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Research article

What is the variant of fractal dimension under addition of functions with same dimension and related discussions

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Abstract: This paper attempts to explore possible box dimension of two added fractal continuous functions with the same dimension. Two interesting and meaningful results are obtained. Let $g(x)$ and $h(x)$ have the same box dimension t $(1 < t \leq 2)$, the box dimension of $g(x) + h(x)$ may or may not exist. If it exists, it can take an arbitrary real number γ satisfying $1 < \gamma \leq t$. If it does not exist, its lower and upper box dimensions can reach arbitrary different real numbers t_1 and t_2 that satisfy $1 < t_1 < t_2 < t \leq 2$. These unexpected conclusions drive us to probe into the characteristics of collection of all fractal continuous functions with the same box dimension under ordinary linear operations (scalar multiplication and addition). Following the known fractal features of some typical fractal functions such as the Weierstrass function $W_t(x)$, we classify the fractal functions into three types: consistent fractal functions, non-consistent fractal functions, and simple fractal functions. By utilizing these classifications and fractal feature descriptions, the causality of the box dimension of two added fractal functions can be partially revealed. We hope that these initial superficial discussions will lead deeper consideration on the essence of variants of fractal dimension under linear combinations of fractal functions. Moreover, these fractal features may be applied further in other fields of fractals.

Keywords: box dimension; fractal continuous function; summation of two fractal functions
Mathematics Subject Classification: 26A33, 28A80

1. Introduction

Up to now Weierstrass function $W_t(x)$ (see Figure 1) has been seemingly "unique fractal continuous function". Naturally we ask how many real fractal functions (fractional dimension lies between 1 and 2, is not equal to 1) exist. Luckily, we got a positive answer for this problem: the cardinality of all fractal continuous functions is the second category by Baire theory [1]. The fractal dimension (mainly box dimension and Hausdorff dimension) of a plane fractal curve was possibly studied firstly by Besicovitch [2]. Then numerous studies on the box dimension of fractal function graphs appeared openly, which are mainly classified into several types. (1) Self-similar set (including plane set of self-affine function graphs) as presented in Refs. [3–5] and physical applications (see, e.g., [6, 7]). (2) Some special constructed fractal functions such as the Weierstrass type and its variant Besicovitch type (see, e.g., [8–13]). (3) Some induced fractal functions (by integration or differentiation of known particular fractal curves), as shown in Refs. [14–18]. It is worth mentioning Shen's study [19]. An unresolved fractal problem is whether the Hausdorff dimension of the Weierstrass function is equal to its box dimension. Although the box dimension of the Weierstrass function has been clearly discussed, it remains unproven. Shen [19] proved that this open problem holds for all integers λ (see in Example 2.7), which may be a significant advancement in calculating fractal dimension. (4) General discussion about fractal dimensions, for instance, estimate box dimension of fractional integral about Hölder function (see, e.g., [14, 20]), some constructed fractal functions (see, e.g., [21, 22]), and fractal interpolation (see, e.g., [23]). All these papers focused on estimating fractal dimension of a fractal set or a plane curve produced by a fractal continuous function.

Write ${}^{t}D_I$ $(1 < t \leq 2)$ for the collection of all fractal continuous functions with the same box dimension t on a closed interval $I = [0, 1]$ and $\Upsilon(h, I)$ for the graph $\{(x, y),\ y = h(x),\ x \in I\}$. Since the box dimension of a plane curve produced by $h(x)$ on I, indicated by $\dim_B \Upsilon(h, I) = t$ $(1 < t \leq 2)$, is greater than 1, $h(x)$ is a real fractal function, not an ordinary function. Our concerned global properties of all fractal continuous functions mean the following aspects: What is cardinality of the collection of all real fractal functions? (discussed in Ref. [1])? What are algebraic properties of the collection of all fractal functions under linear operations of functions (scalar multiplication and addition)? An interesting fundamental problem is estimating the box dimension of sum of two fractal continuous functions, which is a topic that we are currently investigating.

Let C_I be the collection of all fractal continuous functions. $\underline{\dim}_B \Upsilon(h, I)$, $\overline{\dim}_B \Upsilon(h, I)$, and $\dim_B \Upsilon(h, I)$ indicate the lower box dimension, the upper box dimension, and the box dimension of the graph of the function $h(x)$ on $I = [0, 1]$, respectively. Actually, Wen [5] may be the first attempt to estimate the box dimension of sum of two fractal continuous functions, and reached the following conclusion.

Proposition 1.1. [5] *For any* $g(x),\ h(x) \in C_I$,
(1) *If* $\underline{\dim}_B \Upsilon(g, I) > \overline{\dim}_B \Upsilon(h, I)$, *then*

$$\overline{\dim}_B \Upsilon(g + h, I) = \overline{\dim}_B \Upsilon(g, I).$$

(2) *If* $\dim_B \Upsilon(g, I) > \dim_B \Upsilon(h, I)$, *then*

$$\dim_B \Upsilon(g + h, I) = \dim_B \Upsilon(g, I).$$

Following Proposition 1.1, one may question the possible values of $\dim_B \Upsilon(g+h, I)$ given the known $\dim_B \Upsilon(g, I)$ and $\dim_B \Upsilon(h, I)$? Wen [5] mentioned that it is a hard problem during a Chinese fractal conference.

Since Proposition 1.1 answers the case of $\dim_B \Upsilon(g + h, I)$ when $\dim_B \Upsilon(g, I) \neq \dim_B \Upsilon(h, I)$, this paper discusses the case of $\dim_B \Upsilon(g + h, I)$ while $g(x),\ h(x) \in {}^{t}D_I$. Under conventional function addition and scalar multiplication, we focus on the following problems:

Suppose that $g(x),\ h(x) \in {}^{t}D_I$, i.e., $\dim_B \Upsilon(g, I) = \dim_B \Upsilon(h, I) = t$, then

(1) Does $\dim_B \Upsilon(g + h, I)$ exist?

(2) If $\dim_B \Upsilon(g + h, I)$ exists, what is the value (if it is unique), or what is its range (if it is not unique)?

(3) If $\dim_B \Upsilon(g+h, I)$ does not exist, what are the possible values for $\underline{\dim}_B\Upsilon(g+h, I)$ and $\overline{\dim}_B\Upsilon(g+h, I)$?

(4) What is the essential feature to affect $\dim_B \Upsilon(g + h, I)$? How describe this fractal feature?

This paper takes an initial study and arrives at some interesting results.

For convenience, some notations are listed as follows:

(1) ${}^{t_2}_{t_1}D_I$, all fractal functions on closed interval $I = [0, 1]$ with the lower and upper box dimensions t_1 and t_2 ($1 \le t_1 \le t_2 \le 2$), respectively. When $t_1 = t_2 = t$, this notation indicates ${}^tD_I$.

(2) ${}^tD_I^{cs}$, all consistent fractal functions with the box dimension t (presented in Definition 4.1).

(3) ${}^tD_I^{nc}$, all non-consistent fractal functions with the box dimension t (presented in Definition 4.3).

(4) ${}^tD_I^{sp}$, all simple fractal functions with the box dimension t (presented in Definition 4.5).

(5) $N^{[a,b]}_{h,\delta}$ represents the smallest number of meshes of diameter at most δ which can cover $\Upsilon(h, [a, b])$, particularly, $N_{h,\delta} = N^I_{h,\delta} = N^{[0,1]}_{h,\delta}$.

(6) The oscillation of a function $h(x)$ on a closed interval $[a, b]$ is defined by

$$R_h[a, b] = R_{h,[a,b]} = \sup_{x,y\in[a,b]} |h(x) - h(y)|.$$

(7) C, C_1, and C_2 denote some absolute constants and may be different real numbers even in the same line in different environments.

2. Preliminaries

In this preparatory section, there are three aspects involved: the box dimension of fractal functions, the covering number of a function graph $N_{h,\delta}$, and two examples of typical fractal functions.

2.1. Box-counting dimension

First, we present precise mathematical definitions by the covering number $N_{h,\delta}$ for the lower and upper box dimensions of a function $h(x)$ on its domain $I = [0, 1]$.

Definition 2.1. [4, 5] *The lower and upper box dimensions of a function $h(x)$ on $I = [0, 1]$ are defined, respectively,*

$$\underline{\dim}_B\Upsilon(h, I) = \underline{\lim}_{\delta\to 0} \frac{\log N_{h,\delta}}{-\log \delta}$$

and

$$\overline{\dim}_B\Upsilon(h, I) = \overline{\lim}_{\delta\to 0} \frac{\log N_{h,\delta}}{-\log \delta}.$$

If both the lower and upper box dimensions are equal to each other, then the box dimension is defined as

$$\dim_B \Upsilon(h, I) = \underline{\dim}_B\Upsilon(h, I) = \overline{\dim}_B\Upsilon(h, I).$$

From Definition 2.1, some results hold trivially.

Lemma 2.2. *Assume $g(x) \in C_I$ and its box dimension exists. The following statements hold.*

(1) *For a constant function $g(x) = c$,* $\dim_B \Upsilon(g, I) = \dim_B \Upsilon(c, I) = 1$.

(2) $\dim_B \Upsilon(cg, I) = \dim_B \Upsilon(g, I)$ $(c \neq 0)$.

(3) $1 \leq \dim_B \Upsilon(g, I) \leq 2$.

2.2. Coving number $N_{h,\delta}$

From Definition 2.1, $\dim_B \Upsilon(h, I) = s$ means that $N_{h,\delta}$ obeys power law as $\delta \to 0$ (see [1,4,5]), i.e.,

$$N_{h,\delta} \simeq C\delta^{-s}$$

for some absolute constant C. Hence, the following assertions are trivial.

Lemma 2.3. [1,4,5] *The following assertions hold as $\delta \to 0$.*

(1) $C_1\delta^{-s} \leq N_{h,\delta} \leq C_2\delta^{-s}$, *if and only if* $\dim_B \Upsilon(h, I) = s$.

(2) *If $N_{h,\delta} \leq C\delta^{-s}$, then* $\overline{\dim}_B\Upsilon(h, I) \leq s$.

(3) *If $N_{h,\delta} \geq C\delta^{-s}$, then* $\underline{\dim}_B\Upsilon(h, I) \geq s$.

The calculation of $N_{h,\delta}$ is the key to estimate $\dim_B \Upsilon(h, I)$. Given $0 < \delta < 1/2$, assume that the interval $I = [0, 1]$ is divided into m subintervals by δ. Write

$$\Delta_k = [k\delta, (k + 1)\delta], \ k = 0, 1, 2, \cdots, m - 1.$$

Here, $m = [\delta^{-1}]$ denotes the largest integer less than or equal to δ^{-1}. Sometimes suppose $m\delta^{-1} = 1$ without loss of generality. Then the estimation of $N_{h,\delta}$ can be converted into the oscillation on these subintervals. The following assertion is adopted from Refs. [4,5].

Lemma 2.4. [4,5] *The range of $N_{h,\delta}$ can be estimated as follows.*

$$\delta^{-1}\sum_{i=0}^{m-1} R_h[i\delta, (i + 1)\delta] \leq N_{h,\delta} \leq 2m + \delta^{-1}\sum_{i=0}^{m-1} R_h[i\delta, (i + 1)\delta].$$

For arbitrary given two fractal functions $h(x)$ and $g(x)$, we present an estimation of $N_{h+g,\delta}$, which is adopted from our recent paper [1].

Lemma 2.5. [1] *Let $h(x),\ g(x) \in C_I$. Then*

$$\max\{0, \left|N_{h,\delta} - N_{g,\delta}\right| - 2\delta^{-1}\} \leq N_{h+g,\delta} \leq 2\delta^{-1} + N_{h,\delta} + N_{g,\delta}.$$

The evaluation of the upper box dimension is always applied repeatedly. We present a general estimation of the upper box dimension.

Theorem 2.6. *The following two statements hold.*

(1) *If $h(x) \in {}^sD_I,\ g(x) \in {}^tD_I$ $(1 \leq s, t \leq 2)$, then*

$$\overline{\dim}_B\Upsilon(h + g, I) \leq \max\{\dim_B \Upsilon(h, I), \dim_B \Upsilon(g, I)\}.$$

(2) *If $h(x) \in {}^{s_1}_{s_2}D_I,\ g(x) \in {}^{t_1}_{t_2}D_I$, and $s_2 > t_1$, then*

$$\underline{\dim}_B\Upsilon(h + g, I) \geq s_2.$$

Proof. (1) For any $\delta > 0$, we consider the limit as $\delta \to 0$. It is assumed without loss of generality that $\delta \le 1/2$. Given that $h(x) \in {}^{s}D_I, g(x) \in {}^{t}D_I$ $(1 \le s, t \le 2)$, Lemma 2.3 shows that

$$N_{h,\delta} \le C_1\delta^{-s},\ \ N_{g,\delta} \le C_2\delta^{-t}$$

for some absolute constants C_1 and C_2. Then, by Lemma 2.5,

$$\begin{aligned} N_{h+g,\delta} &\le 2\delta^{-1} + N_{h,\delta} + N_{g,\delta} \\ &\le 2\delta^{-1} + C_1\delta^{-s} + C_2\delta^{-t} \\ &\le \left(2\delta^{\max\{s,t\}-1} + C_1\delta^{\max\{s,t\}-s} + C_2\delta^{\max\{s,t\}-t}\right)\delta^{-\max\{s,t\}} \\ &\le C\delta^{-\max\{s,t\}}, \end{aligned}$$

which, in combination with Lemma 2.3, leads to (1) in Theorem 2.6.

(2) Since $h(x) \in {}^{s_1}_{s_2}D_I$ and $g(x) \in {}^{t_1}_{t_2}D_I$, for an arbitrary positive $\epsilon < 1$, there exists $\hat{\delta} > 0$ such that when $\delta \le \hat{\delta}$,

$$N_{h,\delta} > \delta^{s_2} - \frac{1}{2}\epsilon,\ \text{and}\ N_{g,\delta} < \delta^{t_1} + \frac{1}{2}\epsilon,$$

which, by Lemma 2.5, leads to

$$\begin{aligned} N_{h+g,\delta} &\ge \left|N_{h,\delta} - N_{g,\delta}\right| - 2\delta^{-1} \\ &\ge \delta^{-s_2} - \delta^{-t_1} - 2\delta^{-1} - \epsilon \\ &\ge \left(1 - \delta^{s_2-t_1} - 2\delta^{s_2-1} - \epsilon\delta^{s_2}\right)\delta^{-s_2} \\ &\ge C\delta^{s_2}, \end{aligned}$$

for sufficiently small δ. Hence, from Lemma 2.3, we conclude $\underline{\dim}_B \Upsilon(h+g, I) \ge s_2$.

Obviously, Proposition 1.1 is a particular case of Theorem 2.6. Of course, we need to use the analogous method of Theorem 2.6 to make an induction of Proposition 1.1. □

2.3. *Two typical fractal functions*

In view of fractals, the Weierstrass function may be regarded as the first fractal function. For consideration of box dimension, the extreme cases were constructed, called the Besicovitch function. Here, these two typical fractal functions are introduced for later applications.

Example 2.7. [4, 5] *Weierstrass function $W_t(x)$ (see Figure 1).*

Let $1 < t < 2, \lambda > 1$. The Weierstrass function with parameter t is defined as

$$W_t(x) = \sum_{j=1}^{\infty} \lambda^{(t-2)j} \sin(\lambda^j x).$$

For a large enough real number λ, we know that

$$\dim_B \Upsilon(W_t, I) = t.$$

This means

$$W_t(x) \in {}^{t}D_I.$$

Especially, for any subinterval $[c, d] \subset I = [0, 1]$,

$$\dim_B \Upsilon(W_t, [c, d]) = t.$$

The graph of the Weierstrass function $W_t(x)$ looks like Figure 1.

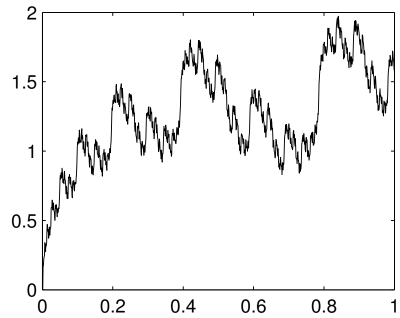

Figure 1. Weierstrass function $W_t(x)$.

Remarks to Example 2.7. From Example 2.7, we can find a real fractal continuous function with a given box dimension t $(1 < t < 2)$. Especially, at any point $x \in [0, 1]$, its box dimension is still t.

Now we give another example the Besicovitch function (Weierstrass type) whose box dimension might not exist on I.

Example 2.8. [8] *Besicovitch function $BW(x)$ provided by Barański.*

Let a periodic Lipschitz function $h(x)$ be monotonic on a subinterval I of its domain, which satisfies

$$|h(x) - h(y)| > C|x - y|$$

for arbitrary $x,\ y \in I$ and some $C > 0$. Given any two positive sequences $\{b_n\}$ and $\{c_n\}$ satisfying that $b_{n+1}/b_n \to 0,\ \ c_{n+1}/c_n \to \infty$ as $n \to \infty$ and $\theta_n \in R$, the Besicovitch function $BW(x)$ is defined by

$$BW(x) = \sum_{n=1}^{\infty} b_n h(c_n x + \theta_n).$$

For arbitrary real numbers L_1 and L_2 with $1 \leq L_1 \leq L_2 \leq 2$, there is always a Besicovitch type function $BW(x)$ satisfying that

$$\dim_H \Upsilon(BW, I) = \underline{\dim}_B \Upsilon(BW, I) = L_1, \qquad \overline{\dim}_B \Upsilon(BW, I) = L_2,$$

where $\dim_H \Upsilon(BW, I)$ indicates Hausdorff dimension of this function $BW(x)$. From this point, if we take two real numbers t_1 and t_2 satisfying $1 \leq t_1 < t_2 \leq 2$, then we can always find a fractal function $BW(x)$ that satisfies $BW(x) \in {}_{t_1}^{t_2}D_I$, which exhibits that there always exists a continuous function taking any different lower and upper box dimensions.

3. Box dimension of summation

Our interesting goal is to study the algebraic properties under function addition and scalar multiplication, or further investigate the algebraic structure of all fractal continuous functions with the same fractal dimension.

For given two fractal functions $h(x)$ and $g(x)$ with the same box dimension, we discuss possible values of $\dim_B \Upsilon(h+g, I)$. We discover that the box dimension of $h(x)+g(x)$ maybe exist or not even when $\dim_B \Upsilon(h, I) = \dim_B \Upsilon(g, I)$. Actually, the following theorem exhibits that the value of the box dimension of their summation $h(x)+g(x)$ can be arbitrary.

Theorem 3.1. *Given a real number t $(1 < t \leq 2)$, for an arbitrary s satisfying $1 < s < t$, there exist two fractal functions $h(x),\ g(x) \in {}^tD_I$ $(1 < t \leq 2)$ such that*

$$\dim_B \Upsilon(h+g, I) = s \in (1, t).$$

That is to say, assume $h(x),\ g(x) \in {}^tD_I$ $(1 < t \leq 2)$. If $\dim_B \Upsilon(h+g, I)$ exists, then it can arrive at any real number between one and t.

Proof. Let $h(x) \in {}^tD_I$ and set $g(x) = -h(x) + W_s(x)$, where $W_s(x)$ is the Weierstrass function with parameter s.

Firstly, we need to verify that $g(x) \in {}^tD_I$. From Example 2.7, we know $\dim_B \Upsilon(W_s, I) = s$. From Lemma 2.2,

$$\dim_B \Upsilon(-h, I) = \dim_B \Upsilon(h, I) = t > s = \dim_B \Upsilon(W_s, I).$$

By Proposition 1.1,

$$\dim_B \Upsilon(g, I) = \dim_B \Upsilon(-h + W_s, I) = \max\{\dim_B \Upsilon(-h, I), \dim_B \Upsilon(W_s, I)\} = t,$$

which means $g(x) = -h(x) + W_s(x) \in {}^tD_I$.

Secondly, $h(x) + g(x) = W_s(x)$, and

$$\dim_B \Upsilon(h+g, I) = \dim_B \Upsilon(W_s, I) = s\ (1 < s < t \leq 2).$$

So, $\dim_B \Upsilon(h+g, I)$ reaches any given number s $(1 < s < t \leq 2)$. Theorem 3.1 is done. □

Remarks to Theorem 3.1. Two points are emphasized.

(1) If $h(x), g(x) \in {}^tD_I$ $(1 < t \leq 2)$, and $\dim_B \Upsilon(h+g, I)$ exists, then $\dim_B \Upsilon(h+g, I)$ can reach at most t. By Theorem 2.6, we have

$$\overline{\dim}_B \Upsilon(h+g, I) \leq t.$$

So, t is the best upper boundary of $\dim_B \Upsilon(h+g, I)$.

(2) For any $1 < t \leq 2$, there exist two fractal functions $h(x),\ g(x) \in {}^tD_I$ such that $h(x)+g(x) \in {}^\gamma D_I$ for any γ $(1 < \gamma < t)$.

Naturally we ask what is the possible value of the lower and upper box dimension of summation whenever its box dimension does not exist? The following Theorem 3.2 answers.

Theorem 3.2. *Given a real number t $(1 < t \leq 2)$, for an arbitrary pair $\langle s_1, s_2 \rangle$ satisfying $1 < s_1 < s_2 < t \leq 2$, there exist two fractal functions $h(x),\ g(x) \in {}^tD_I$ such that $h(x)+g(x) \in {}^{s_2}_{s_1}D_I$. That is to say, if the box dimension of $h(x)+g(x)$, $\dim_B \Upsilon(h+g, I)$, does not exist, then its lower and upper box dimension can take any different values between one and t, i.e.,*

$$1 < \underline{\dim}_B \Upsilon(h+g, I) = s_1 < s_2 = \overline{\dim}_B \Upsilon(h+g, I) < t \leq 2.$$

Proof. Let $h(x) \in {}^tD_I$, and $g(x) = -h(x) + BW(x)$, where $BW(x)$ is a specified Besicovitch type function. From Example 2.9, we just select two appropriate sequences $\{b_n\}$ and $\{c_n\}$ such that

$$1 < \underline{\dim}_B \Upsilon(BW(x), I) = s_1 < \overline{\dim}_B \Upsilon(BW(x), I) = s_2 < t,$$

then this specified $BW(x) \in {}^{s_2}_{s_1}D_I$.

Now we verify $g(x) \in {}^tD_I$. Assertion (1) in Theorem 2.6 tells us

$$\overline{\dim}_B \Upsilon(g, I) = \overline{\dim}_B \Upsilon(-h + BW, I) \le \max\left\{\dim_B \Upsilon(-h, I), \overline{\dim}_B(WB, I)\right\} = t.$$

Since $t > s_2$, assertion (2) in Theorem 2.6 tells us

$$\underline{\dim}_B \Upsilon(g, I) = \underline{\dim}_B \Upsilon(-h + BW, I) \ge t.$$

These two inequalities show that $\dim_B \Upsilon(g, I) = t$, or $g(x) \in {}^tD_I$.

However, $h + g = BW(x) \in {}^{s_2}_{s_1}D_I$, which accomplishes Theorem 3.2. □

Remarks: From Theorems 3.1 and 3.2, we obtain surprising conclusions.

(1) For any two fixed numbers t and s with $1 < s < t \le 2$, there exist two fractal continuous functions $h(x), g(x) \in {}^tD_I$ such that $h(x) + g(x) \in {}^sD_I$.

(2) For any three fixed numbers t, s_1, and s_2 with $1 < s_1 < s_2 < t \le 2$, there exist two fractal continuous functions $h(x)$ and $g(x)$ with $h(x), g(x) \in {}^tD_I$ such that $h + g \in {}^{s_2}_{s_1}D_I$.

These two conclusions show that we cannot talk about the algebraic structure (linear space or linear manifold) of ${}^tD_I$ under conventional function addition and scale multiplication, because $h(x) + g(x) \notin {}^tD_I$ might hold while $h(x), g(x) \in {}^tD_I$. In our recent paper [1], we found a unique subspace, ${}^1D_I$. Hence, we try to know what causes the wide range of $\dim_B \Upsilon(h + g, I)$, such as Theorems 3.1 and 3.2. Or further, explore essential characteristics of fractal functions that controlling the box dimension of summation. This is our motivation.

4. Description of fractal feature

We are motivated to try understanding fractal features on fractal function graphs from the range of $\dim_B \Upsilon(h + g, I)$ after the box dimensions of $h(x)$ and $g(x)$ are known. We naturally ask, why a fractal function behaves with fractal features? What causes the wide range of $\dim_B \Upsilon(f + g, I)$ even when $h(x),\ g(x) \in {}^tD_I$ (see Theorems 3.1 and 3.2)?

According to our viewpoint, there are three aspects that can display the characteristics of fractal functions globally or locally:

(1) box dimension;

(2) total variation;

(3) length of curve.

For $h(x) \in {}^tD_I$, we know $\dim_B \Upsilon(h, I)$ is only one way to totally describe the fractal feature of the function $h(x)$ on I. Then, what are the pointwise property of a real fractal function $h(x)$ (local fractal features) is? It means we need to investigate the fractal behavior at a given point $a \in I = [0, 1]$. For $h(x) \in {}^tD_I$, its total variation is unbounded on I. However, there is possibly a subinterval $[c, d] \subseteq [0, 1]$ where its total variation is bounded. Similarly, in some subintervals of I, the box dimension of $h(x)$

may be less than the total dimension. On the basis of this idea, we try to give some definitions to describe fractal features locally and/or globally.

Definition 4.1. *Let* $h(x) \in {}^tD_I$ $(t \in [1,2])$. *If* $\dim_B \Upsilon(h,[c,d]) = t$ *for all subinterval* $[c,d] \subseteq I$*, then* $h(x)$ *is called a consistent fractal function. A set of all these fractal functions is denoted by* ${}^tD_I^{cs}$.

Remarks to Definition 4.1. Notice the following facts.

(1) Definition 4.1 presents the global fractal behavior of a function, which says that a consistent fractal function exhibits the same fractal behavior (by box dimension) everywhere in I.

(2) The consistent behavior include box dimension and total variation. Actually, if the box dimension is greater than one, the total variation of fractal function is naturally unbounded.

(3) For an arbitrary closed subinterval $[c,d] \subseteq I$, $\dim_B \Upsilon(h,[c,d]) = t$, or we say that the fractal function has the same box dimension at each point of I, which exhibits consistent box dimension in the entire interval I.

(4) For an arbitrary closed subinterval $[c,d] \subseteq I$, the total variation of $h(x)$ is unbounded on $[c,d]$, called uniformly unbounded variation, which shows the uniform property of unbounded variation in the entire interval I.

(5) The Weierstrass function $W_t(x)$ in Example 2.7 is a consistent fractal function, i.e., $W_t(x) \in {}^tD_I^{cs}$ for all $t \in (1,2)$. From the graph of the Weierstrass function (see Figure 1), its consistent fractal feature is exhibited completely.

Except for the Weierstrass function $W_t(x)$, another example of a consistent fractal function is $\Phi_{XZ}(x) \in {}^2D_I$.

Example 4.2. [13] $\Phi_{XZ}(x)$ *with* $\dim_B \Upsilon(\Phi_{XZ}(x), I) = 2$.

Let $\lambda > 1,\ b > a > 1$. Define

$$\varphi(x) = 2x, \quad 0 \le x \le 1/2, \quad \varphi(-x) = \varphi(x) \quad \text{and} \quad \varphi(x+1) = \varphi(x).$$

The graph of the following function

$$\Phi_{XZ}(x) = \sum_{n\ge1} \lambda^{-n^a} \varphi(\lambda^{n^b} x), \quad 0 \le x \le 1$$

has the box dimension two,

$$\dim_B \Upsilon(\Phi_{XZ}, I) = 2.$$

Thus, we know $\Phi_{XZ}(x) \in {}^2D_I$. Particularly, for any subinterval $[c,d] \subset I$, $\dim_B \Upsilon(\Phi_{XZ},[c,d]) = 2$.

If the fractal behaviours are broken in some subinterval of I, or or more precisely at some point $\xi \in I$, we consider these variations as local features. Some fractal behavior hold at a point $\xi \in I$, which always means that they hold on some subinterval $[c,d] \subset I$, not really a point $\xi \in [c,d]$.

The local fractal dimension might be stated as follows, contrasting to the global fractal dimension.

Definition 4.3. *Let* $h(x) \in {}^tD_I$ $(\ 1 \le t \le 2\ )$ *be a function with unbounded variation everywhere on* I*. If there exists some subinterval* $[c,d] \in I$ *such that*

$$\dim_B \Upsilon(h,[c,d]) = s \in [1,t),$$

then $h(x)$ is called a non-consistent fractal function. A set of all these fractal functions is indicated by ${}^{t}D_I^{nc}$.

Remarks to Definition 4.3. Notice the following facts.

(1) Notice that $\dim_B \Upsilon(h, I) = t$ is a total feature (not global), since there may exist a subinterval $[c, d] \subset I = [0, 1]$ such that $\dim_B \Upsilon(h, I) = s < t$.

(2) Definition 4.3 states the local behaviors about the box dimension.

(3) For all subintervals $[c, d] \subset I$, the function $h(x)$ is always of unbounded variation on $[c, d]$ (global behavior), which means that the unbounded variation property keeps unchanged. However, the fractal behavior about the box dimension is broken in some subinterval or saying at some point, which exhibits the local property about the box dimension.

Example 4.4. *A non-consistent fractal continuous function* $f_{nc}(x)$

Take real numbers t and s satisfying $1 < t < s < 2$. Construct a fractal function by Weierstrass function $W_t(x)$,

$$f_{nc}(x) = \begin{cases} W_s(x), & \text{for } x \in [-1, 0], \\ W_t(x), & \text{for } x \in [0, 1]. \end{cases}$$

This function $f_{nc}(x)$ is continuous on $[-1, 1]$, and belongs to ${}^{s}D_I^{nc}$ (note $t < s$). In any subinterval $[c, d] \subset [-1, 1]$, the total variation of $f_{nc}(x)$ is unbounded. So it is a non-consistent fractal function by Definition 4.3.

Remarks to Example 4.4. Example 4.4 is constructed for Definition 4.3. It seems to be an unnatural function. Can somebody construct this type of non-consistent fractal function within the "same framework"?

Definition 4.3 describes one of the local fractal behaviors, the box dimension is broken somewhere. Actually, another fractal feature, total variation, can be broken at some point in I. We present Definition 4.5 to describe this local fractal feature.

Definition 4.5. *If* $h(x) \in {}^{t}D_I (1 \le t \le 2)$*, and its total variation is bounded on some subintervals, then* $h(x)$ *is called a simple fractal function. A set of all these simple fractal functions is denoted by* ${}^{t}D_I^{sp}$.

Remarks to Definition 4.5. Note the following facts.

(1) Definition 4.5 presents another local fractal behavior (about the total variation).

(2) A simple function has two characterizations:

(a) $\dim_B \Upsilon(h, I) = t$;

(b) total variation of the function is bounded on some subinterval $[c, d] \subset I$.

(3) $\dim_B \Upsilon(h, I) = t$ is an entire (not surely consistent) feature, but we do not say the box dimension unchanged everywhere. Obviously, $\dim_B \Upsilon(h, [c, d]) = 1$ if the total variation of $h(x)$ on the subinterval $[c, d] \subset I$ is bounded.

(4) If the function with bounded variation satisfies $\dim_B \Upsilon(h, I) = 1$, it is actually an ordinary function (obviously, its length is finite), which is a member of ${}^{1}D_I^{sp}$.

Example 4.6. [24] *A simple fractal function* Φ_{zh}.

The graph of a function $\Phi_{zh}(x)$ adopted from Ref. [24] is exhibited in Figure 2, which is a simple fractal function, $\Phi_{zh}(x) \in {}^{1}D_I^{sp}$. Obviously, the total variation of the function $\Phi_{zh}(x)$ is bounded in any subinterval $[c, d] \subset [0, 1)$, and it is unbounded for the total variation of $\Phi_{zh}(x)$ in any subinterval

$[a, 1] \subset [0, 1]$. Equivalently speaking, the fractal feature about unbounded total variation for $\Phi_{zh}(x)$ is broken at the right end point of the interval $I = [0, 1]$. Hence, we call this type of function a simple fractal function. It looks "simple".

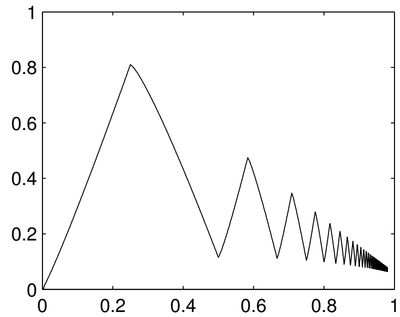

Figure 2. Simple fractal function $\Phi_{zh}(x)$.

Remarks to Example 4.6. Note that this fractal function is constructed from "the same framework". It looks naturally like fractal function. This is why we think that Example 4.4 looks like an unnatural fractal function in contrast to Example 4.6, since it is pieced up. These kinds of fractal functions of unbounded total variation with one dimension can be found more in Refs. [15, 24, 25].

At the end, we present a particular example of the Devil's staircase. Although it is regarded as a fractal curve, by our mathematical definition, it is actually an ordinary function since its box dimension is one and its total variation is bounded on $[0, 1]$, it has a finite length.

Example 4.7. [26] *Devil's staircase* $f_{Devil}(x)$ *(see Figure 3).*

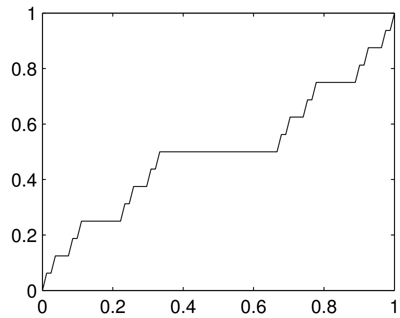

Figure 3. Devil's staircase.

Remarks to Example 4.7. Devil staircase is a simple ordinary function, not a simple fractal function as its total variation is bounded everywhere on $[0, 1]$ (its length is finite). Note the difference.

5. Application of fractal feature

Theorems 3.1 and 3.2 reveal that the box dimension of summation of two fractal functions in ${}^sD_I$ might be any size. However, the box dimension of summation might remain unchanged under certain conditions. Naturally, we ask What are the conditions preserving summation remains in ${}^sD_I$? We have explored the fractal characteristics of a fractal function in the previous section, which can help us investigate the condition that preserves summation in ${}^{s}D_I$.

Here we discuss the summation of two fractal functions from different set (${}^{t}D_I^{cs}$, ${}^{t}D_I^{nc}$, and ${}^{t}D_I^{sp}$), which may help us understand fractal features and discover conditions preserving summation in ${}^{t}D_I$ to some extent.

Theorem 5.1. *Assume $h(x) \in {}^{s}D_I^{cs}$, and $g(x) \in {}^{s}D_I^{nc}$ with $1 < s \leq 2$. It holds*

$$\dim_B \Upsilon(h+g, I) = s.$$

Proof. Since $h(x) \in {}^{s}D_I^{cs}$, and $g(x) \in {}^{s}D_I^{nc}$, by Theorem 2.6,

$$\overline{\dim}_B \Upsilon(h+g, I) \leq \max\{\dim_B \Upsilon(h, I), \dim_B \Upsilon(g, I)\} = s. \tag{5.1}$$

On the other side, since $g(x) \in {}^{s}D_I^{nc}$, by Definition 4.3, there exists a real number t with $1 < t < s \leq 2$ and a subinterval $[c, d] \subset I = [0, 1]$ such that

$$\dim_B \Upsilon(g, [c, d]) = t < s,$$

which indicates that there exists an absolute C_2 such that (from Lemma 2.3)

$$N_{g,\delta}^{[c,d]} \leq C_2 \delta^{-t}.$$

Since $h(x) \in {}^{s}D_I^{cs}$, from Definition 4.1 and Lemma 2.3, there exists an absolute C_1 such that

$$N_{h,\delta}^{[c,d]} \geq C_1 \delta^{-s}.$$

Note that $[c, d] \subset I = [0, 1]$. Applying Lemmas 2.3 and 2.5, for sufficiently small δ, we have

$$\begin{aligned} N_{h+g,\delta} &= N_{h+g,\delta}^{[0,1]} \geq N_{h+g,\delta}^{[c,d]} \\ &\geq \left| N_{h,\delta}^{[c,d]} - N_{g,\delta}^{[c,d]} \right| - 2\delta^{-1} \\ &\geq C_1 \delta^{-s} - C_2 \delta^{-t} - 2\delta^{-1} \\ &= \left(C_1 - C_2 \delta^{s-t} - 2\delta^{s-1} \right) \delta^{-s} \\ &\geq C \delta^{-s}. \end{aligned}$$

According to Lemma 2.3, it can be concluded that,

$$\underline{\dim}_B \Upsilon(h+g, I) \geq s. \tag{5.2}$$

Combination (5.1) with (5.2) leads to $\dim_B \Upsilon(h+g, I) = s$, which indicates $h(x) + g(x) \in {}^{s}D_I$. Theorem 5.1 is done. □

Theorem 5.2. *Assume $h(x) \in {}^{s}D_I^{cs}$ and $g(x) \in {}^{s}D_I^{sp}$ with $1 < s \leq 2$. It holds*

$$\dim_B \Upsilon(h+g, I) = s, \ 1 < s \leq 2.$$

Proof. Because $g(x) \in {}^sD_I^{sp}$, by Definition 4.5, there exists a subinterval $[c, d] \subset I = [0, 1]$ such that the total variation of $g(x)$ on the subinterval $[c, d]$ is bounded, which means $\dim_B \Upsilon(g, [c, d]) = 1$. Analogical inductions like Theorem 5.1 lead to $\dim_B \Upsilon(h + g, I) = s$. Theorem 5.2 is done. □

From our investigation of fractal characteristics, some fractal features are described by Definitions 4.1, 4.3, and 4.5. Applying these fractal features, we discuss possible cases of fractal functions that make the summation is the original box dimension. We hope these conclusions (Theorems 5.1 and 5.2) help us understand the range of the box dimension of summation (Theorems 3.1 and 3.2), and understand when the summation keeps the identical box dimension, i.e., $\dim_B \Upsilon(h+g, I) = s$ holds while $\dim_B \Upsilon(h, I) = \dim_B \Upsilon(g, I) = s$.

6. Conclusions

Under conventional function addition and scale multiplication, the investigation of the algebraic properties of fractal functions is our original motivation. To begin this investigation, we need to know how many real fractal functions exist. The study [1] shows that the collection of fractal continuous functions with the identical box dimension t, indicated by ${}^tD_I$, is the second category according to Baire theory. So it is meaningful to discuss the algebraic properties of fractal functions. For the known result Proposition 1.1 (or see [5]), we naturally focus on the set of fractal functions with the same box dimension ${}^tD_I$. As a result, let $h(x)$, $g(x) \in {}^tD_I$ and then $\dim_B \Upsilon(h + g, I)$ may exist or not. If $\dim_B \Upsilon(h + g, I)$ exists, then $\dim_B \Upsilon(h + g, I)$ can be any real number between one and t (Theorem 3.1). If $\dim_B \Upsilon(h + g, I)$ does not exist, then $h(x) + g(x) \in {}^{t_2}_{t_1}D_I$ for any real number t_1 and t_2 satisfying $1 < t_1 < t_2 < t \leq 2$ (Theorem 3.2). These surprising results drive us to probe the characteristics of fractal functions. According to the fractal features of known fractal functions, we classify fractal functions into three types: consistent fractal functions, non-consistent fractal functions, and simple fractal functions. This classification helps us partially understanding the essence of fractal functions, which may be applied in other fields of fractals.

Author contributions

Ruhua Zhang and Wei Xiao: Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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