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Research article

A comprehensive view of the solvability of non-local fractional orders pantograph equation with a fractal-fractional feedback control

A. M. A. El-Sayed[1], **H. H. G. Hashem**[1,*] **and Sh. M. Al-Issa**[2,3]

[1] Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt
[2] Department of Mathematics, Faculty of Science, Lebanese International University, Lebanon
[3] Department of Mathematics, Faculty of Science, The International University of Beirut, Lebanon

* **Correspondence:** Email: hindhghashem@gmail.com.

Abstract: In this article, the solvability of the pantograph equation of fractional orders under a fractal-fractional feedback control was investigated. This investigation was located in the class of all continuous functions. The necessary conditions for the solvability of that problem and the continuous dependence of the solution on some parameters and the control variable were established with the help of some fixed point theorems. Additionally, the Hyers-Ulam stability of the issue was explored. Finally, some specific problems extended to the corresponding problem with integer orders were illustrated. The theoretical results were supported by numerical simulations and comparisons with existing results in the literature.

Keywords: constrained problem; functional integro-differential equation; fractal-fractional feedback control; Hyers-Ulam stability; pantograph equation; Schauder's fixed point theorem
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1. Introduction

Fractional-order differential and integral equations represent wide applications in physics, engineering, and biomedical engineering. Nonlocal situations often arise in mathematical and physical problems. In which the system's behavior is influenced by multiple elements or parameters. Nonlocal integral conditions are common in mathematical analysis when working with differential equations, sometimes involving constraints or objectives. Fixed point theorems are valuable tools for examining the solvability of differential equation problems extensively covered in various monographs and publications (see [1–5] and the references therein).

Delay systems have been extensively used to describe the evolution of propagation and transportation or population movements [6, 7]. In economic systems, decisions like investment strategies and the dynamics of commodity markets are spread out across time intervals, leading to the natural occurrence of delays. In the 19th century, Euler, Lagrange, and Laplace delved into the realm of delayed differential equations. Subsequently, in the late 1930s and early 1940s, Voltaire introduced various delayed differential equations while researching the predator-prey model, becoming the first to systematically investigate these equations.

The pantograph differential equation is a type of delay differential equation initially developed through the study of an electric locomotive [8]. Mahler introduced pantograph equations in 1940 as part of his Number Theory [9]. The term "pantograph" originated from the work of Ockendon and Taylor, who examined the electric locomotive's catenary system. Their objective was to formulate an equation for analyzing the movement of a pantograph head on an electric locomotive that operates using a trolley overhead wire [8]. Kato and Mcleod [10] investigated the asymptotic properties and stability of solutions to the pantograph equation $u'(t) = au(kt) + bu(t)$.

A pantograph (or "pan" or "panto") is a device installed on the roof of an electric train, tram, or electric bus to gather power by making contact with an overhead line [11]. An important application of the pantograph appears in engineering, particularly in designing machines and mechanisms that require precise scaling and copying of movements (see [12–14]). For instance, a pantograph can be utilized to move in a specific manner. Additionally, pantographs find relevance in electrodynamics [9] and number theory [15]. In 1971, Ockendon and Taylor [8] researched how electric current is collected by the pantograph of an electric locomotive using a delay equation, now known as the pantograph equation. Since then, numerous researchers have explored and applied it across various mathematical and scientific domains such as number theory, probability, electrodynamics, and medicine, as seen in [8, 16, 17] and the references therein.

Much research has been conducted on fractional pantograph equations due to their significance in various research areas. For instance, in [18], Balachandran and Kiruthika examined the existence of solutions. Additionally, in [19], Jalilian and Ghasemi studied a fractional integro-differential equation of pantograph type along with suitable initial conditions.

Let $C(I) = C[0,1]$ be the class of continuous functions defined on $I = [0,1]$, and the norm of $x \in C(I)$ is defined by $\|x\|_C = \sup_{t \in I} |x(t)|$.

Inspired by contemporary literature, we consider the nonlocal issue of the pantograph equation via Caputo fractional-order derivative D^ς, $\varsigma \in \{\alpha, \beta, \rho\}$,

$$\frac{dx}{dt} = f(t, u_x(t), x(t), D^\alpha x(\gamma t)), \quad t \in (0,1], \quad \gamma \in (0,1), \tag{1.1}$$

satisfying

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^\beta x(\gamma s))ds, \quad \tau \in (0,1] \tag{1.2}$$

equipped with the fractal-fractional feedback control

$$\frac{du_x(t)}{dt^\delta} = -\lambda u_x(t) + g(t, x(t), D^\rho x(\gamma t)), \;\; u_0 = u_x(0), \;\lambda \geq 0, \;\delta \in (0,1], \tag{1.3}$$

where $\alpha, \beta, \rho \in (0,1]$, τ is a fixed parameter and $\frac{d}{dt^\delta}$ denotes the fractal derivative of order δ (for more information on fractal derivatives, refer to [20, 21]).

In this study, we explore the presence of solutions $x \in C(0,1]$ to the problem (1.1)–(1.3). The necessary conditions for the uniqueness of the solution will be provided. The continuous dependence of the unique solution $x \in C(0,1]$ of (1.1) and (1.2) on $y(t) = \frac{dx(t)}{dt}$, the function h, and the parameter x_0 will be demonstrated. The Hyers-Ulam stability of the problem (1.1)–(1.3) will be examined. The feedback control problem of the linear version of the pantograph equation (Ambartsumian)

$$\frac{dx}{dt} = au_x(t) + bx(t) + cD^\alpha x(\gamma t), \qquad t \in (0,1], \tag{1.4}$$

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^\beta x(\gamma s))ds, \qquad \tau \in (0,1] \tag{1.5}$$

equipped with the fractal-fractional feedback control (1.3) will be addressed. The continuation of $\alpha, \beta \to 1$ will be established. We outline the key contributions of this article as follows:

- We examine the feedback control problem of the fractional pantograph differential equation with arbitrary fractional orders (1.1) and (1.2) with the fractal-fractional feedback control (1.3).
- We explore the feedback control problem of the linear version of the pantograph (Ambartsumian) equations (1.4) and (1.5) with the fractal-fractional feedback control (1.3).

This paper enhances the qualitative analysis of the feedback control fractional pantograph differential equation problem. The paper's structure is as follows: Section 2 covers key features, and demonstrates the existence-uniqueness of the solution, and Section 3 explores Ulam Hyers stability (UHRS) and the continuous dependence on some data. Furthermore, Section 4 presents a special case and an example. Finally, Section 5 provides a conclusion.

2. Main result

The problem (1.1)–(1.3) will be investigated under the assumptions:

(i) The function $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is measurable in $t \in I$ for all $u_i, v_i, w_i \in \mathbb{R}, i = 1, 2$, and satisfies the Lipschitz condition with a positive Lipschitz constant b

$$|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_1)| \le b(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|).$$

Observation 1. Based on assumption (i), we have

$$|f(t, u_1, v_1, w_1)| - |f(t, 0, 0, 0)| \le |f(t, u_1, v_1, w_1) - f(t, 0, 0, 0)| \le b(|u_1| + |v_1| + |w_1|).$$

This implies that

$$|f(t, u_1, v_1, w_1)| \le a^* + b(|u_1| + |v_1| + |w_1|), \quad where\ \ a^* = \sup_{t\in I} |f(t, 0, 0, 0)|.$$

(ii) $h : I \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in $t \in I$ for every $u, v \in \mathbb{R}$ and continuous in $u, v \in \mathbb{R}$ for every $t \in I$. There exist a function $a_2 \in L_1(I)$ and a positive constant b_2 such that

$$|h(t, u, v)| \le a_2(t) + b_2(|u| + |v|), \qquad \sup_{t\in I} \int_0^t |a_2(s)|ds \le M.$$

(iii) $g : I \times \mathbb{R}^2 \to \mathbb{R}$, is continuous function for every $u, v \in \mathbb{R}$, such that

$$|g(t, u, v)| \leq k[|u| + |v|] + m, \text{ where } m = \sup_{t \in I} |g(t, 0, 0)|.$$

(iv) r_1 and r_2 represent positive solutions of the two simultaneous equations:

$$|x_0| + M + b_2(r_2 + \gamma\, r_1) + 2r_1 = r_1,$$

$$a^* + b_1 r_2 + b_1(w + w_1 + w_2\, r_2 + w_3 + w_4) + 2\gamma r_1 b_1 = r_2.$$

Additionally, the condition $b_1(1 + w_2) < 1$ is satisfied.

Let X be the Banach space consisting of all pairs (x, y) where x and y are functions in the space $C(I)$. The norm of an element (x, y) in X is defined as the sum of the sup norms of x and y in $C(I)$ respectively.

The following lemma shows that the feedback control problem described in Eqs (1.1) and (1.2) equipped with a fractal feedback control (1.3) is equivalent to its respective integral equations.

Lemma 2.1. If the solution of (1.1)–(1.3) exists, it can be expressed by the following coupled system

$$x(t) = x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds, \tag{2.1}$$

$$y(t) = f(t, u_x(s), x(t), \gamma I^{1-\alpha} y(\gamma t)), \tag{2.2}$$

along with the integral feedback equation

$$u_x(t) = u_0 e^{-\lambda t^{\delta}} + \delta \int_0^{t} e^{-\lambda(t^{\delta} - s^{\delta})}\, s^{\delta-1}\, g(s, x(s), \gamma I^{1-\rho} y(\gamma s))ds, \quad u_0 = u_x(0). \tag{2.3}$$

Proof. Let x represent the solution of the problem (1.1)–(1.3). Take $y(t) = \frac{d}{dt}x(t)$, then

$$x(\gamma t) = x(0) + \int_0^{\gamma t} y(s)ds, \tag{2.4}$$

$$\frac{d}{dt}x(\gamma t) = \gamma y(\gamma t). \tag{2.5}$$

Applying the Riemann-Liouville fractional integral operators $I^{1-\alpha}$, $I^{1-\beta}$ and $I^{1-\rho}$ to both sides of (2.5), and we get

$$D^{\alpha} x(\gamma t) = I^{1-\alpha} \frac{dx(\gamma t)}{dt} = \gamma I^{1-\alpha} y(\gamma t), \tag{2.6}$$

$$D^{\beta} x(\gamma t) = I^{1-\beta} \frac{dx(\gamma t)}{dt} = \gamma I^{1-\beta} y(\gamma t), \tag{2.7}$$

and

$$D^{\rho} x(\gamma t) = I^{1-\rho} \frac{dx(\gamma t)}{dt} = \gamma I^{1-\rho} y(\gamma t).$$

Using the substitutions of $D^{\alpha}x(\gamma t)$, $D^{\beta}x(\gamma t)$, and $D^{\rho}x(\gamma t)$ in the problem (1.1)-(1.2), we obtain the representation (2.1)-(2.2).

On the contrary, assume x is a solution of (2.1) and differentiate both sides of (2.1). We get

$$\frac{dx}{dt} = y(t) = f(t, u_x(t), x(t), D^{\alpha}x(\gamma t)), \quad t \in (0, 1]$$

Put $t = \tau$ in Eq (2.1), then we can deduce

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^{\beta}x(\gamma s))ds, \quad \tau \in (0, 1].$$

This demonstrates the equivalence between the problem (1.1)-(1.2) and the problem (2.1)-(2.2).

Now, the fractal-fractional feedback control (1.3), can be expressed as

$$\frac{du_x(t)}{dt}\frac{dt}{dt^{\delta}} = -\lambda u_x(t) + g(t, x(t), D^{\rho}x(\gamma t)),$$

then

$$\frac{1}{\delta \cdot t^{\delta-1}}\frac{du_x(t)}{dt} = -\lambda u_x(t) + g(t, x(t), D^{\rho}x(\gamma t)),$$

therefore,

$$\frac{du_x(t)}{dt} = -\lambda\, \delta \cdot t^{\delta-1}\, u_x(t) + \delta \cdot t^{\delta-1} g(t, x(t), D^{\rho}x(\gamma t)).$$

Multiply both terms by $e^{\lambda t^{\delta}}$

$$e^{\lambda t^{\delta}}\frac{du_x(t)}{dt} + e^{\lambda t^{\delta}}\lambda\, \delta \cdot t^{\delta-1}\, u_x(t) = e^{\lambda t^{\delta}}\delta \cdot t^{\delta-1} g(t, x(t), D^{\rho}x(\gamma t)),$$

and

$$\frac{d}{dt}(u_x(t) \cdot e^{\lambda t^{\delta}}) = e^{\lambda t^{\delta}}\delta \cdot t^{\delta-1} g(t, x(t), D^{\rho}x(\gamma t)).$$

Integrate with respect to t, then

$$u_x(t) \cdot e^{\lambda t^{\delta}} = u_x(0) + \int_0^t \delta \cdot s^{\delta-1}\, e^{\lambda s^{\delta}} g(s, x(s), D^{\rho}x(\gamma s))\, ds.$$

Hence

$$u_x(t) = u_0\, e^{-\lambda t^{\delta}} + \int_0^t \delta \cdot s^{\delta-1}\, e^{-\lambda(t^{\delta}-s^{\delta})} g(s, x(s), D^{\rho}x(\gamma s))\, ds.$$

Substitute for $D^{\rho}x(\gamma t)$, and we obtain

$$u_x(t) = u_0\, e^{-\lambda t^{\delta}} + \int_0^t \delta \cdot s^{\delta-1}\, e^{-\lambda(t^{\delta}-s^{\delta})} g(s, x(s), \gamma I^{1-\rho} y(\gamma s))ds.$$

For any real-valued function x, the solution of the fractal differential feedback control (1.3) denoted as $u_x(t)$ can be expressed as shown in (2.3). □

Lemma 2.2. The control variable $u_x(t)$ satisfies (1.3) and can be expressed by (2.3), then the solution $u_x(t)$ is bounded for $u_0 > 0$.

Proof.

$$\begin{aligned}
|u_x(t)| &\leq u_0|e^{-\lambda t^\delta}| + \delta \int_0^t e^{-\lambda(t^\delta - s^\delta)}\ s^{\delta-1}\ [k\,(|u_x(s)| + |x(s)| + |\gamma I^{1-\rho} y(\gamma s))|) + m]ds \\
&\leq w + k\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1}|u_x(s)|ds + k\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1}||x||ds \\
&+ k\gamma \int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} I^{1-\rho}||y||ds + k\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} m\, ds \\
&\leq w + \sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} k\, |u_x(s)|ds + \sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1}\ k\ ||x||\ ds \\
&+ k\gamma \sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} I^{1-\rho}||y||ds + \sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} m\, ds \\
&\leq w + w_1 + w_2\ r_2 + w_3 + w_4,
\end{aligned}$$

where

$$\begin{aligned}
&\sup_{t\in I} u_0\ e^{-\lambda t^\delta} = w, \\
&\sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)}\ s^{\delta-1}\ k|u_x(s)|ds = w_1, \\
&\sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)}\ s^{\delta-1}\ ||x||ds = w_2, \\
&\sup_{t\in I}\int_0^t e^{-\lambda(t^\delta - s^\delta)}\ s^{\delta-1}\ m\ ds = w_3, \\
&\sup_{t\in I} k\gamma \int_0^t e^{-\lambda(t^\delta - s^\delta)} s^{\delta-1} I^{1-\rho}||y||ds = w_4,
\end{aligned}$$

and

$$\begin{aligned}
&|u_{x_1}(t) - u_{x_2}(t)| \\
&\leq \delta \int_0^t e^{-\lambda(t^\delta - s^\delta)}\ s^{\delta-1}\ |g(s, u_{x_1}(s), x_1(s), \gamma I^{1-\rho} y(\gamma s)) - g(s, u_{x_2}(s), x_2(s), \gamma I^{1-\rho} y(\gamma s))|ds \\
&\leq k\ \int_0^t e^{-\lambda(t^\delta - \tau^\delta)}\ s^{\delta-1}\ (|u_{x_1}(s) - u_{x_2}(s)| + |x_1(s) - x_2(s)|)ds \\
&\leq k\ \frac{e^{-\lambda}}{\delta\lambda}\ (||u_{x_1} - u_{x_2}|| + ||x_1 - x_2||).
\end{aligned}$$

Hence,

$$||u_{x_1} - u_{x_2}|| \leq \Delta\ ||x_1 - x_2||,$$

with $k\ \dfrac{e^{-\lambda}}{\delta\lambda} < 1$ and $\Delta = \dfrac{k\ \frac{e^{-\lambda}}{\delta\lambda}}{(1 - k\ \frac{e^{-\lambda}}{\delta\lambda})}$. □

2.1. Existence of the solution

Here, we demonstrate the existence of a continuous solution for the problem (1.1)-(1.2) equipped with a fractal differential constraint (1.3). To achieve this, we introduce the following theorem.

Theorem 2.3. Assuming conditions (i)–(iv) are met, then the constrained problem (1.1)–(1.3) has at least one solution $x \in C(I)$.

Proof. Define an operator F as $F(x, y) = (F_1 y, F_2 x)$, where

$$\begin{aligned} F_1 y(t) &= x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds, \\ F_2 x(t) &= f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)). \end{aligned}$$

Define set $U \subset X$ as

$$U = \{u = (x, y) \in X : \|x\|_C \le r_1,\ \|y\|_C \le r_2\},$$

then $\|(x, y)\|_X = \|x\|_C + \|y\|_C \le r$ where $r = r_1 + r_2$, for positive values of r_1 and r_2 satisfying condition (iv).

Obviously, U is a closed convex bounded set. For $(x, y) \in U$ and $t \in I$, we have:

$$\begin{aligned} |F_1 y(t)| &= \left| x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \right| \\ &\le |x_0| + \int_0^{1-\tau} |h(s, x(s), \gamma I^{1-\beta} y(\gamma s)|ds + \int_0^{\tau} |y(s)|ds + \int_0^{t} |y(s)|ds \\ &\le |x_0| + \int_0^{1-\tau} \Big(|a_2(s)| + b_2\Big(|x(s)| + \gamma I^{1-\beta}|y(s)|\Big)\Big)ds + \int_0^{\tau} |y(s)|ds + \int_0^{t} |y(s)|ds \\ &\le |x_0| + \int_0^{1-\tau} \Big(|a_2(s)| + b_2\Big(r_2 + r_1 \frac{\gamma}{\Gamma(2-\beta)}\Big)\Big)ds + 2r_1 \\ &\le |x_0| + M + b_2\Big(r_2 + \frac{r_1\,\gamma}{\Gamma(2-\beta)}\Big) + 2r_1, \end{aligned}$$

and

$$\|F_1 y\|_C \le |x_0| + M + b_2(r_2 + \gamma\, r_1) + 2r_1 = r_1.$$

In a similar manner,

$$\begin{aligned} |F_2 x(t)| &= \left| f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)) \right| \\ &\le |a_1(t)| + b_1\Big(|u_x(t)| + |x(t)| + \gamma I^{1-\alpha}|y(\gamma t)|\Big) \\ &\le a^* + b_1 r_2 + b_1(w + w_1 + w_2\, r_2 + w_3 + w_4) + \frac{\gamma r_1 b_1}{\Gamma(2-\alpha)}, \end{aligned}$$

and

$$\|F_2x\| \leq \ a^* + b_1r_2 + b_1(w + w_1 + w_2\, r_2 + w_3 + w_4) + 2\gamma r_1 b_1 = r_2.$$

Therefore,

$$\|FU\|_X = \|F(x,y)\|_X = \|(F_1y, F_2x)\|_X = \|F_1y\|_C + \|F_2x\|_C = r_1 + r_2 = r.$$

For each point $u = (x, y) \in U$, the function $F(u)$ is also in U, showing that F maps the set U into itself. This implies that the set of functions $\{FU\}$ is uniformly bounded on the interval I.

Now, we show that the class F_1y is equi-continuous. Let $t_1, t_2 \in I$ such that $t_2 > t_1$ and $| \ t_1 - t_2 \ |\leq \delta$, then

$$\begin{aligned}
|F_1y(t_2) - F_1y(t_1)| &= \Big|x_0 + \int_0^{1-\tau} h(s, x(s), \lambda I^{1-\beta} y(\lambda s))ds - \int_0^{\tau} |y(s)|ds + \int_0^{t_2} y(s)ds \\
&\quad - x_0 - \int_0^{1-\tau} h(s, x(s), \lambda I^{1-\beta} y(\lambda s))ds + \int_0^{\tau} |y(s)|ds - \int_0^{t_1} y(s)ds\Big| \\
&\leq \int_0^{t_2} |y(s)|ds - \int_0^{t_1} |y(s)|ds \\
&\leq \int_{t_1}^{t_2} |y(s)|ds.
\end{aligned}$$

This demonstrates that the class of functions $\{F_1y\}$ is equi-continuous on the interval I in the space of continuous functions.

Similarly,

$$\begin{aligned}
& |F_2x(t_2) - F_2x(t_1)| \\
= & \left|f\Big(t_2, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_1, x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big)\right| \\
= & \Big|f\Big(t_2, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) \\
+ & f\Big(t_1, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) \\
+ & f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) \\
+ & f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big)\Big| \\
+ & f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_1), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big)\Big| \\
\leq & \left|f\Big(t_2, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big)\right| \\
+ & \left|f\Big(t_1, u_x(t_2), x(t_2), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s)ds\Big)\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left|f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big)\right| \\
&+ \quad \left|f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big) - f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big)\right| \\
&+ \quad \left|f\Big(t_1, u_x(t_2), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big) - f\Big(t_1, u_x(t_1), x(t_1), \gamma \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma s) ds\Big)\right| \\
&\le \quad \epsilon + b_1|x(t_2) - x(t_1)| + \gamma b_1 \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma t) ds + b_1 |u_x(t_2) - u_x(t_1)| \\
&+ \quad \gamma b_1 \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} y(\gamma t) ds \\
&\le \quad \epsilon + b_1|u_x(t_2) - u_x(t_1)| + b_1|x(t_2) - x(t_1)| + \gamma b_1 r_2 \int_{t_1}^{t_2} \frac{1}{\Gamma(1-\alpha)(t_2-s)^{\alpha}} ds \\
&+ \quad \gamma b_1 r_2 \int_0^{t_1} \frac{(t_1-s)^{\alpha} - (t_2-s)^{\alpha}}{\Gamma(1-\alpha)(t_2-s)^{\alpha}(t_1-s)^{\alpha}} ds.
\end{aligned}
$$

This demonstrates that the class $\{F_2 x\}$ is equi-continuous on the interval $C(I)$.

$$
\begin{aligned}
Fu(t_2) - Fu(t_1) &= F(x,y)(t_2) - F(x,y)(t_1) \\
&= (F_1 y(t_2), F_2 x(t_2)) - (F_1 y(t_1), F_2 x(t_1)) \\
&= (F_1 y(t_2) - F_1 y(t_1), F_2 x(t_2) - F_2 x(t_1)),
\end{aligned}
$$

which implies that

$$
\|Fu(t_2) - Fu(t_1)\| \le \|F_1 y(t_2) - F_1 y(t_1)\| + \|F_2 x(t_2) - F_2 x(t_1)\|.
$$

Then, the class of functions FU is equi-continuous on X.

Thus, by the Arzela-Ascoli theorem [22], $\{FU\}$ is relatively compact. Hence the operator F is compact.

Now, we will show that the operator F is continuous.

Let $(x_n, y_n) \in U$ such that $x_n \to x,\ y_n \to y$, then

$$
\begin{aligned}
F_1 y_n &= x_0 + \int_0^{1-\tau} h\Big(s, x_n(s), \lambda I^{1-\beta} y_n(\lambda s)\Big) ds - \int_0^{\tau} y_n(s) ds + \int_0^{t} y_n(s) ds, \\
F_2 x_n &= f\Big(t, u_{x_n}(t), x_n(t), \gamma I^{1-\alpha} y_n(\gamma t)\Big), \\
\lim_{n\to\infty} F_1 y_n &= x_0 + \lim_{n\to\infty} \int_0^{1-\tau} h\Big(s, x_n(s), \lambda I^{1-\beta} y_n(\lambda s)\Big) ds - \lim_{n\to\infty} \int_0^{\tau} y_n(s) ds + \lim_{n\to\infty} \int_0^{t} y_n(s) ds,
\end{aligned}
$$

and

$$
\lim_{n\to\infty} F_2 x_n(t) \quad = \quad \lim_{n\to\infty} f\Big(t, u_{x_n}(t), x_n(t), \gamma I^{1-\alpha} y_n(\gamma t)\Big).
$$

Since f, h are continuous in x, y, then

$$
f\Big(t, u_{x_n}(t), x_n(t), I^{1-\alpha} y_n(s)\Big) \to f\Big(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)\Big),
$$

$$h\Big(s, x_n(s), \lambda I^{1-\beta} y_n(\lambda s)\Big) \to h\Big(s, x(s), \lambda I^{1-\beta} y(\lambda s)\Big).$$

Since

$$|h(t, x, y)| \leq a_2(t) + b_2(|x| + |y|),$$

and regarding to the Lebesgue dominated convergence theorem [22], we can derive

$$\begin{aligned} \lim_{n\to\infty} F_1 y_n(t) &= x_0 + \int_0^{1-\tau} h(s, x(s), \lambda I^{1-\beta} y(\lambda s))ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \\ &= F_1 y(t). \end{aligned}$$

Then F_1 is continuous. Also,

$$\lim_{n\to\infty} F_2 x_n(t) = f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)) = F_2 x(t),$$

and F_2 is continuous. Hence,

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty}(F_1 y, F_2 x) = (F_1 y, F_2 x) = F(x, y).$$

Therefore, the function F is continuous. As all the requirements of Schauder's fixed point theorem [22] are met, it follows that F must have at least one fixed point $u = (x, y) \in U$. Then the coupled system of integral equations

$$\begin{aligned} x(t) &= x_0 + \int_0^{1-\tau} h(s, x(s), \lambda I^{1-\beta} y(\lambda s))ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds, \\ y(t) &= f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)), \end{aligned}$$

has a solution $u = (x, y) \in U$. Consequently, the problem (1.1)-(1.2) with a fractal differential constraint (1.3) has a solution $x \in C(I)$. □

Corollary 2.1. Let the assumptions of Theorem 2.3 be satisfied, then $D^{\alpha} x(\gamma t)$, $D^{\beta} x(\gamma t) \in C(I)$.

Proof. From Theorem 2.3, we have $y \in C(I)$. By utilizing the definition and properties of the fractional operators [23], and making use of Eqs (2.6) and (2.7), we can conclude that

$$\begin{aligned} D^{\alpha} x(\gamma t) &= \gamma I^{1-\alpha} y(\gamma t) \in C(I), \\ D^{\beta} x(\gamma t) &= \gamma I^{1-\beta} y(\gamma t) \in C(I). \end{aligned}$$

□

3. Features of the solution

3.1. Uniqueness of the solution

For the uniqueness of the solution of the problem (1.1)-(1.2) with the fractal-fractional feedback control (1.3), we replace condition (*ii*) by the assumption:

(ii*) The function $h : I \times \mathbb{R}^2 \to \mathbb{R}$ is measurable for $t \in I$ for all $(x_i, y_i) \in \mathbb{R} \times \mathbb{R}$, where $i = 1, 2$, and h satisfies the Lipschitz condition

$$|h(t, x_1, y_1) - h(t, x_2, y_2)| \leq b_2(|x_1 - x_2| + |y_1 - y_2|), \quad t \in I, \quad x_i, y_i \in \mathbb{R}$$

with Lipschitz constant $b_2 > 0$.

Observation 2. From assumption $(ii)^*$, we obtain:

$$|h(t, x, y)| \leq |h(t, 0, 0)| + b_2(|x| + |y|),$$

and

$$|h(t, x, y)| \leq a_2 + b_2(|x| + |y|), \quad where \ \ a_2 = \sup_{t \in I} |h(t, 0, 0)|, \quad t \in I.$$

This demonstrates that assumption (ii) is satisfied.

Theorem 3.1. If the conditions of Theorem 2.3 are met, assumption (ii) is replaced by $(ii)^*$, and if the inequality

$$\frac{b_1(1 + \Delta)(b_2\gamma + 2)}{(1 - 2\gamma b_1)(1 - \ b_2)} < 1$$

holds, then the solution $x \in C(I)$ of problem (1.1)-(1.2) (and, consequently, (1.4)-(1.5)) with the fractal-fractional feedback control (1.3) is unique.

Proof. All conditions of Theorem 2.3 are met, implying that solutions to the problem (2.1)-(2.2) with feedback control (1.3) exists. Consider $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ as two solutions of the problem (2.1)-(2.2) with feedback control (1.3). Then, we can observe

$$\|(x_1, y_1) - (x_2, y_2)\|_X = \|x_1 - x_2, y_1 - y_2\|_X = \|x_1 - x_2\|_C + \|y_1 - y_2\|_C.$$

Now,

$$\begin{aligned}
|x_1 - x_2| &= |x_0 + \int_0^{1-\tau} h(s, x_1(s), \gamma I^{1-\beta} y_1(\gamma s))ds - \int_0^{\tau} y_1(s)ds + \int_0^{t} y_1(s)ds \\
&- x_0 - \int_0^{1-\tau} h(s, x_2(s), \gamma I^{1-\beta} y_2(s))ds + \int_0^{\tau} y_2(s)ds - \int_0^{t} y_2(s)ds| \\
&\leq \int_0^{1-\tau} |h(s, x_1(s), \gamma I^{1-\beta} y_1(s)) - h(s, x_2(s), \gamma I^{1-\beta} y_2(s))|ds \\
&+ \int_0^{\tau} |y_1(s) - y_2(s)|ds + \int_0^{t} |y_1(s) - y_2(s)|ds \\
&\leq b_2 \int_0^{1-\tau} \Big[|x_1(s) - x_2(s)| + \gamma I^{1-\beta} |y_1(\gamma s) - y_2(\gamma s)|\Big]ds \\
&+ \int_0^{\tau} |y_1(s) - y_2(s)|ds + \int_0^{t} |y_1(s) - y_2(s)|ds \\
&\leq b_2\|x_1 - x_2\|_C + \frac{b_2\gamma}{\Gamma(2 - \beta)}\|y_1 - y_2\|_C + 2\|y_1 - y_2\|_C,
\end{aligned}$$

then

$$\|x_1 - x_2\|_C \leq \frac{(b_2\gamma + 2)}{1 - \ b_2}\|y_1 - y_2\|_C.$$

Also

$$\begin{aligned}
|y_1 - y_2| &= \left| f\Big(t, u_{x_1}(t), x_1(t), \gamma I^{1-\alpha} y_1(\gamma t)\Big) - f\Big(t, u_{x_2}(t), x_2(t), \gamma I^{1-\alpha} y_2(\gamma t)\Big) \right| \\
&\leq \left| f\Big(t, u_{x_1}(t), x_1(t), \gamma I^{1-\alpha} y_1(\gamma t)\Big) - f\Big(t, u_{x_2}(t), x_2(t), \gamma I^{1-\alpha} y_1(\gamma t)\Big) \right. \\
&+ \left. f\Big(t, u_{x_1}(t), x_2(t), \gamma I^{1-\alpha} y_1(\gamma t)\Big) - f\Big(t, u_{x_2}(t), x_2(t), \gamma I^{1-\alpha} y_2(\gamma t)\Big) \right| \\
&\leq b_1\|u_{x_1} - u_{x_2}\|_C + b_1\|x_1 - x_2\|_C + b_1\gamma I^{1-\alpha}|y_1(\gamma t) - y_2(\gamma t)| \\
&\leq b_1 \ \Delta \ \|x_1 - x_2\|_C + b_1\|x_1 - x_2\|_C + \frac{\gamma b_1}{\Gamma(2-\alpha)}\|y_1 - y_2\|_C, \\
&\leq b_1(1 + \Delta)\|x_1 - x_2\|_C + 2\gamma b_1\|y_1 - y_2\|_C,
\end{aligned}$$

then

$$\|y_1 - y_2\|_C \leq \frac{b_1(1 + \Delta)\|x_1 - x_2\|_C}{1 - 2\gamma b_1}.$$

Hence,

$$\|x_1 - x_2\|_C \leq \frac{b_1(1 + \Delta)(b_2\gamma + 2)}{(1 - 2\gamma b_1)(1 - \ b_2)}\|x_1 - x_2\|_C,$$

$$\left(1 - \frac{b_1(1 + \Delta)(b_2\gamma + 2)}{(1 - 2\gamma b_1)(1 - \ b_2)}\right)\|x_1 - x_2\|_C \leq 0,$$

this gives $x_1 = x_2$.

In a similar manner,

$$\|y_1 - y_2\|_C \leq \frac{b_1(1 + \Delta)}{1 - 2\gamma b_1}\|x_1 - x_2\|_C \leq \frac{b_1(1 + \Delta)(b_2\gamma + 2)}{(1 - 2\gamma b_1)(1 - \ b_2)}\|y_1 - y_2\|_C,$$

$$\|y_1 - y_2\|_C \left(1 - \frac{b_1(1 + \Delta)(b_2\gamma + 2)}{(1 - 2\gamma b_1)(1 - \ b_2)}\right) \leq 0,$$

then $y_1 = y_2$.

Thus, the solution of the coupled system (2.1)-(2.2) with the fractal-fractional feedback control (1.3) is unique. Consequently, the solution $x \in C(I)$ of the problem (1.1)-(1.2) with a feedback control (1.3) is also unique. □

3.2. Hyers-Ulam stability

Definition 3.1. Let the unique solution $x \in C(I)$ of (1.1)-(1.2) with feedback control (1.3) exist. The problem (1.1)–(1.3) is Hyers-Ulam stable if $\forall \epsilon > 0$, $\exists \ \delta(\epsilon) > 0$ such that for any approximate solution $x_s \in C(I)$ of (1.1)-(1.2) with a feedback control (1.3) satisfying

$$\left| \frac{dx_s}{dt} - f(t, u_{x_s}(t), x_s(t), D^\alpha x_s(\gamma t)) \right| \leq \delta.$$

Then

$$\|x - x_s\|_C \leq \epsilon.$$

Theorem 3.2. Assuming that the hypothesis of Theorem 3.1 is satisfied, the problem (1.1)-(1.2) with feedback control (1.3) is Hyers-Ulam stable. Consequently, (1.4)-(1.5) with feedback control (1.3) is Hyers-Ulam stable.

Proof. Let us improve

$$\left|\frac{dx_s(t)}{dt} - f(t, u_{x_s}(t), x_s(t), D^{\alpha}x_s(\gamma t))\right| \leq \delta,$$

then

$$\begin{aligned} -\delta &\leq \frac{dx_s(t)}{dt} - f(t, u_{x_s}(t), x_s(t), D^{\alpha}x_s(\gamma t)) \leq \delta, \\ -\delta_1 &\leq y_s(t) - f(t, u_{x_s}(t), x_s(t), \gamma I^{1-\alpha}y_s(\gamma t)) \leq \delta_1, \quad y_s(t) = \frac{dx_s(t)}{dt}. \end{aligned}$$

Now,

$$\begin{aligned} |y(t) - y_s(t)| &= \Big| f(t, , u_x(t), x(t), \gamma I^{1-\alpha}y(\gamma t)) - y_s(t) \\ &- f(t, u_{x_s}(t), x_s(t), \gamma I^{1-\alpha}y_s(\gamma t)) + f(t, u_{x_s}(t), x_s(t), \gamma I^{1-\alpha}y_s(\gamma t))\Big| \\ &\leq \left| f(t, u_x(t), x(t), \gamma I^{1-\alpha}y(\gamma t)) - f(t, u_{x_s}(t), x_s(t), \gamma I^{1-\alpha}y_s(\gamma t))\right| \\ &+ \left| y_s(t) - f(t, u_{x_s}(t), x_s(t), \gamma I^{1-\alpha}y_s(\gamma t))\right| \\ &\leq b_1|u_x(t) - u_{x_s}(t)| + b_1|x(s) - x_s(s)| + \gamma b_1 I^{1-\alpha}|y(\gamma t) - y_s(\gamma t)| + \delta \\ &\leq b_1\|u_{x_1} - u_{x_2}\|_C + b_1\|x - x_s\|_C + \frac{(2-\alpha)\gamma b_1}{\Gamma(3-\alpha)}\|y - y_s\|_C + \delta \\ &\leq b_1\Delta\, \|x_1 - x_2\|_C + b_1\|x - x_s\|_C + 2\gamma b_1\|y - y_s\|_C + \delta. \end{aligned}$$

Hence,

$$\|y - y_s\|_C(1 - 2\gamma b_1) \leq \delta + b_1(1 + \Delta)\|x - x_s\|_C$$

and

$$\|y - y_s\|_C \leq \frac{b_1(1 + \Delta)\|x - x_s\|_C}{1 - 2\gamma b_1} + \frac{\delta}{1 - 2\gamma b_1}.$$

Now,

$$\begin{aligned} |x(t) - x_s(t)| &= \Big| x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta}y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \\ &- x_0 - \int_0^{1-\tau} h(s, x_s(s), \gamma I^{1-\beta}y_s(\gamma s))ds + \int_0^{\tau} y_s(s)ds - \int_0^{t} y_s(s)ds\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq b_2 \int_0^{1-\tau} [|x(s) - x_s(s)| + \gamma I^{1-\beta}|y(\gamma s) - y_s(\gamma s)|]ds \\
&+ \int_0^{\tau} |y(s) - y_s(s)|ds + \int_0^{t} |y(s) - y_s(s)|ds, \\
&\leq b_2\|x - x_s\|_C + \frac{b_2\gamma}{\Gamma(2-\beta)}\|y - y_s\|_C + 2\|y - y_s\|_C \\
&\leq b_2\|x - x_s\|_C + b_2\gamma\|y - y_s\|_C + 2\|y - y_s\|_C. \\
&\leq b_2\|x - x_s\|_C + (2 + b_2\gamma)\|y - y_s\|_C,
\end{aligned}
$$

and

$$\|x - x_s\|_C \leq \frac{(2 + b_2)\|y - y_s\|_C}{1 - b_2}.$$

Substituting by $\|y - y_s\|_C$, we obtain

$$
\begin{aligned}
\|x - x_s\|_C &\leq (2 + b_2\gamma)\left(\frac{b_1(1+\Delta)\|x - x_s\|_C}{1 - 2\gamma b_1} + \frac{\delta}{1 - 2\gamma b_1}\right) \\
&\leq \frac{(2 + b_2)\delta}{1 - 2\gamma b_1} + \frac{(2 + b_2)b_1(1+\Delta)\|x - x_s\|_C}{1 - 2\gamma b_1},
\end{aligned}
$$

$$\left[1 - \left(\frac{(2 + b_2)b_1(1+\Delta)}{1 - 2\gamma b_1}\right)\right] \|x - x_s\|_C \leq \frac{(2 + b_2)\delta}{1 - 2\gamma b_1},$$

and

$$\|x - x_s\|_C \leq \frac{\frac{(2 + b_2)\delta}{1 - 2\gamma b_1}}{1 - \left(\frac{(2 + b_2)b_1(1+\Delta)}{1 - 2\gamma b_1}\right)}.$$

Since

$$\frac{(2 + b_2)b_1(1+\Delta)}{1 - 2\gamma b_1} \leq 1,$$

then

$$\|x - x_s\|_C \leq \epsilon.$$

Then, the problem (1.1)-(1.2) with feedback control (1.3) is Hyers-Ulam-stable. □

3.3. Continuous dependence on y, h, and x_0

Definition 3.2. The unique solution $x \in C(I)$ of (1.1)-(1.2) constrained with (1.3) depends continuously on y, h, and x_0, and if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\max\{|y - y^*|,\ |h - h^*|,\ |x_0 - x_0^*| \leq \delta\} \ \Rightarrow \|x - x^*\|_C \leq \epsilon,$$

where x^* and y^* are the solutions of the following respectively

$$x^*(t) = x_0^* + \int_0^{1-\tau} h^*(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s))ds - \int_0^{\tau} y^*(s)ds + \int_0^t y^*(s)ds, \tag{3.1}$$

$$y^*(t) = f(t, u_{x^*}(t), x^*(t), \gamma I^{1-\alpha} y^*(\gamma t)), \tag{3.2}$$

and

$$u_x^*(t) = u_0 e^{-\lambda t^\delta} + \delta \int_0^t e^{-\lambda(t^\delta - s^\delta)}\, s^{\delta-1}\, g(s, x^*(s), \gamma I^{1-\rho} y^*(\gamma s))ds. \tag{3.3}$$

Theorem 3.3. Suppose that the hypotheses of Theorem 3.1, are satisfied; then the solution $x \in C(I)$ of (1.1)-(1.2) with feedback control (1.3) (consequently, (1.4)-(1.5) with feedback control (1.3)) depends continuously on y, h, and x_0).

Proof. Let $x(t)$ and $x^*(t)$ be the two solutions of (1.1) and (1.2), respectively, then

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big| x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \\
&- x_0^* - \int_0^{1-\tau} h^*(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s))ds + \int_0^{\tau} y^*(s)ds - \int_0^t y^*(s)ds \Big| \\
&\leq |x_0 - x_0^*| + \Big| \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - h^*(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s))ds \\
&+ \int_0^{\tau} y(s) - y^*(s)ds + \int_0^t y(s) - y^*(s)ds \Big| \\
&\leq |x_0 - x_0^*| + \int_0^{1-\tau} |h(s, x(s), \gamma I^{1-\beta} y(\gamma s)) - h(s, x(s), \gamma I^{1-\beta} y^*(\gamma s))\; ds \\
&+ \int_0^{1-\tau} |h(s, x(s), \gamma I^{1-\beta} y^*(\gamma s)) - h(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s))| ds \\
&+ \int_0^{1-\tau} |h(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s)) - h^*(s, , x^*(s), \gamma I^{1-\beta} y^*(\gamma s))| ds + 2\|y - y^*\|_C \\
&\leq |x_0 - x_0^*| + b_2 \int_0^{1-\tau} (|x(s) - x^*(s)| + \gamma I^{1-\beta} |y(\gamma s) - y^*(\gamma s)|) ds \\
&+ b_2 \int_0^{1-\tau} \|h - h^*\|_C ds + 2\|y - y^*\|_C \\
&\leq \delta + b_2 \Big(\|x - x^*\|_C + \|y - y^*\|_C \frac{\gamma}{\Gamma(2-\beta)} + \|h - h^*\|_C \Big) + 2\|y - y^*\|_C, \\
\|x - x^*\|_C &\leq \Big(\delta + b_2 \Big(\|y - y^*\|_C \frac{\gamma}{\Gamma(2-\beta)} + \|h - h^*\|_C \Big) + 2\|y - y^*\|_C \Big)(1 - b_2)^{-1}.
\end{aligned}$$

Then

$$\|x - x^*\|_C \leq \Big(\delta + b_2 \delta (2 + \frac{1}{\Gamma(2-\beta)}) + b_2 \delta \Big)(1 - b_2)^{-1} = \epsilon,$$

and, the solution $x \in C(I)$ of (1.1)-(1.2) (consequently, (1.4)-(1.5)) with feedback control (1.3) depends continuously on y, h, *and* x_0. □

3.4. Continuous dependence on the control variable u_x

Definition 3.3. The unique solution $x \in C(I)$ of (1.1)-(1.2) with feedback control (1.3) depends continuously on the control variable u_x, if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\max\{|u_x - u_{x^*}| \leq \delta\} \quad \Rightarrow ||y - y^*||_C \leq \epsilon,$$

where x^* and y^* are the solutions of

$$x^*(t) = x_0^* + \int_0^{1-\tau} h^*(s, x^*(s), \gamma I^{1-\beta} y^*(\gamma s))ds - \int_0^{\tau} y^*(s)ds + \int_0^t y^*(s)ds, \tag{3.4}$$

$$y^*(t) = f\Big(t, u_{x^*}(t), x^*(t), \gamma I^{1-\alpha} y^*(\gamma t)\Big), \tag{3.5}$$

constrained to (1.3), respectively.

Proof. Let $(x(t), y(t))$ and $(x^*(t), y^*(t))$ be the two solutions of the problem (1.1)-(1.2) constrained to (1.3), respectively, then

$$\begin{aligned}
|y - y^*| &= |f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)) - f(t, u_{x^*}(t), x^*(t), \gamma I^{1-\alpha} y^*(\gamma t))| \\
&\leq |f(t, u_x(t), x(t), \gamma I^{1-\alpha} y(\gamma t)) - f(t, u_{x^*}(t), x^*(t), \gamma I^{1-\alpha} y(\gamma t)) \\
&+ f(t, u_x(t), x^*(t), \gamma I^{1-\alpha} y(\gamma t)) - f(t, u_{x^*}(t), x^*(t), \gamma I^{1-\alpha} y^*(\gamma t))| \\
&\leq b_1 ||u_x - u_{x^*}||_C + b_1 ||x - x^*||_C + b_1 \gamma I^{1-\alpha} |y(\gamma t) - y^*(\gamma t)| \\
&\leq b_1\, \delta + b_1 ||x - x^*||_C + \frac{\gamma b_1}{\Gamma(2-\alpha)} ||y - y^*||_C, \\
&\leq b_1 \delta + b_1 ||x - x^*||_C + 2\gamma b_1 ||y - y^*||_C.
\end{aligned}$$

$$||y - y^*||_C \leq \frac{b_1 \delta + b_1 ||x - x^*||_C}{1 - 2\gamma b_1}.$$

Also,

$$\begin{aligned}
|x - x^*| &= |x_0 + \int_0^{1-\tau} h(s, x(s), \gamma I^{1-\beta} y(\gamma s))ds - \int_0^{\tau} y(s)ds + \int_0^t y_1(s)ds \\
&- x_0 - \int_0^{1-\tau} h(s, x^*(s), \gamma I^{1-\beta} y^*(s))ds + \int_0^{\tau} y^*(s)ds - \int_0^t y^*(s)ds| \\
&\leq \int_0^{1-\tau} |h(s, x(s), \gamma I^{1-\beta} y(s)) - h(s, x^*(s), \gamma I^{1-\beta} y^*(s))|ds \\
&+ \int_0^{\tau} |y(s) - y^*(s)|ds + \int_0^t |y(s) - y^*(s)|ds \\
&\leq b_2 \int_0^{1-\tau} [|x(s) - x^*(s)| + \gamma I^{1-\beta} |y(\gamma s) - y^*(\gamma s)|]ds \\
&+ \int_0^{\tau} |y(s) - y^*(s)|ds + \int_0^t |y(s) - y^*(s)|ds \\
&\leq b_2 ||x - x^*||_C + \frac{b_2 \gamma}{\Gamma(2-\beta)} ||y - y^*||_C + 2||y - y^*||_C,
\end{aligned}$$

$$\|x-x^*\|_C = \frac{(b_2\gamma+2)}{1-\ b_2}\|y-y^*\|_C.$$

Therefore

$$\|y-y^*\|_C \leq \frac{b_1\delta}{1-2\gamma b_1} + \frac{b_1(b_2\gamma+2)}{(1-2\gamma b_1)(1-\ b_2)}\|y-y^*\|_C.$$

Then

$$\|y-y^*\|_C \leq \frac{b_1\delta}{1-2\gamma b_1}\Big(1-\frac{b_1(b_2\gamma+2)}{(1-2\gamma b_1)(1-\ b_2)}\Big)^{-1} = \epsilon.$$

Thus, the solution $x \in C(I)$ of the problem (1.1)-(1.2) constrained with (1.3) (consequently, (1.4)-(1.5) with feedback control (1.3)) depends continuously on feedback control u_x. □

4. Special cases and examples

In this section, we analyze cases in the presence and absence of a control variable, addressing the integer orders issue with an illustrative example.

4.1. In the presence of control variable

Here, we pinpoint specific instances that are valuable for qualitatively analyzing the nonlocal issue of the fractional pantograph differential equation with the fractal feedback control and are essential for various models and practical applications.

- For $\lambda = 0,\ \delta \to 1$, then the fractional pantograph differential equation

$$\frac{dx}{dt} = f(t, u_x(t), x(t), D^\alpha x(\gamma t)), \quad t \in (0,1], \quad \gamma \in (0,1), \tag{4.1}$$

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^\beta x(\gamma s))ds, \quad \tau \in (0,1], \tag{4.2}$$

equipped with the fractal feedback control

$$\frac{du_x(t)}{dt} = g(t, x(t), D^\rho x(\gamma s)),\ \ u_0 = 0,\ \lambda \geq 0,\ \delta \in (0,1], \tag{4.3}$$

which gives

$$\frac{dx}{dt} = f\Big(t, \int_0^t g(s, x(s), D^\rho x(\gamma s))\, ds, x(t), D^\alpha x(\gamma t)\Big), \quad t \in (0,1], \quad \gamma \in (0,1), \tag{4.4}$$

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^\beta x(\gamma s))ds, \quad \tau \in (0,1], \tag{4.5}$$

under the assumptions of Theorem 3.3. The problem (4.4)-(4.5) depends continuously on the functions on $y,\ h$, and x_0. This case is the same result discussed in [24].

- If we put $\tau = 1$ in Eq (1.2), then the pantograph problem has the form

$$\frac{dx}{dt} = f(t, u_x(t), x(t), D^\alpha x(\gamma t)), \quad t \in (0, 1], \quad \gamma \in (0, 1), \tag{4.6}$$

with backward boundary condition

$$x(1) = x_0 \tag{4.7}$$

equipped with the fractal-fractional feedback control

$$\frac{du_x(t)}{dt^\delta} = -\lambda u_x(t) + g(t, x(t), D^\rho x(\gamma t)), \;\; u_0 = u_x(0), \; \lambda \geq 0, \; \delta \in (0, 1], \tag{4.8}$$

under the assumptions of Theorem 3.3. The pantograph backward problem (4.6)-(4.7) equipped with the fractal-fractional feedback control (4.8) depends continuously on the functions y, h, and on the parameter x_0.

4.2. *In absence of control variable*

We derive specific cases in the absence of the control variable, which are valuable for the qualitative analysis of certain functional integral equations and essential for various models and real problems.

- The pantograph problem becomes

$$\frac{dx}{dt} = f(t, x(t), D^\alpha x(\gamma t)), \quad t \in (0, 1], \quad \gamma \in (0, 1), \tag{4.9}$$

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), D^\beta x(\gamma s))ds, \qquad \tau \in (0, 1], \tag{4.10}$$

under the assumptions of Theorem 3.3. The problem (4.9)-(4.10) depends continuously on y, h, and x_0.

4.3. *Integer orders problem*

Assuming the conditions of Theorem 2.3 are met, utilizing the characteristics of the fractional order derivative [23], we derive

$$\lim_{\alpha \to 1} \frac{dx}{dt} = \lim_{\alpha \to 1} f(t, u_x(t), x(t), D^\alpha x(\gamma t)),$$

yielding

$$\frac{dx}{dt} = f(t, u_x(t), x(t), \lim_{\alpha \to 1} D^\alpha x(\gamma t))$$

and

$$\frac{dx}{dt} = f(t, u_x(t), x(t), \frac{dx(\gamma t)}{dt}), \quad \gamma \in (0, 1]. \tag{4.11}$$

Moreover,

$$\lim_{\beta\to 1} x(\tau) = x_0 + \lim_{\beta\to 1}\int_0^{1-\tau} h(s, x(s), D^{\beta}x(\gamma s))ds.$$

However,

$$|h(t, x(t), D^{\beta}x(\gamma t))| \le |a_2(t)| + b_2(|x(t)| + |D^{\beta}x(\gamma t)|),$$

thus

$$\lim_{\beta\to 1} h(t, x(t), D^{\beta}x(\gamma t)) = h(t, x(t), \lim_{\beta\to 1} D^{\beta}x(\gamma t)) = h(t, x(t), \frac{dx(\gamma t)}{dt}),$$

and

$$x(\tau) = x_0 + \int_0^{1-\tau} \lim_{\beta\to 1}(h(s, x(s), D^{\beta}x(\gamma s)))ds.$$

$$x(\tau) = x_0 + \int_0^{1-\tau} h(s, x(s), \frac{dx(\gamma s)}{ds})ds. \tag{4.12}$$

Therefore, we have established the subsequent corollary.

Corollary 4.1. Assuming the conditions of Theorem 2.3 are met, then both of the two integer order problems:

$$\begin{aligned}\frac{dx}{dt} &= f(t, u_x(t), x(t), \frac{d}{dt}x(\gamma t)), \quad t \in (0, 1], \quad \gamma \in (0, 1],\\ x(\tau) &= x_0 + \int_0^{1-\tau} h(s, x(s), \frac{d}{ds}x(\gamma s))ds, \quad \tau \in (0, 1],\end{aligned}$$

equipped with the fractal-fractional feedback control

$$\frac{du_x(t)}{dt^{\delta}} = -\lambda u_x(t) + g(t, x(t), \frac{d}{dt}x(\gamma t))), \;\; u_0 = u_x(0), \;\; \lambda \ge 0, \;\; \delta \in (0, 1],$$

and

$$\begin{aligned}\frac{dx}{dt} &= a\, u_x(t) + b\, x(t) + c\, \frac{d}{dt}x(\gamma t), \quad \gamma \in (0, 1],\\ x(\tau) &= x_0 + \int_0^{1-\tau} h(s, x(s), \frac{d}{ds}x(\gamma s))ds, \quad \tau \in (0, 1],\end{aligned}$$

equipped with the fractal-fractional feedback control

$$\frac{du_x(t)}{dt^{\delta}} = -\lambda u_x(t) + g(t, x(t), \frac{d}{dt}x(\gamma t))), \;\; u_0 = u_x(0), \;\; \lambda \ge 0, \;\; \delta \in (0, 1],$$

are guaranteed to have at least one solution $x \in C(I)$.

Corollary 4.2. Under the assumption that Theorem 2.3 is valid, the linear pantograph (Ambartsumian) problem (1.4) and (1.5) possesses at least one solution $x \in C(I)$.

Proof. By considering

$$f(t, u_x(t), x(t), D^\alpha x(\gamma t)) = a\, u_x(t) + b\, x(t) + c\, D^\alpha x(\gamma t),$$

the conclusions can be derived. □

Corollary 4.3. Let the hypothesis of Theorem 2.3 be valid; if we put $\tau = 1$ in (1.2), then the backward problem

$$\frac{dx}{dt} = f(t, u_x(t), x(t), D^\alpha x(\gamma t)), \quad t \in (0, 1], \tag{4.13}$$

with

$$x(1) = x_0, \tag{4.14}$$

with a fractal feedback control (1.3) has a solution $x \in C(I)$. Consequently, if the hypotheses of Theorem 3.1 are valid, then it has a unique solution $x \in C(I)$.

Corollary 4.4. Let the hypothesis of Corollary 4.2 be valid; if we put $\tau = 1$ in (1.5), then the backward problem

$$\frac{dx}{dt} = a\, u_x(t) + bx(t) + cD^\alpha x(\gamma t), \qquad t \in (0, 1], \tag{4.15}$$

$$x(1) = x_0, \tag{4.16}$$

with a fractal feedback control (1.3) has a solution $x \in C(I)$. Consequently, if the hypotheses of Corollary 4.2 are valid, it has a unique solution $x \in C(I)$.

Example 1. Consider the problem

$$\frac{dx}{dt} = (\frac{t}{2})^2 + u_x(t) + \frac{1}{3}x(t) + \frac{1}{3}D^{\frac{1}{2}}x(\frac{t}{2}), \qquad t \in (0, \frac{1}{4}], \tag{4.17}$$

$$x(\tau) = \frac{1}{4} + \int_0^{1-\tau} \Big(\frac{\sqrt{t}}{3} + x(s) + \frac{1}{3}D^{\frac{1}{2}}x(\frac{s}{2})\Big)ds, \tag{4.18}$$

with fractal-fractional feedback control

$$\frac{du(t)}{dt^{\frac{1}{2}}} = -0.4u(t) + e^{-\frac{7}{2}t}(\cos t + \frac{1}{3}t^3\ D^{\frac{1}{2}}x(t)). \tag{4.19}$$

Note that, this issue is a specific instance of a feedback control problem (1.1)–(1.3) as shown below

$$\alpha = \beta = \gamma = \rho = \frac{1}{2},\ x_0 = \frac{1}{4}.$$

Set

$$f(t, u_x(t), x(t), D^\alpha x(\gamma t)) = (\frac{t}{2})^2 + u_x(t) + \frac{1}{3}x(t) + \frac{1}{3}D^{\frac{1}{2}}x(\frac{t}{2}),$$

$$h(t,x(t),D^{\beta}x(\gamma t)) \;=\; \frac{\sqrt{t}}{3}+x(t)+\frac{1}{3}D^{\frac{1}{2}}x(\frac{t}{2}).$$

Thus, conditions $(i),(ii)$ are satisfied with $a^* = \frac{1}{4}$, $b_1 = b_2 = \frac{1}{3}$, $M = \frac{1}{6}$. It is evident that all conditions of Theorem 2.3 are met as follows $b_1(1+w_2) = 0.333 < 1$. Therefore, there is at least one solution $x \in C(I)$ for (4.17)–(4.19). Additionally, we have

$$\frac{b_1(1+\Delta)(b_2\gamma+2)}{(1-2\gamma b_1)(1-\ b_2)} \approx 0.1981 < 1.$$

Thus, all assumptions of Theorem 3.1 are satisfied, then the solution of the problem (4.17)–(4.19) is unique.

5. Conclusions

In some applicable differential equations, the state variable appears with a delayed argument. This kind of differential equation, which appears in many fields of science, is well-known as a delay differential equation. This equation involving fractional orders has been examined by several authors in [25, 26]. In some studies, fractional-order delay differential equations have been found to exhibit interesting dynamical behaviors that differ from their integer-order counterparts. The presence of delays introduces memory effects into the system, leading to rich and complex dynamics. Researchers have explored various analytical and numerical techniques to study the stability, bifurcations, and oscillatory behavior of such systems. The investigation of fractional delay differential equations is an active area of research with applications in physics, engineering, biology, and other fields [27].

In this research, the solvability of the fractional pantograph differential equation (1.1) with an integro-differential boundary condition (1.2) constrained to a fractal-fractional feedback control (1.3) was established. The existence of solutions to the problem (1.1)–(1.3) was proved, some sufficient conditions for the uniqueness of the solution were provided, and then the Hyers Ulam stability of the problem (1.1)–(1.3) was derived. Additionally, some continuous dependency results of the solution x on the fractional-order derivative $y(t)$, the parameter x_0, the function h, and on the control variable u_x were established. Finally, a few special cases and examples were presented.

Author contributions

All authors contributed equally and significantly to writing this article. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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