mathfrak{y}^2\} = \frac{1}{16}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}.$$

Thus,

$$\begin{aligned}
\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y})) &\leq \frac{3}{32}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}\\
&= \frac{1}{32}\max\{\mathfrak{u}^2,\mathfrak{y}^2\} + \frac{1}{16}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}\\
&\leq \zeta(\hbar(M(\mathfrak{u},\mathfrak{y})))\hbar(M(\mathfrak{u},\mathfrak{y})) + \mathrm{L}\tau(N(\mathfrak{u},\mathfrak{y})).
\end{aligned}$$

It follows that all conditions of Theorem 3.5 are satisfied. It is obvious that 0 is the unique common fixed point of $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}$.

If $s = 1$, $\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y}) = 1$, for all $\mathfrak{u},\mathfrak{y}\in\mathcal{X}$, $\xi(\iota) = \varrho(0<\varrho<1)$, and $\hbar(\kappa)=\kappa, \tau(\kappa)=\kappa$, we get a corollary as follows.

Corollary 3.7. Let $(\mathcal{X}, d_p)$ be a partial metric space and $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be given mappings with $\mathfrak{f}(\mathcal{X})\subseteq\mathcal{T}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X})\subseteq\mathcal{S}(\mathcal{X})$. If the following conditions are satisfied:

(i) $(\mathfrak{f},\mathfrak{g})$ and $(\mathcal{Q},\mathcal{T})$ are weakly compatible,

(ii) one of $\mathfrak{f}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is p-complete,

(iii) for all $\mathfrak{u},\mathfrak{y}\in\mathcal{X}$, $\mathrm{L}\geq 0$, such that

$$d_p(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y}) \leq \varrho M'(\mathfrak{u},\mathfrak{y}) + \mathrm{L}N'(\mathfrak{u},\mathfrak{y}),$$

where

$$\begin{aligned}
M'(\mathfrak{u},\mathfrak{y}) = \max\{&d_p(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y}), d_p(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y}), d_p(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}), d_p(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}), \frac{d_p(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y})+d_p(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})}{2},\\
&\frac{d_p(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})d_p(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})}{1+d_p(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y})}, \frac{1+d_p(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})+d_p(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y})}{1+d_p(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})+d_p(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})}d_p(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})\},
\end{aligned}$$

and

$$N'(\mathfrak{u},\mathfrak{y}) = \min\left\{d_p^w(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}), d_p^w(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}), d_p^w(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y}), d_p^w(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})\right\},$$

$$d_p^w(\mathfrak{u},\mathfrak{y}) = d_p(\mathfrak{u},\mathfrak{y}) - \min\{d_p(\mathfrak{u},\mathfrak{u}), d_p(\mathfrak{y},\mathfrak{y})\},$$

then $\mathfrak{f},\mathfrak{g},\mathcal{T},\mathcal{S}$ have a unique common fixed point.

If $\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y}) = 1$, $\xi(\iota) = 1-\varrho (0<\varrho<1)$, $\theta(\kappa) = \varrho\hbar(\kappa)$, $\tau(\kappa) = \kappa$, then we get the following corollary.

Corollary 3.8. Let $(\mathcal{X},\partial_b)$ be a p_b-complete partial b-metric space with parameter $s \geq 1$ and $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}: X \to X$ be given mappings with $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. If the following conditions are satisfied:

(i) $(\mathfrak{f},\mathcal{T})$ and $(\mathfrak{g},\mathcal{S})$ are weakly compatible,

(ii) for all $\mathfrak{u},\mathfrak{y} \in \mathcal{X}$, $\mathrm{L} \geq 0$, we have

$$\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y})) \leq \hbar(M(\mathfrak{u},\mathfrak{y})) - \theta(M(\mathfrak{u},\mathfrak{y})) + \mathrm{L}N(\mathfrak{u},\mathfrak{y}),$$

where $M(\mathfrak{u},\mathfrak{y})$ and $N(\mathfrak{u},\mathfrak{y})$ are same as in Theorem 3.5, then $\mathfrak{f},\mathfrak{g},\mathcal{T},\mathcal{S}$ have a unique common fixed point.

If $\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y}) = s, \mathrm{L} = 0, \xi(\iota) = \varrho(0<\varrho<1)$, $k = \frac{\varrho}{s}(0<k<1)$ and $\partial_b(\mathfrak{u},\mathfrak{u}) = 0$, then a partial b-metric space $(\mathcal{X},\partial_b)$ is a b-metric space. One can get

Corollary 3.9. Let $(\mathcal{X},d_b)$ be a b-complete b-metric space with parameter $s \geq 1$ and $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}: X \to X$ be given mappings with $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. If the following conditions are satisfied:

(i) $(\mathfrak{f},\mathcal{T})$ and $(\mathfrak{g},\mathcal{S})$ are weakly compatible,

(ii) one of $\mathfrak{f}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is b-complete,

(iii) for all $\mathfrak{u},\mathfrak{y} \in \mathcal{X}$, such that

$$\hbar(s^2 d_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y})) \leq k\hbar(M^*(\mathfrak{u},\mathfrak{y})),$$

where

$$\begin{aligned}M^*(\mathfrak{u},\mathfrak{y}) = \max\{&d_b(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y}), d_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y}), d_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}), d_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}), \frac{d_b(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y}) + d_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})}{2s},\\ &\frac{d_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})d_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})}{1+d_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{y})}, \frac{1+d_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})+d_b(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y})}{1+sd_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})+sd_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})}d_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u})\},\end{aligned}$$

then $\mathfrak{f},\mathfrak{g},\mathcal{T},\mathcal{S}$ have a unique common fixed point.

Remark 3.10. The corollaries obtained by restricting the conditions of Theorem 3.5 are extensions of the theorems in the literature.

(1) Corollary 3.7 covers Theorem 2.1 in [22];

(2) Corollary 3.8 is an extension of Theorem 4 in [28] on partial b-metric spaces.

If $s = 1$ and $\partial_b(\mathfrak{u},\mathfrak{u}) = 0$, for all $\mathfrak{u} \in \mathcal{X}$, then we get the following corollary.

Corollary 3.11. Let $(\mathcal{X},d)$ be a complete metric space and let $\alpha_s: \mathcal{X}\times\mathcal{X} \to [0,+\infty)$, $\mathfrak{f},\mathfrak{g},\mathcal{T},\mathcal{S}: \mathcal{X} \to \mathcal{X}$ be given mappings and $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ and $\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. Suppose $\hbar \in \mathcal{H}$ and $\tau \in \mathfrak{I}$. If the following conditions are satisfied:

(i) $(\mathfrak{f},\mathfrak{g})$ is mixed $(\mathcal{T},\mathcal{S})$-$\alpha_s$-admissible,

(ii) there is $\mathfrak{u}_0 \in \mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0,\mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{u}_1,\mathcal{S}\mathfrak{u}_2) \geq 1$,

(iii) properties (a), (b) and (c) are satisfied,

(iv) $(\mathfrak{f},\mathcal{S})$ and $(\mathfrak{g},\mathcal{T})$ are weakly compatible,

(v) one of $\mathfrak{f}(\mathcal{X})$, $\mathfrak{g}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is p_b-complete,

(vi) for any $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$ and $\mathrm{L} \geq 0$,

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})) \leq \xi(\hbar(M_*(\mathfrak{u}, \mathfrak{y})))\hbar(M_*(\mathfrak{u}, \mathfrak{y})) + \mathrm{L}\tau(N_*(\mathfrak{u}, \mathfrak{y})),$$

where

$$M_*(\mathfrak{u}, \mathfrak{y}) = \max\{d(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y}), d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}), \frac{d(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})}{2}, \frac{d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})}{1 + d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y})}, \frac{1 + d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}) + d(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y})}{1 + d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}) + d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})}d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})\},$$

and

$$N_*(\mathfrak{u}, \mathfrak{y}) = \min\{d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}), d(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}), d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})\},$$

then $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S}$ have a unique common fixed point.

Definition 3.12. Let $(\mathcal{X}, \partial_b)$ be a partial b-metric space with parameter $s \geq 1$, and let $\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T} : X \to X$ and $\alpha_s : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ be given mappings. The mapping $\mathfrak{f}$ is said to be interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha_s$-admissible if, for all $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$,

(1) $\alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{y}, \mathcal{S}\mathfrak{u}) \geq s$ implies $\alpha_s(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{g}\mathfrak{y}) \geq s$,

(2) $\alpha_s(\mathfrak{u}, \mathfrak{y}) = \alpha_s(\mathfrak{y}, \mathfrak{u})$,

(3) $\alpha_s(\mathfrak{u}, \mathfrak{y}) \geq s$ and $\alpha_s(\mathfrak{y}, \mathfrak{z}) \geq s$ imply $\alpha_s(\mathfrak{u}, \mathfrak{z}) \geq s$.

Remark 3.13. For $\mathfrak{g} = I_X$ and $\mathcal{S} = \mathcal{T}$, the Definition 3.12 reduces to the definition of interspersed $\mathcal{T}$-α_s-admissible mapping in a partial b-metric space.

Let $(\mathcal{X}, \partial_b)$ be a p_b-complete partial b-metric space with parameter $s \geq 1$ and let $\alpha_s : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$. Then,

(a') If $\{\mathfrak{f}\mathfrak{u}_{2n}\}$ is a sequence in $\mathcal{X}$ such that $\mathfrak{f}\mathfrak{u}_{2n} \to \mathfrak{w}$ as $n \to \infty$, then there exists a subsequence $\{\mathfrak{f}\mathfrak{u}_{2n_i}\}$ of $\{\mathfrak{f}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_{2n_i}, \mathfrak{w}) \geq s$ for all $i \in \mathbb{N}$.

(b') If $\{\mathfrak{f}\mathfrak{u}_{2n}\}$ is a sequence in $\mathcal{X}$ such that $\mathfrak{f}\mathfrak{u}_{2n} \to \mathfrak{w}$ as $n \to \infty$, then there exists a subsequence $\{\mathfrak{f}\mathfrak{u}_{2n_k}\}$ of $\{\mathfrak{f}\mathfrak{u}_{2n}\}$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{w}) \geq s$ for all $k \in \mathbb{N}$.

(c') For all $\mathfrak{w}, \mathfrak{v} \in C(\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T})$, we have the condition of $\alpha_s(\mathfrak{w}, \mathfrak{v}) \geq s$.

Lemma 3.14. Let $(\mathcal{X}, \partial_b)$ be a partial b-metric space with parameter $s \geq 1$ and $\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T}$ be four self-mappings such that $\mathfrak{f}$ is interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha_s$-admissible. Assume that there exists $\mathfrak{u}_0 \in \mathcal{X}$ satisfying $\mathfrak{y}_0 = \mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{g}\mathfrak{u}_1$, $\mathfrak{y}_1 = \mathfrak{f}\mathfrak{g}\mathfrak{u}_1 = \mathcal{S}\mathfrak{u}_2$ such that $\alpha_s(\mathfrak{y}_0, \mathfrak{y}_1) = \alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$. Define two sequences $\{\mathfrak{u}_n\}$, $\{\mathfrak{y}_n\}$ in $\mathcal{X}$ by $\mathfrak{y}_{2n} = \mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}$ and $\mathfrak{y}_{2n+1} = \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}$, where $n = 1, 2, 3, \cdots$. Then for $n, m \in \mathbb{N} \cup \{0\}$, $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$.

Proof. Similar to the proof of Lemma 3.4, we get the following results:

$$\alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) = \alpha_s(\mathfrak{y}_0, \mathfrak{y}_1) \geq s$$

implies

$$\alpha_s(\mathfrak{f}\mathfrak{u}_2, \mathfrak{f}\mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_2) = \alpha_s(\mathfrak{y}_1, \mathfrak{y}_2) \geq s,$$
$$\alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_2) \geq s$$

implies

$$\alpha_s(\mathfrak{f}\mathfrak{u}_2, \mathfrak{f}\mathfrak{g}\mathfrak{u}_3) = \alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_3, \mathcal{S}\mathfrak{u}_4) = \alpha_s(\mathfrak{y}_2, \mathfrak{y}_3) \geq s.$$

Applying the above argument repeatedly, we obtain $\alpha_n(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \geq s$, and $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$ for all $n, m \in \mathbb{N} \cup \{0\}$. □

Theorem 3.15. Let $(\mathcal{X}, \partial_b)$ be a p_b-complete partial b-metric space with parameter $s \geq 1$ and let $\alpha_s : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$, $\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T}$ be given self-mappings and $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}\mathfrak{g}(\mathcal{X})$ and $\mathfrak{f}\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. Suppose $\hbar \in \mathcal{H}$ and $\tau \in \mathfrak{I}$. If the following conditions are satisfied:

(i) $\mathfrak{f}$ is interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha_s$-admissible,
(ii) there is $\mathfrak{u}_0 \in \mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{g}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0, \mathfrak{f}\mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$,
(iii) properties (a'), (b') and (c') are satisfied,
(iv) $\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{u}) \leq \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{u})$, for all $\mathfrak{u} \in \mathfrak{g}(\mathcal{X})$,
(v) $(\mathfrak{f}, \mathcal{S})$, $(\mathfrak{g}, \mathfrak{f})$, $(\mathfrak{f}\mathfrak{g}, \mathcal{T}\mathfrak{g})$ are weakly compatible,
(vi) one of $\mathfrak{f}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is p_b-complete,
(vii) for any $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$ and $\mathrm{L} \geq 0$,

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) \leq \xi(\hbar(M_1(\mathfrak{u}, \mathfrak{y})))\hbar(M_1(\mathfrak{u}, \mathfrak{y})) + \mathrm{L}\tau(N_1(\mathfrak{u}, \mathfrak{y})), \tag{3.7}$$

where

$$\begin{aligned} M_1(\mathfrak{u}, \mathfrak{y}) = \max\{&\partial_b(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \frac{\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{S}\mathfrak{u})}{6s}, \\ &\frac{\partial_b(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{S}\mathfrak{u})}{6s}, \frac{\partial_b(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})}{4s^2}, \\ &\frac{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})\partial_b(\mathfrak{f}\mathfrak{y}, \mathfrak{g}\mathfrak{y})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}, \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}) + \partial_b(\mathcal{S}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}) + s\partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y})}\partial_b(\mathfrak{f}\mathfrak{y}, \mathfrak{g}\mathfrak{y})\}, \end{aligned}$$

and

$$N_1(x, y) = \min\left\{\partial_b^w(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \partial_b^w(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b^w(\mathcal{S}\mathfrak{u}, \mathfrak{f}\mathfrak{y}), \partial_b^w(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})\right\},$$

then $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S}$ have a unique common fixed point.

Proof. Let $\mathfrak{u}_0$ be an arbitrary point in $\mathcal{X}$ and meet condition (ii), since $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}\mathfrak{g}(\mathcal{X})$, we can find $\mathfrak{u}_1 \in \mathcal{X}$ such that $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{g}\mathfrak{u}_1$, at the same time, since $\mathfrak{f}\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$, there exists $\mathfrak{u}_2 \in \mathcal{X}$ such that $\mathfrak{f}\mathfrak{g}\mathfrak{u}_1 = \mathcal{S}\mathfrak{u}_2$. In general, $\{\mathfrak{u}_{2n+1}\} \subseteq \mathcal{X}$ is chosen such that $\mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}$ and $\{\mathfrak{u}_{2n+2}\} \subseteq \mathcal{X}$ such that $\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}$. Define a sequence $\{\mathfrak{y}_n\} \subseteq \mathcal{X}$ such that

$$\mathfrak{y}_{2n} = \mathfrak{f}\mathfrak{u}_{2n} = \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathfrak{y}_{2n+1} = \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1} = \mathcal{S}\mathfrak{u}_{2n+2}.$$

According to condition (i) and (ii), we get $\alpha_s(\mathfrak{y}_n, \mathfrak{y}_m) \geq s$.

Step 1. Suppose $\partial_b(\mathfrak{y}_{2m}, \mathfrak{y}_{2m+1}) = 0$ for some m. Furthermore,

$$\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1}) \leq \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}) = 0 \Rightarrow \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1} = \mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1} = \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1}.$$

Thus, $\mathfrak{f}, \mathfrak{g}, \mathcal{T}$ have a coincidence point.

From Lemma 3.14, we have

$$\alpha_s(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}) = \alpha_s(\mathfrak{y}_{2m}, \mathfrak{y}_{2m+1}) \geq s.$$

If $\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1}) > 0$, applying (3.7), we get

$$\begin{aligned} \hbar(\partial_b(\mathfrak{y}_{2m+2}, \mathfrak{y}_{2m+1})) &= \hbar(\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1})) \\ &\leq \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1})) \\ &\leq \alpha_s(\mathcal{S}\mathfrak{u}_{2m+2}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1})) \\ &\leq \xi(\hbar(M_1(\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1})))\hbar(M_1(\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1})) + \mathrm{L}\tau(N_1(\mathfrak{u}_{2m+2}, \mathfrak{g}\mathfrak{u}_{2m+1})), \end{aligned} \tag{3.8}$$

where

$$
\begin{aligned}
M_1(\mathfrak{u}_{2m+2},\mathfrak{g}\mathfrak{u}_{2m+1}) &= \max\{\partial_b(\mathcal{S}\mathfrak{u}_{2m+2},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}),\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}),\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}),\\
&\qquad \partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{S}\mathfrak{u}_{2m+2}),\frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{S}\mathfrak{u}_{2m+2})}{6s},\\
&\qquad \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{S}\mathfrak{u}_{2m+2})}{6s},\\
&\qquad \frac{\partial_b(\mathcal{S}\mathfrak{u}_{2m+2},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})}{4s^2},\\
&\qquad \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{S}\mathfrak{u}_{2m+2})\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})}{1+\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1})},\\
&\qquad \frac{1+\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathcal{S}\mathfrak{u}_{2m+2},\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1})}{1+s\partial_b(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{S}\mathfrak{u}_{2m+2})+s\partial_b(fgx_{2m+1},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})}\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})\}\\
&= \max\{\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m}),\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m}),\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m}),\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m}),\\
&\qquad \frac{\partial_b(\mathfrak{y}_{2m+2},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m+1})}{6s},\frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1},\mathfrak{y}_{2m})+\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m+1})}{6s},\\
&\qquad \frac{\partial_b(\mathfrak{y}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})+\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m})}{4s^2},\frac{\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})\partial_b(\mathfrak{y}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})}{1+\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})},\\
&\qquad \frac{1+\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m})+\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m+1})}{1+s\partial_b d(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})+s\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m})}\partial_b(\mathfrak{y}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})\}\\
&\le \max\{\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}),\partial_b(\mathfrak{y}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})\},
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
N_1(\mathfrak{u}_{2m+2},\mathfrak{g}\mathfrak{u}_{2m+1}) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{S}\mathfrak{u}_{2m+2}),\partial_b^w(\mathfrak{f}\mathfrak{u}_{2m+2},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1}),\partial_b^w(\mathcal{S}\mathfrak{u}_{2m+2},\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1}),\\
&\qquad \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})\}\\
&= \min\{\partial_b^w(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}),\partial_b^w(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m}),\partial_b^w(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m+1}),\partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1},\mathfrak{y}_{2m})\}\\
&=0.
\end{aligned}
$$

By condition (iv), we obtain

$$
\partial_b(\mathfrak{y}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})=\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2m+1})\le\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2m+1},\mathcal{T}\mathfrak{g}\mathfrak{u}_{2m+1})=\partial_b(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m}),
$$

so,

$$
M_1(\mathfrak{u}_{2m+2},\mathfrak{g}\mathfrak{u}_{2m+1})=\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}).
$$

Thus, from (3.8) and $\xi(\varepsilon)<\zeta<1$, we access

$$
\begin{aligned}
\hbar(\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})) &\le \alpha_s(\mathfrak{y}_{2m+1},\mathfrak{y}_{2m})\hbar(s^2\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}))\\
&\le \xi(\hbar(\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})))\hbar(\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}))\\
&< \zeta\hbar(\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1}))<\hbar(\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})),
\end{aligned}
$$

which is a contradiction, then $\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})=0$ and which implies $\mathfrak{y}_{2m+2}=\mathfrak{y}_{2m+1}$. Therefore, $\mathfrak{f}$ and $\mathcal{S}$ have a coincidence point. Similarly, when $\partial_b(\mathfrak{y}_{2m+2},\mathfrak{y}_{2m+1})=0$, we can also obtain $\partial_b(\mathfrak{y}_{2m+3},\mathfrak{y}_{2m+2})=0$, that is, $\mathfrak{y}_{2m+2}=\mathfrak{y}_{2m+1}$ implies $\mathfrak{y}_{2m+3}=\mathfrak{y}_{2m+2}$. Hence, $\{\mathfrak{y}_n\}$ is a Cauchy sequence in $\mathcal{X}$.

Step 2. Suppose $\partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) > 0$ for all $n \geq 0$. It is obvious that $\alpha_s(\mathcal{S}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}) = \alpha_s(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n}) \geq s$. It follows from (3.7) that

$$\begin{aligned}\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})) &= \hbar(\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \alpha_s(\mathcal{S}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}))\\ &\leq \xi(\hbar(M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1})))\hbar(M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1})) + \mathrm{L}\tau(N_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1})).\end{aligned} \tag{3.10}$$

In view of condition (iv), we have

$$\begin{aligned}M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}) &= \max\Big\{\partial_b(\mathcal{S}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}), \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{S}\mathfrak{u}_{2n}),\\ &\qquad \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{S}\mathfrak{u}_{2n})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{S}\mathfrak{u}_{2n})}{6s},\\ &\qquad \frac{\partial_b(\mathcal{S}\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1})}{4s^2}, \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{S}\mathfrak{u}_{2n})\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1})},\\ &\qquad \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathcal{S}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{S}\mathfrak{u}_{2n}) + s\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1})}\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1})\Big\}\\ &= \max\Big\{\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n}), \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n}), \partial_b(\mathfrak{y}_{2n+1}, \mathfrak{y}_{2n}), \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n-1}),\\ &\qquad \frac{\partial_b(\mathfrak{y}_{2n}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathfrak{y}_{2n+1}, \mathfrak{y}_{2n-1})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathfrak{y}_{2n}) + \partial_b(\mathfrak{y}_{2n+1}, \mathfrak{y}_{2n-1})}{6s},\\ &\qquad \frac{\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}) + \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n})}{4s^2}, \frac{\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n-1})\partial_b(\mathfrak{y}_{2n+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1})}{1 + \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})},\\ &\qquad \frac{1 + \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n}) + \partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n+1})}{1 + s\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n-1}) + s\partial_b(\mathfrak{y}_{2n+1}, \mathfrak{y}_{2n})}\partial_b(\mathfrak{y}_{2n+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1})\Big\}\\ &= \max\{\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n}), \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})\},\end{aligned}$$

and

$$\begin{aligned}N_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{S}\mathfrak{u}_{2n}), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2n}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}), \partial_b^w(\mathcal{S}\mathfrak{u}_{2n}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}),\\ &\qquad \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}\}\\ &= \min\{\partial_b^w(\mathfrak{y}_{2n}, \mathfrak{y}_{2n-1}), \partial_b^w(\mathfrak{y}_{2n}, \mathfrak{y}_{2n}), \partial_b^w(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n+1}), \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n+1}, \mathfrak{y}_{2n})\}\\ &=0.\end{aligned}$$

Hence, $M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}) = \max\{\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n}), \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})\}$. If $M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}) = \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})$, then by (3.10) and the property of the function $\xi(\varepsilon)$, we obtain

$$\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})) \leq \xi(\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})))\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})) < \hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})),$$

which is a contradiction. Therefore, $M_1(\mathfrak{u}_{2n}, \mathfrak{g}\mathfrak{u}_{2n+1}) = \partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n})$. Utilizing (3.10), we get

$$\hbar(\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})) \leq \zeta\hbar(\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n})) < \hbar(\partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n})). \tag{3.11}$$

The monotonicity of $\hbar$ ensures that $\partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1}) < \partial_b(\mathfrak{y}_{2n-1}, \mathfrak{y}_{2n})$. Similarly, one can deduce that $\partial_b(\mathfrak{y}_{2n+1}, \mathfrak{y}_{2n+2}) < \partial_b(\mathfrak{y}_{2n}, \mathfrak{y}_{2n+1})$. In summary, $\partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) < \partial_b(\mathfrak{y}_{n-1}, \mathfrak{y}_n)$, for all $n \in \mathbb{N}$. Therefore, $\{\partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1})\}$ is a decreasing sequence, and $\lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = \omega \geq 0$.

If $\omega > 0$, letting $n \to \infty$ in (3.11), we obtain that $\hbar(\omega) \leq \zeta\hbar(\omega)$, which implies $\omega = 0$. It is impossible. Hence, $\lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = 0$.

We next prove that $\{\mathfrak{y}_n\}$ is a Cauchy sequence in the partial b-metric space $(\mathcal{X}, \partial_b)$. Suppose that is not the case. Then using Lemma 2.10, one can get that exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and the following four sequences

$$\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1}), \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1})$$

satisfy

$$\begin{aligned}
\epsilon \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}) \leq s\epsilon,\\
\frac{\epsilon}{s} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1}) \leq s^2\epsilon,\\
\frac{\epsilon}{s} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \leq s\epsilon,\\
\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}) \leq \limsup_{k\to\infty}\ \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}) \leq s^2\epsilon.
\end{aligned}$$

Setting $\mathfrak{u} = \mathfrak{u}_{2m_k}$ and $\mathfrak{y} = \mathfrak{g}\mathfrak{u}_{2n_k+1}$ in (3.7), we have

$$\begin{aligned}
&\alpha_s(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}))\\
&= \alpha_s(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}))\\
&\leq \xi(\hbar(M_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})))\hbar(M_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) + \mathrm{L}\tau(N_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
&M_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})\\
&= \max\{\partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}),\\
&\frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{S}\mathfrak{u}_{2m_k})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{S}\mathfrak{u}_{2m_k})}{6s},\\
&\frac{\partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1})}{4s^2}, \frac{\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k})\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1})},\\
&\frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}) + s\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1})}\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1})\}\\
&= \max\{\partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}),\\
&\frac{\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2m_k-1})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathfrak{y}_{2n_k}) + \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2m_k-1})}{6s},\\
&\frac{\partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}) + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k})}{4s^2}, \frac{\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1})\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1})}{1 + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1})},\\
&\frac{1 + \partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}) + \partial_b(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1})}{1 + s\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}) + s\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k})}\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1})\},
\end{aligned}$$

and

$$\begin{aligned}
&N_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})\\
&= \min\{\partial_b^w(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{S}\mathfrak{u}_{2m_k}), \partial_b^w(\mathfrak{f}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b^w(\mathcal{S}\mathfrak{u}_{2m_k}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}), \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1})\}\\
&= \min\{\partial_b^w(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2m_k-1}), \partial_b^w(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k}), \partial_b^w(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k+1}), \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_k+1}, \mathfrak{y}_{2n_k})\}.
\end{aligned}$$

Taking the upper limit as $k \to \infty$ in (3.12), one can obtain

$$\hbar(s\epsilon) \leq \limsup_{k\to\infty} \hbar(s^2\partial_b(\mathfrak{y}_{2m_k}, \mathfrak{y}_{2n_k+1})),$$

$$\begin{aligned}\limsup_{k\to\infty} \hbar(M_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) &= \hbar(\limsup_{k\to\infty} M_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1}))\\ &\leq \hbar(\max\{s\epsilon, s\epsilon, 0, 0, \frac{s\epsilon}{3}, \frac{s\epsilon}{6}, \frac{s^2\epsilon+\epsilon}{4s}, 0, 0\})\\ &= \hbar(s\epsilon).\end{aligned}$$

Since $\partial_b^w(\mathfrak{y}_n, \mathfrak{y}_{n+1}) = \partial_b(\mathfrak{y}_n, \mathfrak{y}_{n+1}) - \min\{\partial_b(\mathfrak{y}_n, \mathfrak{y}_n), \partial_b(\mathfrak{y}_{n+1}, \mathfrak{y}_{n+1})\}$, we have $\partial_b^w(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \to 0$ as $n \to \infty$. It follows that

$$\limsup_{k\to\infty} \tau(N_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) \leq \tau(\limsup_{k\to\infty} N_1(\mathfrak{u}_{2m_k}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) = 0.$$

In light of condition (i) and Lemma 3.14, we arrive at

$$\alpha(\mathcal{S}\mathfrak{u}_{2m_k}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}) = \alpha(\mathfrak{y}_{2m_k-1}, \mathfrak{y}_{2n_k}) \geq s.$$

Thus,

$$s\hbar(s\epsilon) \leq \zeta\hbar(s\epsilon) < \hbar(s\epsilon),$$

which is a contradiction. It follows that $\{\mathfrak{y}_{2n}\}$ is a Cauchy sequence and which implies $\{\mathfrak{y}_n\}$ is a Cauchy sequence both in $(\mathcal{X}, \partial_b)$ and in $(\mathcal{X}, \partial_b^s)$.

Similar to Step 3 of Theorem 3.5, there exists $\mathfrak{w} \in \mathcal{X}$ such that $\lim_{n,m\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{y}_m) = \lim_{n\to\infty} \partial_b(\mathfrak{y}_n, \mathfrak{w}) = \partial_b(\mathfrak{w}, \mathfrak{w}) = 0$. Since $\mathfrak{y}_n \to \mathfrak{w}$, then the sequences $\{\mathfrak{f}\mathfrak{u}_{2n}\}$, $\{\mathcal{T}\mathfrak{g}\mathfrak{u}_{2n+1}\}$, $\{\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n+1}\}$, $\{\mathcal{S}\mathfrak{u}_{2n+2}\}$ converge to $\mathfrak{w}$.

Step 3. Suppose that $\mathcal{S}(\mathcal{X})$ is p_b−complete. It follows that there exists $\tilde{\mathfrak{p}} \in X$ satisfying $\mathfrak{w} = \mathcal{S}\tilde{\mathfrak{p}}$. Now we shall prove that $\mathfrak{w} = \mathfrak{f}\tilde{\mathfrak{p}}$. If it is not true, then assume that $\partial_b(\mathfrak{w}, \mathfrak{f}\tilde{\mathfrak{p}}) > 0$. At first, in view of condition (a'), one can obtain a subsequence $\{\mathfrak{u}_{2n_i}\}$ of $\{\mathfrak{u}_{2n}\}$ such that $\{\mathfrak{f}\mathfrak{u}_{2n_i}\} \to \mathfrak{w}$ with $\alpha_s(\mathcal{S}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}) = \alpha_s(\mathfrak{w}, \mathfrak{f}\mathfrak{u}_{2n_i}) \geq s$. Applying condition (3.7) to elements $\mathfrak{u} = \tilde{\mathfrak{p}}$ and $\mathfrak{y} = \mathfrak{g}\mathfrak{u}_{2n_i+1}$, we have

$$\begin{aligned}&\alpha_s(\mathcal{S}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1})\hbar(s^2\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}))\\ &= \alpha_s(\mathfrak{w}, \mathfrak{f}\mathfrak{u}_{2n_i})\hbar(s^2\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i+1}))\\ &\leq \xi(\hbar(M_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1})))\hbar(M_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1})) + \mathrm{L}\tau(N_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1})),\end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
M_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1}) = \max\{&\partial_b(\mathcal{S}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}), \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}), \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}), \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{S}\tilde{\mathfrak{p}}),\\
&\frac{\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{S}\tilde{\mathfrak{p}})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{S}\tilde{\mathfrak{p}})}{6s},\\
&\frac{\partial_b(\mathcal{S}\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1})}{4s^2}, \frac{\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{S}\tilde{\mathfrak{p}})\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1})}{1 + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1})},\\
&\frac{1 + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathcal{S}\tilde{\mathfrak{p}}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1})}{1 + s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{S}\tilde{\mathfrak{p}}) + s\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1})}\partial_b(\mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1})\}\\
= \max\{&\partial_b(\mathfrak{w}, \mathfrak{y}_{2n_i}), \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i}), \partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{y}_{2n_i}), \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}),\\
&\frac{\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{w})}{6s}, \frac{\partial_b(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathfrak{y}_{2n_i}) + \partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{w})}{6s},\\
&\frac{\partial_b(\mathfrak{w}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}) + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i})}{4s^2}, \frac{\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})\partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1})}{1 + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i+1})},\\
&\frac{1 + \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i}) + \partial_b(\mathfrak{w}, \mathfrak{y}_{2n_i+1})}{1 + s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}) + s\partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{y}_{2n_i})}\partial_b(\mathfrak{y}_{2n_i+1}, \mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1})\},
\end{aligned}
$$

and

$$
\begin{aligned}
N_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1}) &= \min\{\partial_b^w(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{S}\tilde{\mathfrak{p}}), \partial_b^w(\mathfrak{f}\tilde{\mathfrak{p}}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}), \partial_b^w(\mathcal{S}\tilde{\mathfrak{p}}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_i+1}), \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_i+1}\}\\
&= \min\{\partial_b^w(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}), \partial_b^w(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i}), \partial_b^w(\mathfrak{w}, \mathfrak{y}_{2n_i+1}), \partial_b^w(\mathfrak{g}\mathfrak{g}\mathfrak{u}_{2n_i+1}, \mathfrak{y}_{2n_k})\}.
\end{aligned}
$$

Taking the upper limit as $i \to \infty$ and by condition (iv), we have

$$\hbar(s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})) \leq \limsup_{i\to\infty} \hbar(s^2\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{y}_{2n_i+1})),$$

$$
\begin{aligned}
\limsup_{i\to\infty} \hbar(M_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1}) &= \hbar(\limsup_{i\to\infty} M_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1}))\\
&\leq \hbar(\max\{0, s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}), 0, \partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}), \frac{s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})}{6}, 0, \frac{\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})}{4s}, 0, 0\})\\
&\leq \hbar(s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})),
\end{aligned}
$$

$$\limsup_{i\to\infty} \tau(N_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1})) \leq \tau(\limsup_{i\to\infty} N_1(\tilde{\mathfrak{p}}, \mathfrak{g}\mathfrak{u}_{2n_i+1})) = 0.$$

It follows that

$$s\hbar(s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})) \leq \zeta\hbar(s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})) < \hbar(s\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w})),$$

a contradiction. Thus, $\partial_b(\mathfrak{f}\tilde{\mathfrak{p}}, \mathfrak{w}) = 0$ and which implies $\mathfrak{f}\tilde{\mathfrak{p}} = \mathfrak{w} = \mathcal{S}\tilde{\mathfrak{p}}$. Since $(\mathfrak{f}, \mathcal{S})$ is weakly compatible, $\mathfrak{f}\mathfrak{w} = \mathfrak{f}\mathcal{S}\tilde{\mathfrak{p}} = \mathcal{S}\mathfrak{f}\tilde{\mathfrak{p}} = \mathcal{S}\mathfrak{w}$.

Next, we will prove that $\mathfrak{f}\mathfrak{w} = \mathfrak{w}$. On the contrary, assume that $d(\mathfrak{w}, \mathfrak{f}\mathfrak{w}) > 0$. First, in light of condition (b'), there exists a subsequence $\{\mathfrak{u}_{2n_k}\}$ of $\{\mathfrak{u}_{2n}\}$ such that $\{\mathfrak{f}\mathfrak{u}_{2n_k}\} \to \mathfrak{w}$ with $\alpha_s(\mathcal{S}\mathfrak{w}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1}) = \alpha_s(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{w}) \geq s$. Applying condition (3.7) to elements $\mathfrak{u} = \mathfrak{w}$ and $\mathfrak{y} = \mathfrak{g}\mathfrak{u}_{2n_k+1}$, we obtain

$$
\begin{aligned}
&\alpha_s(\mathcal{S}\mathfrak{w}, \mathcal{T}\mathfrak{g}\mathfrak{u}_{2n_k+1})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{g}\mathfrak{u}_{2n_k+1}))\\
&= \alpha_s(\mathfrak{f}\mathfrak{u}_{2n_k}, \mathcal{S}\mathfrak{w})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k+1}))\\
&\leq \xi(\hbar(M_1(\mathfrak{w}, \mathfrak{g}\mathfrak{u}_{2n_k+1})))\hbar(M_1(\mathfrak{w}, \mathfrak{g}\mathfrak{u}_{2n_k+1})) + \mathrm{L}\tau(N_1(\mathfrak{w}, \mathfrak{g}\mathfrak{u}_{2n_k+1})),
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
M_1(\mathfrak{w}, \mathfrak{gu}_{2n_k+1}) &= \max\{\partial_b(\mathcal{S}\mathfrak{w}, \mathcal{T}\mathfrak{gu}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{T}\mathfrak{gu}_{2n_k+1}), \partial_b(\mathfrak{fgu}_{2n_k+1}, \mathcal{T}\mathfrak{gu}_{2n_k+1}), \partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{S}\mathfrak{w}),\\
&\quad \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{ggu}_{2n_k+1}) + \partial_b(\mathfrak{fgu}_{2n_k+1}, \mathcal{S}\mathfrak{w})}{6s}, \frac{\partial_b(\mathfrak{ggu}_{2n_k+1}, \mathcal{T}\mathfrak{gu}_{2n_k+1}) + \partial_b(\mathfrak{fgu}_{2n_k+1}, \mathcal{S}\mathfrak{w})}{6s},\\
&\quad \frac{\partial_b(\mathcal{S}\mathfrak{w}, \mathfrak{ggu}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{T}\mathfrak{gu}_{2n_k+1})}{4s^2}, \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{S}\mathfrak{w})\partial_b(\mathfrak{fgu}_{2n_k+1}, \mathfrak{ggu}_{2n_k+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{fgu}_{2n_k+1})},\\
&\quad \frac{1 + \partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{T}\mathfrak{gu}_{2n_k+1}) + \partial_b(\mathcal{S}\mathfrak{w}, \mathfrak{fgu}_{2n_k+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{w}, \mathcal{S}\mathfrak{w}) + s\partial_b(\mathfrak{fgu}_{2n_k+1}, \mathcal{T}\mathfrak{gu}_{2n_k+1})}\partial_b(\mathfrak{fgu}_{2n_k+1}, \mathfrak{ggu}_{2n_k+1})\}\\
&= \max\{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k}), \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{w}),\\
&\quad \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{ggu}_{2n_k+1}) + \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{f}\mathfrak{w})}{6s}, \frac{\partial_b(\mathfrak{ggu}_{2n_k+1}, \mathfrak{y}_{2n_k}) + \partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{f}\mathfrak{w})}{6s},\\
&\quad \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{ggu}_{2n_k+1}) + \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k})}{4s^2}, \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{w})\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{ggu}_{2n_k+1})}{1 + \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k+1})},\\
&\quad \frac{1 + \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k}) + \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k+1})}{1 + s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{w}) + s\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{y}_{2n_k})}\partial_b(\mathfrak{y}_{2n_k+1}, \mathfrak{ggu}_{2n_k+1})\},
\end{aligned}
$$

and

$$
\begin{aligned}
N_1(\mathfrak{w}, \mathfrak{gu}_{2n_k+1}) &= \min\{\partial_b^w(\mathfrak{f}\mathfrak{w}, \mathcal{S}\mathfrak{w}), \partial_b^w(\mathfrak{f}\mathfrak{w}, \mathcal{T}\mathfrak{gu}_{2n_k+1}), \partial_b^w(\mathcal{S}\mathfrak{w}, \mathfrak{fgu}_{2n_k+1}), \partial_b^w(\mathfrak{ggu}_{2n_k+1}, \mathcal{T}\mathfrak{gu}_{2n_k+1})\}\\
&= \min\{\partial_b^w(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{w}), \partial_b^w(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k}), \partial_b^w(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k+1}), \partial_b^w(\mathfrak{ggu}_{2n_k+1}, \mathfrak{y}_{2n_k})\}\\
&=0.
\end{aligned}
$$

Taking the upper limit as $k \to \infty$ and by condition (iv), we get

$$\hbar(s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})) \leq \limsup_{k\to\infty} \hbar(s^2\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{y}_{2n_k+1})),$$

$$
\begin{aligned}
\limsup_{k\to\infty} \hbar(M_1(\mathfrak{w}, \mathfrak{gu}_{2n_k+1})) &= \hbar(\limsup_{k\to\infty} M_1(\mathfrak{w}, \mathfrak{gu}_{2n_k+1}))\\
&\leq \hbar(\max\{s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w}), s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w}), 0, \partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{f}\mathfrak{w}), \frac{(s+1)\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})}{6}, \frac{\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})}{6},\\
&\quad \frac{(s+1)\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})}{4s}, 0, 0\}\\
&\leq \hbar(s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})).
\end{aligned}
$$

It follows that

$$s\hbar(s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})) \leq \zeta\hbar(s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})) < \hbar(s\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w})),$$

which is impossible. Hence, $\partial_b(\mathfrak{f}\mathfrak{w}, \mathfrak{w}) = 0$, which implies $\mathfrak{f}\mathfrak{w} = \mathfrak{w} = \mathcal{S}\mathfrak{w}$.

Since $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}\mathfrak{g}(\mathcal{X})$, there exists $\mathfrak{q} \in \mathcal{X}$ such that $\mathcal{T}\mathfrak{g}\mathfrak{q} = \mathfrak{w}$. Similarly, we show that $\mathfrak{w}$ is also fixed point of $\mathfrak{fg}$ and $\mathcal{T}\mathfrak{g}$. After that, we deduce $\mathfrak{fg}\mathfrak{w} = \mathcal{T}\mathfrak{g}\mathfrak{w} = \mathfrak{w}$, and by condition (iv), one can learn that $\partial_b(\mathfrak{fg}\mathfrak{w}, \mathfrak{gg}\mathfrak{w}) \leq \partial_b(\mathfrak{fg}\mathfrak{w}, \mathcal{T}\mathfrak{g}\mathfrak{w}) = \partial_b(\mathfrak{w}, \mathfrak{w}) = 0 \implies \mathfrak{gg}\mathfrak{w} = \mathfrak{fg}\mathfrak{w} = \mathcal{T}\mathfrak{g}\mathfrak{w} = \mathfrak{w}$. Since $(\mathfrak{g}, \mathfrak{f})$ is weakly compatible, we get $\mathfrak{g}\mathfrak{w} = \mathfrak{gfg}\mathfrak{w} = \mathfrak{fgg}\mathfrak{w} = \mathfrak{f}\mathfrak{w}$, $\mathcal{T}\mathfrak{g}\mathfrak{w} = \mathcal{T}\mathfrak{w} = \mathfrak{f}\mathfrak{w} = \mathfrak{fg}\mathfrak{w}$ and which imply $\mathfrak{g}\mathfrak{w} = \mathfrak{f}\mathfrak{w} = \mathcal{T}\mathfrak{w} = \mathcal{S}\mathfrak{w} = \mathfrak{w}$.

The proof for the case in which $\mathfrak{f}(\mathcal{X})$ is p_b-complete are similar.

Step 4. Suppose on the contrary that there is another common fixed point $\mathfrak{v}$ of $\mathfrak{f}$, $\mathfrak{g}$, $\mathcal{S}$ and $\mathcal{T}$. According to condition (c'), one can obtain $\alpha_s(\mathfrak{v},\mathfrak{w}) = \alpha_s(\mathfrak{w},\mathfrak{v}) \geq s$. It follows that

$$\alpha_s(\mathfrak{v},\mathfrak{w})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{v},\mathfrak{g}\mathfrak{w})) \leq \xi(\hbar(M_1(\mathfrak{v},\mathfrak{w})))\hbar(M_1(\mathfrak{v},\mathfrak{w})) + \mathrm{L}\tau(N_1(\mathfrak{v},\mathfrak{w})),$$

where $M_1(\mathfrak{v},\mathfrak{w}) = \partial_b(\mathfrak{v},\mathfrak{w})$ and $N_1(\mathfrak{v},\mathfrak{w}) = 0$. Thus,

$$s\hbar(s^2\partial_b(\mathfrak{v},\mathfrak{w})) \leq \xi(\hbar(M_1(\mathfrak{v},\mathfrak{w})))\hbar(M_1(\mathfrak{v},\mathfrak{w})) < \hbar(\partial_b(\mathfrak{v},\mathfrak{w})),$$

which is a contradiction. That is, $\mathfrak{w} = \mathfrak{v}$.

In conclusion, $\mathfrak{f},\mathfrak{g},\mathcal{T},\mathcal{S}$ have a unique common fixed point. □

Example 3.16. Let $\mathcal{X} = [0,2]$, $\partial_b(\mathfrak{u},\mathfrak{y}) = \max\{\mathfrak{u}^2,\mathfrak{y}^2\} + (\mathfrak{u}-\mathfrak{y})^2$, $\partial_b^w(\mathfrak{u},\mathfrak{y}) = \max\{\mathfrak{u}^2,\mathfrak{y}^2\} - \min\{\mathfrak{u}^2,\mathfrak{y}^2\} + (\mathfrak{u}-\mathfrak{y})^2 = |\mathfrak{u}^2-\mathfrak{y}^2| + (\mathfrak{u}-\mathfrak{y})^2$, $\mathrm{L} = 2$ and $s = 2$. Define mappings $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}$ by

$$\mathfrak{f}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{6}, & \mathfrak{u} \in [0,1] \\ \frac{1}{4}, & \mathfrak{u} \in (1,2] \end{cases}, \mathcal{T}\mathfrak{u} = \begin{cases} \frac{3\mathfrak{u}}{2}, & \mathfrak{u} \in [0,1] \\ \frac{9}{5}, & \mathfrak{u} \in (1,2] \end{cases},$$

$$\mathfrak{g}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{2}, & \mathfrak{u} \in [0,1] \\ \frac{3}{2}, & \mathfrak{u} \in (1,2] \end{cases}, \mathcal{S}\mathfrak{u} = \begin{cases} \frac{3\mathfrak{u}}{4}, & \mathfrak{u} \in [0,1] \\ \frac{3}{16}, & \mathfrak{u} \in (1,2] \end{cases},$$

and since $\mathfrak{g}(\mathcal{X}) = [0,\frac{1}{2}] \cup \{\frac{3}{2}\}$, we get

$$\mathcal{T}\mathfrak{g}\mathfrak{u} = \begin{cases} \frac{3\mathfrak{u}}{4}, & \mathfrak{u} \in [0,1] \\ \frac{9}{5}, & \mathfrak{u} \in (1,2] \end{cases}, \mathfrak{f}\mathfrak{g}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{12}, & \mathfrak{u} \in [0,1] \\ \frac{1}{4}, & \mathfrak{u} \in (1,2] \end{cases}.$$

Since $\mathfrak{f}\mathcal{S}0 = \mathcal{S}\mathfrak{f}0 = 0$, $\mathfrak{f}\mathfrak{g}0 = \mathfrak{g}\mathfrak{f}0 = 0$, $\mathfrak{f}\mathfrak{g}\mathcal{T}\mathfrak{g}0 = \mathcal{T}\mathfrak{g}\mathfrak{f}\mathfrak{g}0 = 0$, $(\mathfrak{f},\mathcal{S})$, $(\mathfrak{g},\mathfrak{f})$ and $(\mathfrak{f}\mathfrak{g},\mathcal{T}\mathfrak{g})$ are weakly compatible. Moreover, when $\mathfrak{u} \in \mathfrak{g}(\mathcal{X})$, we have

Case 1. $\mathfrak{u} \in [0,\frac{1}{2}]$. It follows that

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{u}) = \max\{\frac{\mathfrak{u}^2}{36},\frac{\mathfrak{u}^2}{4}\} + (\frac{\mathfrak{u}}{2}-\frac{\mathfrak{u}}{6})^2 = \frac{\mathfrak{u}^2}{4} + \frac{\mathfrak{u}^2}{9} = \frac{13}{36}\mathfrak{u}^2,$$

and

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}) = \max\{\frac{\mathfrak{u}^2}{36},\frac{9}{4}\mathfrak{u}^2\} + (\frac{3\mathfrak{u}}{2}-\frac{\mathfrak{u}}{6})^2 = \frac{9\mathfrak{u}^2}{4} + \frac{16\mathfrak{u}^2}{9} = \frac{145}{36}\mathfrak{u}^2.$$

Thus,

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{u}) = \frac{13}{36}\mathfrak{u}^2 \leq \frac{145}{36}\mathfrak{u}^2 = \partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}).$$

Case 2. $\mathfrak{u} = \frac{3}{2}$. It is easy to show that

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{u}) = \max\{\frac{1}{16},\frac{9}{4}\} + (\frac{1}{4}-\frac{3}{2})^2 = \frac{61}{16},$$

and

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}) = \max\{\frac{1}{16},\frac{81}{25}\} + (\frac{1}{4}-\frac{9}{5})^2 = \frac{2257}{400}.$$

So,

$$\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{u})=\frac{61}{16}\leq\frac{2257}{400}=\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}).$$

Consequently, we have $\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{g}\mathfrak{u})\leq\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u})$.

Define mappings $\alpha_s:\mathcal{S}(\mathcal{X})\times\mathcal{T}(\mathcal{X})\to[0,+\infty)$ and $\alpha_s(\mathfrak{u},\mathfrak{y})=\alpha_s(\mathfrak{y},\mathfrak{u})$ by

$$\alpha_s(\mathfrak{u},\mathfrak{y})=\begin{cases}s, & \mathfrak{u},\mathfrak{y}\in[0,\frac{3}{4}]\\ 0, & \text{otherwise}\end{cases},$$

and

$$\hbar(\kappa)=\kappa,\tau(\kappa)=\begin{cases}0, & \kappa=0\\ 6t+1, & \kappa\in(0,+\infty)\end{cases},\ \beta(\iota)=\begin{cases}\frac{24}{65}+\frac{23}{50}\iota, & \iota\in[0,1]\\ 0, & \iota>1\end{cases}.$$

It is easy to show that $\mathfrak{f}(\mathcal{X})\subseteq\mathcal{T}\mathfrak{g}(\mathcal{X})$, $\mathfrak{f}\mathfrak{g}(\mathcal{X})\subseteq\mathcal{S}(\mathcal{X})$ and $\mathfrak{f}(\mathcal{X})$ is p_b-complete. For $\mathfrak{u},\mathfrak{y}\in\mathcal{X}$ such that $\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{g}\mathfrak{y})\geq s$, one can deduce that $\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{g}\mathfrak{y}\in[0,\frac{3}{4}]$ and which implies that $\mathfrak{u},\mathfrak{y}\in[0,1]$. Therefore, $\mathfrak{f}$ is mixed $(\mathcal{S},\mathfrak{g},\mathcal{T})$-$\alpha_s$-admissible.

For $\mathfrak{u},\mathfrak{y}\in[0,1]$ and $\mathfrak{u}=\mathfrak{y}$, we have

$$\alpha_s(\mathcal{S}\mathfrak{y},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}))=2\hbar(4\partial_b(\frac{y}{6},\frac{y}{6}))=\frac{2}{9}\mathfrak{y}^2,$$

and

$$\xi(\hbar(M_1(\mathfrak{y},\mathfrak{y})))\hbar(M_1(\mathfrak{y},\mathfrak{y}))\geq\xi(\hbar(\partial_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})))\hbar(\partial_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}))\geq\frac{11}{6}\mathfrak{y}^2.$$

So, $\alpha_s(\mathcal{S}\mathfrak{y},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}))y\leq\xi(\hbar(M_1(\mathfrak{y},\mathfrak{y})))\hbar(M_1(\mathfrak{y},\mathfrak{y}))+\mathrm{L}\tau(N_1(\mathfrak{y},\mathfrak{y}))$. If $\mathfrak{u}\neq\mathfrak{y}$, we obtain

$$\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y}))=2\hbar(4\partial_b(\frac{\mathfrak{u}}{6},\frac{\mathfrak{y}}{6}))=8(\max\{\frac{\mathfrak{u}^2}{36},\frac{\mathfrak{y}^2}{36}\}+(\frac{\mathfrak{u}}{6}-\frac{\mathfrak{y}}{6})^2)\leq\frac{2}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\},$$

$$\hbar(M_1(\mathfrak{u},\mathfrak{y}))\geq\hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}),\partial_b(\mathfrak{f}\mathfrak{y},\mathcal{T}\mathfrak{y})\})=\max\{\frac{65}{72}\mathfrak{u}^2,\frac{145}{36}\mathfrak{y}^2\}\geq\frac{65}{72}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}.$$

So,

$$\begin{aligned}\xi(\hbar(M_1(\mathfrak{u},\mathfrak{y})))\hbar(M_1(\mathfrak{u},\mathfrak{y}))&\geq\xi(\hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}),\partial_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})\}))\hbar(\max\{\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}),\partial_b(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y})\})\\&\geq\frac{1}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\},\end{aligned}$$

$$\begin{aligned}N_1(\mathfrak{u},\mathfrak{y})&=\min\{\partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{S}\mathfrak{u}),\partial_b^w(\mathfrak{g}\mathfrak{y},\mathcal{T}\mathfrak{y}),\partial_b^w(\mathcal{S}\mathfrak{u},\mathfrak{g}\mathfrak{y}),\partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})\}\\&=\min\{\frac{7}{8}\mathfrak{u}^2,3\mathfrak{y}^2,|\frac{9\mathfrak{u}^2}{16}-\frac{\mathfrak{y}^2}{4}|+(\frac{3\mathfrak{u}}{4}-\frac{\mathfrak{y}}{2})^2,|\frac{\mathfrak{u}^2}{36}-\frac{9\mathfrak{y}^2}{4}|+(\frac{\mathfrak{u}}{6}-\frac{3\mathfrak{y}}{2})^2\}\\&\geq\frac{1}{36}|\mathfrak{u}^2-\mathfrak{y}^2|=\frac{1}{36}(\max\{\mathfrak{u}^2,\mathfrak{y}^2\}-\min\{\mathfrak{u}^2,\mathfrak{y}^2\}),\end{aligned}$$

$$\mathrm{L}\tau(N_1(\mathfrak{u},\mathfrak{y}))\geq2\tau(\frac{1}{36}(\max\{\mathfrak{u}^2,\mathfrak{y}^2\}-\min\{\mathfrak{u}^2,\mathfrak{y}^2\}))\geq2\frac{1}{6}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}=\frac{1}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}.$$

It follows that

$$\begin{aligned}\alpha_s(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y})) &\le \frac{2}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}\\ &= \frac{1}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}+\frac{1}{3}\max\{\mathfrak{u}^2,\mathfrak{y}^2\}\\ &\le \xi(\hbar(M_1(\mathfrak{u},\mathfrak{y})))\hbar(M_1(\mathfrak{u},\mathfrak{y}))+\mathrm{L}\tau(N_1(\mathfrak{u},\mathfrak{y})).\end{aligned}$$

Hence, all conditions of Theorem 3.15 are satisfied. It is obvious that 0 is the unique common fixed point of $\mathfrak{f},\mathfrak{g},\mathcal{S},\mathcal{T}$.

Remark 3.17. (1) For Example 3.6, it is known through calculation that it does not satisfy the condition (iii) of Theorem 3.15;

(2) For Example 3.16, we can get $\mathfrak{g}(\mathcal{X})\not\subseteq\mathcal{S}(\mathcal{X})$, it is known through calculation that it does not satisfy the conditions of Theorem 3.5.

That is to say, the conditions for Theorems 3.5 and 3.15 are independent of each other.

If $\mathfrak{g}=I_{\mathcal{X}}$ and $\mathcal{S}=\mathcal{T}$ in Theorem 3.15, then we obtain that

Corollary 3.18. Let $(\mathcal{X},\partial_b)$ be a p_b-complete partial b-metric space with parameter $s\ge 1$ and let $\alpha_s:\mathcal{X}\times\mathcal{X}\to[0,+\infty)$, $\mathfrak{f},\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be given mappings and $\mathfrak{f}(\mathcal{X})\subseteq\mathcal{T}(\mathcal{X})$. Suppose $\hbar\in\mathcal{H}$. If the following conditions are satisfied:

(i) $\mathfrak{f}$ is interspersed $\mathcal{T}$-α_s-admissible mapping,

(ii) there is $\mathfrak{u}_0\in\mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0=\mathcal{T}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_1)=\alpha_s(\mathcal{T}\mathfrak{u}_1,\mathcal{T}\mathfrak{u}_2)\ge s$,

(iii) properties (a'), (b') and (c') are satisfied when $\mathfrak{g}=I_{\mathcal{X}}$ and $\mathcal{S}=\mathcal{T}$,

(iv) $\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{u})\le\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u})$, for all $\mathfrak{u}\in\mathcal{X}$,

(v) $(\mathfrak{f},\mathcal{T})$ is weakly compatible,

(vi) one of $\mathfrak{f}(\mathcal{X}),\mathcal{T}(\mathcal{X})$ is p_b-complete,

(vii) for any $\mathfrak{u},\mathfrak{y}\in\mathcal{X}$ and $\mathrm{L}\ge 0$,

$$\alpha_s(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y}))\le\xi(\hbar(M_2(\mathfrak{u},\mathfrak{y})))\hbar(M_2(\mathfrak{u},\mathfrak{y}))+\mathrm{L}\tau(N_2(\mathfrak{u},\mathfrak{y})),\tag{3.15}$$

where

$$\begin{aligned}M_2(\mathfrak{u},\mathfrak{y})=\max\{&\partial_b(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{y}),\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y}),\partial_b(\mathfrak{f}\mathfrak{y},\mathcal{T}\mathfrak{y}),\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}),\frac{\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{y})+\partial_b(\mathfrak{f}\mathfrak{y},\mathcal{T}\mathfrak{u})}{6s},\\ &\frac{\partial_b(\mathfrak{y},\mathcal{T}\mathfrak{y})+\partial_b(\mathfrak{f}\mathfrak{y},\mathcal{T}\mathfrak{u})}{6s},\frac{\partial_b(\mathcal{T}\mathfrak{u},\mathfrak{y})+\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})}{4s^2},\\ &\frac{\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u})\partial_b(\mathfrak{f}\mathfrak{y},\mathfrak{y})}{1+\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y})},\frac{1+\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y})+\partial_b(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{y})}{1+s\partial_b(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u})+s\partial_b(\mathfrak{f}\mathfrak{y},\mathcal{T}\mathfrak{y})}\partial_b(\mathfrak{f}\mathfrak{y},\mathfrak{y})\},\end{aligned}$$

and

$$N_2(\mathfrak{u},\mathfrak{y})=\min\left\{\partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}),\partial_b^w(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{y}),\partial_b^w(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{y}),\partial_b^w(\mathfrak{y},\mathcal{T}\mathfrak{y})\right\},$$

then $\mathfrak{f},\mathcal{T}$ have a unique common fixed point.

Proof. The proof of Theorem 3.15 is similar to that of Theorem 3.15, we omit it. □

If $\xi(\varepsilon) = \tilde{\delta}(0 < \tilde{\delta} < 1)$, $\mathrm{L} = 0$, we get the following result.

Corollary 3.19. Let $(\mathcal{X}, \partial_b)$ be a p_b -complete partial b-metric space with parameter $s \geq 1$ and let $\mathfrak{f}, \mathcal{T}$ be given self-mappings $(\mathcal{X}, \partial_b)$ with $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$. If the following conditions are satisfied:

(i) $\mathfrak{f}$ is interspersed $\mathcal{T}$-α_s-admissible mapping,

(ii) there is $\mathfrak{u}_0 \in \mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0, \mathfrak{f}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{u}_1, \mathcal{T}\mathfrak{u}_2) \geq s$,

(iii) properties (a'), (b') and (c') are satisfied when $\mathfrak{g} = I_{\mathcal{X}}$ and $\mathcal{S} = \mathcal{T}$,

(iv) $(\mathfrak{f}, \mathcal{T})$ is weakly compatible,

(v) one of $\mathfrak{f}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$ is p_b-complete,

(vi) for any $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$,

$$\alpha_s(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) \leq \tilde{\delta}\hbar(M_3(\mathfrak{u}, \mathfrak{y})), \tag{3.16}$$

where

$$\begin{aligned} M_3(\mathfrak{u}, \mathfrak{y}) = \max\Big\{&\partial_b(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y}), \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{u}), \frac{\partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{u})}{6s}, \\ &\frac{\partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{u})}{6s}, \frac{\partial_b(\mathcal{T}\mathfrak{u}, \mathfrak{f}\mathfrak{y}) + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})}{4s^2}, \\ &\frac{\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{u})\partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y})}{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}, \frac{1 + \partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}) + \partial_b(\mathcal{T}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}{1 + s\partial_b(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{u}) + s\partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y})}\partial_b(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y})\Big\}. \end{aligned}$$

then $\mathfrak{f}, \mathcal{T}$ have a unique common fixed point.

If $s = 1$ and $\partial_b(\mathfrak{u}, \mathfrak{u}) = 0$, for all $\mathfrak{u} \in \mathcal{X}$, then we get the following corollary.

Corollary 3.20. Let $(\mathcal{X}, d)$ be a complete metric space and let $\alpha_s : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$, $\mathfrak{f}, \mathfrak{g}, \mathcal{S}, \mathcal{T}$ be given self-mappings and $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}\mathfrak{g}(\mathcal{X})$ and $\mathfrak{f}\mathfrak{g}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. Suppose $\hbar \in \mathcal{H}$ and $\tau \in \mathfrak{I}$. If the following conditions are satisfied:

(i) $\mathfrak{f}$ is interspersed $(\mathcal{S}, \mathfrak{g}, \mathcal{T})$-$\alpha_s$-admissible,

(ii) there is $\mathfrak{u}_0 \in \mathcal{X}$ and $\mathfrak{f}\mathfrak{u}_0 = \mathcal{T}\mathfrak{g}\mathfrak{u}_1$ with $\alpha_s(\mathfrak{f}\mathfrak{u}_0, \mathfrak{f}\mathfrak{g}\mathfrak{u}_1) = \alpha_s(\mathcal{T}\mathfrak{g}\mathfrak{u}_1, \mathcal{S}\mathfrak{u}_2) \geq s$,

(iii) properties (a'), (b') and (c') are satisfied,

(iv) $d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{u}) \leq d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{u})$, for all $\mathfrak{u} \in \mathfrak{g}(\mathcal{X})$,

(v) $(\mathfrak{f}, \mathcal{S})$, $(\mathfrak{g}, \mathfrak{f})$, $(\mathfrak{f}\mathfrak{g}, \mathcal{T}\mathfrak{g})$ are weakly compatible,

(vi) one of $\mathfrak{f}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is p_b-complete,

(vii) for any $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$ and $\mathrm{L} \geq 0$,

$$\alpha_s(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y})\hbar(d(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) \leq \xi(\hbar(M_1^*(\mathfrak{u}, \mathfrak{y})))\hbar(M_1^*(\mathfrak{u}, \mathfrak{y})) + \mathrm{L}\tau(N_1^*(\mathfrak{u}, \mathfrak{y})),$$

where

$$\begin{aligned} M_1^*(\mathfrak{u}, \mathfrak{y}) = \max\Big\{&d(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{y}), d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), d(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y}), d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), \frac{d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + d(\mathfrak{f}\mathfrak{y}, \mathcal{S}\mathfrak{u})}{6}, \\ &\frac{d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y}) + d(\mathfrak{f}\mathfrak{y}, \mathcal{S}\mathfrak{u})}{6}, \frac{d(\mathcal{S}\mathfrak{u}, \mathfrak{g}\mathfrak{y}) + d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y})}{4}, \\ &\frac{d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u})d(\mathfrak{f}\mathfrak{y}, \mathfrak{g}\mathfrak{y})}{1 + d(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}, \frac{1 + d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}) + d(\mathcal{S}\mathfrak{u}, \mathfrak{f}\mathfrak{y})}{1 + d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}) + d(\mathfrak{f}\mathfrak{y}, \mathcal{T}\mathfrak{y})}d(\mathfrak{f}\mathfrak{y}, \mathfrak{g}\mathfrak{y})\Big\}, \end{aligned}$$

and

$$N_1^*(x, y) = \min\left\{d(\mathfrak{f}\mathfrak{u}, \mathcal{S}\mathfrak{u}), d(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{y}), d(\mathcal{S}\mathfrak{u}, \mathfrak{f}\mathfrak{y}), d(\mathfrak{g}\mathfrak{y}, \mathcal{T}\mathfrak{y})\right\},$$

then $\mathfrak{f}, \mathfrak{g}, \mathcal{T}, \mathcal{S}$ have a unique common fixed point.

4. Application

As is well known, automotive suspension systems, shock absorbers, etc. are practical applications of mass spring damping systems in mechanical engineering problems. When the car is driving on rough and uneven roads, the car shock absorber provides assistance for smooth driving, where the damping cylinder provides damping. External forces may be gravity, collision, ground vibration, tension, etc. If m is the mass of the object, l is the damping coefficient, k is the elastic coefficient of the spring, and $f(\ell)$ is the input external force, then the critical damping motion of the system under the action of external force $f(\ell)$ is controlled by the following initial value problem:

$$\begin{cases} m\frac{d^2\mathfrak{u}}{d\ell^2} + l\frac{d\mathfrak{u}}{d\ell} + k\mathfrak{u}(\ell) = f(\ell), \\ \mathfrak{u}(0) = 0, \\ \mathfrak{u}'(0) = 0, \end{cases} \tag{4.1}$$

among them, it is equivalent to $m\frac{d^2\mathfrak{u}}{d\ell^2} + l\frac{d\mathfrak{u}}{d\ell} - mF(\ell, \mathfrak{u}(\ell)) = 0$, and $F : [0, \mathrm{L}] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous function.

It is easy to show that the problem (4.1) is equivalent to the integral equation:

$$\mathfrak{u}(\ell) = \int_0^{\mathrm{L}} \gamma(\ell, \hat{s}) F(\hat{s}, \mathfrak{u}(\hat{s})) d\hat{s}, \ell \in [0, \mathrm{L}], \tag{4.2}$$

where $\gamma(\ell, \hat{s})$ is Green's function given by

$$\gamma(\ell, \hat{s}) = \begin{cases} \frac{1-e^{\eth(\ell-\hat{s})}}{\eth}, & 0 \leq \hat{s} \leq \ell \leq \mathrm{L} \\ 0, & 0 \leq \ell \leq \hat{s} \leq \mathrm{L}, \end{cases}$$

where $\eth = \frac{l}{m}$ is a constant.

In this section, we will give a theorem with an existential solution to the following integral equation by Corollary 3.19.

$$\mathfrak{u}(\ell) = \int_0^{\mathrm{L}} \mathcal{K}(\ell, \hat{s}, \mathfrak{u}(\hat{s})) d\hat{s}, \tag{4.3}$$

where $\ell \in [0, \mathrm{L}]$, $\mathrm{L} > 0$, $\mathcal{K} : [0, \mathrm{L}] \times [0, \mathrm{L}] \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{u} : \mathbb{R} \to \mathbb{R}$.

Let $\mathcal{X} = C[0, \mathrm{L}]$. Define

$$\partial_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$$

by

$$\partial_b(\mathfrak{u}, \mathfrak{y}) = \sup_{\ell \in [0,\mathrm{L}]} |\mathfrak{u}(\ell) - \mathfrak{y}(\ell)|^{\mathfrak{p}} + (\max\{|\sup_{\ell \in [0,\mathrm{L}]} \mathfrak{u}(\ell)|, |\sup_{\ell \in [0,\mathrm{L}]} \mathfrak{y}(\ell)|\})^{\mathfrak{p}}.$$

It is obvious that $(\mathcal{X}, \partial_b)$ is a p_b-complete partial b-metric space with $s = 2^{\mathfrak{p}-1}$.

Consider the mapping $\mathfrak{f}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathfrak{f}\mathfrak{u}(\ell) = \int_0^{\mathrm{L}} \mathcal{K}_1(\ell, \hat{s}, \mathfrak{u}(\hat{s})) d\hat{s},$$

and

$$\mathcal{T}\mathfrak{u}(\ell) = \int_0^{\mathrm{L}} \mathcal{K}_2(\ell, \hat{s}, \mathfrak{u}(\hat{s}))d\hat{s}.$$

Let $\tilde{\xi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function.

Theorem 4.1. Let $\mathfrak{f}, \mathcal{T}$ be self-mappings on a partial b-metric space $(\mathcal{X}, d)$. Suppose the following hypotheses hold:

(i) $\mathcal{K}_1, \mathcal{K}_2 : [0, \mathrm{L}] \times [0, \mathrm{L}] \times \mathbb{R} \to \mathbb{R}$,

(ii) $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X})$ is p_b-complate,

(iii) $\mathfrak{f}\mathcal{T}\mathfrak{u} = \mathcal{T}\mathfrak{f}\mathfrak{u}$, whenever $\mathfrak{f}\mathfrak{u} = \mathcal{T}\mathfrak{u}$ for some $\mathfrak{u} \in C[0, \mathrm{L}]$,

(iv) there exists $\mathfrak{u}_0 \in \mathcal{X}$ such that $\tilde{\xi}(\mathfrak{f}\mathfrak{u}_0(\ell), \mathcal{T}\mathfrak{u}_0(\ell)) \geq 0$ for all $\ell \in [0, \mathrm{L}]$,

(v) for all $\ell \in [0, \mathrm{L}]$ and $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$, $\tilde{\xi}(\mathcal{T}\mathfrak{u}(\ell), \mathcal{T}\mathfrak{y}(\ell)) \geq 0$ implies $\tilde{\xi}(\mathfrak{f}\mathfrak{u}(\ell), \mathfrak{f}\mathfrak{y}(\ell)) \geq 0$,

(vi) properties (a'), (b') and (c') are satisfied when $\mathfrak{g} = I_{\mathcal{X}}$ and $\mathcal{S} = \mathcal{T}$,

(vii) there exists a continuous function $\mathfrak{h} : [0, \mathrm{L}] \times [0, \mathrm{L}] \to \mathbb{R}^+$ such that $\int_0^{\mathrm{L}} \mathfrak{h}(\ell, \hat{s}) \leq 1$,

(viii) for each $\hat{\mathfrak{w}}, \hat{\mathfrak{v}} \in \mathcal{X}$, $0 < \tilde{\delta} < 1$ and each $\ell, \hat{s} \in [0, \mathrm{L}]$, we have

$$|\mathcal{K}_1(\ell, \hat{s}, \hat{\mathfrak{w}}(\hat{s})) - \mathcal{K}_1(\ell, \hat{s}, \hat{\mathfrak{v}}(\hat{s})| \leq \sqrt[\mathrm{p}]{\frac{\tilde{\delta}}{s^3}}\mathfrak{h}(\ell, \hat{s})|\mathcal{T}\mathfrak{w} - \mathcal{T}\mathfrak{v}|$$

and

$$\max\{|\mathcal{K}_1(\ell, \hat{s}, \hat{\mathfrak{w}}(\hat{s}))|, |\mathcal{K}_1(\ell, \hat{s}, \hat{\mathfrak{v}}(\hat{s})|\} \leq \sqrt[\mathrm{p}]{\frac{\tilde{\delta}}{s^3}} \max\{|\mathcal{K}_1(\ell, \hat{s}, \hat{\mathfrak{w}}(\hat{s}))|, |\mathcal{K}_2(\ell, \hat{s}, \hat{\mathfrak{v}}(\hat{s}))|\},$$

then the integral Eq (4.3) has a unique solution $\mathfrak{z} \in C[0, \mathrm{L}]$.

Proof. Define $\hbar(\kappa) = \kappa$ and $\alpha_s : \mathcal{X} \times \mathcal{X} \longrightarrow [0, +\infty)$ by

$$\alpha_s(\mathfrak{u}, \mathfrak{y}) = \begin{cases} s, & \text{if } \tilde{\xi}(\mathfrak{u}(\ell), \mathfrak{y}(\ell)) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to prove that $\mathfrak{f}$ is interspersed $\mathcal{T}$-α_s-admissible mapping and $\mathfrak{f}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{X})$ is p_b-complate. For $\mathfrak{u}, \mathfrak{y} \in \mathcal{X}$, by virtue of assumptions (i)–(viii), we have

$$
\begin{aligned}
s^3\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y}) =& s^3 \sup_{\ell\in[0,\mathrm{L}]} |\mathfrak{f}\mathfrak{u}(\ell)-\mathfrak{f}\mathfrak{y}(\ell)|^{\mathfrak{p}} + s^3(\max\{|\sup_{\ell\in[0,\mathrm{L}]}\mathfrak{f}\mathfrak{u}(\ell)|,|\sup_{\ell\in[0,\mathrm{L}]}\mathfrak{f}\mathfrak{y}(\ell)|\})^{\mathfrak{p}}\\
=& s^3 \sup_{\ell\in[0,\mathrm{L}]} |\int_0^{\mathrm{L}}\mathcal{K}_1(\ell,\hat{s},\mathfrak{u}(\hat{s}))d\hat{s} - \int_0^{\mathrm{L}}\mathcal{K}_1(\ell,\hat{s},\mathfrak{y}(\hat{s}))d\hat{s}|^{\mathfrak{p}}\\
&+ s^3(\max\{|\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\mathcal{K}_1(\ell,\hat{s},\mathfrak{u}(\hat{s}))d\hat{s})|,|\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\mathcal{K}_1(\ell,\hat{s},\mathfrak{y}(\hat{s}))d\hat{s})|\})^{\mathfrak{p}}\\
\leq& s^3 \sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}|\mathcal{K}_1(\ell,\hat{s},\mathfrak{u}(\hat{s}))-\mathcal{K}_1(\ell,\hat{s},\mathfrak{y}(\hat{s}))|d\hat{s})^{\mathfrak{p}}\\
&+ s^3(\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\max\{|\mathcal{K}_1(\ell,\hat{s},\mathfrak{u}(\hat{s}))|,|\mathcal{K}_1(\ell,\hat{s},\mathfrak{y}(\hat{s}))|\}d\hat{s}))^{\mathfrak{p}}\\
\leq& s^3 \sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\sqrt[\mathfrak{p}]{\frac{\tilde{\delta}}{s^3}}\mathfrak{h}(\ell,\hat{s})d\hat{s})^{\mathfrak{p}}\sup_{\hat{s}\in[0,\mathrm{L}]}|\mathcal{T}\mathfrak{u}(\hat{s})-\mathcal{T}\mathfrak{y}(\hat{s})|^{\mathfrak{p}}\\
&+ s^3(\sqrt[\mathfrak{p}]{\frac{\tilde{\delta}}{s^3}}\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\max\{|\mathcal{K}_2(\ell,\hat{s},\mathfrak{u}(\hat{s}))|,|\mathcal{K}_2(\ell,\hat{s},\mathfrak{u}(\hat{s}))|\}ds))^{\mathfrak{p}}\\
\leq& s^3 \sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\sqrt[\mathfrak{p}]{\frac{\tilde{\delta}}{s^3}}\mathfrak{h}(\ell,\hat{s})d\hat{s})^{\mathfrak{p}}\sup_{\hat{s}\in[0,\mathrm{L}]}|\mathcal{T}\mathfrak{u}(\hat{s})-\mathcal{T}\mathfrak{y}(\hat{s})|^{\mathfrak{p}}\\
&+ \tilde{\delta}(\max\{|\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\mathcal{K}_2(\ell,\hat{s},\mathfrak{u}(\hat{s}))d\hat{s})|,|\sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\mathcal{K}_2(\ell,\hat{s},\mathfrak{y}(\hat{s}))d\hat{s})|\})^{\mathfrak{p}}\\
=& s^3 \sup_{\ell\in[0,\mathrm{L}]}(\int_0^{\mathrm{L}}\sqrt[\mathfrak{p}]{\frac{\tilde{\delta}}{s^3}}\mathfrak{h}(\ell,\hat{s})d\hat{s})^{\mathfrak{p}}\sup_{\hat{s}\in[0,\mathrm{L}]}|\mathcal{T}\mathfrak{u}(\hat{s})-\mathcal{T}\mathfrak{y}(\hat{s})|^{\mathfrak{p}}\\
&+ \tilde{\delta}(\max\{|\sup_{\hat{s}\in[0,\mathrm{L}]}\mathcal{T}\mathfrak{u}(\hat{s})|,|\sup_{\hat{s}\in[0,\mathrm{L}]}\mathcal{T}\mathfrak{y}(\hat{s})|\})^{\mathfrak{p}}.
\end{aligned}
$$

By hypothesis (vii), one obtain

$$
\begin{aligned}
s^3\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y}) \leq& \tilde{\delta}\{\sup_{\ell\in[0,\mathrm{L}]}|\mathcal{T}\mathfrak{u}(\hat{s})-\mathcal{T}\mathfrak{y}(\hat{s})|^{\mathfrak{p}} + (\max\{|\sup_{\ell\in[0,\mathrm{L}]}\mathcal{T}\mathfrak{u}(\hat{s})|,|\sup_{\ell\in[0,\mathrm{L}]}\mathcal{T}\mathfrak{y}(\hat{s})|\})^{\mathfrak{p}}\}\\
=& \tilde{\delta}\partial_b(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{y}) \leq \tilde{\delta}M_3(\mathfrak{u},\mathfrak{y}),
\end{aligned}
$$

that is,

$$
\alpha_s(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{y})\hbar(s^2\partial_b(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y})) \leq \tilde{\delta}\hbar(M_3(\mathfrak{u},\mathfrak{y})).
$$

Therefore, all the conditions of Corollary 3.16 hold. As a result, the mappings $\mathfrak{f}$ and $\mathcal{T}$ have a unique point $\mathfrak{z} \in C[0,\mathrm{L}]$, which is a solution of the integral Eq (4.1). □

5. Conclusions

In this manuscript, we introduced two concepts named mixed $(\mathcal{S},\mathcal{T})$-$\alpha$-admissible mappings and interspersed $(\mathcal{S},\mathfrak{g},\mathcal{T})$-$\alpha$-admissible mappings and gave the sufficient conditions for the existence and uniqueness of common fixed point of generalized $(\alpha_s,\xi,\hbar,\tau)$-Geraghty contractive mapping in partial

b-metric spaces. Also, we provided examples that elaborated the useability of our results. Furthmore, we presented an application to the existence of solutions to an integral equation by means of one of our results. It is of interest to further consider whether we can modify the elements in $M(\mathfrak{u}, \mathfrak{y})$ and $N(\mathfrak{u}, \mathfrak{y})$ mentioned in this article or combine them with Meir-Keeler type contraction to establish a new type of contraction on a partial b-metric space.

Author contributions

Ying Chang and Hongyan Guan: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing-original draft, Writing-review & editing. The authors contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was financially supported by the Science and Research Project Foundation of Liaoning Province Education Department (No: JYTMS20231700).

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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